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A Production Lot-Size Model for Perishable Items Under Two Level Trade Credit Policy for a Retailer with a Powerful Position in a Supply Chain System

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Abstract This paper investigates a production lot-size inventory model for perishable items under two levels of trade credit for a retailer to reflect the supply chain management situation. We assume that the retailer maintains a powerful position and can obtain full trade credit offered by supplier yet retailer just offers the partial trade credit to customers. Under these conditions, retailer can obtain the most benefits. Then, we investigate the retailer's inventory policy as a cost minimization problem to determine the retailer's inventory policy. A rigorous mathematical analysis is used to prove that the annual total variable cost for the retailer is convex, that is, unique and global-optimal solution exists. Mathematical theorems are developed to efficiently determine the optimal ordering policies for the retailer. The results in this paper generalize some already published results. Finally, numerical examples are given to illustrate the theorems and obtain a lot of managerial phenomena.

Keywords EPQ · Deteriorating items · Partial trade credit

1 Introduction

Achieving effective coordination among suppliers and retailers has become a pertinent research issue in supply chain management. A profitable decision policy between a supplier and the retailers can be characterized by an agreement on the trade credit scenario such as permissible delay in payments. The trade credit financing produces two benefits to the supplier: (1) it should attract new customers who consider it to be a type of price reduction; and (2) it should cause a reduction

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in sales outstanding, since some established customers will pay more promptly in order to take advantage of trade credit more frequently. In real life business via share marketing, trade credit financing becomes a powerful tool to improve sales and profits in an industry.

In this regard, a number of research papers appeared which deal with the EOQ problem under the condition of permissible delay in payments. Goyal [1] is the first person to consider the economic order quantity (EOQ) inventory model under the condition of trade credit. Shinn et al. [2] extended Goyal's [1] model and considered quantity discount for freight cost. Chung [3] presented the DCF (discounted cash flow) approach for the analysis of the optimal inventory policy in the presence of trade credit. Teng [4] assumed that the selling price is not equal to the purchasing price to modify Goyal's model [1]. Chung and Huang [5] extended this problem within the economic production quantity (EPQ) framework and developed an efficient procedure to determine the retailer's optimal ordering policy.

However, the perishability of goods is a realistic phenomenon. In real-life situations there are certain products like volatile liquids, medicines, food stuff, blood bank, materials, etc., in which the rate of deterioration due to vaporization, damage, spoilage, dryness etc. is very large. Therefore, the loss due to deterioration should not be ignored. Aggarwal and Jaggi [6] developed inventory model with an exponential deterioration rate under the condition of permissible delay in payments. Chu et al. [7] and Chung et al. [8] extended Goyal's [1] model to allow for deteriorating items. Chang et al. [9] and Chung and Liao [10] dealt with the problem of determining the EOQ for exponentially deteriorating items under permissible delay in payments depending on the ordering quantity. There are several interesting and relevant papers related to trade credit such as Jamal et al. [11], Arcelus et al. [12], Abad and Jaggi [13], and their references.

All the above articles assumed that the supplier would offer the retailer a delay period and the retailer could sell the goods and accumulate revenue and earn interest within the trade credit period. They implicitly assumed that the customer would pay for the items as soon as the items are received from the retailer. That is, they assumed that the supplier would offer the retailer a delay period but the retailer would not offer any delay period to his/her customer. That is one level of trade credit. In most business transactions, this assumption is unrealistic. Usually the supplier offers a credit period to the retailer and the retailer, in turn, passes on this credit period to his/her customers. Recently, Huang [14] modified this assumption to assume that the retailer will adopt the trade credit policy to stimulate his/her customer demand to develop the retailer's replenishment model. That is two levels of trade credit. This new viewpoint is more matched to real-life situations in the supply chain model. Therefore, we want to extend Huang's model [14] to investigate the situation under which the retailer has the powerful decision-making right. That is, we want to assume that the retailer can obtain the full trade credit offered by the supplier and the retailer just offers the partial trade credit to his/her customer. The path of the trade credit policy is illustrated in Fig. 1. In practice, this circumstance is very realistic.

The main purpose of this paper is to amend the paper of Huang [14] and Goyal [1] with a view of making their model more relevant and so applicable to practice. Here, we are taking into account the following factors: (1) the supplier is willing to provide the retailer a full trade credit period for payments and the retailer offers the

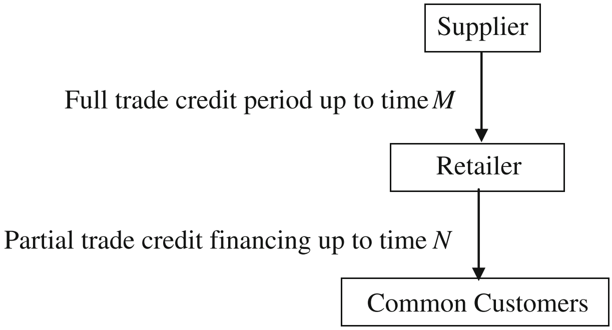


Fig. 1 Two-level trade credit policy with partial trade credit financing to common customers

partial trade credit period to his/her customers; (2) the replenishment rate is finite; (3) the selling items are perishable such as fruits, fresh fishes, gasoline, photographic films, etc. At first, this model shows that there exists a unique optimal cycle time to minimize the annual total variable cost for the retailer. Then, some theorems are developed to determine the optimal ordering policies. We deduce some previously published results of other authors as special cases. Finally, the theorems and the algorithms are illustrated with the help of numerical examples.

2 Assumptions and Notation

The mathematical model of the inventory system is developed on the basis of the following assumptions and notation.

(i) Replenishment rate, P, is known and constant.
(ii) Demand rate, D, is known and constant and always $P > D$.
(iii) Shortage is not allowed.
(iv) Time horizon is infinite.
(v) A constant fraction, θ, assumed to be small, of the on-hand inventory gets deteriorated per unit time, where $0<\theta \leq 1$.
(vi) h: inventory holding cost per item per unit time; A: the replenishment (ordering) cost per order; c: the unit purchase cost; and s: the unit selling price of items of good quality, where $s \geq c$.
(vii) I_c: the interest charged per \$ in stocks per year by the supplier.
(viii) I_e: the interest earned per \$per year, where $I_c \geq I_e$.
(ix) M: the retailer's trade credit period offered by supplier in years and N: the customer's trade credit period offered by retailer in years. It is assumed that $M \geq N$.
(x) α: the customer's fraction of the total amount owed payable at the time of placing an order within the delay period to retailer, where $0 \leq \alpha \leq 1$.
(xi) T and T^* denote the cycle time in years and the optimal cycle time respectively and $Q^* = DT^*$ is the optimal order quantity.
(xii) The supplier offers the full trade credit to the retailer. When $T \geq M$, the account is settled at $T = M$, the retailer pays off all units sold and keeps

his/her profits and the retailer starts paying for the interest charges on the items in stock with rate I_c. When $T < M$, the account is settled at time $T = M$ and the retailer does not pay any interest charge.

(xiii) The retailer just offers the partial trade credit to his/her customer. Hence, his/her customer must make a partial payment to the retailer when the item is sold. Then his/her customer must pay off the remaining balance at the end of the trade credit period offered by the retailer. That is, the retailer can accumulate interest from his/her customer partial payment on $(0,N]$ and from the total amount of payment on $[N,M]$ with rate I_e.

3 Model Formulation

A constant production rate starts at $t = 0$, and continues up to $t = t_1$ where the inventory level reaches the maximum level. Production then stops at $t = t_1$, and the inventory gradually depletes to zero at the end of the production cycle $t = T$ due to deterioration and consumption. Thereafter, during the time interval $(0,t_1)$ the system is subject to the effect of production, demand and deterioration.

The graphical representation of this inventory system is clearly depicted in Fig. 2. Then, the change in the inventory level can be described by the following differential equation:

$$\frac{dI_1(t)}{dt} + \theta\, I_1(t) = P - D, 0 \le t \le t_1 \tag{1}$$

with the initial condition $I_1(0) = 0$.

On the other hand, in the time interval $(t_1,\ T)$, the system is affected by the combined effect of demand and deterioration. Hence, the change in the inventory level is governed by the following differential equation:

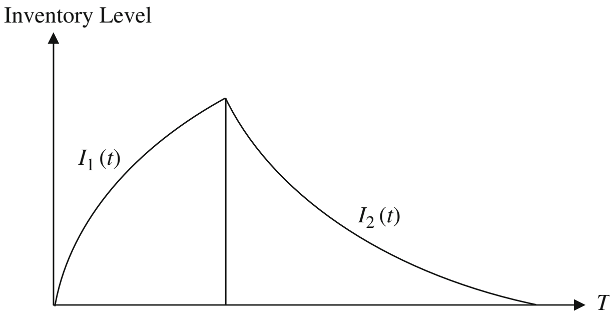
$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -Dt_1 \le t \le T, \tag{2}$$

with the ending condition $I_2(T) = 0$.

The solution of the differential Eqs. 1 and 2 are respectively represented by

$$I_1(t) = \frac{P - D}{\theta}\left(1 - e^{-\theta t}\right), 0 \le t \le t_1, \tag{3}$$

Fig. 2 Graphical representation of inventory system



$$I_2(t) = \frac{D}{\theta}\left(e^{\theta(T-t)} - 1\right), t_1 \le t \le T. \tag{4}$$

In addition, using the boundary condition $I_1(t_1) = I_2(t_1)$, we obtain the following equations:

$$(P-D)\left(1-e^{-\theta t_1}\right) = D\left(e^{\theta(T-t_1)} - 1\right), t_1 = \frac{1}{\theta}\ln\left\{1 + \frac{D}{P}\left(e^{\theta T} - 1\right)\right\} \tag{5}$$

4 Determination of Annual Total Cost Function

We now derive the annual total cost function for the retailer. The annual total relevant cost consists of the following elements: ordering cost, holding cost, deterioration cost, interest payable and interest earned. These components are evaluated as in the following:

(a) Annual ordering cost $= \frac{A}{T}$.
(b) Annual stock holding cost (excluding interest charges)

$$= \frac{h}{T}\left\{\int_0^{t_1} I_1(t)\,dt + \int_{t_1}^{T} I_2(t)\,dt\right\} \tag{6}$$

$$= \frac{h}{\theta^2 T}\left(\theta t_1 + e^{-\theta t_1} - 1\right).P + \frac{h}{\theta^2 T}\left(e^{\theta(T-t_1)} - \theta T - e^{-\theta t_1}\right) D$$

Since $I_1(t_1) = I_2(t_1)$, which implies Eq. 6 can be rearranged as follows:
Annual stock holding cost $\left(\text{excluding interest charges}\right) = \frac{h}{\theta T}\left(Pt_1 - DT\right)$.
(c) Annual cost due to deteriorated units $= \frac{c}{T}\left(Pt_1 - DT\right)$.
(d) According to assumption (xii), there are three cases to occur in interest charged for the items kept in stock per year.

Case 1 T $\ge M$ (shown in Fig. 3)

$$\text{Annual interest payable} = \frac{cI_c}{T}\int_M^T I_2(t).dt$$

$$= \frac{cI_c D}{\theta^2 T}\left\{e^{\theta(T-M)} - \theta(T-M) - 1\right\}.$$

Case 2 $N \le T \le M$ (shown in Fig. 4)

In this case, annual interest payable $= 0$.

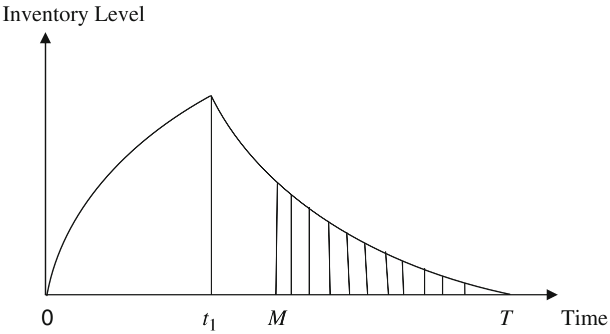


Fig. 3 Total accumulation of interest payable when $T \geq M$

Case 3 $0 < T \leq N$.

Similar as Case 2, annual interest payable = 0.

(e) According to assumption (xiii), three cases will occur in interest earned per year.

Case 1 $T \geq M$ (shown in Fig. 5)

$$\text{Annual interest earned} = \frac{sI_e}{T}\left[\int_0^N \alpha D t dt + \int_N^N D t dt\right]$$
$$= \frac{sI_e}{2T}\left[M^2 - (1-\alpha)\, N^2\right].$$

Case 2 $N \leq T \leq M$ (shown in Fig. 6)

$$\text{Annual interest earned} = \frac{sI_e}{T}\left[\int_0^N \alpha D t dt + \int_N^T D t dt + DT\,(M - T)\right]$$
$$= \frac{sI_e D}{2T}\left[2MT - (1-\alpha)N^2 - T^2\right].$$

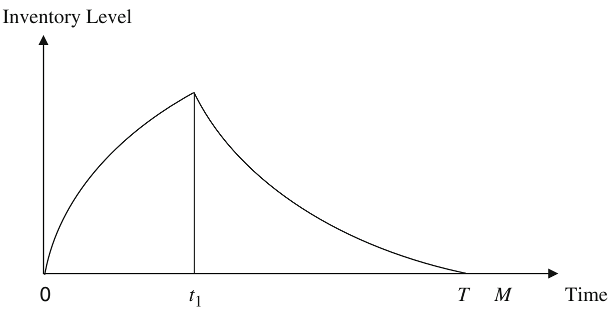


Fig. 4 Total accumulation of interest payable when $T < M$

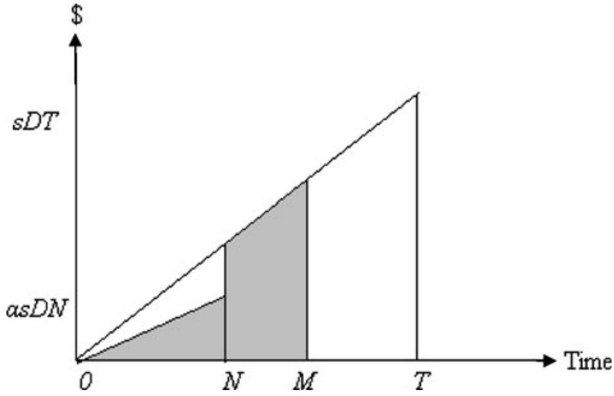


Fig. 5 Total amount of interest earned when $T \geq M$

Case 3 $0 < T \leq N$ (shown in Fig. 7)

$$\text{Annual interest earned} = \frac{sI_e}{T}\left[\int_0^T \alpha Dtdt + \alpha DT(N-T) + DT(M-N)\right]$$

$$= \frac{sI_e D}{2T}\left[M - (1-\alpha)N - \frac{\alpha T}{2}\right].$$

From the above arguments, the annual total relevant cost for the retailer can be expressed as $TRC(T)$ = ordering cost + stock-holding cost + deterioration cost + interest payable-interest earned.

$$TRC(T) = \begin{cases} TRC_1(T); \quad T \geq M & (a) \\ TRC_2(T);\ N \leq T \leq M & (b) \\ TRC_1(T); 0 < T \leq N & (c) \end{cases} \tag{7}$$

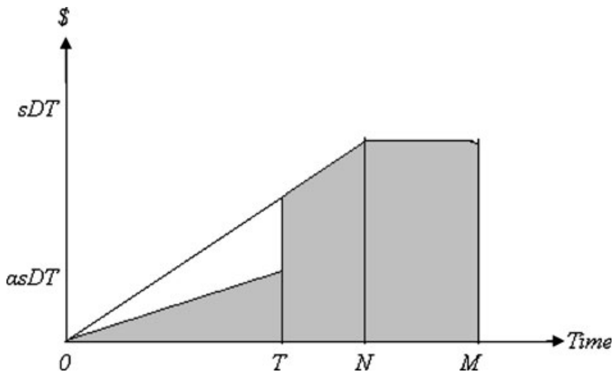
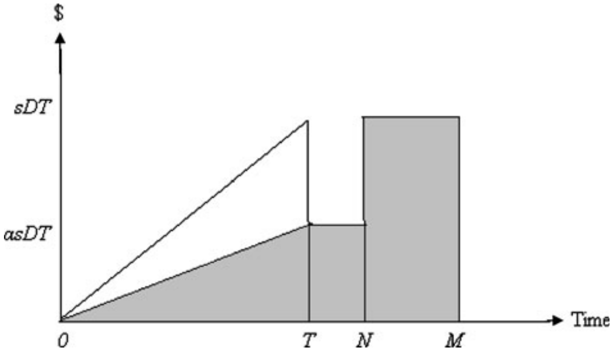


Fig. 6 Total amount of interest earned when $N \leq T \leq M$

Fig. 7 Total amount of interest earned when $0 < T \le N$



Where

$$TRC_1(T) = \frac{A}{T} + \frac{h+\theta\, c}{\theta\, T}(P t_1 - DT) + \frac{cI_c D}{\theta^2 T}\left\{e^{\theta(T-M)} - \theta(T-M) - 1\right\} - \frac{sI_e}{2T}\left[M^2 - (1-\alpha)N^2\right], \quad (8)$$

$$TRC_2(T) = \frac{A}{T} + \frac{h+\theta\, c}{\theta\, T}(P t_1 - DT) - \frac{sI_e D}{2T}\left[2MT - (1-\alpha)N^2 - T^2\right], \quad (9)$$

$$TRC_3(T) = \frac{A}{T} + \frac{h+\theta\, c}{\theta\, T}(P t_1 - DT) - \frac{sI_e D}{2T}\left[M - (1-\alpha)N - \frac{\alpha T}{2}\right]. \quad (10)$$

Since $TRC_1(M) = TRC_2(N)$ and $TRC_2(N) = TRC_3(N)$, $TRC(T)$ is continuous and well-defined on $T > 0$. All $TRC_1(T)$, $TRC_2(T)$ and $TRC_3(T)$ are defined on $T > 0$.

5 Solution Procedure

To find the optimal solution (T^*, Q^*, TRC (T^*)), the following procedure are consider.

Definition 1 A function $f(x)$ defined on an open interval (a,b) is said to be convex if for $x, y \in (a, b)$ and each λ, $0 \le \lambda \le 1$, we have $f(\lambda x + (1-\lambda)\, y) \le \lambda\, f(x) + (1-\lambda)\, f(y)$.

Intermediate Value Theorem (In Real Analysis) Let g be a continuous function on the closed interval $[a,b]$ and let $g(a).g(b) < 0$. Then there exists a number $c \in (a, b)$ such that $g(c) = 0$.

Lemma 1 *If $f(t)$ is a continuous function on (a,b) and if $\frac{df}{dt}$ is non-decreasing, then $f(t)$ is convex.*

Proof Given x,y with $a < x < y < b$, define a function g on [0,1] by $g(t) = tf(y) + (1-t)\, f(x) - f(ty + (1-t)\, x)$. Our goal is to show that g is non-negative on [0,1]. Now g is continuous and g(0) = g (1) = 0. Moreover, $\frac{dg(t)}{dt} = f(y) - f(x) - (y-x)\frac{df}{dt}$.

For $t+h > t$, $\frac{dg(t+h)}{dt} - \frac{dg(t)}{dt} = -(y-x)\left[\frac{df(t+h)}{dt} - \frac{df(t)}{dt}\right]$. Since $\frac{df}{dt}$ is non decreasing, $\frac{df(t+h)}{dt} - \frac{df(t)}{dt} > 0$. It implies that $\frac{dg(t)}{dt}$ is non-increasing on [0,1]. Let c be a point where g assumes its minimum on [0,1] If $c = 1$, then g (t) $\geq$ g (1) = 0 on [0,1]. In this case, g is non-negative. Suppose that c $\in (a, b)$. Since g has a local minimum at c, we have $\frac{dg(c)}{dt} \geq 0$. But $\frac{dg(t)}{dt}$ is non-increasing and so $\frac{dg(t)}{dt} \geq 0$ on [0,c]. Consequently, g is non-decreasing on [0,c] and hence g (c) $\leq$ g (0) = 0, then the minimum of g on [0,1] is non-negative and so $g > 0$ on [0,1]. That is $f(ty + (1-t)\, x) \leq tf(y) + (1-t)\, f(x)$ on [0,1]. This implies $f(t)$ is convex. □

6 Determination of the Optimal Cycle Length

Case 1 $T \geq M$.

The first derivative of $TRC_1(T)$ with respect to T is $\frac{dTRC_1(T)}{dT} = \frac{f_1(T)}{T^2}$, where

$$f_1(T) = -A + \frac{P(h+\theta c)}{\theta}\left\{T\frac{dt_1}{dT} - t_1\right\} + \frac{cI_cD}{\theta^2}\left\{\theta\, Te^{\theta(T-M)} - e^{\theta(T-M)} - \theta\, M + 1\right\} + \frac{sI_eD}{2}\left\{M^2 - (1-\alpha)N^2\right\}. \tag{11}$$

Then both $f_1(T)$ and $TRC_1^*(T)$ have the same sign and domain. The optimal value of T, say , can be obtained by solving the equation $f_1(T) = 0$. We also have $\frac{df_1(T)}{dT} = \frac{P(h+\theta c)}{\theta}T\frac{d^2t_1}{dT^2} + cI_cDTe^{\theta(T-M)} > 0$, if $T > 0$. Hence $f_1(T)$ is increasing on $(0,\infty)$ and so $\frac{dTRC_1}{dT}$ is increasing. From Lemma 1, $TRC_1(T)$ is a convex function on $(0,\infty)$. Also $\lim_{T\to\infty} f_1(T) = \infty > 0$ and $f_1(0) = -\left[A + \frac{sI_eD}{2}\left\{(1-\alpha)\, N^2 - M^2\right\}\right] + \frac{cI_cD}{\theta^2}\left(1 - \theta\, M - e^{-\theta\, M}\right)$. Since $1 - \theta\, M < e^{-\theta\, M}$, so we restrict attention to the condition $A + \frac{sI_eD}{2}\left\{(1-\alpha)N^2 - M^2\right\} > 0$. Then we have $f_1(0) < 0$. Hence we see that

$$\frac{dTRC_1(T)}{dT}\begin{cases} < 0; T \in \left(0, T_1^*\right), & (a)\\ = 0; T = T_1^* & (b)\\ > 0; T \in \left(T_1^*, \infty\right), & (c)\end{cases} \tag{12}$$

Provided that $f_1(0) < 0$. Based upon the above arguments, the intermediate value theorem yields that the optimal solution , not only exists but also is unique.

The similar procedure as described in Case 1 can be applied to the remaining two cases.

Case 2 $N \leq T \leq M$.

The first derivative of $TRC_2(T)$ with respect to $T$is $\frac{dTRC_2(T)}{dT} = \frac{f_2(T)}{T^2}$, where

$$f_2(T) = -A + \frac{P(h+\theta c)}{\theta}\left\{T\frac{dt_1}{dT} - t_1\right\} + \frac{sI_eD}{2}\left\{T^2 - (1-\alpha)\, N^2\right\}. \tag{13}$$

Then both $f_2(T)$ and $TRC_2^*(T)$ have the same sign and domain. The optimal value of T, say T_2^*, can be obtained by solving the equation $f_2(T) = 0$. We also have

$\frac{df_2(T)}{dT} = \frac{P(h+\theta c)}{\theta} T \frac{d^2 t_1}{dT^2} + sI_e DT > 0$, if $T > 0$. Hence $f_2(T)$ is increasing on $(0,\infty)$ and so $\frac{dTRC_2(T)}{dT}$ is increasing. From Lemma 1, $TRC_2(T)$ is a convex function on $(0,\infty)$. Also $\lim\limits_{T\to\infty} f_2(T) = \infty > 0$ and $f_2(0) = -\left[A + \frac{sI_e D(1-\alpha)N^2}{2}\right] < 0$. Hence we have

$$\frac{dTRC_2(T)}{dT}\begin{cases} < 0;\ T \in \left(0,\ T_2^*\right), & (a)\\ = 0;\ T = T_2^* & (b)\\ > 0;\ T \in \left(T_2^*, \infty\right), & (c)\end{cases} \tag{14}$$

Based upon the above arguments, the intermediate value theorem yields that the optimal solution, T_2^*, not only exists but also is unique.

Case 3 $0 < T \le N$.

We have $\frac{dTRC_3(T)}{dT} = \frac{f_3(T)}{T^2}$, where

$$f_3(T) = -A + \frac{P(h+\theta c)}{\theta}\left\{T\frac{dt_1}{dT} - t_1\right\} + \frac{sI_e\alpha D}{2}T^2. \tag{15}$$

Then both $f_3(T)$ and $TRC_3^{/}(T)$ have the same sign and domain. The optimal value of T say T_3^*, can be obtained by solving the equation $f_3(T) = 0$. We also have $\frac{df_3(T)}{dT} = \frac{P(h+\theta c)}{\theta} T \frac{d^2 t_1}{dT^2} + sI_e \alpha DT > 0$ if $T > 0$. Hence $f_3(T)$ is increasing on $(0,\infty)$ and so $\frac{dTRC_3(T)}{dT}$ is increasing. From Lemma 1, $TRC_3(T)$ is a convex function on $(0,\infty)$. Also $\lim\limits_{T\to\infty} f_3(T) = \infty > 0$ and $f_3(0) = -A < 0$. Hence we see that

$$\frac{dTRC_3(T)}{dT}\begin{cases} < 0;\ T \in \left(0,\ T_3^*\right), & (a)\\ = 0;\ T = T_3^* & (b)\\ > 0;\ T \in \left(T_3^*, \infty\right), & (c)\end{cases} \tag{16}$$

Based upon the above arguments, the intermediate value theorem yields that the optimal solution, T_3^*, not only exists but also is unique.

Finally, combining the above three cases we have the following Theorem 1.

Theorem 1

(i) *If* $A + \frac{sI_e D}{2}\left\{(1-\alpha)N^2 - M^2\right\} > 0$, *then* T_1^* *is the unique optimal solution to the cost function* $TRC_1(T)$.
(ii) $TRC_i(T)$ $(i = 2, 3)$ *has the unique optimal solution* T_i^* $(i = 2, 3)$ *on the interval* $(0,\infty)$.

7 Decision Rule of the Optimal Replenishment Cycle Time

In this section, we develop efficient decision rules to find the optimal cycle time for the retailer. From the definition of $TRC(T)$, we have

$$TRC(T) = \begin{cases} TRC_1(T);\ T \ge M, & (a)\\ TRC_2(T);\ N \le T \le M & (b)\\ TRC_3(T);\ 0 < T \le N, & (c)\end{cases} \tag{17}$$

Fortunately, at $T = M$, $TRC_1(M) = TRC_2(M)$ and at $T = N$, $TRC_2(N) = TRC_3(N)$, then $TRC(T)$ is continuous and well defined on $T > 0$. Since $TRC(T)$ is continuously differentiable function of T with a derivative that changes sign only once at

T_i^* $(i = 1, 2, 3)$ from negative to positive, it follows that $TRC(T)$ assumes its global minimum at the point T^*. However, a closed-form solution is not readily available from Eq. 17a–c, but by the Intermediate value theorem, we can establish a fairly straightforward procedure to determine the optimal replenishment time to simplify the solution procedure. Let $T = M$ and N, we obtain from Eqs. 11, 13 and 15 that

$$TRC_1^{/}(M) = TRC_2^{/}(M) = \frac{1}{M^2}\left[-A + \frac{P(h+\theta c)}{\theta}\left(T\frac{dt_1}{dT} - t_1\right)_{T=M} + \frac{sI_eD}{2}\left\{M^2 - (1-\alpha)N^2\right\}\right],$$

$$TRC_2^{/}(M) = TRC_3^{/}(M) = \frac{1}{N^2}\left[-A + \frac{P(h+\theta c)}{\theta}\left(T\frac{dt_1}{dT} - t_1\right)_{T=N} + \frac{sI_eD}{2}\alpha N^2\right].$$

For convenience, let

$$\Delta_1 = -A + \frac{P(h+\theta c)}{\theta}\left(T\frac{dt_1}{dT} - t_1\right)_{T=M} + \frac{sI_eD}{2}\left\{M^2 - (1-\alpha)N^2\right\}, \tag{18}$$

$$\Delta_2 = -A + \frac{P(h+\theta c)}{\theta}\left(T\frac{dt_1}{dT} - t_1\right)_{T=N} + \frac{sI_eD}{2}\alpha N^2. \tag{19}$$

We have $\Delta_1 - \Delta_2 = \frac{P(h+\theta c)}{\theta}\left[\left(T\frac{dt_1}{dT} - t_1\right)_{T=M} - \left(T\frac{dt_1}{dT} - t_1\right)_{T=N}\right] + \frac{sI_eD}{2}\left\{M^2 - N^2\right\}$.
Using Lemma 2, we shall find that $\Delta_1 \geq \Delta_2$.

Lemma 2

(i) $T\frac{dt_1}{dT} - t_1 > 0$ *and (ii)* $\left(T\frac{dt_1}{dT} - t_1\right)_{T=M} > \left(T\frac{dt_1}{dT} - t_1\right)_{T=N}$.

Proof Let $h(T) = T\frac{dt_1}{dT} - t_1$, then $h^{/}(T) = T\frac{d^2t_1}{dT^2} > 0$ if $T > 0$. Hence $h(T)$ is increasing for all $T > 0$. Consequently, $h(T) > h(0) = 0$ if $T > 0$ and also $h(M) > h(N)$ as $M > N$. Thus, we have $T\frac{dt_1}{dT} - t_1 > 0$ and $\left(T\frac{dt_1}{dT} - t_1\right)_{T=M} > \left(T\frac{dt_1}{dT} - t_1\right)_{T=N}$. This completes the proof. Now, we give the following theorems for the decision rule of the optimal replenishment cycle time T^*.

(i) Suppose that $A + \frac{sI_eD}{2}\left\{(1-\alpha)N^2 - M^2\right\} < 0$, then we obtain $\Delta_1 > 0$ from Eq. 18. From 14 and 16, we notice that for $i = 2,3$

$$\frac{dTRC_i(T)}{dT}\begin{cases} < 0;\ T \in \left(0, T_i^*\right), & (a) \\ = 0;\ T = T_i^* & (b) \\ > 0;\ T \in \left(T_i^*, \infty\right), & (c) \end{cases} \tag{20}$$

Then we have the following theorem to determine the optimal cycle time.

□

Theorem 2

(a) *If* $\Delta_2 \geq 0$*, then* $TRC(T^*) = TRC(T_3^*)$ *and* $T^* = T_3^*$.
(b) *If* $\Delta_2 < 0$*, then and* $T^* = T_2^*$.

Proof

(a) If $\Delta_2 \geq 0$, then we have $f_2(N) = f_3(N) \geq 0$; that is $TRC_2^{/}(N) = TRC_3^{/}(N) \geq 0$. Equation 20a–c imply that (i) $TRC_2(T)$ is increasing on $[N, \infty]$, (ii) $TRC_3(T)$ is decreasing on $(0,\ T_3^*]$ and increasing on $[T_3^*,\ N]$. Combining (i), (ii) and Eq. 17a–c, we see that $TRC(T)$ is decreasing on $(0,\ T_3^*]$ and increasing on $[T_3^*,\ \infty)$. Consequently, $T^* = T_3^*$.
(b) If $\Delta_2 < 0$, then we have $f_2(N) = f_3(N) < 0$; that is $TRC_2^{/}(N) = TRC_3^{/}(N) < 0$. Equation 20a–c imply that (i) $TRC_2(T)$ is decreasing on $[N,\ T_2^*]$ and increasing on $[T_2^*,\ \infty)$, (ii) $TRC_3(T)$ is decreasing on $(0,\ N]$. Combining (i), (ii) and Eq. 17a–c, we see that $TRC(T)$ is decreasing on $(0,\ T_2^*]$ and increasing on $[T_2^*,\ \infty)$. Consequently, $T^* = T_2^*$. □

(ii) Suppose that $A + \frac{sI_eD}{2}\left\{(1-\alpha)\,N^2 - M^2\right\} > 0$, then all T_i^* $(i = 1,\ 2,\ 3)$ are well-defined. From (12), (14) and (16), we notice that for $i = 1,2,3$.

$$\frac{dTRC_i\,(T)}{dT}\begin{cases} < 0;\ T \in \left(0,\ T_i^*\right), & (a)\\ = 0;\ T = T_i^* & (b)\\ > 0;\ T \in \left(T_i^*, \infty\right), & (c)\end{cases} \tag{21}$$

Then we have the following theorem to determine the optimal cycle time.

Theorem 3

(a) *If $\Delta_1 > 0$ and $\Delta_2 \geq 0$, then $TRC(T^*) = TRC(T_3^*)$ and $T^* = T_3^*$.*
(b) *If $\Delta_1 > 0$ and $\Delta_2 < 0$, then $TRC(T^*) = TRC(T_2^*)$ and $T^* = T_2^*$.*
(c) *If $\Delta_1 \leq 0$ and $\Delta_2 < 0$, then and $T^* = T_1^*$.*

Proof

(a) If $\Delta_1 > 0$ and $\Delta_2 \geq 0$, then we have $f_1\,(M) = f_2\,(M) > 0$ and $f_2\,(N) = f_3\,(N) \geq 0$; that is, $TRC_1^{/}\,(M) = TRC_2^{/}\,(M) > 0$ and $TRC_2^{/}\,(N) = TRC_3^{/}\,(N) \geq 0$. So $T_1^* < M$, $T_2^* < M$, $T_3^* < N$and $T_2^* < N$. Equation 21a–c imply that (i) $TRC_1(T)$ is increasing on $[M,\ \infty)$, (ii) $TRC_2(T)$ is increasing on $[N,M]$, (iii) $TRC_3(T)$ is decreasing on $(0,\ T_3^*]$ and increasing on $[T_3^*,\ N]$. Combining (i)–(iii) and Eq. 17a–c, we see that $TRC(T)$ is decreasing on $(0,\ T_3^*]$ and increasing on $[T_3^*, \infty)$. Consequently, $T^* = T_3^*$ and $TRC\,(T^*) = TRC\left(T_3^*\right)$.
(b) If $\Delta_1 > 0$ and $\Delta_2 < 0$, then we have $f_1\,(M) = f_2\,(M) > 0$ and $f_2\,(N) = f_3\,(N) < 0$; that is, $TRC_1^{/}\,(M) = TRC_2^{/}\,(M) > 0$ and $TRC_2^{/}\,(N) = TRC_3^{/}\,(N) < 0$. So $T_1^* < M$, $T_2^* < M$, $T_3^* > N$and $T_2^* > N$. Equation 21a–c imply that (i) $TRC_1(T)$ is increasing on $[M,\infty)$, (ii) $TRC_2(T)$ is decreasing on $[N,\ T_2^*]$ and increasing on $[T_2^*,\ M]$, (iii) $TRC_3(T)$ is decreasing on $[0,N]$. Combining (i)–(iii) and Eq. 17a–c, we see that $TRC(T)$ is decreasing on $(0,\ T_2^*]$ and increasing on $[T_2^*, \infty)$. Consequently, $T^* = T_2^*$ and $TRC(T^*) = TRC(T_2^*)$
(c) If $\Delta_1 \leq 0$ and $\Delta_2 < 0$, then we have $f_1\,(M) = f_2\,(M) \leq 0$and $f_2\,(N) = f_3\,(N) < 0$; that is, $TRC_1^{/}\,(M) = TRC_2^{/}\,(M) \leq 0$ and $TRC_2^{/}\,(N) = TRC_3^{/}\,(N) < 0$. So $T_1^* > M$, $T_2^* > MT_3^* > N$and $T_2^* > N$. Equation 21a–c imply that (i) $TRC_1(T)$ is decreasing on $[M,\ T_1^*]$ and increasing on $[T_1^*, \infty)$, (ii) $TRC_2(T)$ is decreasing on $[N,M]$, (iii) $TRC_3(T)$ is decreasing on $[0,N]$. Combining (i)–(iii) and

Eq. 17a–c, we see that $TRC(T)$ is decreasing on $(0, T_1^*]$ and increasing on $[T_1^*, \infty)$. Consequently, $T^* = T_1^*$ and $TRC(T^*) = TRC(T_1^*)$

Generally speaking, Theorem 2 explains that after computing, we can immediately determine which one of T_1^*, T_2^* or T_3^* is optimal. □

8 Special Case

In this section, we obtain some previously published results of other authors as special cases.

a) Huang's model [14]

Here, we let $\theta \to 0$, $P \to \infty$, $s = c$ and $\alpha = 0$. Applying the above conditions, Eqs. 8–10 yield that

$$TRC_4(T) = \frac{A}{T} + \frac{DTh}{2} + \frac{cI_cD}{2T}(T-M)^2 - \frac{cI_eD}{2T}\left(M^2 - N^2\right),$$

$$TRC_5(T) = \frac{A}{T} + \frac{DTh}{2} - \frac{cI_eD}{2T}\left(2MT - N^2 - T^2\right),$$

$$TRC_5(T) = \frac{A}{T} + \frac{DTh}{2} - cI_eD(M-N).$$

Equation 17a–c will be reduced as follows:

$$TRC(T) = \begin{cases} TRC_4(T); T \geq M, & (a) \\ TRC_5(T); N \leq T \leq M & (b) \\ TRC_6(T); 0 < T \leq N, & (c) \end{cases} \tag{22}$$

Let $T_4^* = \sqrt{\frac{2A + cD\left\{M^2(I_c - I_e) + N^2 I_e\right\}}{D(h + cI_e)}}$, $T_5^* = \sqrt{\frac{2A + cDN^2 I_e}{D(h + cI_e)}}$, $T_6^* = \sqrt{\frac{2A}{Dh}}$. Then $TRC_i^{/}\left(T_i^*\right) = 0$ for $i = 4,5,6$. Equation 22a–c are consistent with Eq. 1a–c in Huang's model [14], respectively. Hence, Huang's model [14] is a special case of this model.

b) Goyal's model [1]

When $\theta \to 0$, $P \to \infty$, $s = c$, $N = 0$ and $\alpha = 0$, let $TRC_7(T) = \frac{A}{T} + \frac{DTh}{2} + \frac{cI_cD}{2T}(T-M)^2 - \frac{cI_eDM^2}{2T}$, $TRC_8(T) = \frac{A}{T} + \frac{DTh}{2} - cI_eD\left(M - \frac{T}{2}\right)$, $T_7^* = \sqrt{\frac{2A + cDM^2(I_c - I_e)}{D(h + cI_c)}}$, $T_8^* = \sqrt{\frac{2A}{D(h + cI_e)}}$. Then $TRC_i^{/}\left(T_i^*\right) = 0$ for $i = 7,8$. Equation 17a–c will be reduced as follows:

$$TRC(T) = \begin{cases} TRC_7(T); T \geq M, & (a) \\ TRC_8(T); 0 \leq T \leq M & (b) \end{cases} \tag{23}$$

Equation 23a, b will be consistent with Eqs. 1 and 4 in Goyal [1] model, respectively. Hence, Goyal model [1] will be a special case of this paper.

9 Numerical Example

The purposes of the numerical analysis are as follows:

1. To obtain the optimal solutions for two cases of the cost functions for the retailer.
2. To use sensitivity analysis to highlight the influence of the parameters associated with the model.

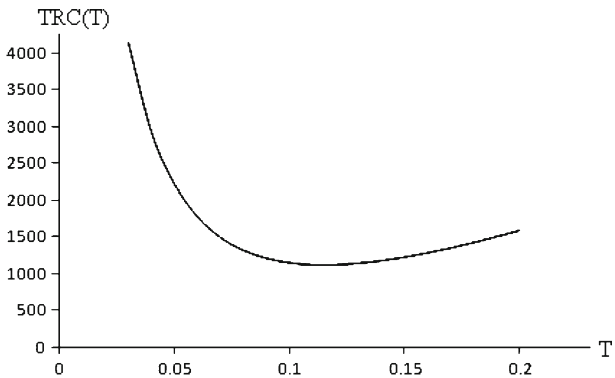
9.1 Numerical Examples

The following numerical examples are given to illustrate the above solution procedure.

Example 1 Let $A = \$150$/order, $D = 250$units/year, $P = 3000$units/year, $s = \$75$/ unit, $c = \$50$/unit, $h = \$15$/unit/year, $I_c = \$0.15/\$$/year, $I_e = \$0.10/\$$/year , $M = 0.1$ year, $N = 0.05$ year, $\theta = 0.05$, $\alpha = 0.05$, then $\Delta_1 = -92.138 < 0$, . These conditions satisfy Theorem 3(c). Hence solving equation $f_1(T) = 0$ by Newton–Raphson method, we obtain $T^* = T_1^* = 0.115$, $Q^* = 287.61$ and $TRC(T^*) = TRC\left(T_1^*\right) = 1119.83$. The two dimensional graph of the annual total cost function $TRC(T^*)$ of the retailer is presented in Fig. 8. The graph reveals that there exists a corresponding optimal solution T^* which minimizes the annual total cost for the retailer. Also, Fig. 8 shows that $TRC(T)$ is strictly convex function of T. As a result, we are sure that the optimum solution obtained is indeed the global optimum solution for the annual total cost function for the retailer.

Example 2 Let $A = \$100$/order, $D = 2500$units/year, $P = 4000$units/year, $s = \$75$/ unit, $c = \$50$/unit, $h = \$15$/unit/year, $I_c = \$0.15/\$$/year, $I_e = \$0.10/\$$/year, $M = 0.1$ year, $N = 0.05$ year, $\theta = 0.05$, $\alpha = 0.05$, then Δ_1=53.45 > 0, . These conditions satisfy Theorem 3(b). Hence solving equation $f_2(T) = 0$ by Newton–Raphson method, we obtain $T^* = T_2^* = 0.0834$, $Q^* = 208.50$ and $TRC(T^*) = TRC\left(T_2^*\right) = 1056.79$. The two dimensional graph of the annual total cost function $TRC(T^*)$ of the retailer is presented in Fig. 9. The graph reveals that there exists a corresponding optimal solution T^* which minimizes the annual total cost for the retailer. Also, Fig. 9

Fig. 8 The optimal annual total cost for various values of T



shows that $TRC(T)$ is strictly convex function of T. As a result, we are sure that the optimum solution obtained is indeed the global optimum solution for the annual total cost function for the retailer.

Example 3 Let A = \$50/order, D = 2500units/year, P = 4000units/year, s = \$100/ unit, c = \$50/unit, h = \$15/unit/year, I_c = \$0.15/\$/year, I_e = \$0.10/\$/year, $M = 0.1$ year, $N = 0.08$ year, $\theta = 0.05$, $\alpha = 0.05$, then $\Delta_1 = 80.96 > 0$, $\Delta_2 = 6.46 > 0$. These conditions satisfy Theorem 3(a). Hence solving equation $f_3(T) = 0$ by Newton–Raphson method, we obtain $T^* = T_3^* = 0.0752$, $Q^* = 188.12$ and $TRC(T^*) = TRC(T_3^*) = 728.57$. The two dimensional graph of the annual total cost function $TRC(T^*)$ of the retailer is presented in Fig. 10. The graph reveals that there exists a corresponding optimal solution T^* which minimizes the annual total cost for the retailer. Also, Fig. 10 shows that $TRC(T)$ is strictly convex function of T. As a result, we are sure that the optimum solution obtained is indeed the global optimum solution for the annual total cost function for the retailer.

9.2 Effect of Changing the Inventory Model Parameters

Here, we consider the following example. Let A = \$150/order, D = 2500units/year, P = 3000units/year, s = \$75/ unit, c = \$50/unit, h = \$15/unit/year, I_c = \$0.15/\$/year, I_e = \$0.10/\$/year , $M = 0.1$ year, $N = 0.05$ year, $\theta = 0.05$, $\alpha = 0.05$.

The sensitivity analysis is performed by varying different parameters and is given in Table 1. It is important to discuss the influence of key model parameters on the optimal solutions. The effect of changing the parameters is shown in Table 1. Based on Table 1, we have the following comments.

(a) The larger the value of s, the smaller value of the optimal cycle time, the optimal order quantity and the smaller the value of the annual total relevant cost. That is, when the unit selling price is increasing, the retailer will order less quantity to take the benefits of the trade credit more frequently.

(b) As production rate increases, $TRC(T^*)$ increases; so it is not advisable to increase the production rate without the prior knowledge about the demands.

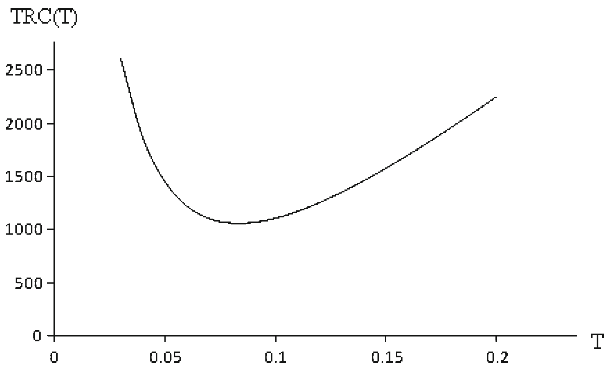


Fig. 9 The optimal annual total cost for various values of T

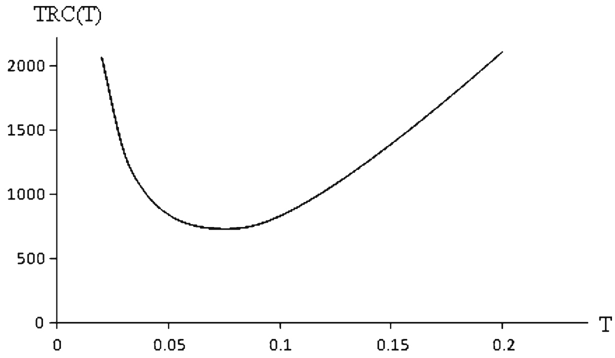


Fig. 10 The optimal annual total cost for various values of T

(c) A higher value of the deteriorating rate θ results in lower values for the optimal cycle time, T^* and the optimal economic order quantity, Q^* and a higher value for the annual total relevant cost, $TRC(T^*)$. It tells us that the retailer will order less quantity to avoid the items deteriorating when the deterioration rate θ increases.

(d) The larger the value of N, the larger the value of the optimal cycle time and the higher the value of the annual total relevant cost. That is, when the customer's trade credit period offered by the retailer is increasing, the retailer will order more quantity to accumulate more interest to compensate the loss of interest earned when longer trade credit period is offered to his/her customer.

(e) The larger the value of α, the smaller the value of the optimal cycle time T^*, the smaller value of order quantity Q^* and the lower the value of the optimal annual total relevant cost $TRC(T^*)$. That is, when the customer's fraction of the

Table 1 Sensitivity analysis for various inventory parameters

Parameter		T^*	Q^*	$TRC\ (T^*)$
S	50	0.127	317.07	1,318.99
	75	0.115	287.61	1,119.83
	100	0.107	266.97	905.03
	125	0.099	246.36	672.37
P	3000	0.115	287.61	1,119.83
	3500	0.105	262.54	1,405.76
	4000	0.098	244.89	1,566.26
	4500	0.094	234.64	1,702.13
θ	0.02	0.116	290.43	1,090.01
	0.04	0.115	288.73	1,107.94
	0.06	0.115	286.49	1,131.66
	0.08	0.114	284.29	1,155.19
N	0.02	0.109	273.71	975.14
	0.04	0.112	280.83	1,049.33
	0.06	0.118	295.67	1,203.79
	0.08	0.135	337.17	1,402.31
α	0.02	0.115	288.19	1,125.93
	0.04	0.114	287.92	1,123.08
	0.06	0.113	287.41	1,117.79
	0.07	0.111	286.72	1,112.14

total amount owed payable at the time of placing an order offered by the retailer is increasing, the retailer will order less quantity and increase order frequency. The retailer can accumulate more interest under higher order frequency and higher customer's fraction of the total amount owed payable at the time of placing an order offered by the retailer.

10 Summary and Conclusion

The results of this paper not only provide a valuable reference for decision-makers in planning and controlling the inventory but also provide a useful model for many organizations that use the decision rule to improve their total operation cost. In this paper, we formulated a production lot-size inventory model for deteriorating items that investigates retailer's decision making right in a supply chain under some realistic features. First, the supplier is willing to provide the retailer a full trade credit period for payments and the retailer offers the partial trade credit period to his/her customers Second, the replenishment rate is finite. Lastly, the selling items are perishable such as fruits, fresh fishes, gasoline, photographic films, etc. These assumptions are consistent with economic senses. We develop some effective and easy-to-use theorems to help the decision maker to find the optimal replenishment policy. Theorems 1, 2 and 3 give the decision rules of the optimal ordering policy for the retailer. Then we deduce Haung's model [14] and Goyal's model [1] as particular cases of this paper. Numerical examples are given to illustrate all effective theorems and obtained a lot of managerial insights.

A future study will further incorporate the proposed model into more realistic assumptions, such as probabilistic demand, allowable shortages, or quantity discounts.

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References

1. Goyal, S.K.: Economic order quantity under conditions of permissible delay in payments. J. Oper. Res. Soc. **36**, 335–338 (1985)
2. Shinn, S.W., Hwang, H.P., Sung, S.: Joint price and lot size determination under conditions of permissible delay in payments and quantity discounts for freight cost. Eur. J. Oper. Res. **91**, 528–542 (1996)
3. Chung, K.J.: A theorem on the determination of economic order quantity under conditions of permissible delay in payments. Comput. Oper. Res. **25**, 49–52 (1998)
4. Teng, J.T.: On the economic order quantity under conditions of permissible delay in payments. J. Oper. Res. Soc. **53**, 915–918 (2002)
5. Chung, K.J., Huang, Y.F.: The optimal cycle time for EPQ inventory model under permissible delay in payments. Int. J. Prod. Econ. **84**, 307–318 (2003)
6. Aggarwal, S.P., Jaggi, C.K.: Ordering policies of deteriorating items under permissible delay in payments. J. Oper. Res. Soc. **46**, 658–662 (1995)
7. Chu, P., Chung, K.J., Lan, S.P.: Economic order quantity of deteriorating items under permissible delay in payments. Comput. Oper. Res. **25**, 817–824 (1998)
8. Chung, K.J., Chang S.L., Yang, W.D.: The optimal cycle time for exponentially deteriorating products under trade credit financing. Eng. Econ. **46**, 232–242 (2001)
9. Chang, C.T. Ouyang, L.Y., Teng, J.T.: An EOQ model for deteriorating items under supplier credits linked to ordering quantity. Appl. Math. Model. **27**, 983–996 (2003)

10. Chung, K.J., Liao, J.J.: The optimal ordering policy in a DCF analysis for deteriorating items when trade credit depends on the order quantity. Int. J. Prod. Econ. **100**, 116–130 (2006)
11. Jamal, A.M.M., Sarker, B.R., Wang, S.: Optimal payment time for a retailer under permitted delay of payment by the wholesaler. Int. J. Prod. Econ. **66**, 59–66 (2000)
12. Arcelus, F.J., Shah, N.H., Srinivasan, G.: Retailer's pricing, credit and inventory policies for deteriorating items in response to temporary price/credit incentives. Int. J. Prod. Econ. **81–82**, 153–162 (2003)
13. Abad, P.L., Jaggi, C.K.: A joint approach for setting unit price and the length of the credit period for a seller when end demand is price sensitive. Int. J. Prod. Econ. **83**, 115–122 (2003)
14. Huang, Y.F.: Optimal retailer's ordering policies in the EOQ model under trade credit financing. J. Oper. Res. Soc. **54**, 1011–1015 (2003)