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The *k*-Centrum Straight-line Location Problem

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Abstract A line is sought in the plane which minimizes the sum of the k largest (Euclidean) weighted distances from n given points. This problem generalizes the known straight-line center and median problems and, as far as the authors are aware, has not been tackled up to now. By way of geometric duality it is shown that such a line may always be found which either passes through two of the given points or lying at equal weighted distance from three of these. This allows construction of an algorithm to find all t-centrum lines for $1 \leq t \leq k$ in $O((k + \log n)n^3)$. Finally it is shown that both, the characterization of an optimal line and the algorithm, can be extended to any smooth norm.

Keywords Line-location · k-centrum line · Geometric duality

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1 Introduction

The two most commonly-applied criteria in Locational Analysis are the minimization of the sum of distances and the maximum distance to a set of points, leading to the concepts of Median and Center of a set of points. When the facility cannot be modeled as an isolated point but needs some geometrical figure to be represented, extensive facility location and location of structures arise. The median straight-line problem, i.e. finding a line minimizing the sum of weighted distances to the points of a set, appeared within the context of transportation in the paper [11]. The first exact algorithm for this problem was devised by Wesolowsky [20]. As far as the authors are aware, the current lowest time algorithm is that provided by Lee and Ching [9], that finds the weighted median straight line in $O(n^2)$ time. For the unweighted version, subquadratic algorithms have been proposed (see for example [8]). The center straight-line problem, i.e. finding a line minimizing the maximum weighted distance to the points of a given set, was first addressed by Morris and Norback [13] and an $O(n \log n)$ optimal time algorithm was proposed by Edelsbrunner [6]. More information about median and center line location problems can be found in [17] and [5].

While the median takes into account all the points, the center objective function only takes into consideration the (weighted) distance corresponding to the worst possible situation. The median criteria are concerned with the spatial efficiency and often provide solutions where remote population areas are discriminated in terms of accessibility as compared with centrally situated areas. For this reason when locating public services the center solution concept is usually applied to minimize the maximum distance between the facility and the farthest consumer. Nevertheless, locating a facility at the center may cause a large increase in the total (average) distance, generating a substantial loss in spatial efficiency.

A criterion which at the same time offers a compromise and generalizes these two classical models is the k-centrum which minimizes the sum of the k largest weighted distances to points. The particular cases of $k = 1$ and $k = n$ correspond to the center and the median, respectively. However, the k-centrum criterion has only been applied to point location, (see [19] for point facility location problems in networks with the k-centrum criterion) but not to dimensional structures location [5].

Due to the averaging within the group of the worst outcomes the k-Centrum criterion reduces the flaws of the center and median criteria and it is a good approach to locate services in which the total (average) and maximum time to serve (distances) should be minimized and it may not be acceptable to have zones that are poorly served while some zones are extremely well served. Therefore, the k-Centrum is an adequate criterion to locate, for example, emergency services or health care facilities. In a different context, Romeijn et al. [16] considered the problem of designing a treatment plan for a technique considered to be the most effective radiation therapy for many forms of cancer. They propose a model that has the potential to achieve most of the goals with respect to the quality of the treatment plan. In the restrictions of the model the k-Centrum criterion, expressed in terms of dose of radiation, is used in order to establish bounds on the average dose received by the subset of the structure of volume relative $1 - \alpha$ receiving the highest amount of dose.

In this paper we investigate the properties of the k-centrum objective for straight lines by considering it in dual space, leading to a quite efficient algorithm. The problem and its geometrical dual are stated in Section 2. In Section 3 we study the tesselations induced by the dual of the given points and their bisectors. This allows us to derive a characterization of the solutions in Section 4, leading to a finite set containing a solution. An algorithm based on the results given in Sections 2, 3 and 4, is provided in Section 5, together with an analysis of its complexity. Finally, in Section 6 an alternative procedure to compute a more reduced finite dominating set is given and in Section 7 several particular cases, extensions and further research topics are considered.

2 The Problem and its Geometrical Dual

A point set $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ and its associated (positive) weight set $W = \{w_1, \ldots, w_n\}$ are given. For $1 \leq k \leq n$, the k-Centrum straight-line problem consists in finding a straight line ℓ that minimizes the sum of the k largest weighted Euclidean distances to the points of P or, equivalently, which minimizes the function

$$f(\ell) = \max_{\substack{Q \subset P \\ |Q| = k}} \sum_{p \in Q} w_p d(\ell, p) \tag{1}$$

where $d(\cdot, \cdot)$ is the point-line Euclidean distance: $d(p, \ell) = \min_{q \in l} d(p, q)$.

In order to solve it, the problem will be transformed into an equivalent one stated in the geometrical dual of the Euclidean plane. For this purpose, let us consider the duality map that associates with each point $p = (p_x, p_y) \in \mathbb{R}^2$ the dual (non-vertical) straight line $p^* : y = p_x x - p_y$, and with the non-vertical straight line $\ell : y = mx + n$ the point $\ell^* = (m, -n)$. This map has the following relevant properties [4]:

1. Preserves incidence between points and straight-lines: $p \in \ell$ if and only if $\ell^* \in p^*$.
2. Preserves the relative position between points and straight lines: p is above ℓ if and only if ℓ^* is above p^*.
3. The map is idempotent, i.e. the bi-dual of either a point or a straight line coincides with the original.
4. For the Euclidean distance

$$d(p, \ell) = \frac{d_v(\ell^*, p^*)}{\sqrt{1 + \ell_x^{*2}}} \tag{2}$$

where ℓ_x^* denotes the abcissa of ℓ^* and $d_v(\cdot, \cdot)$ is the vertical distance, which for the points $p = (p_x, p_y), q = (q_x, q_y) \in \mathbb{R}^2$ is defined as:

$$d_v(p, q) = \begin{cases} |p_y - q_y|, & \text{when} \quad p_x = q_x \\ \infty, & \text{when} \quad p_x \neq q_x \end{cases}$$

and for a point $p = (p_x, p_y)$ and a non-vertical straight line $\ell : y = mx - n$ as

$$\begin{aligned} d_v(p, \ell) &= \min_{q \in \ell} d_v(p, q) \\ &= |p_y - mp_x + n| \end{aligned} \tag{3}$$

The duality map is a bijection between the primal and the dual planes transforming points in non-vertical straight lines and vice-versa. This fact along with the above properties allow us to state an equivalent problem in the dual. If $P^* = \{p_1^*, \ldots, p_n^*\}$ is the dual image of the set P, then finding a minimum of the function of Eq. 1 is equivalent to finding a minimum of the function:

$$f^*(\ell^*) = \max_{\substack{Q^* \subset P^* \\ |Q^*| = k}} \sum_{p^* \in Q^*} \frac{w_p d_v(p^*, \ell^*)}{\sqrt{1 + \ell_x^{*2}}} \tag{4}$$

except when the minimum of Eq. 1 is a vertical straight line, in which case it corresponds to some point at infinity of the dual plane.

3 Tessellations of the Dual Plane

The following two tessellations of the dual plane will be useful for solving the straight-line k-Centrum problem.

First define the bisector of the weighted straight lines $p_i^*, p_j^* \in P^*$ by

$$bis_v(p_i^*, p_j^*) = \{\ell^* : w_i d_v(p_i^*, \ell^*) = w_j d_v(p_j^*, \ell^*)\}$$

Using Eqs. 2 and 3 we can see that $\ell^* = (m, n) \in bis_v(p_i^*, p_j^*)$ corresponds to the equations

$$w_i|(p_i)_y - m(p_i)_x + n| = w_j|(p_j)_y - m(p_j)_x + n|$$

It follows that the bisector of two straight lines p_i^* and p_j^* with different weights consists of two non-vertical straight lines in the dual plane with the same intersection point as p_i^* and p_j^*, and when p_i^* and p_j^* are parallel the two branches of the bisector are also parallel to them. When the weights are equal, one of these bisector branches is vertical in case p_i^* and p_j^* intersect, and when they are parallel the bisector is reduced to a single line (the second being the line at infinity).

Lemma 1 *The set $B_v(P^*) = \{bis_v(p_i^*, p_j^*) : i \neq j, 1 \leq i \leq n, 1 \leq j \leq n\}$ induces a partition of the dual plane into polygonal regions. All points within each such region consist of dual images of straight lines for which the ordering of the weighted distances to the elements of P remains constant.*

Proof Since the vertical distance function is continuous with respect to ℓ^*, the order between $w_i d_v(p_i^*, \ell^*)$ and $w_j d_v(p_j^*, \ell^*)$ may only be inverted by crossing some line of $bis_v(p_i^*, p_j^*)$. Therefore, for each region R^* determined by $B_v(P^*)$, there exists a permutation σ of the set $\{1, 2, \ldots, n\}$ so that

$$w_{\sigma(1)} d_v(p_{\sigma(1)}^*, \ell^*) \geq w_{\sigma(2)} d_v(p_{\sigma(2)}^*, \ell^*) \geq \cdots \geq w_{\sigma(n)} d_v(p_{\sigma(n)}^*, \ell^*), \quad \forall \ell^* \in R^*$$

from which it follows that

$$w_{\sigma(1)} \frac{d_v(p_{\sigma(1)}^*, \ell^*)}{\sqrt{1 + \ell_x^{*2}}} \geq w_{\sigma(2)} \frac{d_v(p_{\sigma(2)}^*, \ell^*)}{\sqrt{1 + \ell_x^{*2}}} \geq \cdots \geq w_{\sigma(n)} \frac{d_v(p_{\sigma(n)}^*, \ell^*)}{\sqrt{1 + \ell_x^{*2}}}$$

or equivalently,

$$w_{\sigma(1)} d((\ell^*)^*, p_{\sigma(1)}) \geq w_{\sigma(2)} d((\ell^*)^*, p_{\sigma(2)}) \geq \cdots \geq w_{\sigma(n)} d((\ell^*)^*, p_{\sigma(n)})$$

□

Definition 2 The tesselation of the dual plane determined by $B_v(P^*)$ induces the *Vertical Distance Completely-Ordered Dual Voronoi Diagram of P*, VDCODVD(P). Each region of VDCODVD(P) is the union of polygonal cells, not necessarily bounded, of the tessellation induced by $B_v(P^*)$ in which the ordering of weighted distances to the elements of P^* remains constant.

Definition 3 The set P^*, in the dual plane, induces a partition in regions forming a Dual Tessellation that will be denoted by $DT(P)$.

The vertices of the regions of VDCODVD(P) are described as follows:

Lemma 4 *Extreme points of regions of VDCODVD(P) are (at least) of one the three following types :*

– `Type 1 points`: *Vertices of a bisector of two elements of P^*, or equivalently duals of a line connecting two points of P.*
– `Type 2 points`: *Intersections of two edges of VDCODVD(P) corresponding to the bisectors of two pairs of P^* with a common element. In fact, in such extreme points six edges of VDCODVD(P) are concurrent.*
– `Type 3 points`: *Intersections of two edges of VDCODVD(P) corresponding to the bisectors of two pairs of P^* without common elements. In fact, in such extreme points four edges of VDCODVD(P) are concurrent.*

Proof Let v be a vertex of a region of VDCODVD(P). Since such a region is limited by bisectors of pairs of elements of P^*, three possibilities arise:

1. v is the intersection of the two branches of a bisector $bis_v(p_i^*, p_j^*)$, $p_i^*, p_j^* \in P^*$ and therefore $v \in p_i^* \cap p_j^*$. As a consequence, v is the dual point of a straight-line passing through p_i and p_j.
2. v is the intersection of two bisectors $bis_v(p_i^*, p_j^*)$ and $bis_v(p_i^*, p_t^*)$, $p_i^*, p_j^*, p_t^* \in P^*$ and therefore v is at the same weighted vertical distance of p_i^*, p_j^* and p_t^*. As a consequence, $bis_v(p_j^*, p_t^*)$ also passes through v and, since the three bisectors coincide in v, (at least) six edges of VDCODVD(P) are concurrent in v.
3. v is the intersection of two bisectors $bis_v(p_i^*, p_j^*)$ and $bis_v(p_s^*, p_t^*)$ where $p_i^*, p_j^*, p_s^*, p_t^*$ are four different elements of P^*. As a consequence, (at least) four edges of VDCODVD(P) are concurrent in v.

□

In Fig. 1, where $P = \{(1, -1), (1/2, -2), (-1, 2), (1/8, 2)\}$ with weights $W = \{1, 2, 1, 4\}$, a connected component of the Voronoi region $(1, 4, 3, 2)$ of the VDCODVD(P) is partially depicted. Vertex v_3 is a `type 1 point`, vertex v_4 is a `type 3 point` and vertices v_1 and v_2 are `type 2 points`.

In order to simplify the following proofs we will ignore for the moment details that would clutter our understanding of the geometric concepts we are dealing with. Therefore in case $k < n$ we will assume from now on that the points in P are in general position in the dual plane, in the sense that `type # points` $(\# = 1, 2, 3)$ are uniquely determined and that for each `type 3 point` there is no p_i^* line which crosses it. Similarly when $k = n$, we assume that vertices of $DT(P)$ are uniquely determined. These restrictions will be removed in the last section (more information

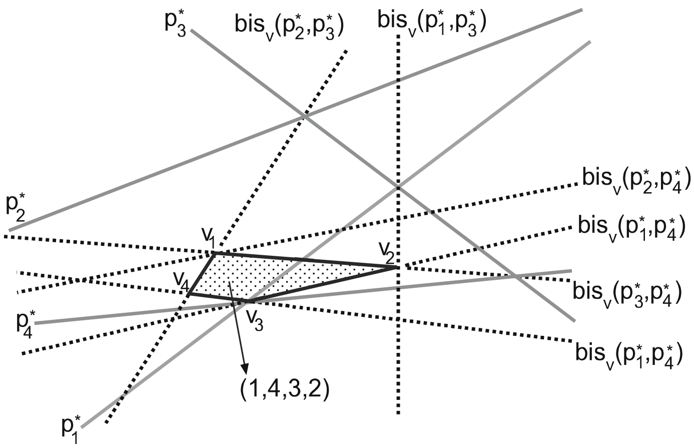


Fig. 1 Connected component of a Voronoi region

about the general treatment of degenerate cases in geometrical algorithms can be found in [4]). In particular, this general position assumption in the dual plane is equivalent to the following four conditions on the primal plane:

1. There are neither three aligned points in P nor four points at equal weighted distance from a straight line.
2. If ℓ crosses p_i, $p_j \in P$ then for each pair $\{p_s, p_t\}$, $p_s, p_t \in P \setminus \{p_i, p_j\}$, $w_s\, d(p_s, \ell) \neq w_t\, d(p_t, \ell)$ holds.
3. If a straight line ℓ satisfies $w_i\, d(p_i, \ell) = w_j\, d(p_j, \ell)$ and $w_s\, d(p_s, \ell) = w_t\, d(p_t, \ell)$ for four points $p_i, p_j, p_s, p_t \in P$ then $w_h\, d(p_h, \ell) \neq w_m\, d(p_m, \ell)$ for any $p_h, p_m \in P \setminus \{p_i, p_j, p_s, p_t\}$ and no point $p \in P \setminus \{p_i, p_j, p_s, p_t\}$ lies on ℓ.
4. Given $p_h, p_i, p_j \in P$, if there exists a straight line ℓ so that $w_h\, d(p_h, \ell) = w_i\, d(p_i, \ell) = w_j\, d(p_j, \ell)$, then $w_s\, d(p_s, \ell) \neq w_t\, d(p_t, \ell)$ for any pair of points $p_s, p_t \in P \setminus \{p_h, p_i, p_j\}$.

Lemma 5 *Each of the connected components of a region R^* of VDCODVD(P), is crossed by a single dual straight line $p^* \in P^*$ at most. In this case p^* is the element of P^* closest to the points of R^*, in terms of the weighted vertical distance.*

Proof Contrarily let us assume that there are two such straight lines $p_i^*, p_j^* \in P^*$ crossing R^*. In case $p_i^* \cap p_j^* \neq \emptyset$, $p_i^* \cup p_j^*$ partitions the dual plane into four regions, and each of the half straight lines composing $bis_v(p_i^*, p_j^*)$ crosses just one of them. Otherwise, when $p_i^* \cap p_j^* = \emptyset$ and $w_i \neq w_j$, then one of the straight lines composing $bis_v(p_i^*, p_j^*)$ separates p_i^* from p_j^*; if $w_i = w_j$ instead, then $bis_v(p_i^*, p_j^*)$ is composed of only one straight line separating p_i^* from p_j^*. Consequently, if p_i^* and p_j^* cross the same connected component of R^* then $bis_v(p_i^*, p_j^*)$ crosses R^* also, contradicting the fact that R^* is a region of VDCODVD(P).

For the second part, let us assume that $p^* \in P^*$ crosses R^*. According to the definition of R^*, the order of the weighted vertical distances to the elements of P^* remains fixed in this region. Since $d_v(\ell^*, p^*) = 0$ for all $\ell^* \in p^* \cap R^*$ and no other element of P^* crosses R^* (thus the distance from any line of $P^* \setminus \{p^*\}$ to all the inner points of R^* is strictly positive), p^* is necessarily the closest element of P^* to all the points of R^*. □

4 A Finite Dominating Set

Lemma 6 *For $k < n$ the objective function f^* in the dual plane is quasiconcave in each cell of $VDCODVD(P)$. It is also strongly quasiconcave on each segment of a non-vertical bisector in which the subset of points of P defining the function remains constant. When $k = n$, f^* is quasiconcave in each region of $DT(P)$.*

Proof

– Case $k < n$: Let R^*_C be a connected component of VDCODVD(P); the ordering of the weighted vertical distances from lines p^*_i to points in it remains constant, i.e. there is a fixed permutation σ of P such that $\forall \ell^* \in R^*_C$:

$$w_{\sigma(1)} d_v(\ell^*, p^*_{\sigma(1)}) \geq \cdots \geq w_{\sigma(n)} d_v(\ell^*, p^*_{\sigma(n)})$$

It follows that for any $\ell^* \in R^*_C$ the k largest weighted vertical distances are always obtained at the same straight lines $p^*_{\sigma(i)}$ $(i = 1, \ldots, k)$. Since $k < n$ this does not involve $p^*_{\sigma(n)}$ which, by Lemma 5, is the only straight line that can cross R^*_C. Therefore, the vertical point-line distance given in Eq. 3 becomes a linear function and the function

$$f^*(\ell^*) = \frac{\sum_{i=1}^{k} w_{\sigma(i)}\, d_v(\ell^*, p^*_{\sigma(i)})}{\sqrt{1 + \ell^{*2}_x}}$$

on R^*_C is a sum of linear functions divided by a positive convex function, yielding a quasiconcave function. Furthermore, the same argument can be used to show that the function is explicitly quasiconcave as can be checked using Theorem 51 of [12]. That is, $f^*(\lambda\ell^*_{(1)} + (1 - \lambda)\ell^*_{(2)}) > \min\{f^*(\ell^*_{(1)}), f^*(\ell^*_{(2)})\}$, $\forall \ell^*_{(1)}, \ell^*_{(2)} \in R^*_C$, $f^*(\ell^*_{(1)}) \neq f^*(\ell^*_{(2)})$, $\lambda \in (0, 1)$.

Let sg be a segment of a non-vertical bisector, say $bis_v(p^*_i, p^*_j)$, for which neither $w_i d_v(\ell^*, p^*_i)$ nor $w_j d_v(\ell^*, p^*_j)$ occupy the k-th and the $(k+1)$-th position in the descending order of weighted distances for all $\ell^* \in sg$. Since the numerator is a sum of linear functions, there exist $A, B, C \in \mathbb{R}$ such that

$$f^*(\ell^*) = \frac{A\ell^*_x + B\ell^*_y + C}{\sqrt{1 + \ell^{*2}_x}}, \quad \ell^* = (\ell^*_x, \ell^*_y) \in sg.$$

Since sg is a line segment the coordinates of ℓ^* are related by $\ell^*_y = a\ell^*_x + b$. Therefore,

$$f^*(\ell^*) = \frac{A\ell^*_x + B(a\ell^*_x + b) + C}{\sqrt{1 + \ell^{*2}_x}} = \frac{A'\ell^*_x + C'}{\sqrt{1 + \ell^{*2}_x}}.$$

Since the numerator is the sum of weighted vertical distances to the farthest straight lines then $(A', C') \neq (0, 0)$ and the function $f^*(\ell^*)$ is strictly monotone on sg. Therefore, $f^*(\lambda\ell^*_{(1)} + (1 - \lambda)\ell^*_{(2)}) > \min\{f^*(\ell^*_{(1)}), f^*(\ell^*_{(2)})\}$, $\forall \lambda \in (0, 1)$, and f^* is strongly quasiconcave on sg.

– Case $k = n$: In each region R^* of DT(P) all the vertical distances to lines of P^* are linear, since none of the terms inside absolute values in Eq. 3 changes sign. Therefore, on R^* f^* is again a sum of linear functions divided by a positive convex function, yielding a quasiconcave function.

□

From the Lemma 6, a finite dominating set for the k-Centrum straight-line problem (i.e. a set containing at least a solution) is composed, when $k < n$, of the primals of the extreme points of the regions of VDCODVD(P) and, when $k = n$, of the primals of the extreme points of the regions of $DT(P)$. In the following theorem it is shown that this set can be reduced: when $k < n$ type 3 points can be removed from the finite dominating set and, when $k = n$, type 2 and type 3 points can be removed from this set.

Theorem 7 *A finite dominating set for the k-Centrum straight-line problem is composed of the primals of type* 1 *and type* 2 *points, i.e. all straight lines passing through two points of P and all straight lines at equal weighted distances from three points of P. When* $k = n$ *it may be restricted to the primals of type* 1 *points only.*

Proof First we will consider the case $k < n$. Candidate straight lines correspond under the geometrical duality map to candidate points for the dual objective function. By Lemma 6, and since a quasiconcave function attains its minimum in an extreme point of any polygonal region, the set of extreme points (possibly at infinity) of each region of VDCODVD(P), i.e. all points of type 1, 2 or 3 and extreme points at infinity, form a dominating set for f^*.

(i) Let us assume that the minimum is attained in a vertex of VDCODVD(P). However, type 3 points are dominated by other extreme points or there exists a type 1 or 2 point whose objective function value is at least equal to that of the type 3 point. To prove this, let $\overline{\ell}^*$ be such a point, i.e. $\overline{\ell}^*$ is a point of intersection of the bisectors of two disjoint pairs $\{p_i^*, p_j^*\}$ and $\{p_s^*, p_t^*\}$. Therefore, $w_i d_v(\overline{\ell}^*, p_i^*) = w_j d_v(\overline{\ell}^*, p_j^*)$ and $w_s d_v(\overline{\ell}^*, p_s^*) = w_t d_v(\overline{\ell}^*, p_t^*)$. From our assumption of the general position we know that all $p_m \in P \setminus \{p_i, p_j, p_s, p_t\}$ satisfy $w_i d_v(\overline{\ell}^*, p_i^*) \neq w_m d_v(\overline{\ell}^*, p_m^*) \neq w_s d_v(\overline{\ell}^*, p_s^*)$ and we may assume that $w_i d_v(\overline{\ell}^*, p_i^*) < w_s d_v(\overline{\ell}^*, p_s^*)$. Let H_{ij}^* and H_{ji}^* be the two half planes defined by the branch of $bis_v(p_i^*, p_j^*)$ passing through $\overline{\ell}^*$, and let U^* be a small enough $\overline{\ell}^*$-neighborhood so that its intersection with any bisector different from $bis_v(p_i^*, p_j^*)$ and $bis_v(p_s^*, p_t^*)$, as well as with any element of P^*, is empty (Fig. 2). If the distances to the elements of P^* are ordered from greater to lesser and in case of tie the order of the subindexes, two possibilities arise:

a. If none of the pairs of distances $\{w_i d_v(\overline{\ell}^*, p_i^*), w_j d_v(\overline{\ell}^*, p_j^*)\}$ and $\{w_s d_v(\overline{\ell}^*, p_s^*), w_t d_v(\overline{\ell}^*, p_t^*)\}$ occupy the k-th and $(k+1)$-th positions, then the set of k straight lines the farthest away from $\overline{\ell}^*$ does not change in U^*, and neither does the function f^* which is therefore quasiconcave in U^* and strongly quasiconcave on at least one bisector crossing $\overline{\ell}^*$, so it cannot be a local minimum of f^*.

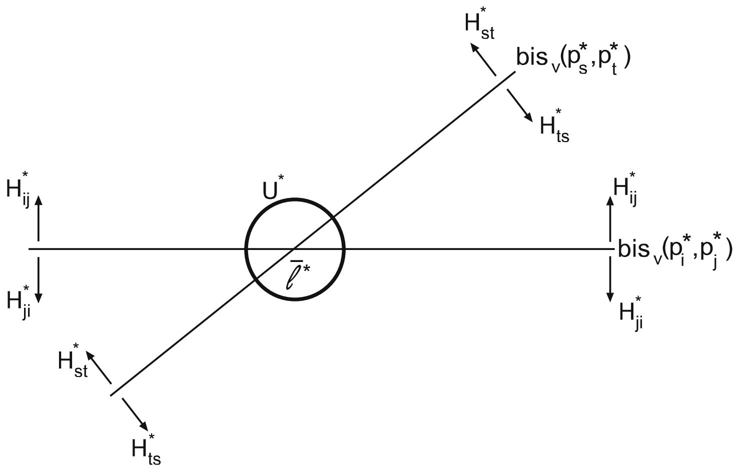
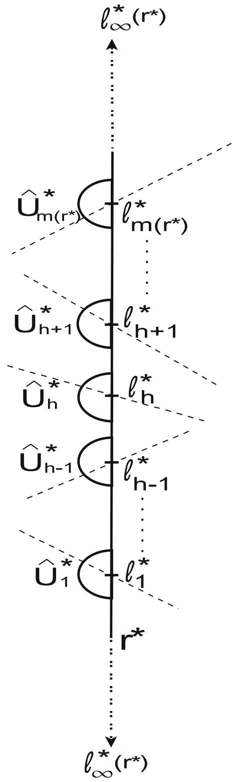


Fig. 2 Theorem 7 proof

b. If one of the pairs of distances $\{w_i d_v(\overline{\ell}^*, p_i^*), w_j d_v(\overline{\ell}^*, p_j^*)\}$ and $\{w_s d_v(\overline{\ell}^*, p_s^*), w_t d_v(\overline{\ell}^*, p_t^*)\}$ occupies the k-th and $(k+1)$-th positions, say $w_i d_v(\overline{\ell}^*, p_i^*) = w_j d_v(\overline{\ell}^*, p_j^*)$, consider the half neighborhood $\hat{U}^* = U^* \cap H_{ij}^*$ of $\overline{\ell}^*$, in which the set of k straight lines of P^* farthest away remains constant. Let r^* denote the boundary of the half plane defining $\hat{U}^*$ (from here on it will be called the straight line support of $\hat{U}^*$).
If r^* is a non-vertical straight line then f^* is a quasiconcave function in $\hat{U}^*$ and strongly quasiconcave in $r^* \cap \hat{U}^*$; thus since ℓ^* is not an extreme point of $\hat{U}^*$ it cannot be a local optimum of f^*.
If r^* is a vertical straight line, let $\ell_1^*, \ell_2^*, \ldots, \ell_{m(r^*)}^* \in r^*$ be the vertices of VDCODVD(P) in r^*, ordered from lesser to greater y-coordinate. Let $\ell_\infty^*(r^*)$ denote the infinity point of r^*. For each type 3 point $\ell_s^* \in \{\ell_1^*, \ell_2^*, \ldots, \ell_{m(r^*)}^*\}$, let $\hat{U}_s^*$ denote the half neighborhood of this point which is defined in a similar way to $\hat{U}^*$.
Let us first assume that $\ell_h^* = \overline{\ell}^*$ for some $h,\ 2 \leq h \leq m(r^*) - 1$. Then, according to the definition of $\hat{U}_h^*$ the set of k elements of P^* farthest away from ℓ_h^* remains constant on $\hat{U}_h^*$ and therefore f^* is a linear function on the segment $\overline{\ell_{h-1}^* \ell_{h+1}^*}$. There are two possibilities:

- f^* is constant on $\overline{\ell_{h-1}^* \ell_{h+1}^*}$. In this case (see Fig. 3), if all the vertices $\ell_1^*, \ell_2^*, \ldots, \ell_{m(r^*)}^*$ were type 3 points and the corresponding half neighborhoods $\hat{U}_1^*, \hat{U}_2^*, \ldots, \hat{U}_{m(r^*)}^*$ were supported by r^*, then the set of k elements of P^* farthest away would remain constant along the vertical straight line r^* and, given that f^* is linear on r^* and constant on $\overline{\ell_{h-1}^* \ell_{h+1}^*}$, it would also be constant in r^*. This cannot happen since, when moving from ℓ_1^* to $\ell_\infty^*(r^*)$ along r^*, the function f^* is strictly increasing. Therefore some ℓ_d^* with $1 \leq d \leq h-1$ exists which is either not a type 3 point or its neighborhood is not supported by r^*. In a similar way, when moving from $\ell_{m(r^*)}^*$ to $\ell_\infty^*(r^*)$ along r^*, the function f^* is

Fig. 3 Vertices in a vertical line



strictly increasing and some $\ell^*_{d'}$ with $h+1 \le d' \le m(r^*)$ exists which is either not a type 3 point or its neighborhood is not supported by r^*. Since the arguments are similar we will only deal with the first case, i.e. with ℓ^*_d. If $\hat{U}^*_d$ is not supported by r^* then f^* is quasiconcave in $\hat{U}^*_d$ and strongly quasiconcave on the supporting straight line, and it follows that ℓ^*_d is not a local optimum. Since $f^*(\ell^*_h) = f^*(\ell^*_d)$, $\ell^*_h(=\overline{\ell}^*)$ it is not a global optimum either. If ℓ^*_d is not a type 3 point, we have found a non-type 3 point providing the same value for the objective function than $\ell^*_h(=\overline{\ell}^*)$.

- f^* is not constant on $\overline{\ell^*_{h-1}\ell^*_{h+1}}$. Let us assume, without loss of generality, that f^* decreases when moving from ℓ^*_{h+1} towards ℓ^*_{h-1}; then, as in the former case with the points ℓ^*_d; $1 \le d \le h-1$, we obtain the same result.

In case $h=1$ or $h=m(r^*)$ then it is sufficient to take $\ell^*_0 = \ell^*_{m(r^*)+1}(= \ell^*_\infty(r^*))$, respectively, and since f^* is decreasing when moving from $\ell^*_\infty(r^*)$ towards ℓ^*_1 on r^* and from $\ell^*_\infty(r^*)$ towards $\ell^*_{m(r^*)}$, we can proceed similarly to the previous case.

(*ii*) Let us assume that the minimum of f^* is not attained in a vertex of DVCODVD(P), then there exists a minimum of f^* reached at the infinity point of a half straight line of DVCODVD(P). In this case the solutions to the *k*-Centrum problem are vertical straight lines in the primal plane. Let us denote V as the set of such vertical lines and G_α as a rotation whose vertex is the origin; its angle is chosen in order that the transformed set $G_\alpha(V)$ does not contain any vertical line. Then, since the straight lines of $G_\alpha(V)$ are optima of the problem associated to $G_\alpha(P)$ and are not vertical, the dual points are type 1 or 2 extreme points of DVCODVD(P) and, therefore, at least one line of $G_\alpha(V)$ crosses two points of $G_\alpha(P)$ or it is at equal weighted distance from three points.

Finally, let us consider the case in which $k = n$. Using Lemma 6, if R^* is a region of DT(P), then f^* is a quasiconcave function in R^* and, therefore, at least a minimum of f^* exists which is a vertex of DT(P) or an infinity point of any of the half straight lines forming the edge set of DT(P). In the first case, it follows that a solution corresponding to a non vertical line crossing two points of P exists. In the second case, the solution corresponds to a vertical straight line, which, by means of a similar case (ii) rotation argument, can be proven to cross two points of P. □

5 An Algorithm

The dual finite dominating set derived in the last section consists of intersections between elements of $B_v(P^*)$ when $k < n$ and straight lines of P^* when $k = n$, respectively. Since there are specific and efficient algorithms for the latter case [9], we will discuss here only the first. In order to examine the candidates we will consider each pair $\{p_i, p_j\}$ of points and the bisector $bis_v(p_i^*, p_j^*)$ of their dual points, taking each of the two possible lines of the bisector $bis_v^m(p_i^*, p_j^*)$; $m = 1, 2$ separately.

Let

$$C^*_{ij,m} = (p_i^* \cap p_j^*) \cup \{bis_v^m(p_i^*, p_j^*) \cap bis_v(p_i^*, p_{j'}^*) : j' \neq i, j\}$$

be the candidate set on the branch m of $bis_v(p_i^*, p_j^*)$, and $C_{ij}^m = \{(C^*_{ij,m})^*\} \cup \{\ell_{ij}^v\}$ be the corresponding straight-line set on the primal plane to which the vertical straight line ℓ_{ij}^v passing through the common point $O_{ij}^m = (bis_v^m(p_i^*, p_j^*))^*$ has been added.

Then the straight-line set

$$\cup_{i=1}^n \cup_{j=i+1}^n \cup_{m=1}^2 C_{ij}^m$$

contains the dominating set described in Section 4, and we may solve the problem by repeated restriction to each C_{ij}^m.

The problem of finding the line in C_{ij}^m that minimizes the sum of the k largest weighted distances may be transformed in an equivalent non-weighted one,

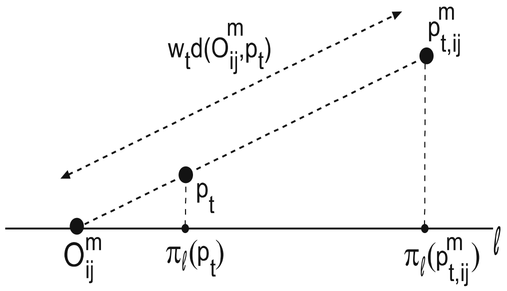
$$\min_{\ell \in C_{ij}^m} \max_{\substack{Q \subset P_{ij}^m \\ |Q| = k}} \sum_{p^m \in Q} d(\ell, p^m) \tag{5}$$

with given point set

$$P_{ij}^m = \{p_{t,ij}^m = O_{ij}^m + w_t \overrightarrow{O_{ij}^m p_t} \mid p_t \in P\}.$$

Note that under this transformation the weighted distance from a point p_t of P to any straight line ℓ of C_{ij}^m is equal to the unweighted distance from $p_{t,ij}^m$ to ℓ (see Fig. 4.)

Fig. 4 Triangles similarity



In order to determine the k farthest points in P^m_{ij} from the straight line $\ell \in C^m_{ij}$, the algorithm described in [14] will be applied. When k is fixed, after an $O(kn + n \log n)$ preprocessing time, the k ordered farthest points to any line ℓ are computed in $O(k + \log n)$. If k is part of the input, the preprocessing time is $O(n^2)$.

A short description of the algorithm follows:

Input: The sets P of points and W of associated non-negative weights to the points in P.

Output: A t-centrum straight line for each $t = 1, 2, \ldots, k$.

For each pair $\{p_i, p_j\} \in P,\ m = 1, 2$, repeat:

- Step 1: Obtain the sets C^m_{ij}
- Step 2: Construct the set of non-weighted points P^m_{ij} and preprocess it as in [14].
- Step 3: For each straight line $\ell \in C^m_{ij}$, determine the k farthest points (in decreasing order of distances) from ℓ by way of the algorithm described in [14] and for each t-centrum problem ($t = 1, 2, \ldots, k$) evaluate the objective function, and update if necessary the best solution found.

Theorem 8 *All t-centrum* ($1 \le t \le k$) *straight lines may be computed in* $O(kn^3 + n^3 \log n)$ *time. When k is part of the input, this complexity is* $O(n^4)$.

Proof The algorithm described above has the announced time-complexity. Indeed, Step 1 and the construction of the set P^m_{ij} takes $O(n)$ time. The preprocessing of P^m_{ij} in Step 2 requires $O(kn + n \log n)$ time. The search of the k farthest points from each straight line $\ell \in C^m_{ij}$ and the computation of the corresponding objective functions use $O(k + \log n)$ time; since there are $O(n)$ straight lines in C^m_{ij}, Step 3 requires $O(k\,n + n\ \log n)$ time. Finally there are $O(n^2)$ C^m_{ij} sets.

The last result follows from $k = O(n)$. □

5.1 A Short Example

A set P and its associated weight set W, where

$$P = \{(0.5, 3), (2.5, .8), (3.7, 3.8), (6, 5), (7, 1.2), (8.5, 2.9), (8.6, 4.6), (12, 1.6)\}$$

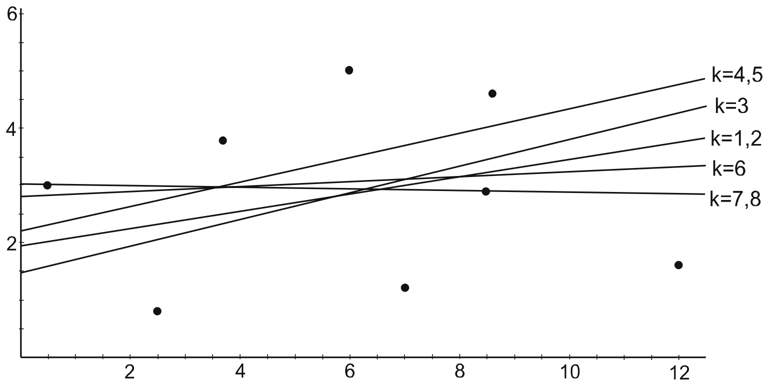


Fig. 5 Solutions for all the k-values

and

$$W = \{1/145, 1/130, 1/704, 1/322, 1/560, 1/237, 1/116, 1/185\}$$

are considered. The points of P represent the capitals of Andalusia and the associated weights are the inverse of the populations (in thousand inhabitants) of these cities. In the Fig. 5 the solutions for all the k-values have been represented.

6 An Alternative Procedure

In the last section an easily implementable algorithm based in the finite dominating set given in Section 4 has been described. Nevertheless this finite dominating set could be too large when it is desired to find a k-centrum straight-line only for a fixed k-value. In what follows we will describe a procedure (suggested by the authors in [10]) in order to reduce the finite dominating set and obtain an alternative algorithm, with less complexity, when only a fixed k-value is considered.

For each point $p = (p_x, p_y) \in P$ and its dual $p^* : y^* = p_x x^* - p_y$ let us consider the dual planes defined by

$$\alpha_1^*(p^*) : z^* = w_p(p_x x^* - p_y - y^*) \text{ and } \alpha_2^*(p^*) : z^* = -w_p(p_x x^* - p_y - y^*)$$

It is clear that for any dual point $\ell^* = (\ell_x^*, \ell_y^*)$ and each $p^* \in P^*$

$$w_p d_v(\ell^*, p^*) = \max\{w_p d_v((\ell_x^*, \ell_y^*, 0), \alpha_1^*(p)), w_p d_v((\ell_x^*, \ell_y^*, 0), \alpha_2^*(p))\}$$

Let us consider the set of $2n$ dual planes $\mathcal{H} = \{\alpha_1^*(p), \alpha_2^*(p) : p \in P\}$ and $\mathcal{A}(\mathcal{H})$ the arrangement of $\mathcal{H}$ (note that $\mathcal{A}(\mathcal{H})$ is an arrangement in the three-dimensional dual space). Then for each dual point, the k highest planes correspond to the k-farthest lines of P^*. Once the $2n$ planes have been computed, the procedure is similar to those used by Chazelle and Edelsbrunner [3] in order to compute the k-order Voronoi diagram: compute the $n-k$ level of $\mathcal{A}(\mathcal{H})$, i.e. the closure of the points that lie in some plane and have $n-k$ planes strictly below them (and therefore have k planes above them, including the one they lie on) and project the edges and vertices of the $n-k$ level onto the plane $z^*=0$. This projection (in what follows $\pi_{n-h}(\mathcal{H})$), induces a tesselation in the dual plane such that for each cell all its points have the same set of k highest planes of $\mathcal{H}$ and are duals to lines that have the same set of k farthest points of P. Therefore, in each cell, the sum in the objective function is over a fixed set.

Moreover, the sign of each term in the numerator is fixed and, therefore, the objective function is concave over each cell of the tesselation and the optimum values are obtained in the vertices of the cells and infinite points (corresponding to vertical primal lines) of the dual straight lines.

The k-level of an arrangement of n planes can be computed in $O(n \log f + f^{1+\varepsilon})$ [2] where f is the complexity of the k-level. Since the best upper bound on this complexity is $O(nk^{3/2})$ [18], when k is a fixed value $\pi_{n-h}(\mathcal{H})$ and therefore a finite dominating set can be computed in $O(n \log n + n^{1+\varepsilon})$.

7 Particular Cases, Extensions and Further Research

7.1 Strong Characterization of Solutions

If weights are different pairwise (i.e. $w_i \neq w_j$, for $i \neq j$) then there is no vertical bisector in the dual plane. Therefore, according to the proof of Theorem 7, no type 3 point can be optimal. The strong property of the solutions follows:

Corollary 9 *If the weights of points in P are different pairwise then any optimal k-Centrum straight-line ℓ_{opt} satisfies one of the following conditions:*

1. *ℓ_{opt} crosses two points $p_i, p_j \in P$.*
2. *ℓ_{opt} is at equal weighted distance from three points in P:*

$$w_i d(\ell_{opt}, p_i) = w_j d(\ell_{opt}, p_j) = w_h(\ell_{opt}, p_h).$$

7.2 Degeneracies

In this subsection the case where points are not in general position is discussed. In order to include cases of degeneracy the classification of the vertices of VDCODVD(P) will be reformulated:

Definition 10 If points in P are not in general position we will say that a vertex of VDCODVD(P) is

- a `type 1 point` if it is vertex of a bisector of two elements of P^*, or, equivalently, dual of a line connecting two points of P.

– a `type 2 point` if it is the intersection of two edges of VDCODVD(P) corresponding to the bisectors of two pairs of P^* with a common element and it is not a `type 1 point`.
– a `type 3 point` if it is the intersection of two edges of VDCODVD(P) corresponding to the bisectors of two pairs of P^* without common elements and it is neither a `type 1` nor a `type 2 point`.

The finite dominating set for the straight line k-Centrum problem remains valid when points are not in general position. The only issue to be checked is that any `type 3 point` can be removed from the finite dominating set, either because it is dominated or because there is a `type 1 or 2 point` with the same objective function value. Let $\overline{\ell}^*$ be a `type 3 point`. The first observation is that only one dual straight line $p_t^* \in P^*$ can cross it because, otherwise, it would be a `type 1 point`. For the same reason, for any bisector $bis_v(p_i^*, p_j^*)$ crossing $\overline{\ell}^*$, if some p_t^* exists also crossing it, then $t \notin \{i, j\}$. The only straight line $p_t^* \in P^*$ that can cross $\overline{\ell}^*$ is the nearest one, and therefore p_t^* does not appear in the objective function. From there, the application of the rest of the proof of Theorem 7 does not need any further explanation.

7.3 More General Norms

For general norms, the expression Eq. 2 becomes:

$$d_\mu(p, \ell) = \frac{d_v(\ell^*, p^*)}{\mu^0(u)} \tag{6}$$

where μ^0 is the polar norm of μ, $u = (\ell_x^*, -1)$ with ℓ_x^* the slope of the straight line ℓ^*. If μ is a smooth norm then μ^0 is strictly convex. Therefore, the dual objective function is:

$$f_\mu^*(\ell^*) = \max_{\substack{Q^* \subset P^* \\ |Q^*| = k}} \sum_{p^* \in Q^*} \frac{w_p d_v(p^*, \ell^*)}{\mu^0(\ell_x^*, -1)}. \tag{7}$$

This is a quasiconcave function on each cell of VDCODVD(P) and, therefore, attains its minimum at one of its vertices or in the infinite point associated to one of the edges. Furthermore, since the function $\mu^0(\ell_x^*, -1)$ is strictly convex on each non-vertical segment, it is possible to define a half-neighbourhood where f_μ^* is strongly quasiconcave on the segment of the supporting straight line. Therefore, the finite dominating set of Theorem 7 remains the same for smooth norms.

In order to justify the validity of the algorithm for the case of smooth norms, some comments are needed. On the one hand, the following lemma proves that in the case of a general norm the weighted problem can be transformed into an unweighted one such as Eq. 5.

Lemma 11 *Given two points p and x_0, a general norm μ, a positive value w and the point $p' = x_0 + w\overrightarrow{x_0 p}$, for each straight line ℓ crossing x_0 the following holds: $d_\mu(p', \ell) = w d_\mu(p, \ell)$.*

Proof Denoting the scalar product by $< \cdot, \cdot >$, consider the straight line $\ell = \{x \in \mathbb{R}^2 :< u, x >= c\}$, then the distance between p' and ℓ is:

$$\begin{aligned} d_\mu(p', \ell) &= \frac{| < u, p' > -c|}{\mu^0(u)} \\ &= \frac{| < u, x_0 + w(p - x_0) > -c|}{\mu^0(u)} \\ &= \frac{| < u, x_0 > +w < u, p > -w < u, x_0 > -c|}{\mu^0(u)} \\ &= \frac{w| < u, p > -c|}{\mu^0(u)} \\ &= w d_\mu(p, \ell). \end{aligned}$$

□

Finally, given a set of points P, the algorithm proposed in [14] computes the k farthest points of P to a query line ℓ using the arrangement of the set of lines P^*, the dual point ℓ^* and the vertical distance in the dual plane. Since all these structures are independent of the norm μ considered, this algorithm can be applied in this case for computing the k farthest points from each candidate line in the set C_{ij}^m. However, the norm itself will have to be used for evaluating the objective function.

7.4 Further Research

Several k-Centrum type problems are interesting but lie out of the scope of this paper. For example, both the problems of finding a median and a center hyperplane in $\mathbb{R}^3$ have been studied and $O(n^3)$ and $O(n^{(17/11)+\epsilon})$ algorithms have been suggested in [7] and [8], and [1], respectively. As far as the authors are aware no paper on the k-Centrum Hyperplane has been published. For other geometrical structures such as segments, half-lines and circles, the extreme cases of $k = 1$ and $k = n$ have also been studied. Finally, as in [17] and [15], the application of non-smooth norms and more generally that of gauges would be of particular interest in location problems in which these kinds of distances are appropriate.

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