J Math Model Algor (2007) 6:547–561
DOI 10.1007/s10852-006-9056-z

On the Computational Complexity of the Minimum Committee Problem

Mikhail Yu. Khachay

Received: 3 October 2006 / Accepted: 31 October 2006 /
Published online: 26 January 2007

Abstract Two special cases of the Minimum Committee Problem are studied, the Minimum Committee Problem of Finite Sets (MCFS) and the Minimum Committee Problem of a System of Linear Inequalities(MCLE). It is known that the first of these problems is NP-hard (see (Mazurov et al., *Proc. Steklov Inst. Math.*, 1:67–101, 2002)). In this paper we show the NP-hardness of two integer optimization problems connected with it. In addition, we analyze the hardness of approximation to the MCFS problem. In particular, we show that, unless $NP \subset TIME(n^{O(\log\log n)})$, for every $\varepsilon > 0$ there are no approximation algorithms for this problem with approximation ratio $(1-\varepsilon)\ln(m-1)$, where m is the number of inclusions in the MCFS problem. To prove this bound we use the SET COVER problem, for which a similar result is known (Feige, *J. ACM*, 45:634–652, 1998). We also show that the Minimum Committee of Linear Inequalities System (MCLE) problem is NP-hard as well and consider an approximation algorithm for this problem.

Key words computational complexity • NP-completeness • set cover problem • graph 3-colorability problem • minimum committee problem • approximation algorithms.

Mathematics Subject Classifications (2000) 90C27 • 68Q17 • 68Q32.

1 Introduction

We consider a combinatorial optimization problem known as the Minimum Committee (MC) problem. This problem is closely connected with three areas of the operations research: voting theory, optimization, and pattern recognition. In the voting theory [3–5], several voting procedures based on different logics (democracies) are

M. Y. Khachay (✉)
Institute of Mathematics and Mechanics, Russian Academy of Sciences,
Ural Branch. S.Kovalevskoj 16, 620219 Ekaterinburg, Russia
e-mail: mkhachay@imm.uran.ru

studied. A committee is just a mathematical model for the voting procedure based on the simple majority rule.

In pattern recognition, different collections of empirical algorithms are considered [6, 7]. So called *perceptron algorithms* should be distinguished among them. As one can prove , the 2-layer perceptron with non-negativity constraint for all weights of its second layer is just another formulation for the committee discrimination rule.

Finally, in optimization a problem to be solved is often *inconsistent* [8]. There are several reasons for this fact. In terms of linear programming, e.g., the problem is inconsistent when its primal or dual (or both) constraints systems are infeasible. To correct this situation, one can utilize several technics, e.g. Chebyshev approximation. The committee solutions technique is one of them [1, 8, 9].

In all above cases it is desirable to find the most simple committee construction that leads us to the Minimum Committee Problem.

2 Minimum Committee Problem

Let some set X and m nonempty subsets $D_1, D_2, \ldots, D_m$ of X be given. Consider an abstract system of inclusions in X

$$x \in D_j \qquad (j \in \mathbb{N}_m = \{1, 2, \ldots, m\}) \tag{1}$$

If there is an element $x \in \bigcap\limits_{j \in \mathbb{N}_m} D_j$ then system (1) is called feasible and x is called a solution of (1). Otherwise system (1) is called infeasible in the ordinary sense.

We call [1] a finite sequence $Q = (x^1, x^2, \ldots, x^q)$ a *committee solution with q elements* (or just a *committee*) of system (1) if

$$\left|\{i \;:\; x^i \in D_j\}\right| > \frac{q}{2}, \quad \text{for all } j \in \mathbb{N}_m.$$

A committee solution of system (1) with a minimum number of elements q is called a *minimum committee solution*. It is evident that if system (1) is feasible then every minimum committee solution $Q = (x)$ where $x \in \bigcap\limits_{j \in \mathbb{N}_m} D_j$. Therefore the notion of the minimum committee solution is a generalization of the notion of the ordinary solution.

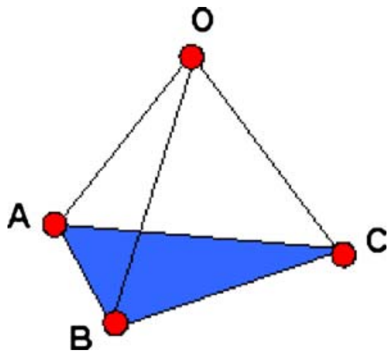
The Minimum Committee (MC) Problem:

Let a ground set X and a finite collection of subsets $D_1, D_2, \ldots, D_m$ be given. Find a committee solution for system (1) *with a minimum number of elements q.*

The MC problem is called *feasible* if and only if system (1) has any committee solution. Consider the following example (see Fig. 1). Here $X = \mathbb{R}^3$, $m = 4$, D_1 is the plane ABC and D_2, D_3, D_4 are the segments AO, BO and CO, correspondingly. It can be easily verified that in this case system (1) is infeasible but has committee solutions and the sequence $Q = (A, B, C, O, O)$ is a minimum committee solution.

It is useful to reformulate the MC problem in terms of the integer linear programming. As usual, we call a maximal, by inclusion, feasible subsystem of an infeasible system a *maximal feasible subsystem* (or *mfs*). Let $J_1, \ldots, J_T$ be the index sets of

Fig. 1 Example of a committee solution



all maximal feasible subsystems of system (1). Let us consider two $m \times T$ incidence matrices A and B, where

$$\begin{array}{lll} a_{ji} = 1, & b_{ji} = 1 & \text{if } j \in J_i, \\ a_{ji} = 0, & b_{ji} = -1 & \text{otherwise} \end{array}$$

and two programs

$$\min \left\{ \mathbf{1}'t \mid Bt \geq \mathbf{1}, t \in \mathbb{Z}_+^T \right\}, \tag{2}$$

$$\min \left\{ s \ : \ \begin{array}{ll} At \geq s\mathbf{1}, & t \in \mathbb{Z}_+^T \\ \mathbf{1}'t \leq 2s - 1, & s \in \mathbb{N} \end{array} \right\}. \tag{3}$$

The following theorem is known.

Theorem 1 ([9]) *Problems MC, (2) and (3) are simultaneously feasible or infeasible. The sets of optimal solutions of problems (2) and (3) are isomorphically embedded into the solution set of the MC problem.*

In this paper, we consider two special cases of the MC problem:

1. The case of the problem, where the set X and all its subsets D_j are finite (we shall call this problem the MCFS). We shall show that the MCFS problem is NP-hard and the equivalent problems (2) and (3) have the same property. Also, we shall estimate an approximation threshold of this problem.
2. The case, where the ground set X is an n-dimensional rational vector space $\mathbb{Q}^n$ and subsets $_j$ are open halfspaces. This problem will be called the MCLE. As shown below, it is also NP-hard. We shall consider a new approximation algorithm, its approximation ratio and computational complexity.

3 Minimum Committee of Finite Sets Problem

In this section, we consider a special case of MC problem when all sets in system (1) are finite.

The Minimum Committee of Finite Sets (MCFS) Problem:

A finite set $X = \{x^1, x^2, \ldots, x^p\}$ and a collection of its subsets $D_1, D_2, \ldots, D_m$ are given. It is required to find a committee solution for system (1) with a minimum number of elements q.

3.1 Computational Complexity

Let us agree to encode an instance of the MCFS by $m \times p$ matrix C, where

$$c_{ji} = \begin{cases} 1, & \text{if } x^i \in D_j, \\ -1, & \text{otherwise.} \end{cases}$$

Further, without loss of a generality, we can assume that for any committee $Q = (y^1, y^2, \ldots, y^q)$ there are natural numbers $k \le p$, and $q_1, q_2, \ldots, q_k$, where

$$q_1 + q_2 + \ldots + q_k = q,$$

and

$$1 \le i_1 < i_2 < \ldots < i_k \le p$$

such that

$$\begin{aligned} y^1 = y^2 = \ldots = y^{q_1} &= x^{i_1}, \\ y^{q_1+1} = \ldots = y^{q_1+q_2} &= x^{i_2}, \\ &\ldots \\ y^{q_1+\ldots+q_{k-1}+1} = \ldots = y^q &= x^{i_k}. \end{aligned}$$

Therefore, we can represent the sequence Q as a multiset in the form

$$\left\{(x^{i_1}, q_1), (x^{i_2}, q_2), \ldots, (x^{i_k}, q_k)\right\}.$$

The MCFS problem is a combinatorial one and there is no efficient algorithm for this problem unless $P = NP$.

Theorem 2 ([1]) *The MCFS problem is NP-hard.*

Let us notice that the MCFS problem remains NP-hard even when every set D_j except, maybe, one, satisfies the condition $|D_j| \le 3$.

Theorem 3 *Problems MCFS, (2), and (3) are polynomially equivalent.*

Proof Let us prove that the MCFS and the integer program (2) can be polynomially reduced to each other. Let an instance of the MCFS problem be given by the $m \times p$ matrix C. Let matrix B consist of all undominatable columns of C. Matrix B can be constructed in a polynomial time of m and p. Without loss of a generality, we assume that B consists of the first T columns of C. The instance of problem (2) determined by the matrix B is a required one. Indeed, let $\bar{t} = [\bar{t_1}, \bar{t_2}, \ldots, \bar{t_T}]$ be the optimal solution of problem (2). Then, by virtue of the construction of the matrix B, the sequence

$$\bar{Q} = \left\{(x^i, \bar{t_i}) \mid i \in \mathbb{N}_T,\ \bar{t_i} > 0\right\}$$

is the required solution of the initial problem MCFS (minimum committee).

On the other hand, let the matrix B define an individual problem (2). Let us consider the instance of the MCFS problem with $C = B$. Let

$$\bar{Q} = \left\{(x^{i_1}, q_1), (x^{i_2}, q_2), \ldots, (x^{i_k}, q_k)\right\}$$

be the minimum committee in the MCFS problem, then the vector $\bar{t} \in Z_+^T$, where

$$t_l = \begin{cases} q_j, & \text{if } l = i_j, \\ 0, & \text{otherwise,} \end{cases}$$

is the optimal solution of problem (2). □

Corollary 1 *Problems* (2) *and* (3) *are NP-hard.*

It is interesting that a statement like Theorem 2 does not hold for the general MC problem.

3.2 Approximation Threshold

A general issue in studying the NP-hard optimization problem is designing so-called *approximation* algorithms. As usual, we call an algorithm for a combinatorial optimization problem an approximation algorithm (with approximation ratio r) if for each instance

$$f^* = \min\{ f(x) \mid x \in M\}$$

of length L of the problem under consideration this algorithm finds a feasible solution x_{app} with

$$\frac{f(x_{app})}{f^*} \leq r$$

in a polynomial time in L.

Another problem that should be considered in studying the NP-hard problem is to find a threshold for which it can be proved that there are no approximation algorithms with ratios less then this bound (under some reasonable assumptions, e.g. $NP \neq P$). To prove the existence such a threshold for the MCFS problem, we shall take advantage of known results for the famous SET COVER problem.

The SET COVER Problem:

Let a finite set $S = \{s_1, s_2, \ldots, s_m\}$ *and a nonempty collection of its subsets* $C = \{c_1, c_2, \ldots, c_l\} \subseteq 2^S$ *be given. It is required to find a minimum cardinality subset* $C' \subseteq C$ *that covers* S (*i.e.,* $\bigcup_{c_i \in C'} c_i = S$).

For the SET COVER problem the following results are known.

Theorem 4 ([10]) *Unless $P = NP$ there is no polynomial time algorithm that approximates the set cover within ratio $\frac{1}{4}\log_2 m$.*

Theorem 5 ([2]) *If there is some $\varepsilon > 0$ such that a polynomial time algorithm can approximate the set cover within $(1-\varepsilon)\ln m$, then*

$$NP \subset TIME(n^{O(\log\log n)}).$$

A similar bound may be proved for the MCFS problem as well.

Lemma *The existence of an approximation algorithm with ratio r for the MCFS problem implies the existence of a polynomial time algorithm that approximates the set cover within the same ratio.*

Proof

1. Let us reduce the SET COVER problem to the MCFS. Let the sets $S = \{s_1, s_2, \ldots, s_m\}$ and $C = \{c_1, c_2 \ldots, c_l\}$ be fixed. We are going to formulate an appropriate instance of the MCFS problem in a polynomial time in l and m and demonstrate that for natural k there is a cover $C' \subseteq C$, $|C'| = k$, if and only if constructed instance of the MCFS problem has a feasible committee solution of $2k-1$ elements.
   Let us introduce the $m \times l$ incidence matrix A corresponding to S and C. As above,

$$a_{ji} = \begin{cases} 1, & \text{if } s_j \in c_i, \\ 0, & \text{otherwise.} \end{cases}$$

   Now we shall consider a new $(m+1)\times(l+1)$ matrix A' obtained from A by bordering with a row and a column consisting of ones (see Table 1). We will put its element in the right-bottom corner equal to zero.
   Let us consider the MCFS problem corresponding to the matrix A'. Namely, let us take a ground set $X = \{x^1, x^2, \ldots, x^{l+1}\}$ and its subsets according to

$$D_j = \left\{x^{l+1}\right\} \cup \left\{x^i \;:\; s_j \in c_i, i \in \mathbb{N}_l\right\} \ (j \in \mathbb{N}_m)$$

$$D_{m+1} = \left\{x^1, x^2, \ldots, x^l\right\}.$$

Table 1 The construction of the instance of the MCFS problem

		x^1	x^2	$\ldots$	x^l	x^{l+1}
		c_1	c_2	$\ldots$	c_l	S
D_1	s_1					1
D_2	s_2					1
$\vdots$	$\vdots$			A		$\vdots$
D_m	s_m					1
D_{m+1}	C	1	1	$\ldots$	1	0

Let $C' = \{c_{i_1}, c_{i_2}, \ldots, c_{i_k}\}$ be a cover, that is, for each $j \in \mathbb{N}_m$ there is

$$\mu(j) \in \mathbb{N}_k \ : \ s_j \in c_{i_{\mu(j)}}$$

or $x^{i_{\mu(j)}} \in D_j$ due to the construction. Hence, the sequence

$$Q = \left(x^{i_1}, x^{i_2}, \ldots, x^{i_k}, \underbrace{x^{l+1}, \ldots, x^{l+1}}_{k-1} \right)$$

is a committee solution for system (1) because each D_j contains at least k elements of Q.
On the other hand, let us consider the committee solution Q of system (1) with $2k-1$ elements, where

$$Q = \left(x^{i_1}, x^{i_2}, \ldots, x^{i_{2k-1-\lambda}}, \underbrace{x^{l+1}, \ldots, x^{l+1}}_{\lambda} \right).$$

By the choice of the set D_{m+1}, we have $\lambda < k$. Thus, we take a k-subsequence of Q: $(x^{i_1}, \ldots, x^{i_k})$. For each $j \in \mathbb{N}_m$ there is

$$\mu(j) \in \mathbb{N}_k \ : \ x_{i_{\mu(j)}} \in D_j,$$

since Q is a committee and, hence, $s_j \in c_{i_{\mu(j)}}$ by the construction of sets D_j. Therefore, the set $C' = \{c_{i_\mu} \ : \ \mu \in \mathbb{N}_k\}$ is the required cover.

2. Let us suppose that there is an approximation algorithm $\mathcal{A}$ with ratio r for the MCFS problem. It is known [1] that each committee with an even number of elements $2k$ can be reduced to a committee of $2k-1$ elements removing any element. Let us consider an arbitrary instance of the Set Cover problem; let t be the cardinality of its minimum cover. According to procedure described above in a polynomial time one can construct an appropriate instance of the MCFS problem such that

   (1) The number of elements of the minimum committee in this instance equals $2t-1$;
   (2) For every committee solution with $2k-1$ elements there is a cover with cardinality not greater than k that can be found using the known committee in a polynomial time.

Suppose the algorithm $\mathcal{A}$ has found a committee solution of system (1) with $2k-1$ elements. By the assumption about the ratio of the algorithm, we have

$$1 \leq \frac{2k-1}{2t-1} \leq r.$$

Consequently, we obtain the following estimates:

$$\frac{k}{t} \leq r\left(1-\frac{1}{2t}\right)+\frac{1}{2t} \leq r\left(1-\frac{1}{2t}\right)+\frac{r}{2t} \leq r.$$

Lemma is proved. □

Theorem 6 *Unless $P = NP$ there is no approximation algorithm for the MCFS problem with ratio $\frac{1}{4}\log_2(m-1)$.*

Proof Suppose the contrary. Let an algorithm $\mathcal{A}$ find a feasible solution of the MCFS problem with accuracy $\frac{1}{4}\log_2(m-1)$. According to Lemma, there is an approximation algorithm for the SET COVER problem that can find for $|S| = m-1$ a cover not exceeding the optimal one more than $\frac{1}{4}\log(m-1)$ times, which according to Theorem 4 implies $P = NP$. □

The following theorem is proved in a similar fashion.

Theorem 7 *If $NP \not\subset TIME(n^{O(\log\log n)})$, then for each $\varepsilon > 0$ there is no polynomial time algorithm for MCFS problem with approximation ratio $(1-\varepsilon)\ln(m-1)$.*

4 Minimum Committee of Linear Inequalities Problem

Let a ground set be $X = \mathbb{Q}^n$ and let its subsets D_j be open halfspaces

$$D_j = \left\{x \in X \mid (a_j, x) > 0\right\} \quad 0 \neq a_j \in X.$$

In this case, system (1) becomes

$$(a_j, x) > 0 \qquad (j \in \mathbb{N}_m). \tag{4}$$

Minimum Committee of Linear Inequalities System (MCLE) Problem:

Let naturals m and n and vectors $a_1, a_2, \ldots, a_m \in \mathbb{Q}^n$ be given. It is required to find a committee solution of system (4) with a minimum number of elements.

The MCLE problem is interesting for at least two reasons. On the one hand, it has the obvious applications in the statistical learning theory. On the other hand, it cannot be efficiently solved using the reduction to equivalent integer linear programs (2) and (3). Indeed, to make such a reduction, it is required to list all mfs of system (4) to be analyzed. However, this enumeration problem is NP-hard according to the following Theorem 8.

The Densest Hemisphere (DH) Problem:

Let naturals m and n and vectors $a_1, a_2, \ldots, a_m \in \mathbb{Q}^n$ be given. It is required to find the greatest mfs of system (4).

Theorem 8 ([11]) *The DH problem is NP-hard.*

Thus, the traditional computational complexity analysis scheme in the case of the MCLE problem is ineffective. Let us notice that the technique based on the reduction to programs (2) and (3) can be successfully utilized for solving the following closely related problem.

The Optimal Committee Improvement (COMIMP) Problem:

Let naturals m and n and vectors

$$a_1, a_2, \ldots, a_m,\ x^1, x^2, \ldots, x^q \in \mathbb{Q}^n$$

be given. It is required to find a subcommittee $Q' = (y^1, y^2, \ldots, y^{q'})$ with the least possible $q' \leq q$, where

$$y^i \in \left\{x^1, x^2, \ldots, x^q\right\} \quad (i \in \mathbb{N}_{q'}).$$

4.1 Computational Complexity

It is known [1] that the number of elements of the minimum committee solution of system (4) can be used as a measure of its infeasibility. Therefore, the whole set of all systems of linear inequalities can be covered by a countable set of concentric classes. The most narrow class consists of feasible systems, each of them has a one element minimum committee solution. This class is a subclass of the class of systems with 3-elements minimum committee solutions, and so on.

It is important to design a fast algorithm that can find the most narrow class for each system (4) containing this system. It is known that the problem of checking up the feasibility of system (4) has a polynomial time algorithm. But, as is shown below, the problem of checking up the existence of a 3-element committee solution of system (4) is NP-complete.

The 3-element Committee of the Linear Inequalities System (3-COMLE) Problem:

Let naturals m and n and vectors $a_1, a_2, \ldots, a_m \in \mathbb{Q}^n$ be given. Does there exist a committee solution of system (4) that consists of three elements?

Consider another combinatorial problem. Let $G = (V, E)$ be a finite graph. As usual, we say that a function $f : V \to \mathbb{N}_k$ is a coloring of G with k colors if there is no 'monochromatic' edge. That is $|\{f(v) \mid v \in e\}| = 2$ for every $e \in E$.

The Colorability of a Graph with Three Colors (GRAPH 3-COLORABILITY) Problem:

Let the finite graph $G = (V, E)$ with $V = \{1, \ldots, n\}$, be given. Does there exist a coloring of the graph G with 3 colors?

It is known [12] that the GRAPH 3-COLORABILITY problem is NP-complete. We shall prove that this problem can be reduced (by Karp) to the 3-COMLE problem.

Theorem 9 *The* 3-*COMLE problem is NP-complete.*

Proof The 3-COMLE problem belongs to the class NP, since it is possible to check up whether the sequence $Q = (x^1, x^2, x^3)$ is a committee solution of system (4) in a polynomial time in its length.

Let us consider the finite graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ setting the instance of the GRAPH 3-COLORABILITY problem. Let us introduce the following system of linear inequalities in $\mathbb{Q}^n$:

$$\begin{cases} x_i + x_j > 0 & (\{i, j\} \in E) \\ x_i < 0 & (i \in V). \end{cases} \tag{5}$$

System (5), obviously, can be constructed in a polynomial, in n, time. We shall prove that the graph G can be colored with 3 colors if and only if system (5) has a committee solution of 3 elements. In the trivial cases ($n < 3$ or $E = \emptyset$), it is evident, that both problems have the same answer 'Yes.'

Further, let the GRAPH 3-COLORABILITY have the answer 'Yes', and let the partition $V_1 \dot\cup V_2 \dot\cup V_3 = V$ set a coloring of G with 3 colors. It can be easily verified that the sequence $Q = (x^{1,2}, x^{1,3}, x^{2,3})$, where

$$x_k^{i,j} = \begin{cases} -1, & \text{if } k \in V_i \cup V_j, \\ 2, & \text{otherwise} \end{cases} \qquad (\{i, j\} \subset \mathbb{N}_3, k \in \mathbb{N}_n),$$

is a committee solution of system (5). Therefore, the 3-COMLE problem have the answer 'Yes', as well.

On the other hand, let the 3-COMLE problem have the answer 'Yes', and let the sequence $Q = (x^1, x^2, x^3)$ be a committee solution of system (5). Let us define the sets V_1, V_2 and V_3 as follows:

$$\begin{aligned} V_1 &= \{i \in V \ : \ x_i^1 < 0, \ x_i^2 < 0\}, \\ V_2 &= \{i \in V \ : \ x_i^2 < 0, \ x_i^3 < 0\}, \\ V_3 &= \{i \in V \ : \ x_i^3 < 0, \ x_i^1 < 0\}. \end{aligned} \tag{6}$$

By the construction of Q, we have $V_1 \cup V_2 \cup V_3 = V$. Without loss of generality, we can assume that $V_i \neq \emptyset$ for $i \in \mathbb{N}_3$ and $V_i \cap V_j = \emptyset$ for every $i \neq j$. We shall prove, that the sets V_1, V_2, V_3 determine a coloring of G with 3 colors (and the initial instance of the GRAPH 3-COLORABILITY problem has the answer 'Yes', as well). Indeed, let us assume on the contrary that there is an edge $e \in E$, $e = \{i, j\}$ such that $e \in V_1$ (the cases of V_2 and V_3 can be considered similarly). According to the definition of V_1, we have

$$x_i^1 < 0, \ x_i^2 < 0, \ x_j^1 < 0, \ x_j^2 < 0,$$

therefore,

$$x_i^1 + x_j^1 < 0 \quad \text{and} \quad x_i^2 + x_j^2 < 0.$$

On the other hand, according to the definition of a committee solution, at least one of the inequalities

$$x_i^1 + x_j^1 > 0 \quad \text{or} \quad x_i^2 + x_j^2 > 0$$

should be valid. The contradiction obtained proves the correctness of the coloring. The theorem is proved. □

Theorem 10 *The MCLE problem is NP-hard.*

The proof of the theorem is obtained as a corollary of Theorem 9 and the following Statement.

Statement *The 3-COMLE problem can be reduced by Turing*[1] *to the MCLE problem.*

Proof Let us consider an arbitrary instance of the 3-COMLE problem and assign to it an appropriate instance of the MCLE problem. Let us solve the later problem with an arbitrary algorithm and analyze the solution to be obtained. If system (4) has no committee solutions, than the answer in the initial 3-COMLE problem is also 'No'. Otherwise, let the sequence $Q = (x^1, \ldots, x^q)$ be the minimum committee of system (4). If $q > 3$ then the answer is 'No' as well. If $q = 3$, then Q is the required solution, and the answer is 'Yes'. Finally, let $q = 1$ and $Q = (x^1)$. Let us take a vector z such that

$$(a_j, z) \neq 0 \qquad (j \in \mathbb{N}_m).$$

This vector can be found, obviously, in a polynomial time. The sequence $Q = (x^1, z, -z)$ is the required committee solution, and the answer is 'Yes', as well. □

Remark 1 As is seen from the proof, the MCLE (3-COMLE) problem remains NP-hard (NP-complete) if all coefficients of system (4) belong to the set $\{-1, 0, 1\}$ and every inequality has at most 3 nonzero coefficients.

Remark 2 The result of Theorems 9 and 10 can be extended to the case of a more general system

$$(a_j, x)\ \mathcal{R}_j\ b_j, \quad \mathcal{R}_j \in \{>, <, \geq, \leq\}\ (j \in \mathbb{N}_m).$$

Remark 3 Theorems 9 and 10 are true provided that n can take arbitrarily great values. Under an additional upper bound on n, the 3-COMLE has a trivial polynomial time algorithm and the MCLE problem can appear to be solvable in a polynomial time as well. For instance, it is known [1] that the MCLE problem with the constraint $n = 2$ has a polynomial time algorithm.

[1] In a polynomial time.

4.2 Approximation Algorithm

In this section, we consider a polynomial time approximation algorithm for the MCLE problem. Let us introduce some additional constraints on system (4):

(1) $m > n > 2$ and every subsystem of n inequalities is feasible;
(2) $|a_j| = 1$ for every $j \in \mathbb{N}_m$;
(3) $m = 2k + n - 1$ for some natural k.

The last constraint is introduced only for convenience of the further estimates (the case of $m = 2k + n$ can be considered similarly). Let us assign the following sets to vector $x \in \mathbb{Q}^n$:

$$J_>(x) = \left\{ j \in \mathbb{N}_m \ : \ (a_j, x) > 0 \right\},$$
$$J_<(x) = \left\{ j \in \mathbb{N}_m \ : \ (a_j, x) < 0 \right\},$$
$$J_=(x) = \left\{ j \in \mathbb{N}_m \ : \ (a_j, x) = 0 \right\}.$$

Algorithm [13]

Step 1 Find any nontrivial solution z^1 of the system

$$(a_j, z) = 0 \qquad (j \in \mathbb{N}_{n-1})$$

and the sets $J_>(z^1)$, $J_<(z^1)$ and $J_=(z^1)$. Let x^1 be any solution to a subsystem J_1 of system (4), where

$$J_1 = \begin{cases} J_>(z^1) \cup J_=(z^1), & \text{if } |J_>(z^1)| \geq |J_<(z^1)|, \\ J_<(z^1) \cup J_=(z^1), & \text{otherwise.} \end{cases}$$

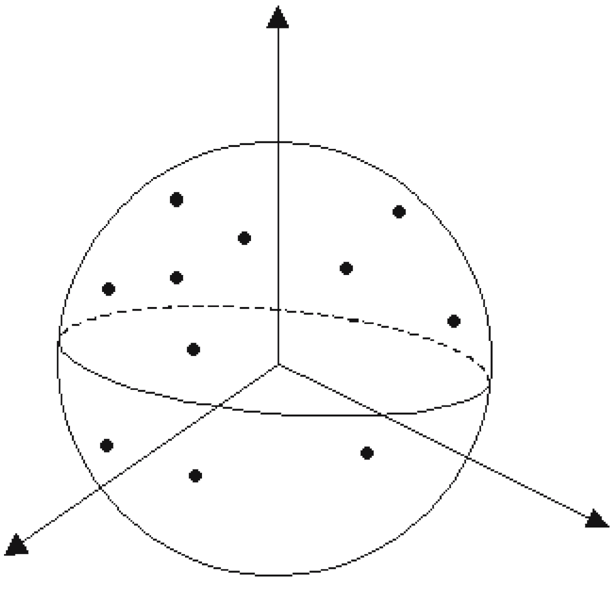


Fig. 2 Starting phase of the algorithm

Set $J = \mathbb{N}_m \setminus J_1$ and $i = 1$.

Step 2 If $J = \emptyset$, then STOP; the sequence $(x^1, x^2, \ldots, x^i)$ is the required committee solution of system (4).

Step 3 Take any subset

$$L' \subseteq J : \quad |L'| = \min\{|J|, n-1\},$$

find a nontrivial solution z^{i+1} of system

$$(a_j, z) = 0 \qquad (j \in L').$$

Set $L = J_=(z^{i+1})$ and find solutions x^{i+1}, x^{i+2} of subsystems with indices $J_>(z^{i+1}) \cup L$ and $J_<(z^{i+1}) \cup L$ of system (4), respectively.

Step 4 Set $J = J \setminus L$, $i = i + 2$ and return to Step 2.

Let us illustrate the algorithm in the case of $n = 3$. Since $|a_j| = 1$, it is convenient to depict the system as the set of points distributed on the unit sphere of the conjugate space (see Fig. 2).

In this example, each element x^i of the required committee defines a hemisphere $\{a \in S_2 \ : \ (a, x^i) > 0\}$.

Figure 3 corresponds to Step 1 of the first iteration of the algorithm where the sequence (x^1) has been chosen as the approximation of the required committee solution. The hemisphere containing the points from J_1 is filled with dark grey.

Figure 4 corresponds to completion of Step 4 of the first iteration, where the current approximation is (x^1, x^2, x^3). The grayed part of the sphere contains points related to the subsystem of system (4), which has current approximation as a committee solution.

Let us agree to call one iteration of the algorithm the sequence of Steps 2–4 (the first iteration includes also Step 1 executed by the algorithm once).

Fig. 3 Step 1 is complete

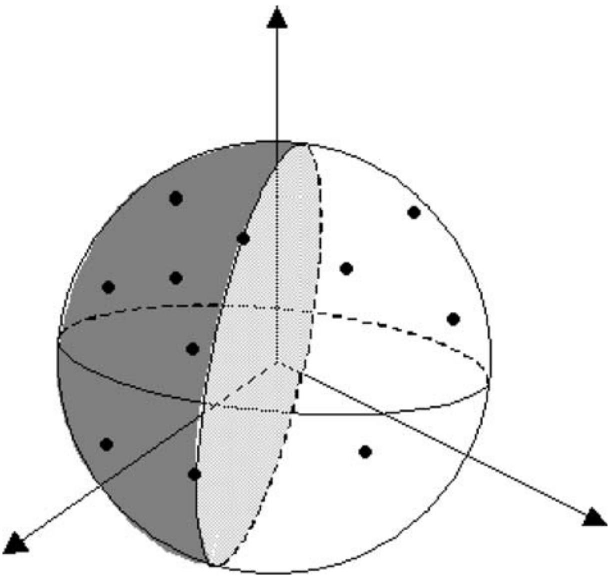
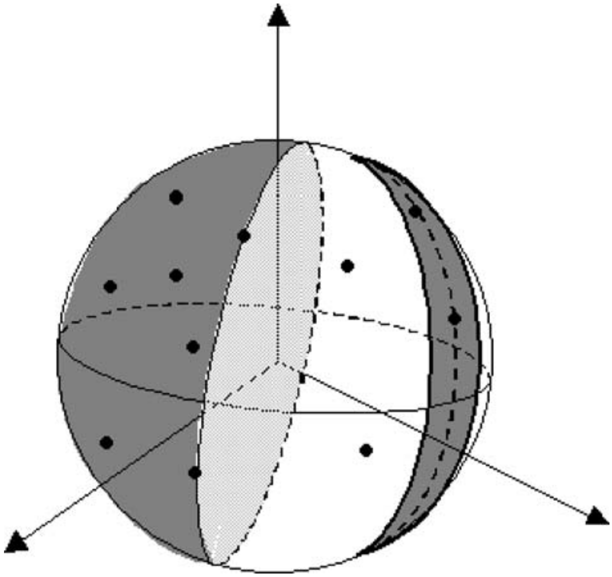


Fig. 4 The first iteration is complete



Theorem 11 ([13])

1. *The algorithm is correct and has at most*

$$\left\lceil \frac{k}{n-1} \right\rceil$$

iterations.

2. *Let the cardinality of the greatest feasible subsystem of system* (4) *have the upper bound of* $k + (n-1) + t$ *for some natural* t. *Then the approximation ratio* r *of the algorithm satisfies the condition*

$$1 \leq r \leq \frac{2\lceil \frac{k}{n-1} \rceil + 1}{2\lceil \frac{k-t}{2t+n-1} \rceil + 1} \approx 1 + \frac{2t}{n-1}.$$

Remark 4 [13] The algorithm finds the optimal solution of the MCLE problem in the class of uniformly distributed inequalities systems [14].

5 Conclusion

In this paper, two special cases, the MCFS and the MCLE, of the Minimum Committee combinatorial optimization problem are considered. It is proved that both problems are NP-hard. For the MCFS problem, an inapproximability threshold similar to the one of the SET COVER problem is established. In particular, the existence of approximation algorithm for the MCFS problem with ratio $\frac{1}{4}\log_2(m-1)$ implies $P = NP$, and if the ratio $(1-\varepsilon)\ln(m-1)$, is guaranteed for some $\varepsilon > 0$, this implies $NP \subset TIME(n^{O(\log\log n)})$. In addition, an approximation algorithm for the MCLE problem is discussed.

Acknowledgements The research was supported by grants of Russian president, no. NS-5595.2006.1 and MD-6768.2006.1.

References

1. Mazurov, Vl.D., Khachai, M.Yu., Rybin, A.I.: Committee constructions for solving problems of selection, diagnostics and prediction. Proc. Steklov Inst. Math. **1**, 67–101 (2002)
2. Feige, U.: A threshold of $\ln n$ for approximating set cover. J. ACM **45**(4), 634–652 (1998)
3. Arrow, K.J.: Social Choice and Individual Values, 2nd edn. Wiley, New York (1963)
4. Saari, D.G.: Basic Geometry of Voting. Springer, Berlin Heidelberg New York (1995)
5. Black, D.: The Theory of Committees and Elections, 2nd edn. Kluwer, New York (1998)
6. Vapnik, V.N.: Statistical Learning Theory. Wiley, New York (1998)
7. Hastie, T., Tibshirani, R., Friedman, J.: Elements of Statistical Learning. Springer, Berlin Heidelberg New York (2001)
8. Eremin, I.I.: Theory of Linear Optimization. BVM, Offenbach (2002)
9. Eremin, I.I., Mazurov, Vl.D.: Nonstable Processes of Mathematical Programming. Nauka, Moscow (1979)
10. Lund, C., Yannakakis, M.: On the hardness of approximating minimization problems. In: Proceedings of the 33rd IEEE Symposium on Foundations of Computer Science, pp. 960–981. IEEE Computer Society, New York (1992)
11. Johnson, D.S., Preparata, F.P.: The densest hemisphere problem. Theor. Comp. Sci. **6**, 93–107 (1978)
12. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. Freeman, San Francisco, CA (1979)
13. Khachay, M.Yu.: On approximate algorithm of a minimum committee of a linear inequalities system. Pattern Recognit. Image Anal. **13**(3), 459–464 (2003)
14. Gale, D.: Neighboring vertices on a convex polyhedron. In: Kuhn, H.W., Tucker, A.W. (eds.) Linear Inequalities and Related Systems, pp. 255–263. Princeton University Press, Princeton, NJ (1956)