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On Some Properties of Calibrated Trifocal Tensors

E. V. Martyushev[1]

Abstract In two-view geometry, the essential matrix describes the relative position and orientation of two calibrated images. In three views, a similar role is assigned to the calibrated trifocal tensor. It is a particular case of the (uncalibrated) trifocal tensor, and thus it inherits all its properties but, due to the fewer degrees of freedom, satisfies a number of additional algebraic constraints. Some of them are described in this paper. More specifically, we define a new notion—the trifocal essential matrix. On the one hand, it is a generalization of the ordinary (bifocal) essential matrix, while, on the other hand, it is closely related to the calibrated trifocal tensor. We prove the two necessary and sufficient conditions that characterize the set of trifocal essential matrices. Based on these characterizations, we propose three necessary conditions on a calibrated trifocal tensor. They have the form of 15 quartic and 99 quintic polynomial equations. We show that in the practically significant real case the 15 quartic constraints are also sufficient.

Keywords Multiple view geometry · Calibrated trifocal tensor · Trifocal essential matrix · Quartic and quintic constraints on calibrated trifocal tensor

1 Introduction

In multiview geometry, the fundamental matrix and the trifocal tensor describe the relative orientation of two and three

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(uncalibrated) images respectively. If the cameras are calibrated, i.e., we are given the calibration matrices for each view, the fundamental matrix is transformed to the so-called essential matrix. It was first introduced by Longuet-Higgins in [14]. The essential matrix has fewer degrees of freedom and additional algebraic properties, compared to the fundamental matrix. A detailed investigation of these properties is given by Demazure, Faugeras, Maybank and other researchers in [6,7,12,13,15]. We shortly recall the most important of them in the next section.

The object of study of the present paper is the trifocal tensor for calibrated cameras. We call this entity the calibrated trifocal tensor (CTFT for short), although some authors use the term Euclidean trifocal tensor. It first appeared as a tool of scene reconstruction from line correspondences in the papers by Spetsakis and Aloimonos [23] and Weng et al. [27]. Later, Hartley [10] generalized the trifocal tensor for uncalibrated cameras and showed that it can be applied for projective scene reconstruction from different combinations of point and line correspondences. Afterward, the properties and applications of (uncalibrated) trifocal tensors have been intensively investigated by Hartley, Shashua, Triggs, and many other researchers in [3,11,21,22,25,26].

One of the most important problems regarding multifocal tensors is their algebraic characterization, i.e., a set of necessary and sufficient conditions under which an arbitrary tensor of suitable order becomes the multifocal tensor. The conditions are usually given in a form of polynomial equations. While the fundamental matrix is completely characterized by the single equation—its determinant must be zero—the set of essential matrices is characterized by the nine cubic constraints, cf. [6,15] or Eq. (7) below. The significance of this result lies in its practical applications. A few efficient algorithms for some metric structure-from-motion problems

have been developed based on the characterization of essential matrices; see, e.g., [16,24].

The investigation of constraints characterizing *uncalibrated* trifocal tensors has received considerable attention in the last two decades: see [4,17,20] and recent works [1,2,18] for details. We would like to mention that the structure of the set of trifocal tensors is much more complicated compared to the bifocal case. In particular, the characterizations from [2] and [1] consist of 36 and 2071 (!) polynomials, respectively.

On the other hand, constraints on trifocal tensors specific to the calibrated case have not previously been studied in the published literature. The present paper is a first step in this direction. Its main contribution is the characterization of the set of *real* CTFTs. Namely, we show that the necessary and sufficient conditions for a real trifocal tensor to be calibrated consist of 15 quartic polynomial equations. Most of them arise from the study of certain complex 3×3 matrix associated with a CTFT. Alternatively, we propose the characterization consisting of 99 quintic polynomial equations.

The rest of the paper is organized as follows. In Sect. 2, we recall some definitions and results from multiview geometry. In Sect. 3, we introduce a new notion—the trifocal essential matrix. On the one hand, it is a generalization of the ordinary (bifocal) essential matrix, and, on the other hand, it is closely related to the calibrated trifocal tensor. We prove the two necessary and sufficient conditions that characterize the set of trifocal essential matrices. In Sect. 4, we define the trifocal essential matrix associated with a CTFT and, based on the characterizations from Sect. 3, we propose our three necessary conditions. They have the form of 15 quartic and 99 quintic homogeneous polynomial equations in the entries of a CTFT. In Sect. 5, we show that in the practically significant real case the 15 quartic constraints are also sufficient. Combining this result with the existing characterization of trifocal tensors, we give a complete set of constraints characterizing a real CTFT. In Sect. 6, we discuss the paper's results and propose further directions of research. In Appendix, we have included some auxiliary lemmas used in the paper.

2 Preliminaries

2.1 Notation

We preferably use $\alpha, \beta, \ldots$ for scalars, $a, b, \ldots$ for column 3-vectors or polynomials, and $A, B, \ldots$ both for matrices and column 4-vectors. For a matrix A, the entries are $(A)_{ij}$, the transpose is A^{T}, the determinant is $\det A$, and the trace is $\operatorname{tr} A$. For two 3-vectors a and b the cross product is $a \times b$. For a vector a the entries are $(a)_i$, the notation $[a]_\times$ stands for the skew-symmetric matrix such that $[a]_\times b = a \times b$ for any vector b. We use I for the identity matrix.

The group of $n \times n$ matrices subject to $R^{\mathrm{T}} R = I$ and $\det R = 1$ is denoted by $\mathrm{SO}(n)$ in case R is real and $\mathrm{SO}(n, \mathbb{C})$ if R is allowed to have complex entries.

2.2 Pinhole Cameras

We briefly recall some definitions and results from multiview geometry; see [7,8,11,15] for details.

A *pinhole camera* is a triple (O, Π, P), where Π is the image plane, P is a central projection of points in three-dimensional Euclidean space onto Π, and $O \notin \Pi$ is the camera center (center of projection P).

Let there be given coordinate frames in 3-space and in the image plane Π. Let Q be a point in 3-space represented in homogeneous coordinates as a 4-vector, and q be its image in Π represented as a 3-vector. Projection P is then given by a 3×4 homogeneous matrix, which is called the *camera matrix* and is also denoted by P. We have

$$q \sim PQ,$$

where $\sim$ means an equality up to a scale. For the sake of brevity, we identify further the camera (O, Π, P) with its camera matrix P.

The *focal length* is the distance between O and Π, the orthogonal projection of O onto Π is called the *principal point*. All intrinsic parameters of a camera (such as the focal length and the principal point offsets) are combined into a single upper-triangular matrix, which is called the *calibration matrix*. A camera is called *calibrated* if its calibration matrix is known.

By changing coordinates in the image plane, the calibrated camera can be represented in the form

$$P = \begin{bmatrix} R & t \end{bmatrix},$$

where $R \in \mathrm{SO}(3)$ and $t \in \mathbb{R}^3$ are called the *rotation matrix* and *translation vector*, respectively.

2.3 Two-View Case

Let there be given two cameras $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} A & a \end{bmatrix}$, where A is a 3×3 matrix and a is a 3-vector. Let Q be a point in 3-space, and q_k be its kth image. Then,

$$q_k \sim P_k Q, \quad k = 1, 2.$$

The *incidence relation* for a pair (q_1, q_2) says

$$q_2^{\mathrm{T}} F q_1 = 0, \tag{1}$$

where matrix $F = [a]_\times A$ is called the *fundamental matrix*. It is important that relation (1) is linear in the entries of F,

so that given a number of point correspondences (eight or more), one can estimate the entries of F by solving a linear system.

It follows from the definition of matrix F that $\det F = 0$. One easily verifies that this condition is also sufficient. Thus we have

Theorem 1 ([11]) *A real nonzero* 3×3 *matrix* F *is a fundamental matrix if and only if*

$$\det F = 0. \tag{2}$$

The *essential matrix* E is the fundamental matrix for calibrated cameras $\hat{P}_1 = \begin{bmatrix} I & 0 \end{bmatrix}$ and $\hat{P}_2 = \begin{bmatrix} R & t \end{bmatrix}$, where $R \in \mathrm{SO}(3)$, t is a 3-vector, that is

$$E = [t]_\times R. \tag{3}$$

The matrices F and E are related by

$$F \sim K_2^{-\mathrm{T}} E K_1^{-1}, \tag{4}$$

where K_k is the calibration matrix of the kth camera. It follows that the incidence relation (1) for the essential matrix becomes

$$\hat{q}_2^\mathrm{T} E \hat{q}_1 = 0,$$

where $\hat{q}_k = K_k^{-1} q_k$ are the so-called *normalized coordinates*.

Equality (3) can be thought of as the definition of the essential matrix, i.e., it is a 3×3 nonzero skew-symmetric matrix post-multiplied by a special orthogonal matrix. Moreover, we can even consider complex essential matrices assuming that in (3) vector $t \in \mathbb{C}^3$ and matrix $R \in \mathrm{SO}(3, \mathbb{C})$.

The real fundamental matrix has seven degrees of freedom, whereas the real essential matrix has only five degrees of freedom. It is translated into the following property [7,11, 13]: Two of the singular values of matrix E are equal, and the third is zero. The condition is also sufficient. An equivalent form of this result is given by

Theorem 2 ([6,7]) *A real* 3×3 *matrix* E *is an essential matrix if and only if*

$$\det E = 0, \tag{5}$$
$$\mathrm{tr}(E E^\mathrm{T})^2 - 2\,\mathrm{tr}((E E^\mathrm{T})^2) = 0. \tag{6}$$

It should be mentioned here that constraints (5) and (6) characterize only *real* essential matrices. There exist "non-essential" complex 3×3 matrices which nevertheless satisfy both conditions (5) and (6). The most general form of such matrices will be given in the next section.

The following theorem gives another characterization constraint on the entries of essential matrix E. It is also valid in case of complex E.

Theorem 3 ([6,7,15]) *A real or complex* 3×3 *matrix* E *of rank two is an essential matrix if and only if*

$$(\mathrm{tr}(E E^\mathrm{T})I - 2 E E^\mathrm{T})E = 0_{3\times 3}. \tag{7}$$

We note that there are "non-essential" rank one matrices which satisfy (7). However, it can be shown that all of them are limits of sequences of essential matrices [15]. Thus the closure of the set of essential matrices constitutes an algebraic variety generated by (7).

It is interesting to note that Theorem 3 is a key for developing efficient algorithms of the essential matrix estimation from five-point correspondences in two views [16].

2.4 Three-View Case

A $(2, 1)$ tensor is a valency-3 tensor with two contravariant and one covariant indices. For a $(2, 1)$ tensor T, we write $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$, where T_k are 3×3 matrices corresponding to the covariant index.

Let there be given three cameras $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} A & a \end{bmatrix}$ and $P_3 = \begin{bmatrix} B & b \end{bmatrix}$, where A and B are 3×3 matrices, a and b are 3-vectors. The *trifocal tensor* $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ is a $(2, 1)$ tensor defined by

$$T_k = A e_k b^\mathrm{T} - a e_k^\mathrm{T} B^\mathrm{T}, \tag{8}$$

where e_1, e_2, e_3 constitute the standard basis in $\mathbb{R}^3$. For a trifocal tensor T matrices T_k are called the *correlation slices*.

It is clear that $\det T_k = 0$. If matrices T_k are of rank two, then let l_k and r_k be the left and right null vectors of T_k, respectively. It follows from (8) that $l_k = [a]_\times A e_k$ and $r_k = [b]_\times B e_k$. Therefore, the two (sextic in the entries of T_1, T_2, T_3) *epipolar constraints* hold [11,17]:

$$\det \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} = \det([a]_\times A) = 0, \\ \det \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} = \det([b]_\times B) = 0. \tag{9}$$

Moreover, for any scalars α, β, γ, the matrix $\alpha T_1 + \beta T_2 + \gamma T_3$ is also degenerate (its right null vector is $[b]_\times B(\alpha e_1 + \beta e_2 + \gamma e_3)$) meaning that

$$\det(\alpha T_1 + \beta T_2 + \gamma T_3) = 0. \tag{10}$$

This equality is referred to as the *extended rank constraint*. It is equivalent to ten (cubic in the entries of T_1, T_2, T_3) equations each of which is the coefficient in $\alpha^i \beta^j \gamma^k$ with $i + j + k = 3$.

Theorem 4 ([9,17]) *Let* $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ *be a real* $(2, 1)$ *tensor such that* $\mathrm{rank}\, T_k = 2$, $k = 1, 2, 3$. *Let* T *satisfy the two epipolar* (9) *and ten extended rank* (10) *constraints. Let the ranks of matrices* $\begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}$ *and* $\begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$ *equal two. Then* T *is a trifocal tensor.*

Remark 1 The additional rank constraints from Theorem 4 are sometimes referred to as the "general viewpoint assumption." The following example demonstrates that they cannot be omitted. The (2, 1) tensor

$$T = \left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

satisfies both the epipolar and extended rank constraints. However it is not a trifocal tensor, i.e., it cannot be represented in form (8). On the other hand, there exist degenerate trifocal tensors such that at least one of matrices $\begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}$, $\begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$ or even T_k is of rank less than two.

Also, it should be mentioned that the characterization from Theorem 4 is not unique. There exist other sets of constraints characterizing trifocal tensors, cf. [1,2,4,18,20].

Let q_k be the kth image of a point Q in 3-space. The *trifocal incidence relation* for a triple (q_1, q_2, q_3) says [11]

$$[q_2]_\times \sum_{j=1}^{3} (q_1)_j T_j [q_3]_\times = 0_{3\times3}. \tag{11}$$

Note that matrix relation (11) is linear in the entries of T. It consists of nine scalar equations, but only four of them are linearly independent.

The *calibrated trifocal tensor* (CTFT) $\hat{T}$ is the trifocal tensor for calibrated cameras $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} R_2 & t_2 \end{bmatrix}$ and $P_3 = \begin{bmatrix} R_3 & t_3 \end{bmatrix}$, where $R_2, R_3 \in \mathrm{SO}(3)$, $t_2, t_3 \in \mathbb{R}^3$, i.e.,

$$\hat{T}_k = R_2 e_k t_3^{\mathrm{T}} - t_2 e_k^{\mathrm{T}} R_3^{\mathrm{T}}. \tag{12}$$

The CTFT is an analog of the essential matrix in three views. The tensors T and $\hat{T}$ are related by

$$T_j \sim K_2 \sum_{k=1}^{3} (K_1^{-\mathrm{T}})_{jk} \hat{T}_k K_3^{\mathrm{T}}, \tag{13}$$

where K_k is the calibration matrix of the kth camera.

For any invertible 3×3 matrix M and 3-vector t, the following identity holds:

$$[M^{-1}t]_\times = \det(M^{-1}) M^{\mathrm{T}} [t]_\times M.$$

Therefore, the trifocal incidence relation (11) for the CTFT becomes

$$[\hat{q}_2]_\times \sum_{j=1}^{3} (\hat{q}_1)_j \hat{T}_j [\hat{q}_3]_\times = 0_{3\times3},$$

where $\hat{q}_k = K_k^{-1} q_k$ are the normalized coordinates.

The tensors T and $\hat{T}$ have 18 and 11 degrees of freedom, respectively. It follows that matrices $\hat{T}_k$ must satisfy a number of additional algebraic constraints. Some of them are described in Sect. 4 below.

3 The Trifocal Essential Matrix and Its Characterization

The *trifocal essential matrix* is, by definition, a 3×3 matrix S which can be represented in form

$$S = s_1 t_1^{\mathrm{T}} + t_2 s_2^{\mathrm{T}}, \tag{14}$$

where $t_1, t_2, s_1, s_2 \in \mathbb{C}^3$, and vectors s_1, s_2 are nonzero and such that $s_k^{\mathrm{T}} s_k = 0$, $k = 1, 2$. It is clear that matrices S, S^{T} and RSQ, where $R, Q \in \mathrm{SO}(3, \mathbb{C})$, simultaneously are (or are not) the trifocal essential matrices.

Theorem 5 *Let a 3×3 matrix S be a trifocal essential matrix. Then SS^{T} has one zero and two other equal eigenvalues.*

Proof Let S be a trifocal essential matrix, i.e., it can be represented in form (14). The matrix SS^{T} has zero eigenvalue, as $\det S = 0$. Taking into account that $s_2^{\mathrm{T}} s_2 = 0$, we get

$$SS^{\mathrm{T}} = s_1(\mu s_1^{\mathrm{T}} + \nu t_2^{\mathrm{T}}) + \nu t_2 s_1^{\mathrm{T}}, \tag{15}$$

where we have denoted $\mu = t_1^{\mathrm{T}} t_1$, $\nu = s_2^{\mathrm{T}} t_1$. By Lemma 7 (see the Appendix), the potentially nonzero eigenvalues of (15) are equal to the ones of 2×2 matrix

$$\begin{bmatrix} \nu t_2^{\mathrm{T}} s_1 & \nu(\mu s_1^{\mathrm{T}} + \nu t_2^{\mathrm{T}}) t_2 \\ 0 & \nu s_1^{\mathrm{T}} t_2 \end{bmatrix},$$

and the eigenvalues of the latter matrix are both equal to $\nu s_1^{\mathrm{T}} t_2 = (s_1^{\mathrm{T}} t_2)(s_2^{\mathrm{T}} t_1)$. Theorem 5 is proved. $\square$

Theorem 6 *A 3×3 matrix S is a trifocal essential matrix if and only if*

$$\det S = 0, \tag{16}$$
$$\mathrm{tr}(SS^{\mathrm{T}})^2 - 2\,\mathrm{tr}((SS^{\mathrm{T}})^2) = 0. \tag{17}$$

Proof The "only if" part is due to Theorem 5 and Lemma 8 (see the Appendix). To prove the "if" part, let S be a 3×3 matrix satisfying (16), (17). We denote c_k the kth column of matrix S. Because S is degenerate, there exists a nonzero vector a such that $Sa = 0$. There are two possibilities.

Case 1: $a^{\mathrm{T}} a \neq 0$. Scaling a and post-multiplying S by an appropriate matrix from $\mathrm{SO}(3, \mathbb{C})$, we assume without loss of generality that $a = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$. Therefore $c_3 = 0$.

Suppose first that either $c_1^T c_1 \neq 0$ or $c_2^T c_2 \neq 0$. Without loss of generality we assume $c_2^T c_2 \neq 0$. Pre-multiplying S by an appropriate rotation, we obtain

$$S = \begin{bmatrix} \lambda & \mu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The substitution of S into (17) gives

$$((\mu + \nu)^2 + \lambda^2)((\mu - \nu)^2 + \lambda^2) = 0.$$

It follows that $\lambda = i(\epsilon_1 \mu + \epsilon_2 \nu)$, where $\epsilon_k = \pm 1$. Thus,

$$S = \begin{bmatrix} i(\epsilon_1 \mu + \epsilon_2 \nu) & \mu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} i\epsilon_2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \nu & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} i\epsilon_1 & 1 & 0 \end{bmatrix}.$$

Consider the case $c_1^T c_1 = c_2^T c_2 = 0$. Due to Lemma 9 (see Appendix), we can pre-multiply S by an appropriate rotation to get

$$S = \begin{bmatrix} \alpha & 1 & 0 \\ \beta & i & 0 \\ \gamma & 0 & 0 \end{bmatrix},$$

where $\alpha^2 + \beta^2 + \gamma^2 = 0$. The substitution of S into (17) yields

$$4(i\alpha - \beta)^2 = 0.$$

It follows that $\beta = i\alpha$ and $\gamma = 0$. Therefore, matrix S has rank one and

$$S = \begin{bmatrix} \alpha & 1 & 0 \\ i\alpha & i & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \begin{bmatrix} \alpha & 1 & 0 \end{bmatrix} + 0 s^T,$$

where s is an arbitrary 3-vector satisfying $s^T s = 0$. Thus in either case S is a trifocal essential matrix, as required.

Case 2: $a^T a = 0$. Due to Lemma 9, we can post-multiply S by an appropriate matrix from $SO(3, \mathbb{C})$ and suppose without loss of generality that $a = \begin{bmatrix} 0 & 1 & i \end{bmatrix}^T$. Therefore $c_3 = ic_2$.

By direct computation, equality (17) becomes $(c_1^T c_1)^2 = 0$, i.e., $c_1^T c_1 = 0$. This yields

$$S = \begin{bmatrix} \alpha & \lambda & i\lambda \\ \beta & \mu & i\mu \\ \gamma & \nu & i\nu \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} \begin{bmatrix} 0 & 1 & i \end{bmatrix},$$

where $\alpha^2 + \beta^2 + \gamma^2 = 0$, i.e., S is a trifocal essential matrix. Theorem 6 is proved. □

We notice that constraints (16), (17) coincide with constraints (5), (6) from Theorem 2. Hence, if a trifocal essential matrix is real, then it is an essential matrix.

In general, a trifocal essential matrix does not satisfy cubic constraint (7). The proof consists in exhibiting a counterexample. Let $s_1 = s_2 = \begin{bmatrix} 1 & i & 0 \end{bmatrix}^T$, $t_1 = t_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Then $S = \begin{bmatrix} 2 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the eigenvalues of SS^T are 0, 1, 1. However,

$$(\mathrm{tr}(SS^T)I - 2SS^T)S = -4 \begin{bmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0_{3 \times 3}.$$

Nevertheless, there exists an analog of identity (7) for trifocal essential matrices.

Theorem 7 *A 3×3 matrix S is a trifocal essential matrix if and only if*

$$(\mathrm{tr}(SS^T)I - 2SS^T)^2 S = 0_{3 \times 3}. \tag{18}$$

Proof Let us denote

$$\Phi(M) = (\mathrm{tr}(MM^T)I - 2MM^T)^2 M$$

and

$$\varphi(M) = \mathrm{tr}(MM^T)^2 - 2\,\mathrm{tr}((MM^T)^2).$$

Then it is straightforward to show that, for arbitrary 3×3 matrix M, the following identity holds:

$$\Phi(M) = 4M^* \det M - M\varphi(M), \tag{19}$$

where M^* is meant the matrix of cofactors of M.

Let matrix S be a trifocal essential matrix. By Theorem 6, $\det S = \varphi(S) = 0$. Then it follows from (19) that $\Phi(S) = 0_{3 \times 3}$, i.e., (18) holds. The "only if" part is proved.

Conversely, let a 3×3 matrix S satisfy (18), i.e., $\Phi(S) = 0_{3 \times 3}$. It suffices to show that $\det S = 0$. Suppose, by hypothesis, that $\det S \neq 0$. Then, post-multiplying (18) by S^{-1}, we get

$$(\mathrm{tr}(SS^T)I - 2SS^T)^2 = 0_{3 \times 3}.$$

It follows that all the eigenvalues of $\mathrm{tr}(SS^T)I - 2SS^T$ are zeroes and

$$\mathrm{tr}(\mathrm{tr}(SS^T)I - 2SS^T) = \mathrm{tr}(SS^T) = 0.$$

The substitution of this into (18) yields $(\det S)^5 = 0$ in contradiction to the hypothesis $\det S \neq 0$. Thus, $\det S = 0$ and, by (19), $\varphi(S) = 0$. By Theorem 6, matrix S is a trifocal essential matrix. Theorem 7 is proved. $\qquad \square$

To summarize, the above theorems imply the following statements.

– The pair of scalar constraints (16), (17) of degrees 3 and 4 respectively is equivalent to the single matrix constraint (18) of degree 5.
– The most general form of a 3×3 matrix satisfying Eqs. (16) and (17) is the trifocal essential matrix given by (14).
– If a trifocal essential matrix is real, then it is an essential matrix.
– Every essential matrix is a trifocal essential matrix, but the converse is not true in general.

4 Three Necessary Conditions on a Calibrated Trifocal Tensor

A new notion of trifocal essential matrix, introduced in the previous section, turns out to be closely related to calibrated trifocal tensors. The connection is established by the following lemma.

Lemma 1 *Let* $\hat{T} = \begin{bmatrix} \hat{T}_1 & \hat{T}_2 & \hat{T}_3 \end{bmatrix}$ *be a CTFT. Then a* 3×3 *matrix* $S_{\hat{T}} = \alpha \hat{T}_1 + \beta \hat{T}_2 + \gamma \hat{T}_3$, *where numbers* α, β, γ *are such that* $\alpha^2 + \beta^2 + \gamma^2 = 0$, *is a trifocal essential matrix, i.e., it can be represented in form* (14).

Proof We notice that

$$S_{\hat{T}} = \alpha \hat{T}_1 + \beta \hat{T}_2 + \gamma \hat{T}_3 = R_2 s t_3^{\mathrm{T}} - t_2 s^{\mathrm{T}} R_3^{\mathrm{T}}$$
$$= s_2 t_3^{\mathrm{T}} + (-t_2) s_3^{\mathrm{T}},$$

where $s = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}^{\mathrm{T}}$, and $s_k = R_k s$ are 3-vectors satisfying

$$s_k^{\mathrm{T}} s_k = s^{\mathrm{T}} R_k^{\mathrm{T}} R_k s = s^{\mathrm{T}} s = 0.$$

It follows that $S_{\hat{T}}$ is a trifocal essential matrix. Lemma 1 is proved. $\qquad \square$

The *trifocal essential matrix associated with a CTFT* $\hat{T} = \begin{bmatrix} \hat{T}_1 & \hat{T}_2 & \hat{T}_3 \end{bmatrix}$ is defined by

$$S_{\hat{T}}(s) = \sum_{j=1}^{3} (s)_j \hat{T}_j = \alpha \hat{T}_1 + \beta \hat{T}_2 + \gamma \hat{T}_3, \qquad (20)$$

where $s = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}^{\mathrm{T}}$ and $s^{\mathrm{T}} s = 0$. It is worth emphasizing that $S_{\hat{T}}(s)$ is a 3×3 degenerate matrix which generically has complex entries.

We are going to apply the two characterizations from Sect. 3 to derive the necessary conditions on the entries of CTFT $\hat{T}$.

In the rest of this section, calibrated trifocal tensors are allowed to have complex entries, that is in (12) matrices R_2, R_3 belong to SO(3, $\mathbb{C}$), and vectors t_2, t_3 are in $\mathbb{C}^3$.

Let us introduce six symmetric matrices ($k = 1, 2, 3$):

$$\begin{aligned} U_k &= \hat{T}_k \hat{T}_k^{\mathrm{T}}, \\ V_k &= \hat{T}_k \hat{T}_{k+1}^{\mathrm{T}} + \hat{T}_{k+1} \hat{T}_k^{\mathrm{T}}. \end{aligned} \qquad (21)$$

Here $k + 1$ should be read as $k \pmod 3 + 1$, i.e., $V_3 = \hat{T}_3 \hat{T}_1^{\mathrm{T}} + \hat{T}_1 \hat{T}_3^{\mathrm{T}}$.

Theorem 8 (First necessary condition) *Let* $\hat{T} = \begin{bmatrix} \hat{T}_1 & \hat{T}_2 & \hat{T}_3 \end{bmatrix}$ *be a CTFT, matrices* U_k, V_k *be defined in* (21). *Then the entries of* $\hat{T}$ *are constrained by the following equations:*

$$\psi(U_3 - U_1, U_3 - U_1) - \psi(V_3, V_3) = 0, \qquad (22)$$
$$\psi(U_3 - U_1, V_1) + \psi(V_2, V_3) = 0, \qquad (23)$$
$$\psi(U_1 - U_2, V_1) = 0, \qquad (24)$$

where $\psi(X, Y) = \mathrm{tr}(X) \mathrm{tr}(Y) - 2 \mathrm{tr}(XY)$. *Six more equations are obtained from* (22) *to* (24) *by a cyclic permutation of indices* $1 \to 2 \to 3 \to 1$. *The resulting nine equations are linearly independent.*

Proof Let $S_{\hat{T}} = \alpha \hat{T}_1 + \beta \hat{T}_2 + \gamma \hat{T}_3$ be a trifocal essential matrix associated with $\hat{T}$. By Theorem 6, the following equation holds:

$$\mathrm{tr}(S_{\hat{T}} S_{\hat{T}}^{\mathrm{T}})^2 - 2 \mathrm{tr}((S_{\hat{T}} S_{\hat{T}}^{\mathrm{T}})^2) = 0. \qquad (25)$$

The definition of matrices U_k, V_k (see (21)) permits us to write

$$S_{\hat{T}} S_{\hat{T}}^{\mathrm{T}} = \alpha^2 U_1 + \beta^2 U_2 + \gamma^2 U_3 + \alpha\beta V_1 + \beta\gamma V_2 + \gamma\alpha V_3.$$

Substituting this into (25), we find the coefficients in $\alpha^4, \alpha^3\beta$ and $\alpha\beta^3$ taking into account that $\gamma^2 = -\alpha^2 - \beta^2$. Because α and β are arbitrary, these coefficients must vanish:

$$\alpha^4: \psi(U_3 - U_1, U_3 - U_1) - \psi(V_3, V_3) = 0, \qquad (26)$$
$$\alpha^3\beta: \psi(U_1 - U_3, V_1) - \psi(V_2, V_3) = 0, \qquad (27)$$
$$\alpha\beta^3: \psi(U_2 - U_3, V_1) - \psi(V_2, V_3) = 0. \qquad (28)$$

Thus we get (22) = (26), (23) = −(27), and (24) = (27) − (28). It is clear that we can get six more constraints on $\hat{T}_k$ from (22) to (24) by a cyclic permutation of the indices.

Finally, the resulting nine polynomials can not be linearly dependent, since each of them contains monomials that are not contained in all the other polynomials. Examples of such monomials for (22), (23) and (24) are $(\hat{T}_3)_{11}^2(\hat{T}_1)_{11}^2$, $(\hat{T}_3)_{11}^2(\hat{T}_1)_{11}(\hat{T}_2)_{11}$ and $(\hat{T}_1)_{11}^3(\hat{T}_2)_{11}$ respectively. Theorem 8 is proved.

From now on, the nine equalities from Theorem 8 will be referred to as the *eigenvalue constraints*.

Lemma 2 *Let $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ be a (2, 1) tensor satisfying the ten extended rank and nine eigenvalue constrains. Then matrix $S_T = \alpha T_1 + \beta T_2 + \gamma T_3$ with $\alpha^2 + \beta^2 + \gamma^2 = 0$ is a trifocal essential matrix.*

Proof The extended rank constraints imply $\det S_T = 0$. Taking into account that $\gamma^2 = -\alpha^2 - \beta^2$, we conclude that the expression

$$\varphi(S_T) = \operatorname{tr}(S_T S_T^{\mathrm{T}})^2 - 2\operatorname{tr}((S_T S_T^{\mathrm{T}})^2)$$

contains 9 monomials:

$$\alpha^4, \alpha^3\beta, \alpha^3\gamma, \alpha^2\beta^2, \alpha^2\beta\gamma, \alpha\beta^3, \alpha\beta^2\gamma, \beta^4, \beta^3\gamma.$$

It is directly verified that the coefficients in all of them are linear combinations of the nine polynomials from Theorem 8, i.e., $\varphi(S_T) = 0$. By Theorem 6, S_T is a trifocal essential matrix, as required. □

Theorem 9 (Second necessary condition) *Let $\hat{T} = \begin{bmatrix} \hat{T}_1 & \hat{T}_2 & \hat{T}_3 \end{bmatrix}$ be a CTFT. Then the entries of $\hat{T}$ are constrained by the 99 linearly independent quintic (of degree 5) polynomial equations.*

Proof Let $S_{\hat{T}} = \alpha\hat{T}_1 + \beta\hat{T}_2 + \gamma\hat{T}_3$ be a trifocal essential matrix associated with $\hat{T}$. By Theorem 7, the following equation holds:

$$(\operatorname{tr}(S_{\hat{T}}S_{\hat{T}}^{\mathrm{T}})I - 2S_{\hat{T}}S_{\hat{T}}^{\mathrm{T}})^2 S_{\hat{T}} = 0_{3\times3}. \quad (29)$$

We notice that equality (29) is quintic in the entries of matrix $S_{\hat{T}}$. Taking into account that $\gamma^2 = -\alpha^2 - \beta^2$, every entry in the l.h.s. of (29) contains 11 monomials in variables α, β and γ. Because the coefficient in each of these monomials must vanish, there are in total 99 quintic polynomial constraints on the entries of $\hat{T}$. Theorem 9 is proved. □

Remark 2 An explicit form of the quintic polynomial equations from Theorem 9 is as follows:

$$(\Psi_1(U_{13}) - \Psi_1(V_3))\hat{T}_1 - \Psi_2(U_{13}, V_3)\hat{T}_3 = 0_{3\times3}, \quad (30)$$

$$\Psi_2(U_{13}, V_3)\hat{T}_1 + (\Psi_1(U_{13}) - \Psi_1(V_3))\hat{T}_3 = 0_{3\times3}, \quad (31)$$

$$(\Psi_2(U_{13}, V_2) + \Psi_2(V_1, V_3))\hat{T}_1 + \Psi_2(U_{13}, V_3)\hat{T}_2$$
$$+ (\Psi_2(U_{13}, V_1) - \Psi_2(V_2, V_3))\hat{T}_3 = 0_{3\times3}, \quad (32)$$

$$(\Psi_2(U_{13}, V_1) - \Psi_2(V_2, V_3))\hat{T}_1 + (\Psi_1(U_{13}) - \Psi_1(V_3))\hat{T}_2$$
$$- (\Psi_2(U_{13}, V_2) + \Psi_2(V_1, V_3))\hat{T}_3 = 0_{3\times3}, \quad (33)$$

where matrices U_k, V_k are defined in (21), $U_{jk} = U_j - U_k$, and

$$\Psi(X, Y) = (\operatorname{tr}(X)I - 2X)(\operatorname{tr}(Y)I - 2Y),$$
$$\Psi_1(X) = \Psi(X, X),$$
$$\Psi_2(X, Y) = \Psi(X, Y) + \Psi(Y, X).$$

Equations (30)–(33) give $4 \times 9 = 36$ constraints on $\hat{T}$. We get 72 more equations from (30) to (33) by a cyclic permutation of indices $1 \to 2 \to 3 \to 1$. In total, there are 108 quintic constraints. Let M_k be the kth version of the L.H.S. of (32). Then we have

$$M_1 + M_2 + M_3 \equiv 0_{3\times3}.$$

It follows that eqs. (30)–(33) give only $108 - 9 = 99$ constraints. Their linear independence is verified directly.

Theorem 10 *A (2, 1) tensor $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ satisfies the ten extended rank and nine eigenvalue constrains if and only if it satisfies the 99 quintic constraints from Theorem 9.*

Proof The result immediately follows from Lemma 2 and the two characterizations of trifocal essential matrices (Theorems 6 and 7). □

Theorem 10 implies the equivalence of the two sets of constraints. From this point of view the quintic polynomials contribute nothing new and seem to be redundant. Nevertheless, we expect them to be useful in deriving more (quintic) polynomial constraints on CTFTs. We return to this question in Sect. 6.

Finally, we propose the third necessary condition on a CTFT. It seems not to be directly related to the matrix $S_{\hat{T}}$. However, this condition could be useful in applications, since it consists of another set of quartic polynomial equations that are satisfied by a calibrated trifocal tensor.

Theorem 11 (Third necessary condition) *Let $\hat{T} = \begin{bmatrix} \hat{T}_1 & \hat{T}_2 & \hat{T}_3 \end{bmatrix}$ be a CTFT. Then the entries of $\hat{T}$ satisfy the following equations:*

$$\operatorname{tr}(U_2)^2 - \operatorname{tr}(V_3)^2 - \operatorname{tr}(U_2^2 - V_3^2 + (U_3 - U_1)^2) = 0, \quad (34)$$

$$\operatorname{tr}(V_2)\operatorname{tr}(U_1 - 2U_2 - U_3) - \operatorname{tr}(V_1)\operatorname{tr}(V_3) + 2\operatorname{tr}(V_2 U_2) = 0, \quad (35)$$

where matrices U_k, V_k are defined in (21). *Four more equations are obtained from (34) to (35) by a cyclic permutation of indices $1 \to 2 \to 3 \to 1$. The resulting six equations are linearly independent.*

Proof Let tensor $\hat{T}$ be represented in form (12). First we replace $\hat{T}$ with $\hat{T}' = \left[R_2^{\mathrm{T}} \hat{T}_1 R_3 \ R_2^{\mathrm{T}} \hat{T}_2 R_3 \ R_2^{\mathrm{T}} \hat{T}_3 R_3 \right]$. Then the correlation slices of $\hat{T}'$ are simplified to $\hat{T}'_k = e_k t_3^{\mathrm{T}} - t_2 e_k^{\mathrm{T}}$. A straightforward computation proves that $\hat{T}'$ satisfies eqs. (34) – (35) and the four their consequences. Then so does $\hat{T}$, since $U_k = R_2 U'_k R_2^{\mathrm{T}}$ and $V_k = R_2 V'_k R_2^{\mathrm{T}}$.

The resulting six polynomials can not be linearly dependent, as each of them contains monomials that are not contained in all the other polynomials. Examples of such monomials for (34) and (35) are $(\hat{T}_3)_{11}^2 (\hat{T}_1)_{11}^2$ and $(\hat{T}_2)_{11} (\hat{T}_3)_{11}^3$ respectively. Theorem 11 is proved. ∎

The 15 equalities from Theorems 8 and 11 will be further referred to as the *quartic constraints*.

Remark 3 Let us briefly describe the way by which the six constraints from Theorem 11 were initially derived. The nine eigenvalue polynomial equations consist in total of 2160 monomials. We constructed the dense matrix A of size 2160×2160 such that each row of A is the monomial vector taken on a randomly generated CTFT. Using the singular value decomposition of A, we found that the null space of A has dimension 15. It is meant that in addition to the eigenvalue constraints, there must exist six more quartic equations linearly independent with them. The (small integer) coefficients of these equations were derived by the Gauss–Jordan elimination on the basis vectors of the null space of A.

Remark 4 The eigenvalue constraints do not imply the six equalities from Theorem 11. The following trifocal tensor gives a counterexample:

$$T = \left[\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \right].$$

One verifies that T satisfies the eigenvalue constraints, but not the six ones from Theorem 11. Hence T is not calibrated.

Remark 5 Generically, over the field of complex numbers, the quartic constraints are insufficient for a trifocal tensor T to be calibrated. Here is a counterexample. Consider a (2, 1) tensor

$$T = \left[\begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i \\ -i & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -i & 0 & 0 \\ i & 1 & 0 \end{bmatrix} \right]. \tag{36}$$

It is trifocal, as

$$T_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} e_k \begin{bmatrix} i & 1 & 0 \end{bmatrix} - \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} e_k^{\mathrm{T}} \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & -1 \\ i & 0 & 0 \end{bmatrix}.$$

Moreover, T satisfies all the 15 quartic constraints. Suppose that T is calibrated. Then there must exist 3-vectors u_k, v_k, t_2, t_3 such that $u_k^{\mathrm{T}} u_k = v_k^{\mathrm{T}} v_k = 1$ and

$$T_k = u_k t_3^{\mathrm{T}} - t_2 v_k^{\mathrm{T}}.$$

Let us define an ideal:

$$J = \langle T_k - u_k t_3^{\mathrm{T}} + t_2 v_k^{\mathrm{T}}, \ u_k^{\mathrm{T}} u_k - 1, \ v_k^{\mathrm{T}} v_k - 1 \mid k = 1, 2, 3 \rangle.$$

Then $J \subset \mathbb{C}[\xi_1, \ldots, \xi_{24}]$, where ξ_j are the entries of vectors t_2, t_3, u_1, u_2, u_3, v_1, v_2 and v_3. It is straightforward to show by the computation of the Gröbner basis of J that $1 \in J$. It follows that the affine variety defined by J is empty, and thus T cannot be calibrated.

5 A Characterization of Real Calibrated Trifocal Tensors

In this section, we are going to obtain a three-view analog of Theorem 2. First, it will be shown that a real trifocal tensor is calibrated if and only if it satisfies the 15 quartic constraints (Theorem 12). After that, it is straightforward to combine this result with the characterization of trifocal tensors, e.g., from [2] to get a complete set of constraints characterizing a real CTFT (Theorem 13).

First we prove several lemmas.

Lemma 3 *Let $T = \left[T_1 \ T_2 \ T_3 \right]$ be a real or complex (2, 1) tensor and $T' = \left[T'_1 \ T'_2 \ T'_3 \right]$ be a tensor defined by*

$$T'_j = Q_2 \sum_{k=1}^{3} (Q_1)_{jk} T_k Q_3^{\mathrm{T}}, \tag{37}$$

where Q_1, Q_2, $Q_3 \in \mathrm{SO}(3, \mathbb{C})$. Then T and T' simultaneously

1. *are (or are not) calibrated trifocal tensors;*
2. *satisfy (or do not satisfy) the 10 extended rank and 15 quartic constraints.*

Proof 1. Let T be a CTFT, so that its correlation slices can be represented in form (12). Then,

$$T_j' = Q_2 \sum_{k=1}^{3} (Q_1)_{jk} T_k Q_3^{\mathrm{T}}$$
$$= (Q_2 R_2 Q_1^{\mathrm{T}}) e_j (Q_3 t_3)^{\mathrm{T}} - (Q_2 t_2) e_j^{\mathrm{T}} (Q_3 R_3 Q_1^{\mathrm{T}})^{\mathrm{T}},$$

i.e., T' is a CTFT as well. On the other hand, if T is not a CTFT, then so is not T', since

$$T_k = Q_2^{\mathrm{T}} \sum_{j=1}^{3} (Q_1)_{jk} T_j' Q_3. \qquad (38)$$

2. Let T be a $(2, 1)$ tensor satisfying the 10 extended rank and 15 quartic constraints. Let us construct the matrix $S_T(s) = \sum_{k=1}^{3} (s)_k T_k$, where s is an arbitrary 3-vector. The ten extended rank constraints are then equivalent to $\det S_T(s) = 0$. We get

$$S_{T'}(s) = \sum_{j=1}^{3} (s)_j T_j' = Q_2 S_T(Q_1^{\mathrm{T}} s) Q_3^{\mathrm{T}}, \qquad (39)$$

and thus $\det S_{T'}(s) = 0$, i.e., tensor T' satisfies the ten extended rank constraints as well.

Further, if the vector s is such that $s^{\mathrm{T}} s = 0$, then Lemma 2 implies that $S_T(s)$ is a trifocal essential matrix. It follows from (39) that so is $S_{T'}(s)$. After that, using the same arguments as in the proof of Theorem 8, one shows that tensor T' satisfies the nine eigenvalue constraints.

It remains to prove that T' satisfies also the six constraints from Theorem 11. We denote by $p_k(T)$ the l.h.s. of the kth quartic equation on tensor T so that $p_{10}, \ldots, p_{15}$ are the six polynomials from Theorem 11. Then, by a straightforward computation, we get

$$p_j(T') = \sum_{k=1}^{15} \xi_{jk} p_k(T), \qquad j = 10, \ldots, 15, \qquad (40)$$

where ξ_{jk} are polynomial expressions depending only on the entries of matrix Q_1. It follows that if $p_k(T) = 0$ for all k, then also $p_j(T') = 0$ for all j.

Finally, if T does not satisfy the extended rank and quartic equations, then due to (38) so does not T'. This completes the proof of Lemma 3. □

Lemma 4 *Let $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ be a real trifocal tensor. Then there exist matrices $Q_1, Q_2, Q_3 \in \mathrm{SO}(3)$ such that T can be transformed by (37) to the trifocal tensor*

$$T' = \left[\begin{bmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & 0 \\ \nu_1 & \rho_1 & \sigma_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu_2 \\ \nu_2 & \rho_2 & \sigma_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \rho_3 & \sigma_3 \end{bmatrix} \right], \qquad (41)$$

where $\lambda_1, \mu_2, \nu_1, \nu_2, \rho_k, \sigma_k$ are real scalars.

Proof We are going to explicitly construct rotations Q_1, Q_2 and Q_3 such that T is transformed to T' by (37). Since T is a trifocal tensor, we have

$$T_k = A_2 e_k a_3^{\mathrm{T}} - a_2 e_k^{\mathrm{T}} A_3^{\mathrm{T}}.$$

Let $H_j \in \mathrm{SO}(3)$ be the Householder matrix such that $H_j a_j = \begin{bmatrix} 0 & 0 & \gamma_j \end{bmatrix}^{\mathrm{T}}$, $j = 2, 3$. First we pre- and post-multiply each T_k by H_2 and H_3^{T}, respectively, and denote by B_j a 2×3 matrix consisting of the first two rows of $H_j A_j$. After that, we make the singular value decomposition of $\gamma_3 B_2$ to decompose it in form

$$\gamma_3 B_2 = U \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \mu_2 & 0 \end{bmatrix} V^{\mathrm{T}},$$

where $U \in \mathrm{SO}(2)$ and $V \in \mathrm{SO}(3)$. Finally, let $W \in \mathrm{SO}(2)$ be a rotation such that $(W B_3 V)_{13} = 0$. We set

$$Q_1 = V^{\mathrm{T}}, \quad Q_2 = \begin{bmatrix} U^{\mathrm{T}} & 0 \\ 0 & 1 \end{bmatrix} H_2, \quad Q_3 = \begin{bmatrix} W & 0 \\ 0 & 1 \end{bmatrix} H_3. \qquad (42)$$

One verifies that the trifocal tensor T is transformed to (41) by the rotations Q_1, Q_2 and Q_3 defined in (42). Lemma 4 is proved. □

We denote by $p_1, \ldots, p_{15}$ the 15 quartic polynomials on the trifocal tensor T' defined by (41) and consider the ideal

$$J = \langle p_1, \ldots, p_{15} \rangle \subset \mathbb{C}[\lambda_1, \nu_1, \rho_1, \sigma_1, \mu_2, \nu_2, \rho_2, \sigma_2, \rho_3, \sigma_3]. \qquad (43)$$

Let $\sqrt{J}$ be the radical of J. We are going to derive several polynomials that belong to $\sqrt{J}$. For convenience we divide them into two parts presented in Lemmas 5 and 6.

Lemma 5 *The polynomials*

$$(\lambda_1^2 - \mu_2^2)(\lambda_1^2 + \sigma_1^2), \qquad (44)$$

$$(\lambda_1^2 - \mu_2^2)(\mu_2^2 + \sigma_2^2), \qquad (45)$$

$$\rho_3(\nu_1 \sigma_1 + \nu_2 \sigma_2), \qquad (46)$$

$$\rho_3(\nu_1 \rho_1 + \nu_2 \rho_2), \qquad (47)$$

$$\rho_3(\rho_3^2 + \sigma_3^2)(\nu_1^2 + \nu_2^2 - \rho_1^2 - \rho_2^2 - \rho_3^2), \qquad (48)$$

$$(\rho_3^2 + \sigma_3^2)(\rho_1 \sigma_1 + \rho_2 \sigma_2 + \rho_3(\sigma_3 + \mu_2))$$
$$\times (\rho_1 \sigma_1 + \rho_2 \sigma_2 + \rho_3(\sigma_3 - \mu_2)), \qquad (49)$$

$$\rho_3(\rho_3^2 + \sigma_3^2)(\nu_1^2 + \nu_2^2 - \sigma_1^2 - \sigma_2^2 - (\sigma_3 + \mu_2)^2)$$
$$\times (\nu_1^2 + \nu_2^2 - \sigma_1^2 - \sigma_2^2 - (\sigma_3 - \mu_2)^2) \qquad (50)$$

belong to $\sqrt{J}$, where J is defined in (43).

Proof Let p be any polynomial from (44) to (50). We construct an ideal $\tilde{J} = J + \langle 1 - \tau p \rangle \subset \mathbb{C}[\lambda_1, \ldots, \sigma_3, \tau]$, where τ is a new variable. By direct computation of the Gröbner basis of $\tilde{J}$, we get $1 \in \tilde{J}$. Hence, by Lemma 10 (see Appendix), $p \in \sqrt{J}$. Lemma 5 is proved. $\qquad\square$

Remark 6 Surprisingly, the computation of the Gröbner basis of each $\tilde{J}$ takes only a few seconds in Maple even over the field of rationals. In our computations, we used the *graded reverse lexicographic order* [5]:

$$\lambda_1 > \nu_1 > \rho_1 > \sigma_1 > \mu_2 > \nu_2 > \rho_2 > \sigma_2 > \rho_3 > \sigma_3 > \tau.$$

Lemma 6 *The polynomials*

$$(\nu_1^2 + \rho_1^2 + \sigma_1^2)(\sigma_1^2 + \sigma_2^2 - \rho_3^2 + \lambda_1^2 - \mu_2^2), \tag{51}$$
$$(\nu_2^2 + \rho_2^2 + \sigma_2^2)(\sigma_1^2 + \sigma_2^2 - \rho_3^2 - \lambda_1^2 + \mu_2^2), \tag{52}$$
$$\nu_1\nu_2 + \rho_1\rho_2 + \sigma_1\sigma_2, \tag{53}$$
$$\nu_1^2 + \rho_1^2 + \sigma_1^2 - \nu_2^2 - \rho_2^2 - \sigma_2^2 + \lambda_1^2 - \mu_2^2, \tag{54}$$
$$(\nu_1^2 + \rho_1^2 + \sigma_1^2 - \rho_3^2 - (\sigma_3 + \mu_2)^2 + \lambda_1^2 - \mu_2^2)$$
$$\times (\nu_1^2 + \rho_1^2 + \sigma_1^2 - \rho_3^2 - (\sigma_3 - \mu_2)^2 - \lambda_1^2 + \mu_2^2) \tag{55}$$

belong to $\sqrt{J}$, where J is defined in (43).

Proof See the proof of Lemma 5. $\qquad\square$

Theorem 12 *A real trifocal tensor $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ is calibrated if and only if it satisfies the 15 quartic constraints from Theorems 8 and 11.*

Proof The "only if" part is due to Theorems 8 and 11. We now prove the "if" part.

Let $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ be a real trifocal tensor satisfying the 15 quartic equations $p_1 = \ldots = p_{15} = 0$. By Lemmas 3 and 4, there exist matrices $Q_1, Q_2, Q_3 \in SO(3)$ such that the trifocal tensor T' defined by (37) has form (41) and satisfies the 15 quartic equations as well.

First we note that if $\lambda_1^2 - \mu_2^2 \neq 0$, then, as the trifocal tensor is real, we get from Lemma 5:

$$\lambda_1 = \sigma_1 = \mu_2 = \sigma_2 = 0.$$

However, this is in contradiction to $\lambda_1^2 - \mu_2^2 \neq 0$. As a result, a real solution to the 15 quartic equations on T' exists if and only if $\lambda_1^2 = \mu_2^2$. Let us consider two cases: $\rho_3 \neq 0$ and $\rho_3 = 0$.

Case 1: $\rho_3 \neq 0$. Then, $\rho_3^2 + \sigma_3^2 \neq 0$ and, by Lemma 5, the entries of tensor T' are constrained by

$$\nu_1\sigma_1 + \nu_2\sigma_2 = \nu_1\rho_1 + \nu_2\rho_2 = \nu_1^2 + \nu_2^2 - \rho_1^2 - \rho_2^2 - \rho_3^2$$
$$= (\rho_1\sigma_1 + \rho_2\sigma_2 + \rho_3(\sigma_3 + \mu_2))(\rho_1\sigma_1 + \rho_2\sigma_2$$
$$+ \rho_3(\sigma_3 - \mu_2))$$
$$= (\nu_1^2 + \nu_2^2 - \sigma_1^2 - \sigma_2^2 - (\sigma_3 + \mu_2)^2)$$
$$\times (\nu_1^2 + \nu_2^2 - \sigma_1^2 - \sigma_2^2 - (\sigma_3 - \mu_2)^2) = 0. \tag{56}$$

The correlation slices of T' can be represented in form

$$T_k' = A_2 e_k \begin{bmatrix} 0 & 0 & \mu_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} e_k^{\mathrm{T}} A_3^{\mathrm{T}},$$

where

$$A_2 = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon_2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} \nu_1 & \nu_2 & 0 \\ \rho_1 & \rho_2 & \rho_3 \\ \sigma_1 & \sigma_2 & \sigma_3 - \epsilon_2\mu_2 \end{bmatrix},$$

and $\epsilon_k = \pm 1$. It follows that $A_2 = \pm R_2$, where $R_2 \in SO(3)$. Hence it suffices to show that $A_3 = \theta R_3$, where θ is a nonzero scalar and $R_3 \in SO(3)$. If we suppose that

$$\rho_1\sigma_1 + \rho_2\sigma_2 + \rho_3(\sigma_3 - \epsilon_2\mu_2)$$
$$= \nu_1^2 + \nu_2^2 - \sigma_1^2 - \sigma_2^2 - (\sigma_3 - \epsilon_2\mu_2)^2 = 0,$$

then we are done, since due to (56) $A_3 A_3^{\mathrm{T}} = \theta^2 I$ with $\theta^2 = \rho_1^2 + \rho_2^2 + \rho_3^2 \neq 0$.

On the other hand, if

$$\rho_1\sigma_1 + \rho_2\sigma_2 + \rho_3(\sigma_3 - \epsilon_2\mu_2)$$
$$= \nu_1^2 + \nu_2^2 - \sigma_1^2 - \sigma_2^2 - (\sigma_3 + \epsilon_2\mu_2)^2 = 0, \tag{57}$$

then we add these polynomials to J and denote the resulting ideal by J_1. By the computation of the Gröbner basis of J_1, we get $(\rho_3\mu_2\sigma_3)^3 \in J_1$. Since $\rho_3 \neq 0$, it follows that either $\mu_2 = 0$ or $\sigma_3 = 0$. In both cases, equalities (56) and (57) imply $A_3 A_3^{\mathrm{T}} = \theta^2 I$ with $\theta^2 = \rho_1^2 + \rho_2^2 + \rho_3^2 \neq 0$, and hence tensor T' is calibrated.

Case 2: $\rho_3 = 0$. By Lemma 6, the entries of tensor T' are constrained by (we take into account that $\lambda_1^2 = \mu_2^2$).

$$(\nu_1^2 + \rho_1^2 + \sigma_1^2)(\sigma_1^2 + \sigma_2^2) = (\nu_2^2 + \rho_2^2 + \sigma_2^2)(\sigma_1^2 + \sigma_2^2) = 0.$$

Since tensor T' is real, it follows that $\sigma_1 = \sigma_2 = 0$. Again by Lemma 6, we have

$$\nu_1\nu_2 + \rho_1\rho_2 = \nu_1^2 + \rho_1^2 - \nu_2^2 - \rho_2^2$$
$$= (\nu_1^2 + \rho_1^2 - (\sigma_3 + \mu_2)^2)(\nu_1^2 + \rho_1^2 - (\sigma_3 - \mu_2)^2) = 0.$$

Suppose first that $\nu_1^2 + \rho_1^2 \neq 0$. Then the correlation slices of T' can be represented in form

$$T_k' = A_2 e_k \begin{bmatrix} 0 & 0 & \mu_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} e_k^{\mathrm{T}} A_3^{\mathrm{T}},$$

where

$$A_2 = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon_2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} \nu_1 & \nu_2 & 0 \\ \rho_1 & \rho_2 & 0 \\ 0 & 0 & \sigma_3 - \epsilon_2 \mu_2 \end{bmatrix},$$

$\epsilon_k = \pm 1$. So that $A_2 = \pm R_2$ and $A_3 = \theta R_3$, where $R_2, R_3 \in$ SO(3) and $\theta^2 = \nu_1^2 + \rho_1^2 \neq 0$.

Finally, if $\nu_1^2 + \rho_1^2 = 0$, then $\nu_1 = \rho_1 = \nu_2 = \rho_2 = 0$ and T' is calibrated as well, since

$$T'_k = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon_2 \end{bmatrix} e_k \begin{bmatrix} 0 & 0 & \mu_2 \end{bmatrix} - 0 e_k^{\mathrm{T}} R_3^{\mathrm{T}},$$

where R_3 is arbitrary rotation matrix.

Thus we have shown that the trifocal tensor T' is calibrated in either case. By Lemma 3, the tensor T is calibrated too. Theorem 12 is proved. □

Finally, combining Theorem 12 with the characterization of trifocal tensors from [2], we get the following three-view analog of Theorem 2.

Theorem 13 *A real* (2, 1) *tensor is a CTFT if and only if it satisfies the following constraints:*

– *10 extended rank constraints;*
– *20 equations of degree 9 and 6 of degree 12; see [2, Theorem 2.3];*
– *15 quartic constraints.*

6 Discussion

Here are some remarks on the results of the paper and possible further directions of research.

1. We have defined a new notion—the trifocal essential matrix. Algebraically, it is a complex 3×3 matrix associated with a given CTFT $\hat{T}$ by the contraction of $\hat{T}$ and an arbitrary 3-vector whose squared components sum to zero. In this paper, the trifocal essential matrix plays a technical role. However its deeper investigation should help to explain why its properties are so close to the properties of an ordinary (bifocal) essential matrix. It is also interesting to what extent the trifocal essential matrix can be generalized in case of quadrifocal geometry.

2. Based on the characterization of trifocal essential matrices, we have derived the three necessary conditions on a CTFT (Theorems 8, 9 and 11). They have form of 15 quartic and 99 quintic polynomial equations. It is worth emphasizing again that these constraints relate to the calibrated case only and do not hold for arbitrary trifocal

tensors. Moreover, we have shown that the 15 quartic constraints are also sufficient for a real trifocal tensor to be calibrated (Theorem 12). Combining this result with the 36 polynomial equations from [2], we get the characterization of a real CTFT (Theorem 13). On the other hand, some of new 36 constraints may be redundant in the real calibrated case. This question requires more research.

3. As we have seen at the end of Sect. 4, the quartic constraints do not characterize CTFTs over the field of complex numbers. Hence, there must exist other polynomial equations not related to the trifocal essential matrix. In degree 5, we can expect to derive some of them from the 99 quintic constraints using similar arguments as in Remark 3.

4. The characterization of real calibrated trifocal tensors provides a new tool in investigating three-view metric structure-from-motion problems such as relative pose estimation and self-calibration. This could be a promising direction of future research, since we know that similar results for essential matrices have led to the efficient algorithms for analogous two-view problems [16,24].

The approach could be as follows. The linear incidence relations are first used to parameterize the trifocal tensor. Then the system of 51 polynomial equations is constructed and solved using either Gröbner bases or numerical methods. Theorem 13 guarantees that the true solution (which is always real) is a root of that system. However, a number of spurious complex solutions may occur. For example, the problem of relative pose estimation from four points and one line correspondences in three views is known to have a unique solution in general [19]. We constructed the Gröbner basis for that problem and found that it has 192 (!) spurious complex roots.

To improve the practical applicability of the paper's results we need a characterization of CTFTs over the field of complex numbers. The present paper provides a basis for that future research.

Appendix

We collect here some technical results that we used throughout the paper.

Lemma 7 *Let $a, b, c, d \in \mathbb{C}^n$. Then the (possibly) nonzero eigenvalues of matrix $M = ac^{\mathrm{T}} + bd^{\mathrm{T}}$ coincide with the eigenvalues of 2×2 matrix*

$$N = \begin{bmatrix} c^{\mathrm{T}} a & c^{\mathrm{T}} b \\ d^{\mathrm{T}} a & d^{\mathrm{T}} b \end{bmatrix}.$$

Proof The rank of matrix M is at most 2. Let λ_1, λ_2 be the (possibly) nonzero eigenvalues of M. Then,

$$\lambda_1 + \lambda_2 = \mathrm{tr}(M) = c^{\mathrm{T}}a + d^{\mathrm{T}}b = \mathrm{tr}(N),$$

$$2\lambda_1\lambda_2 = (\lambda_1 + \lambda_2)^2 - (\lambda_1^2 + \lambda_2^2) = \mathrm{tr}(M)^2 - \mathrm{tr}(M^2)$$

$$= 2(c^{\mathrm{T}}a)(d^{\mathrm{T}}b) - 2(c^{\mathrm{T}}b)(d^{\mathrm{T}}a) = 2\det N.$$

We see that λ_1, λ_2 are the eigenvalues of matrix N, as required. $\square$

Lemma 8 *Let M be a degenerate 3×3 matrix. Then the two (possibly) nonzero eigenvalues of M coincide if and only if the entries of M are subject to*

$$\mathrm{tr}(M)^2 - 2\,\mathrm{tr}(M^2) = 0. \tag{58}$$

Proof Let $0, \lambda_1, \lambda_2$ be the eigenvalues of M. Then,

$$\mathrm{tr}(M)^2 - 2\,\mathrm{tr}(M^2) = (\lambda_1 + \lambda_2)^2 - 2(\lambda_1^2 + \lambda_2^2) = -(\lambda_1 - \lambda_2)^2.$$

It follows that $\lambda_1 = \lambda_2$ if and only if (58) holds. Lemma 8 is proved. $\square$

Lemma 9 ([15]) *Let $s_1, s_2 \in \mathbb{C}^3$ be any nonzero vectors satisfying $s_1^{\mathrm{T}}s_1 = s_2^{\mathrm{T}}s_2 = 0$. Then there exists a matrix $R \in \mathrm{SO}(3, \mathbb{C})$ such that $Rs_1 = s_2$.*

Recall that the *radical* of an ideal J, denoted $\sqrt{J}$, is given by the set of polynomials which have a power belonging to J:

$$\sqrt{J} = \{p \mid p^k \in J \text{ for some integer } k \geq 1\}.$$

The following lemma gives a convenient tool to check whether a given polynomial is in the radical or not.

Lemma 10 ([5]) *Let $J = \langle p_1, \ldots, p_s \rangle \subset \mathbb{C}[\xi_1, \ldots, \xi_n]$ be an ideal. Then a polynomial $p \in \sqrt{J}$ if and only if $1 \in \tilde{J} = \langle p_1, \ldots, p_s, 1 - \tau p \rangle \subset \mathbb{C}[\xi_1, \ldots, \xi_n, \tau]$.*

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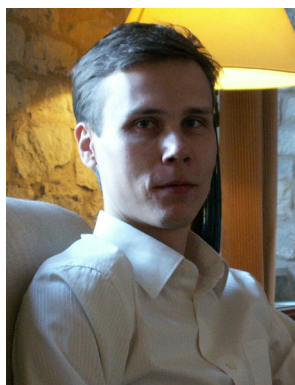
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