

Isometric Embeddings in Imaging and Vision: Facts and Fiction

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Abstract We explore the practicability of Nash’s Embedding Theorem in vision and imaging sciences. In particular, we investigate the relevance of a result of Burago and Zalgaller regarding the existence of *PL* isometric embeddings of polyhedral surfaces in \mathbb{R}^3 and we show that their proof does not extended directly to higher dimensions.

Keywords Nash’s theorem, · *PL*-isometric embedding · Burago-Zalgaller construction · Quasiconformal mapping · Maximal dilatation

1 Introduction

Recently, the attention of the vision community was drawn towards the important problem of isometric embeddings of manifolds in \mathbb{R}^n [43], in particular to the classical case of surfaces and their embedding in \mathbb{R}^3 or \mathbb{R}^5 [9]. This approach was further extended by the employment of metric techniques that achieved fame through the celebrated work of Michael Gromov [29]—see, [9, 10], amongst others. We do not consider here the critique of the metric method in general, since that would be, indeed, against our belief that it represents a basic, essential tool in imaging (see, for reference, [66]), but concentrate rather on the first problem: that of the isometric embedding.

Let us begin by noting that the departure from the entrenched view of surfaces as given only locally by parameterizations¹ certainly has merit, as does the realization that

surfaces, i.e. 2-dimensional manifolds, need often to be embedded, in a metrically controlled manner, in some \mathbb{R}^N . (The preferred value of N is, for obvious reasons, equal to 3.) However, there are some deep and disturbing problems stemming from this approach, that is based upon the celebrated Nash Embedding Theorem.

We begin with the least of these problems: the feasibility of finding an isometric embedding of a given, smooth orientable surface (or, more generally, an orientable manifold) in some \mathbb{R}^N , for N large enough. The root of the difficulty in writing and implementing an algorithm based upon Nash’s Theorem resides in the fact that this theorem² is a obtained via a fixed point method. The impediment here resides not only in the fact that such a method requires, at least theoretically, an infinite number of steps,³ but rather in the disturbing fact that the manifolds in the approximating sequence are not usually submanifolds of the same \mathbb{R}^N , where N represents the dimension of the Euclidean space in which the target (approximated) manifold, is (ideally) to be embedded—see also Remark 1.1 below.⁴ This is not just a theoretical, quasi-philosophical quandary, but rather a serious impediment: indeed, since computers, by their very nature, can perform only a finite number of approximations, the computed manifold will be embedded in a dimension different from the one stipulated (and often depicted—see also Remark 6.9 below).

Even if we can, somehow, surmount this difficulty, one still is faced with the dire specter of dimensionality. To make

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¹Usually via by spline functions.

²Together with other two recently famous methods: the circle packing [35, 36] and the Riemann Mapping Theorem (see, e.g. [4]).

³And that it is far less algorithmic in nature than the Picard Fixed Point Theorem of Differential Equations fame.

⁴A very similar problem arises also in the computational Riemann Mapping Theorem.

this assertion clearer (and more concrete) let us recall a few facts regarding Nash's Theorem.

The celebrated Nash Embedding Theorem [52] assures the existence of an isometric embedding of any \mathcal{C}^k , ($3 \leq k \leq \infty$) orientable⁵ manifold of dimension n into some \mathbb{R}^N , for some N sufficiently large. In contrast with the topological case, were one has the classical Whitney's Theorem (see Appendix 1), the embedding dimension ensured by the Nash theorem is prohibitively large, giving $N = \frac{n(3n+11)}{2}$. (Note, however, that in Nash's original work, the embedding dimension for noncompact manifolds was $N = \frac{3}{2}(n^2 + 5n)(n + 1)$.)

Even with the further dimension reductions of Gromov [26, 28] and Günter [30, 31], the assured embedding dimension, namely $(n^2 + 10n + 3)$, for $n \geq 3$, and $(n + 2)(n + 3)$, if $n \geq 4$ and $\max\{\frac{n(n+3)}{2} + 5, \frac{n(n+5)}{2}\}$, respectively, numerics are not very promising: the embedding dimension N of a surface (i.e. 2-dimensional manifold) provided by the original Nash Theorem is 17, and by Gromov's and Günter's improvements, 10. However, a special method developed by Gromov ([28], p. 298) decreases the embedding dimension for compact surfaces to 5, while for compact 3-manifolds the lowest guaranteed embedding dimension is $N = 13$ —see [28], p. 305. (Moreover, there exists a local isometric embedding of a given M^2 in any 5-dimensional manifold N^5 —see [28].)

Strikingly, there is nothing really known about the case $k = 2$ (its omission in the discussion above being no mistake).⁶ Since curvature (of differentiable surfaces, at least) is essentially a \mathcal{C}^2 notion⁷ one would count on a general Nash embedding theorem for this case, that would allow a straightforward application in imaging.

When one further relaxes the smoothness condition, the embedding dimension decreases dramatically: any \mathcal{C}^1 orientable 2-manifold is isometrically embeddable in \mathbb{R}^3 (see [45, 51]).⁸ Indeed, since the proper (i.e. differentiable) notion of curvature makes sense only for manifolds of class ≥ 2 , the additional dimensions that are required to deal with curvature,⁹ are not necessary in the \mathcal{C}^1 case. The role of the lack of differentiability is further emphasized by the fact that by Nash-Kuiper Theorem, even the *flat* torus is \mathcal{C}^1 isometrically embeddable in \mathbb{R}^3 , in contrast with Tompkins' Theorem [74] (see also [69], pp. 196–197) that asserts that compact flat \mathcal{C}^2 manifolds are not even isometrically immersible

into \mathbb{R}^{2n-1} . Not only this, but also the following result (due to Kuiper [45]) holds: The unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ admits an isometric \mathcal{C}^1 immersion in \mathbb{R}^{n+1} , for any $n \geq 1$. (This being in sharp contrast with the fact that any such \mathcal{C}^2 immersion is congruent to the unit sphere, for $n \geq 2$.)

Moreover, any \mathcal{C}^0 embedding is smoothable to \mathcal{C}^1 , therefore Whitney's Embedding Theorem (see Appendix 1) implies Nash's Embedding Theorem (for $n \geq 2$).

We should note that the case of analytic manifolds is, again, different: In [53] Nash proved that any compact (real) analytic n -dimensional manifold has an isometric embedding in $\frac{3}{2}(n^2 + 5n)$. Gromov [26] extended Nash's result to include noncompact manifolds and also reduced the embedding dimension (for both cases) to $\frac{1}{2}(n^2 + 7n + 10)$.

Remark 1.1 The striking difference between the results for the various degrees of smoothness emphasize once more the delicate manner in which one should approach the embedding problem, and even more so its practical applications. Moreover, even small variations of the metric, especially those producing change in the sign of (Gaussian) curvature (see also Remark 5.2), can abruptly change¹⁰ the embedding dimension (see [33] for a plethora of results in this direction).

It follows that, when using approximating sequences of *PL* manifolds and the isometric embedding technique, one can not ascertain with any degree of certainty that the sequence of embeddings remains in the same (minimal) dimension. Indeed, those familiar with polygonal meshes (e.g. people working in the field of graphics) know that—quite counterintuitively—even polygonal approximations of spheres have vertices of (concentrated) negative curvature (that is saddle points).

Therefore, the only general assured embedding dimension is that guaranteed by the Nash-Gromov-Günter Theorem.

Remark 1.2 Since we perceive shape, rather than distance, and since, by the previous remark (and the discussion preceding it, as well as by Remark 3.5 below), curvature, hence shape, is lost, it follows that \mathcal{C}^1 isometric embeddings and their *PL* counterparts are far less useful in imaging, recognition and matching purposes than hoped for.

Before proceeding any further, we have to add that we are aware that some readers are less familiar with some of the necessary background in differential topology and that, in any case, it would be best to refresh the basic necessary notions regarding immersions and embeddings. Therefore, we have included a glossary of relevant notions as Appendix 1.

⁵In the following, all manifolds are supposed to be orientable, except if otherwise specifically stated.

⁶It is not even known whether \mathcal{C}^2 manifolds admit \mathcal{C}^2 isometric immersions in \mathbb{R}^3 .

⁷Indeed, almost all the corpus of classical differential geometry of surfaces may be developed assuming only this degree of smoothness.

⁸The *PL* version of this result (and its extension to higher dimensions) represents the subject of following 3 sections.

⁹Using, for instance, the Gauss Equation—see, e.g. [1, 18].

¹⁰And, unfortunately, usually increase.

Given its goal, the pace is slow and the tone is rather didactical, therefore many a reader may want to omit it. However, since we also discuss therein one of the common misunderstandings regarding isometric embeddings, we believe it may be useful to all.

2 A Suggested Solution: PL Isometric Embeddings

It has been suggested to us [59] that the disquieting facts regarding the smooth embeddings considered in Nash’s Theorem need not disturb us to much, for, in the imaging and graphics practice, one is faced, in many cases (at least at some intermediary processing stage) with *PL*-flat surfaces (“triangular meshes”)¹¹ and for these a highly surprising and widely unexpected result exists,¹² namely the following theorem due to Burago and Zalgaller:

Theorem 2.1 *Any compact orientable PL 2-manifold admits an isometric embedding in \mathbb{R}^3 .*

The common wisdom regarding the statement above is, of course, that in imaging and graphics such surfaces represent the geometric object under investigation, or at least a “decent” approximation of it.

Unfortunately, this does not represent the solution of the problem in question. Indeed, a number of problems arise as soon as one examines this theorem a bit closer.

To begin with, the formulation above, while convenient and easy to recall, is not the correct one. The correct one can be found in [28], p. 213 (but recall also the title of the paper [15] of Burago and Zalgaller):

Theorem 2.2 (Burago-Zalgaller [28]) *Every compact oriented surfaces with a piecewise linear metric can be piecewise linearly isometrically embedded in \mathbb{R}^3 .*

First thing that strikes us is the apparently cumbersome and futile new terminology. However—as usually is the case in Mathematics, these apparent pointless minutiae and stresses are essential, and not due to just a whim of the mathematician. To comprehend this better in the case at hand, we should first understand the difference between the two (apparently identical) notions:¹³

¹¹Or only slightly more general polyhedral surfaces.

¹²For some more recent, seemingly paradoxical, related results, see e.g. [13, 54, 55].

¹³Given the space limitations and the desire for cohesiveness, we must assume the reader is familiar with the very basic notions of *PL* topology. (For a deep, yet enjoyable and not overly technical source on these notions, see [73]. See also Appendix 1 for additional material on embeddings.)

PL isometric embedding If P is a simplicial polyhedron of dimension n , any simplicial map $f : P \rightarrow \mathbb{R}^m$ (which is, by definition, linear on any simplex of P) induces a flat metric on each such simplex and, in consequence, a (singular) Riemannian metric g on P . More precisely, if P has k edges, then g is uniquely determined by the vector (g_1, \dots, g_k) , where $g_i = (\text{length}(e_i))^2$. Saying that f is a *PL isometric embedding* means that f is as above and, in addition, it is also an embedding.

PL isometric embedding of subdivided polyhedra In this case, the mapping (embedding) $f : P \rightarrow \mathbb{R}^m$ is required to be linearly isometric on the simplices of a simplicial subdivision of P . Evidently, by a sufficient number (albeit practically infinite) number of subdivisions, one can approximate the Riemannian case using *PL* metrics, i.e. such that each simplex of P is isometric to a Euclidean simplex (in \mathbb{R}^n).

So, why is the first definition not adequate? The main problem is its rigidity: Informally put, one “has to work with what he’s got”. That is, further subdivisions (hence approximations) are not allowed. Therefore, this approach rapidly reduces to a largely combinatorial problem, at least in many of its aspects (see, e.g. [38]).

Moreover (and more important) this rigidity is not just of convenience (so to say), quite the contrary—it is essential. Indeed, most¹⁴ *PL isometric maps* are rigid, in the (geo-)metric sense:

Theorem 2.3 [28] *Every small deformation of a n -dimensional polyhedron embedded (immersed) in \mathbb{R}^{n+1} is an isometry.*

In fact, the result just mentioned is more general, but to avoid a further detour, we refer to [28], pp. 210–211.

Obviously, the second notion is far more attractive, both for the geometer/analyst as well as for Computer Graphics and related fields. However, caution should be taken, since this is still a very flexible notion, since just the metric is to be preserved. For instance, one has the following result of Zalgaller [76]:

Theorem 2.4 [76] *Let P be a simplicial polyhedron of dimension n , $n \leq 4$, endowed with a *PL metric*. Then P admits an equidimensional *PL map* into \mathbb{R}^n .*

This is a very surprising and counterintuitive result.¹⁵ So, even though the embedding condition is omitted, it prepares us to understand somewhat better the problematic nature of the definition and of the Burago-Zalgaller theorem.

¹⁴Here “most” has a precise mathematical meaning: More precisely, *generic simplicial mappings are rigid*, where “generic” is a rather technical term (see [28]).

¹⁵However, it represents the *PL* version of the C^1 version (that holds for any n)—see, e.g. [28].

To further elucidate these notions we take below a closer look at Burago and Zalgaller’s proof.

3 The Burago-Zalgaller Construction

Main idea of the proof is—not very surprisingly—to adapt the proof of the C^1 Nash-Kuiper Embedding Theorem.¹⁶ For this, one starts with a (smooth) embedding of the given polyhedron in \mathbb{R}^3 , composed with a contracting homotopy. Then:

- Carry out a sequence of stages, divided in turn into a large number of steps, each of which improves the approximation to isometry and such that the function obtained at each stage is *short*:

Definition 3.1 (Short mappings) Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \rightarrow Y$ is called *C-short* iff

$$\rho(f(x), f(y)) \leq Cd(x, y), \quad \text{for all } x, y \in X.$$

f is called *short* (or *contracting*) iff it is *C-short*, for some $C < 1$.

- Add “ripples”, producing thus a *PL* version of Kuiper’s adaptation [45] of *Nash’s twist* [51] such that one will have “enough space” to isometrically embed the surface in \mathbb{R}^3 .

We won’t dwell too much in the details of the proof, just mention some of the principal “geometric” stages:

(1) *Basic construction element*

- Let $T = \Delta(A_1, A_2, A_3)$ and $t = \Delta(a_1, a_2, a_3)$ be acute triangles;
- let B, b and R, r the centers and radii of their respective circumscribed circles;
- let $E_p = \frac{1}{2}A_kA_l, e_p = \frac{1}{2}a_ka_l; p, k, l \in \{1, 2, 3\}$;
- and let $H_p = BE_p, h_p = be_p$.

Moreover, let $T \simeq t, A_kA_l > a_ka_l, k, l \in \{1, 2, 3\}$. Then T can be isometrically *PL* embedded in \mathbb{R}^3 , as the pleated surface included in the right prism with base t , such that $A_kA_lA_p$ fits $a_ka_lA_p$ where: $B'b \perp t, B'a_p = R$ and E'_p, E'_k, E'_l on the faces of the prism, such that $a_ka_lE'_p = Ei_pa_l = \frac{1}{2}A_kA_l$.

The following variations of the basic construction above are also considered:

- Each angle φ of T satisfies the condition $0 < \alpha < \varphi$ and $C \cdot A_kA_l > a_ka_l, C < 1$. Moreover, $A_kA_l/a_ka_l \approx 1$.

- Each of the lateral faces of the prism—including the broken lines $a_ka_lE'_p$ —can be (independently) slightly rotated around the lines a_ka_l such that the construction still can be performed. (The rotation angle depends upon the constants α and C above.)

(In general, one has to simultaneously construct a large number of the units above.)

(2) *Standard embedding near vertices*

Use the *standard conformal map* (or *folding*) from $K(\theta, \rho) = \{0 \leq \varphi \leq \theta, \rho > 0\}$ to $K(\lambda, r) = \{0 \leq \psi \leq \lambda, r > 0\}$ given by:

$$\psi = \frac{\lambda}{\theta}\varphi, \quad r = a\rho^{\lambda/\theta}.$$

(The most important case for our purposes being: $\lambda = 2\pi$.)

(3) *The triangulation and its refinement*

Let A be a vertex of total angle θ .

- If $\theta < 2\pi$, then encircle A by a small “regular” hexagon composed of 6 triangles of apex angle $\theta/6$. Some small enough neighbourhood of A the will be mapped by the standard conformal mapping onto a planar disk.

Over each triangle included in such a neighbourhood, one can perform the basic construction, obtaining a *PL* isometric embedding of this neighbourhood.

- If $\theta > 2\pi$, proceed analogously to the previous case but

- In a small circular neighbourhood of radius r_1 map (a) isometrically on radial segments and (b) using a $\theta/2\pi$ contraction on circles centered at A ;
- In an annular neighbourhood $\{r_1 < r < r_2\}$ use the standard conformal mapping with the same contraction factor $\theta/2\pi$.

Replace the neighbourhood above with a “cogwheel” (i.e. a circle surrounded by isosceles “triangles” of sides, e.g. 2δ , and having as bases arcs of the same length). The interior of each “cogwheel” is *PL* isometric embedded using “ripples”. (The basic element of each such “ripple” is a pair of congruent triangles, having a common vertex in the center of the “cogwheel”, one side (of each) being a radius, and a second common vertex built over the midpoint of an arc used in the construction of the “cogwheel”—see Fig. 4 of [15]). Away from neighbourhoods of vertices, refine the triangulation using only acute triangles. In particular, at convex vertices subdivide each triangle into n^2 similar triangles, for some large enough n ; while at non-convex vertices into almost regular triangles.

¹⁶We preserve in the following overview of the proof the notation of [15]. For copyright reasons we do not reproduce, however, the figures included therein, but rather refer the reader to the original source.

Remark 3.2 Adaptations [15] of the main technique exposed above ensure the existence of *PL* isometric embeddings of (orientable) *PL* manifolds with boundary and of *PL* immersions of nonorientable *PL* manifolds.

Remark 3.3 A close examination of the arguments of the proof shows that Theorem 2.2 can be extended to include (orientable) non-compact manifolds with bounded (generalized) principal curvatures [63].

Yet one naturally has to ask himself the following

Question 1 *Is the Burago-Zalgaller Theorem applicable for Image Processing/Computer Graphics?*

Unfortunately, the answer is negative, for the following reasons:

- (1) The construction yields an (infinite) approximation process, akin to the original Nash-Kuiper method, hence numerical errors have to contend with and taken into account.
- (2) The geometry¹⁷ of the limiting object is very far from the one of the “target surface”: Not only is the resulting *PL* surface strongly “corrugated” (as evident from the construction), it may also contain “superfluous” vertices, i.e. where the curvature (of the metric) is zero. Moreover, for surfaces of positive *extrinsic curvature*, it is quite possible that the surface admits not even an isometric immersion in \mathbb{R}^3 such that the extrinsic curvature equals the *intrinsic* one. (For the technical definitions and a simple example of a *PL* 2-sphere exhibiting this behavior, see [14], p. 76.)

Also, note that “accidents” in the original embedding can produce widely diverging subdivision schemes—see also Remark 1.1. (To grasp this widely divergent behavior, one should consider, for instance, the examples quoted in footnote 10.)

Example 3.4 (Burago-Zalgaller, Example 1.5) For any $\varepsilon > 0$, the flat torus \mathbb{T}^2 (i.e. the “topologist’s torus”, obtained by “gluing” the opposites sides of a plane square via Euclidean translations) admits a *PL* isometrical embedding ε -close to the rotation (“round”) torus (see, e.g. [19], pp. 434–435).¹⁸

However, one should understand that while the two tori are close in the *metric* sense, they are “far away” in the *geometric* one (see also Remark 3.5 below). If the details given

above of the Burago-Zalgaller construction and one’s intuition still do not suffice, then one should recall, as a further argument, *Mahler’s Compactness Theorem* [49] that, in its geometric form (see, e.g., [28], p. 77), states that a sequence of *n*-dimensional flat tori¹⁹ converges (in the Gromov-Hausdorff metric²⁰) to a flat torus of dimension $m \leq n$. In consequence, the limit of such a sequence can not be a round torus. The apparent contradiction between this fact and the example of Burago and Zalgaller rests in the fact that the aforementioned authors consider, as we have already seen, only converging sequences of *PL-flat* tori, and these fail to be flat on their singular set. Unfortunately, this set tends to be quite large, since its components are produced at each (of the large number of) iterations, by every geometric folding step (basic construction element, “cogwheel”, “ripple”, etc.). Therefore, the flatness of the limit torus will be quite local and the geometry at the points of the singular (non-flatness) set will diverge quite widely from the one of the flat torus—see the following remark for a discussion on this aspect.

Remark 3.5 It is contended in [7], p. 618, that the Burago-Zalgaller embedding method preserves curvature. However, this assertion is not made in [15].²¹ Indeed, this is not possible, as the example above clearly hints and as we shall explain in some detail below.

First, we should understand what type of curvature is preserved. Evidently, not the canonical (“smooth”) one of classical differential geometry, since the considered surfaces are not even C^1 . It may be that the authors of [7] refer to the fact that a piecewise flat surface (or polyhedron) remains piecewise flat during the embedding process and, as such, its curvature is identically zero at all the points that do not belong to the vertices and edges of the triangulation. However, it is explicitly emphasized in the very introduction of [15] that “the metric of a polyhedron is locally flat *except*²² at a finite collection of points; these points are the “true” vertices.”²³

However, it is precisely at these points that Gaussian curvature is concentrated (being the defect of the planar angles (of the faces) incident at any such vertex). This is a known, in fact, since Descartes, but it was introduced in modern Mathematics by Hilbert and Cohn-Vossen [34], and developed first by Polya [60] and then by Banchoff [2, 3] and, more recently by Stone [70] and Fu [21]. (Mean curvature

¹⁷All important in any practical implementation.

¹⁸Again, this result represents the *PL* equivalent of its C^1 counterpart—see the discussion in the Introduction.

¹⁹Algebraically, they can be described as a sequence $\{\mathbb{R}^n / \Lambda_k\}_k$, where Λ_k is a lattice.

²⁰See, e.g., [22, 66]

²¹In fact, the authors of [15] explicitly state (see [15], p. 370) that they “... prove the ... discrete analog of the well-known result of J. Nash and N. Kuiper on C^1 -smooth isometric immersions” (our emphasis).

²²Our emphasis.

²³[15], p. 369.

is, by contrast, concentrated along the edges of a polyhedral mesh, as the *dihedral angle* of the two faces whose intersection is any specific edge—see [47] for a succinct presentation and for the bibliography within.)

4 A Shattered Hope

Having seen that the Burago-Zalgaller construction is not applicable as such, one at least hopes for a positive answer to the following natural

Question 2 *Does Burago-Zalgaller’s Theorem hold in dimension $n \geq 3$?*

Perhaps unexpectedly, and contrary to the unsubstantiated statement of [7],²⁴ the answer to this question is not known! However, there are indications that the answer is negative. These indications emerge from the proof in dimension 2:

- (1) The proof is based on the previous result of Burago and Zalgaller on the existence of acute triangulations. Strangely enough, next to nothing is known about the existence of such triangulations in dimension $n \geq 3$.²⁵
- (2) The proof heavily relies on the use of the standard conformal map to produce a mapping that (around the vertices) is arbitrarily close to conformality (and, in the end) to isometry.

However, in dimension $n \geq 3$ this is not possible: the analogue of the standard conformal map has dilatation bounded away from 1! Indeed, we can be more specific. But first, a few technical²⁶ details:

Definition 4.1 (Wedges) Let $x \in \mathbb{R}^n$ be a point with cylindrical coordinates $x = (r \cos \varphi, r \sin \varphi, z_1, \dots, z_{n-2})$. The set $D_\alpha = \{0 < \varphi < \alpha\}$, ($0 < \alpha \leq 2\pi$) is called a *wedge* of angle α .

Definition 4.2 (Foldings) The homeomorphism $f : D_\alpha \rightarrow D_\beta$, $f(r, \varphi, z) = (r, \frac{\alpha}{\beta}\varphi, z)$, $z = (z_1, \dots, z_{n-2})$ is called a *folding*.

Before proceeding further, the reader should familiarize herself/himself with the technical notions regarding quasiconformal mappings. Not wishing to interrupt the flow of

geometric arguments, we have concentrated these in a short appendix (Appendix 2).

Proposition 4.3 (Gehring-Väisälä [24, 75]) *Let D_α, D_β be wedges, and let $f : D_\alpha \rightarrow D_\beta$ be the respective folding. If $\alpha \leq \beta$, then f is quasiconformal, with dilatations $K_I(f) = \frac{\alpha}{\beta}$, $K_O(f) \geq (\frac{\alpha}{\beta})^{1/(n-1)}$. In particular, for $\beta = \pi$, we obtain $K_I(D_\alpha) = \frac{\pi}{\alpha}$, $K_O(D_\alpha) = (\frac{\pi}{\alpha})^{1/(n-1)}$, whence $K(D_\alpha) = \frac{\pi}{\alpha}$.*

Remark 4.4 Remarkably, the coefficients of quasiconformality for non-convex domains (i.e. $\pi < \alpha \leq 2\pi$) are not known.²⁷

Following [16], we note the following natural generalization of the definition of a wedge:

Definition 4.5 The domain $D_{\alpha k} \subset \mathbb{R}^n$, $D_{\alpha k} = \{(r, \varphi_1, \dots, \varphi_{n-k+1}, z_{n-k+1}, \dots, z_n)\}$, $0 < \varphi_k < \alpha_k$, $1 \leq k \leq n - v - 1$, $\alpha = (\alpha_1, \dots, \alpha_{n-v-1})$, $0 < \alpha_1 \leq 2\pi$, $0 < \alpha_2, \dots, \alpha_{n-v-1} \leq \pi$ is called a *dihedral wedge of type v and angle α* .

Remark 4.6 For $k = n - 2$ we recuperate the classical definition of wedges.

The numbers that allow us to ascertain whether two domains are quasiconformally equivalent, i.e. that one is the quasiconformal (therefore homeomorphic) image of the other, and, if so, which is the smallest possible dilatation of such a mapping, are called the *coefficients of quasiconformality* (see Appendix 2). We have the following

Proposition 4.7 [16] *The coefficients of quasiconformality for $D_{\alpha k}$ are:*

$$\begin{aligned}
 K_I(D_{\alpha k}) &= \frac{\pi^{n-k-1}}{\alpha_1 \cdots \alpha_{n-k-1}}, \\
 K_O(D_{\alpha k}) &\geq \left(\frac{\pi^{n-k-1}}{\alpha_1 \cdots \alpha_{n-k-1}} \right)^{\frac{1}{n-1}}, \\
 K(D_{\alpha k}) &= \frac{\pi^{n-k-1}}{\alpha_1 \cdots \alpha_{n-k-1}}.
 \end{aligned}
 \tag{1}$$

Corollary 4.8 *Let \mathcal{P} be a convex polyhedral domain in \mathbb{R}^n and let m denote the number of faces of \mathcal{P} . Then we have the following estimates:*

$$\begin{aligned}
 K_I(\mathcal{P}) &\geq \frac{m-n+2}{m-n}, & K_O(\mathcal{P}) &\geq \left(\frac{m-n+2}{m-n} \right)^{\frac{1}{n-1}}, \\
 K(\mathcal{P}) &\geq \frac{m-n+2}{m-n}.
 \end{aligned}
 \tag{2}$$

²⁴In fact, the abstract [44] does state the result but does not sustain it was actually obtained.

²⁵The little existing information is summarized by Zamfirescu [77]. (Some additional hope stems from a different method developed recently by Tasmuratov [71].)

²⁶But hopefully not too technical—we shall restrict ourselves to the simplest case.

²⁷At least to the best of our knowledge.

Remark 4.9 Evidently, the same estimates hold for PL -smooth convex manifolds.

Hence, for polyhedra with a very large number of faces, such as encountered in (good) PL approximations of domains D in \mathbb{R}^3 (or, more generally, in \mathbb{R}^n), having smooth (convex) boundaries, $K(D) \gg 1$. Even without considering approximations and without making appeal to Corollary 4.8, one can easily produce (convex) polyhedra \mathcal{P} that require arbitrarily large dilatation $K(\mathcal{P})$, by choosing polyhedra with at least one dihedral angle (between n -faces) π/m , where m is any (arbitrarily large) natural number, and applying Proposition 4.3 directly.²⁸

It is interesting to note in this context, that (at least for polyhedral domains in \mathbb{R}^3) the “primary carrier” of dilatation is the mean curvature H —see Remark 3.5 above.

5 Final Comments

5.1 Glimmers of Hope

We bring below two different approaches to the embedding problem, that both circumvent the intricacies of the Nash Embedding theorem mentioned in the preceding sections.

5.1.1 A Compromise

Reviewing the facts above, it is hard not to reach the conclusion that the situation is quite bleak, as far as the practical use of Nash’s Embedding Theorem is concerned. However, one may quite justifiably sustain that making appeal to global isometric embeddings in general, and to Nash’s Theorem in particular, is to be somewhat overenthusiastic. Indeed, it may be very well claimed, that one is rarely faced, in computer vision, graphics and other related domains, with surfaces (manifolds) globally defined, hence one can restrict himself to *local isometric embeddings*. After all, this is the position already adopted (albeit in a different context, where large amounts of data have to be processed) in the widely quoted work of Roweis and Saul [62]. The method of [72] is also basically local (see, however [20] for a discussion on its possible globality).

This approach is also augmented by the very first result on isometric embeddings, namely the following theorem of Burstin, Janet and Cartan (see, e.g. [69]):

Theorem 5.1 (Burstin-Janet-Cartan) *Any (real) analytic manifold M^n can be locally (real) analytically isometrically embedded into $\mathbb{R}^{\frac{n(n+1)}{2}}$.*

²⁸For a stronger result regarding the nonexistence of isometric embeddings for PL manifolds in dimension ≥ 3 , see [63].

The problem with the result above is the fact that it requires analyticity. In fact, even if conjectured already by Schlaefly in 1873, the proof of the result above for C^∞ manifolds is still elusive. (It is true, however, that weaker forms of this result—that is, with higher embedding dimension—were obtained by Greene [25] and Gromov [28].)²⁹

Remark 5.2 As in the global case, for lower differentiability classes, no general results are even possible. Indeed, there exist a counterexample, due to Pogorelov [58], of a $C^{2,1}$ metric on the unit disk $\mathbb{B}^2 = \mathbb{B}^2(0, 1) \subset \mathbb{R}^2$, such that there exists no C^2 isometric imbedding in \mathbb{R}^3 of $\mathbb{B}^2(0, r)$, for any $0 < r < 1$. (See also [50] for some more recent results in this direction.)

Still, one may argue (rather convincingly) that it is quite common³⁰ in imaging and vision to adopt smooth, even analytic models and consider standard types of approximations (for the manifolds and for various differential operators on these manifolds).

A further incentive to adopt the local point of view as a viable and practical alternative for the Nash embedding theorem, at least as far as surfaces are concerned, is provided by the low embedding dimension (compare with the discussion in the Introduction, regarding the Nash dimension). Indeed, by a result of Jacobowitz [37], any smooth (more specifically of class C^m , $m > 3$) 2-dimensional Riemannian manifold admits a smooth local isometric embedding in \mathbb{R}^4 . The embedding dimension can be further reduced to 3 if the given surface has strictly positive Gaussian curvature (see [37]). Unfortunately, the proof is again based upon an infinite iteration scheme, so the problems raised above in connection with the Nash embedding resurfaces again. However, an alternative, constructive proof, due to Poznyak [61], exists—see [42], p. 82 (also [33], Proposition 41), for an outline of the proof. Moreover, the proof provides a stronger result, by producing an embedding of the given surface into a specific 3-dimensional hypersurface embedded in R^4 . It seems, therefore, that for surfaces, the use of local embeddings represents an approach truly feasible in applications.

5.1.2 More Dimensions

Surprisingly, a very effective (at least from the theoretical viewpoint) alternative embedding method follows the quite opposite direction: Instead of reducing the scope of the embedding, one can extend it by adding dimensions. That is, one can embed M^n not in some \mathbb{R}^N , but in an infinitely dimensional space, more precisely in $L^\infty(M^n)$ —the (Banach)

²⁹The author is not aware of the existence of meaningful, general theorems regarding C^k manifolds, for $2 \leq k < \infty$.

³⁰Even though the author does not subscribe himself to this philosophy.

space of bounded *Borel functions* on M^n , endowed with the “sup” metric, i.e. $d(f, g) = \sup_{x \in M^n} |f(x) - g(x)|$, for any $f, g \in L^\infty(M^n)$ —via the *Kuratowski Embedding* [46]:

Definition 5.3 (Kuratowski embedding) Let M^n be a closed Riemannian manifold. Then

$$K : M^n \rightarrow L^\infty(M^n), \quad K(x) = \text{dist}_x,$$

where

$$\text{dist}_x = \text{dist}(x, \cdot), \quad (3)$$

where “dist” denotes the (intrinsic, Riemannian) distance on M^n , is called the *Kuratowski embedding* (of M^n).

This method is much more powerful than it would appear at first sight. Indeed, the Kuratowski embedding is an isometry, more precisely we have the following Lemma (see, e.g. [32]):

Lemma 5.4 *With the notation above, we have*

$$d(\text{dist}_x, \text{dist}_y) = \text{dist}(x, y).$$

Remark 5.5 This approach is widely divergent from the Riemannian embedding one adopted in Nash’s Theorem. Indeed, the Riemannian and Kuratowski embeddings coincide iff $K(M^n) \subset L^\infty(M^n)$ is a convex, open subset of an affine linear subspace of dimension n .

On behalf of the Kuratowski embedding, one can remark that, albeit being infinite dimensional, it may be quite advantageous when a functional approach is needed or sought for (e.g. when considering *spline functions*, *wavelets*, etc.).³¹ However, usually (and more realistically) the spaces that appear in Computer Science (and even more so in Graphics) are finitely dimensional. Moreover, most people in the said communities find infinitely dimensional spaces as somewhat of an artifice, highly nonintuitive, and of theoretical value at best.

Fortunately, there exists a finitely dimensional version of the Kuratowski embedding: Let X be an ε -net³² in M^n , $|X| = m$. Then, for small enough ε , $K_X : X \rightarrow l_\infty^m$ is an embedding, where $K_X = K|_X$ —the restriction of K to X and l_∞^m denotes the m -dimensional Banach space endowed,

³¹A closely related approach is well known to the Imaging and Vision community: Embedding by using the eigenvalues of the Laplacian or of the Green Kernel. (For applications of these methods in the context of Riemannian Geometry, see, e.g. [5, 39].)

³²Recall that ε -nets are defined as follows:

Definition 5.6 Let (X, d) be a metric space, and let $A \subset X$. A is called an ε -net iff $d(x, A) \leq \varepsilon$, for all $x \in X$.

again, with the “sup” metric: if $\mathbf{x} = (x_1, \dots, x_p)$, then $\|\mathbf{x}\| = \sup_i |x_i|$.

Moreover, we can assure that this “discrete” version of the Kuratowski embedding is *bi-Lipschitz*, more precisely we have the following result [32, 41]:

Theorem 5.7 *Let M^n be a compact Riemannian manifold without boundary. Then, for any $C > 0$, there exists a ε -net X , where $\varepsilon = \varepsilon(C)$, such that*

$$(1 - C)\text{dist}(x, y) \leq |K_X(x) - K_X(y)| \leq \text{dist}(x, y). \quad (4)$$

Due to the theorem above, the finite dimensional version of the Kuratowski embedding is proved to be very useful in Global Differential Geometry: Its use in the study of *sys-toles* was pioneered by Gromov [29] (see also [32, 41]). It was also employed to prove yet another result of Gromov [27] (and Katsuda [40]), namely a *rigidity* theorem: Informally stated, the theorem in question asserts that if two n -dimensional (compact) Riemannian manifolds, having the same lower bound for their volumes, and upper bounds on diameters and sectional curvatures, are sufficiently close one to each other in the Gromov-Hausdorff (metric) topology,³³ then they are diffeomorphic.

Since the manifolds usually encountered in Imaging, Vision, etc., naturally satisfy such bounds, it follows that the result above, as well as the finitely-dimensional Kuratowski embedding in general, are quite relevant for applications in the mentioned fields, in particular for recognition type problems.³⁴ (See also [64] for a different application of the Kuratowski embedding in Imaging, namely to Sampling Theory.)

5.2 A Possible Solution

A more realistic approach (both from the theoretical and implementational viewpoints) would be to obtain a Discrete Nash Embedding Theorem [48]. A certain amount of confidence in the feasibility of obtaining such a result stems, amongst others, from the existence of discrete versions of the required differential operators and invariants (see, e.g. [11, 12]).

Another geometrization approach stems from the differential geometry of metric spaces (see, e.g. [8]). By using a discretization of the (metric) Finsler-Haantjes curvature of curves [65] we can obtain the embedding, via a proper discretization of the Gauss-Bonnet theorem, of any given metric graph, not into \mathbb{R}^n (or l_p , \mathbb{H}^n) but rather into a model space (a model surface, to be more precise)—see [67].

³³See, e.g. [22, 66] for a short overview of the notion.

³⁴For the implementation of the (finitely dimensional) Kuratowski embedding, one can still make appeal, for instance to MDS (Multidimensional Scaling), in one of its many forms (even if, perhaps, it is not very efficient in this case). (See, e.g. [57] for a brief presentation of the MDS method.)

Naturally, one expects the two embedding methods considered above to converge and augment each other, particularly in our purely geometric context, stemming from problems in *PL* differential geometry, computer graphics and image processing, where the graphs considered are skeletal of triangulations of manifolds (or of cell-complexes), and the weights are either edge-weights (i.e. distances between vertices) or/and vertex-weights (i.e. curvature measures)—see, e.g. [66, 68].

However, both methods, applied in a more general context, lend themselves to various practical implementations, in such areas as multicommodity flows in networks (e.g. for the prediction of informational bottlenecks, discovery of holes, etc.), clustering of statistical data (in particular in bioinformatics—see, e.g. [65]) and expanders.

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Appendix 1: Immersions and Embeddings

We presume the reader is familiar with the notions of differentiable manifold and tangent space, as well as with basic concepts of topology (for any eventually needed details see [73]), and we recall only the relevant definitions:

Definition 6.1 (Immersion) Let M^m, N^n be smooth differentiable manifolds and let $f : M^m \rightarrow N^n$ a differentiable map. If $\text{rank } f = m$ at each point of M^m , then f is called an *immersion*.

Definition 6.2 (Embedding) Let M^m, N^n be smooth differentiable manifolds and let $f : M^m \rightarrow F(M^m) \subseteq N^n$ a differentiable homeomorphism. If f is also an immersion, then it is called an *embedding*.

(Note that, in this case, $m = n$, of course.)

The condition that f be a homeomorphism is very strong and shouldn’t considered lightly. Indeed, not even asking that f be injective will suffice, as proven by the (classical)

fact that there exists a injective immersion of \mathbb{R} into the “figure eight” curve, but this is not an embedding, since it is not a homeomorphism: the image is not even a manifold!³⁵

To sum up: The notions of embedding and immersion are not interchangeable—while any embedding is, in particular, an immersion, the opposite is not true.³⁶

If one discards even the differential structure, then topological embeddings (in the sense of that they are homeomorphic on their image) are relatively easily obtained by

Theorem 6.3 (Whitney’s Theorem) *Every (smooth) manifold of dimension n admits a (smooth) embedding in \mathbb{R}^{2n} and a (smooth) immersion in \mathbb{R}^{2n-1} .*

Remark 6.4 It is easy to prove, for compact manifolds, that an embedding in some finite dimension N exists. It is then progressively (much) harder to discard the compactness restriction and to gradually “zero in” to dimension $2n$, via embedding dimensions $(n + 1)^2$ and $2n + 1$.

Up to this point we have dealt with classical (“pure”) differential topology, that is the famous “rubber geometry” of popularization texts (albeit endowed with some “smoothness”—necessary for the “differential” part). At this point, however, we should introduce a bit of “solid” geometry, necessary, e.g. for recognition purposes.³⁷ The idea is to use a specific “measuring yard”, for each manifold, that is a *Riemannian metric*:

Definition 6.5 (Riemannian manifolds) Let M^n be a manifold and let $x = (x_1, \dots, x_n)$ denote a standard coordinate chart. The Riemannian metric g on M^n is defined by the length element

$$ds^2 = g = \sum_{i=1}^n g_{ij} dx^i dx^j,$$

where the functions $g_{ij} = g_{ij}(x_1, \dots, x_n)$ represent the scalar products of the vector fields $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$ associated to the given chart:

$$g_{ij} = g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right).$$

³⁵An even more pathological example can be constructed, where \mathbb{R} is injectively immersed in the (flat) torus T^2 as a dense geodesic (just consider the image under the covering map of T^2 , of a line making an irrational angle with the Ox axis.

³⁶Therefore, even though, for stylistic reasons, one usually tends to avoid repetitions of the same word, one cannot (see e.g. [10]) freely interchange “immersion” and “embedding”...

³⁷Think of the individualization problems amoebas are faced with...

Once an infinitesimal distance is introduced, global ones can also be measured (transforming a Riemannian manifold into a “honest-to-God” metric space) as follows:

Let $c : [0, 1] \rightarrow M^n$ be a curve. Then its length is given by:

$$\text{length}(c) = \int_0^1 \|c'(t)\| dt.$$

(Here is important to recall that $c'(t)$ is just a tangent vector, so $\|c'(t)\| = \sqrt{g(c'(t), c'(t))}$.)

The *intrinsic* (or *inner*) distance between two points $p, q \in M^n$ is defined as

$$d(p, q) = \inf_c \{\text{length}(c) \mid c \text{ is a curve of ends } p \text{ and } q\}.$$

(We have tried here to keep the technical aspects of the definition above to a minimal level; for those insisting on absolute formal correctness, we recommend, for instance, [17].)

Of course, the Riemannian metric induces a topology on M^n (the metric topology), but in fact a much stronger result holds:

Theorem 6.6 (Palais [56]) *The metric of Riemannian manifolds determines its (smooth) manifold structure.*

Thus, Riemannian manifolds are, a fortiori, smooth manifolds, and the discussion above holds for them as well. However, a much more specific notion of embedding is relevant in this case, namely:

Definition 6.7 (Isometric embedding) Let M^n be a Riemannian manifold. An embedding $f : M^n \rightarrow \mathbb{R}^N$ is called isometric iff

$$\left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle = g_{i,j}, \quad 1 \leq i, j \leq n, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbb{R}^N .

Remark 6.8 Of course one can extend the definition above to include embeddings $f : M^n \rightarrow Q^N$, where Q is a general Riemannian manifold with metric h , by imposing, instead of (5), the following condition:

$$\langle \nabla_i f, \nabla_j f \rangle_h = g_{i,j}, \quad 1 \leq i, j \leq n, \tag{6}$$

where $\nabla_i f = Df(\frac{\partial}{\partial x_i})$ and where $\langle \cdot, \cdot \rangle_h$ represents the scalar product defined by the Riemannian metric h on $T_{f(x)}(Q)$, ($x = (x_1, \dots, x_n) \in M^n$).

Remark 6.9 (A common fallacy) Sadly, the notion of isometric embedding (and in particular the result above) are sometimes puzzling even for the professional mathematician (and even, sometimes, for topologists!...) However,

things become much simpler if one keeps in mind Definition 6.7 and remembers that saying that the intrinsic metric “equals” the Euclidean one, means just that the *infinitesimal* length element, as defined by the Riemannian metric coincides with that of the *infinitesimal* one induced by the ambient Euclidean space. In any case this shouldn’t be interpreted as affirming that lengths of curves, as measured on the manifold, equal the Euclidean distance between their ends, as measured in the ambient space \mathbb{R}^N (see Fig. 1). Therefore, it would be redundant and useless (not to say mistaken) to try and isometrically embed such surfaces in \mathbb{R}^N , for some $N > 3$.³⁸

On the positive side, surfaces in \mathbb{R}^3 are already embedded and inherit, therefore, a Riemannian structure from the ambient space, that is the induced metric defines on the surface a Riemannian metric (even if none was supposed (or given) a priori).

Appendix 2: Quasiconformal Mappings

Definition 7.1 (Quasiregular and quasiconformal mappings) Let $D \subseteq \mathbb{R}^n$ be a domain; $n \geq 2$ and let $f : D \rightarrow \mathbb{R}^n$ be a continuous mapping. f is called

- (1) *quasiregular* (*qr*) iff
 - (a) f is locally Lipschitz (and thus differentiable a.e.); and
 - (b) $0 < |f'(x)|^n \leq K J_f(x)$, for any $x \in M^n$; where $|f'(x)| = \sup_{|h|=1} |f'(x)h|$, and where $J_f(x) = \det f'(x)$;
- (2) *quasiconformal* (*qc*) iff $f : D \rightarrow f(D)$ is a quasiregular homeomorphism;

The smallest number K that satisfies condition (b) above is called the *outer dilatation* of f .

Remark 7.2 One can extend the definitions above to mappings between oriented, connected Riemannian n -manifolds, $n \geq 2$, by using coordinate charts (for details see, e.g. [75]).

Remark 7.3 It follows immediately from Condition (1) (b) above, that *qr*-mappings are sense preserving.

If $f : D \rightarrow \mathbb{R}^n$ is quasiregular, then there exists $K' \geq 1$ such that the following inequality holds a.e. in M^n :

$$J_f(x) \leq K' \inf_{|h|=1} |T_x f h|^n. \tag{7}$$

By analogy with the outer dilatation we have the following definition:

³⁸In fact, the only manifolds for which the embedding and intrinsic metric coincide are precisely the piecewise flat ones—see [6].

Fig. 1 Intrinsic vs. ambient distance: The length of the (straight, added) cord never equals the one of the bow, independently of the dimension in which Rodin's bowman resides



Definition 7.4 (*K*-quasiregularity) The smallest number K' that satisfies inequality (7) is called the *inner dilation* $K_I(f)$ of f , and $K(f) = \max(K_O(f), K_I(f))$ is called the *maximal dilatation* of f . If $K(f) < \infty$ we say that f is called *K-quasiregular*.

The dilations are $K(f)$, $K_O(f)$ and $K_I(f)$ are simultaneously finite or infinite. Indeed, the following inequalities hold: $K_I(f) \leq K_O^{n-1}(f)$ and $K_O(f) \leq K_I^{n-1}(f)$.

Definition 7.5 (Coefficients of quasiconformality) Let $D_1, D_2 \subset \mathbb{R}^n$ be domains homeomorphic to each other. The numbers

$$\begin{aligned}
 K_O(D_1, D_2) &= \inf_f K_O(f), K_I(D_1, D_2) \\
 &= \inf_f K_I(f), K(D_1, D_2) = \inf_f K(f),
 \end{aligned}
 \tag{8}$$

where the infima are taken over all the homeomorphisms $f : D_1 \xrightarrow{\sim} D_2$ are called the *outer*, *inner* and *total coefficient of quasiconformality* of D_1 with respect to D_2 , respectively. If D_2 is the unit ball \mathbb{B}^n , then the numbers $K_O(D_1) = K_O(D_1, \mathbb{B}^n)$, $K_I(D_1) = K_I(D_1, \mathbb{B}^n)$, $K(D_1) = K(D_1, \mathbb{B}^n)$ are simply called the (inner, resp. outer, resp. total) coefficients of conformality of D_1 .

Again, the numbers $K_O(D_1, D_2)$, $K_I(D_1, D_2)$ and $K(D_1, D_2)$ are simultaneously finite or infinite. However, it is not always guaranteed that there actually exists a homeomorphism f as above, such that $K_I(f) = K_I(D_1, D_2)$ or $K_O(f) = K_O(D_1, D_2)$, nor that if existing, it is unique. However, in the following important cases such an *extremal* mapping (for K_I or K_O) is known to exist:

Theorem 7.6 (Gehring-Väisälä [24], Gehring [23]) *The extremal mappings for K_1 and K_O exist if*

1. D_1 or D_2 is a ball;
2. The boundary of D_1 , ∂D_1 has k components, where $2 \leq k < \infty$;
3. D_1, D_2 are tori in \mathbb{R}^3 .

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