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A Variational Model to Remove the Multiplicative Noise in Ultrasound Images

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Abstract In this paper we study a variational model to deal with the speckle noise in ultrasound images. We prove the existence and uniqueness of the minimizer for the variational problem, and derive the existence and uniqueness of weak solutions for the associated evolution equation. Furthermore, we show that the solution of the evolution equation converges weakly in BV and strongly in L^2 to the minimizer as $t \to \infty$. Finally, some numerical results illustrate the effectiveness of the proposed model for multiplicative noise removal.

Keywords Calculus of variation · Weak solutions · BV · Variational model · Multiplicative noise · Image restoration

1 Introduction

Multiplicative noise removal problems have attracted much attention in recent years. In a multiplicative noise model, a recorded image f defined on a rectangle $\Omega \subset \mathbb{R}^2$, is the multiplication of an original image u and a noise n:

$$f = un. \tag{1.1}$$

Without loss of generality, we can assume that u and n are positive in the noise model. Unlike additive noises, these noises are much more difficult to be removed from the corrupted images, mainly not only because of their multiplicative nature, but also because of their distributions which are generally not Gaussian. In many real world image processing applications, multiplicative noises are commonly found, such as in laser images, microscope images, synthetic aperture radar (SAR) images and medical ultrasonic images. The additive noise removal problems have been extensively studied over the last decades, such as the PDE-based variational methods including the Rudin-Osher-Fatemi (ROF) model [16] and Lysaker-Lundervold-Tai (LLT) model [14]. The classical ROF model can be described to solve the following minimization problem:

$$\min_u \left\{ J(u) + \lambda \int_\Omega (f-u)^2 \right\}, \tag{1.2}$$

where $J(u) = \int_\Omega |Du|$ is the TV regularization term, the last term is the data fitting term, λ is the weighted parameter. The ROF model has been proved to be an invaluable tool for preserving sharp edges in image denoising. However, the multiplicative noise has not yet been studied thoroughly. As far as we know, the variational approach devoted to multiplicative noise is firstly proposed by Rudin, Lions and Osher [15] as follows:

$$\min_u \left\{ J(u) + \lambda_1 \int_\Omega \frac{f}{u} + \lambda_2 \int_\Omega \left(\frac{f}{u} - 1\right)^2 \right\}, \tag{1.3}$$

where the last two terms are the data fitting terms, λ_1 and λ_2 are the weighted parameters. We call this model as RLO

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model and it is particularly effective for Gaussian multiplicative noise removal. Recently, several variational approaches are devoted to the multiplicative noise removal problems (see [1, 7, 8]).

It is know that medical ultrasonic images are strongly corrupted by a noise called speckle [17]. In [18], the authors pointed out that when the scatter density (the number of scatter per resolution cell) was more than 10, this speckle noise followed a Rayleigh distribution and satisfied multiplicative model (1.1). The density function of the Rayleigh distribution is given by

$$p(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

where $\sigma > 0$ is a parameter. However, the displayed images from the ultrasound device have differential properties. One property of these images is logarithmic compression [9]. In [4, 13], the authors show that the speckle noise in the displayed images on longer follows the Rayleigh distribution. Experimental measurements in [2] indicate that the displayed ultrasonic images can be modeled as corrupted with signal-dependent noise of the form:

$$f = u + \sqrt{u}n, \tag{1.4}$$

where n the is a zero-mean Gaussian variable. Based on the model (1.4) and the characteristics of the Gaussian distribution, Krissian, Kikinis, Westin and Vosburgh in [11] further develop a new data fitting term:

$$E_1(u) = \int_\Omega \frac{(f-u)^2}{u}. \tag{1.5}$$

In this paper, motivated by the ROF model (1.2) and the new fitting term (1.5), we study the following model for removing the speckle noise in the ultrasonic images:

$$\min_u \left\{ J(u) + \lambda \int_\Omega \frac{(f-u)^2}{u} \right\}, \tag{1.6}$$

where λ is the weighted parameter. Furthermore, we investigate the following initial boundary value problem of the evolution equation corresponding to (1.6):

$$\partial_t u = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda\left(\frac{f^2}{u^2} - 1\right) \quad \text{on } \Omega_T, \tag{1.7}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \times [0, T], \tag{1.8}$$

$$u(0) = f \quad \text{on } \Omega. \tag{1.9}$$

For the problem (1.6) we prove the existence and uniqueness of the minimizer in $S(\Omega) := \{u \in BV(\Omega), u > 0\}$. For the problem (1.7)–(1.9), we establish the existence, uniqueness and long-time behavior of the weak solution. Furthermore, we prove that as $t \to \infty$ the solution of (1.7)–(1.9) converges strongly in L^2 and weakly in BV to the minimizer of the variational problem (1.6). Here our method is to study firstly the approximation problem of (1.7)–(1.9). Then some uniform estimates of the approximation solutions are derived, which enable us to pass to the limit in the approximate problem to get the existence of weak solutions to (1.7)–(1.9).

This paper is organized as follows. In Sect. 2 we give some classical theory for the space of $BV(\Omega)$ and present the definition of weak solutions to the problem (1.7)–(1.9). In Sect. 3, the existence and uniqueness of the minimizer to the problem (1.6) is proved. In Sect. 4 we study the associated evolution equation and get the existence, uniqueness and long-time behavior of weak solutions to the problem (1.7)–(1.9). Finally, in Sect. 5 some numerical experiments are demonstrated to illustrate the effectiveness of the proposed model for multiplicative noise removal.

It is worth mentioning that the interested domain for image processing problem is in $\mathbb{R}^2$. However, our results of this paper hold for a generic domain in $\mathbb{R}^n$.

2 Preliminaries

Let Ω be an open, bounded Lipschitz domain in $\mathbb{R}^n$, and write $\Omega_T := \Omega \times [0, T]$ with $T > 0$. For now on we always write

$$\begin{aligned} E(u) &:= \int_\Omega |Du| + \lambda \int_\Omega \frac{(f-u)^2}{u}, \\ h(s) &:= \frac{(f-s)^2}{s}. \end{aligned} \tag{2.1}$$

In the following we recall some basic notations and facts on the space of $BV(\Omega)$ (see [5, 6, 12]).

Definition 2.1 Define $BV(\Omega)$ as a space of functions $u \in L^1(\Omega)$ such that the following quantity

$$\int_\Omega |Du| := sup\left\{ \int_\Omega u\, div(\varphi)\, dx | \varphi \in C_0^1(\Omega; \mathbb{R}^n), |\varphi| \le 1 \right\}$$

is finite. $BV(\Omega)$ is a Banach space with the norm $\|u\|_{BV(\Omega)} = \int_\Omega |Du| + \|u\|_{L^1(\Omega)}$.

About the lower semicontinuity and compactness, we state the following theorems [5].

Theorem 2.1 *suppose $u_k \in BV(\Omega)$ $(k = 1, \dots)$ and $u_k \to u$ in $L^1_{loc}(\Omega)$. Then*

$$\int_\Omega |Du| \le \liminf_{k\to\infty} \int_\Omega |Du_k|.$$

Theorem 2.2 *Assume $\{u_k\}_{k=1}^{\infty}$ is a sequence in $BV(\Omega)$ satisfying $\sup_k \|u_k\|_{BV(\Omega)} < \infty$. Then there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and a function $u \in BV(\Omega)$ such that*

$$u_{k_j} \to u \quad \text{in } L^1(\Omega)$$

as $j \to \infty$.

Let us give the definition of weak solutions to problem (1.7)–(1.9) following the idea of [3, 19].

Definition 2.2 A function $u \in L^2([0,T]; BV(\Omega) \cap L^2(\Omega))(0 < T < \infty)$ is called a weak solution of (1.7)–(1.9) if $\partial_t u \in L^2(\Omega_T)$, $u(0) = u_0$ and u satisfies $u > 0$ in Ω_T, and

$$\int_0^s \int_\Omega (\partial_t u)(v-u)\,dx\,dt + \int_0^s E(v)\,dt \geq \int_0^s E(u)\,dt \quad (2.2)$$

for all $v \in L^2([0,T]; BV(\Omega) \cap L^2(\Omega))$ and a.e. $s \in [0,T]$.

Remark 1 If (2.2) holds, by selecting $v = u + \epsilon\phi$ for $\phi \in C^\infty(\Omega)$ and noticing that the left-hand side of (2.2) has a minimum at $\epsilon = 0$, we can show that u is a solution of (1.7)–(1.9) in the sense of distribution by computing

$$\frac{d}{d\epsilon}\left(\int_\Omega (\partial_t u)(\epsilon\phi)\,dx + E(u+\epsilon\phi)\right)$$

at $\epsilon = 0$.

3 The Variational Problem

In this section we discuss the following denoising model:

$$\inf_{u \in S(\Omega)} \left\{ \int_\Omega |Du| + \lambda \int_\Omega \frac{(f-u)^2}{u} \right\}, \quad (3.1)$$

where $S(\Omega) := \{u \in BV(\Omega), u > 0\}$, $f \in L^\infty(\Omega)$ is the given data and $\lambda > 0$ is the weighted parameter.

Theorem 3.1 *Let $f \in L^\infty(\Omega)$ with $\inf_\Omega f > 0$, then problem (3.1) admits a unique solution $u \in S(\Omega)$ satisfying*

$$\inf_\Omega f \leq u \leq \sup_\Omega f. \quad (3.2)$$

Proof Denote by $\alpha = \inf_\Omega f$, $\beta = \sup_\Omega f$. It is obvious that $E(u) \geq 0$ for all $u \in S(\Omega)$. Therefore, we consider a minimizing sequence $\{u_n\}_{n=1}^{\infty} \subset S(\Omega)$ for (3.1).

First, we show that $\alpha \leq u_n \leq \beta$. Since $f \in L^\infty(\Omega)$ with $\inf_\Omega f > 0$, we can choose a sequence $\{f_n\} \subset C^\infty(\overline{\Omega})$ such that $f_n \to f$ in $L^1(\Omega)$ and a.e. in Ω as $n \to \infty$, and

$$\inf_\Omega f \leq f_n \leq \sup_\Omega f. \quad (3.3)$$

If in (2.1) f is replaced by f_n, we see that $h(s)$ is decreasing as $s \in (0, f_n)$ and increasing as $s \in (f_n, +\infty)$ for $n \in \mathbb{N}$. Therefore, if $A \geq f_n$, one always has

$$\frac{(f - \min(s,A))^2}{\min(s,A)} \leq \frac{(f-s)^2}{s}$$

for $x \in \Omega$ and $n \in \mathbb{N}$. Note that $\beta = \sup_\Omega f \geq f_n$ from (3.3). Hence, if we let $A = \beta$, we have

$$\int_\Omega \left(\frac{(f_n - \min(u,\beta))^2}{\min(u,\beta)}\right) \leq \int_\Omega \left(\frac{(f_n - u)^2}{u}\right).$$

Letting $n \to \infty$ in the above inequality, using Lebesgue Convergence Theorem and (3.3), we deduce

$$\int_\Omega \left(\frac{(f - \min(u,\beta))^2}{\min(u,\beta)}\right) \leq \int_\Omega \left(\frac{(f-u)^2}{u}\right). \quad (3.4)$$

Moreover, by using the results of [10] (see Lemma 1 in Sect. 4.3), we obtain

$$\int_\Omega |D(\min(u,\beta))| \leq \int_\Omega |Du|. \quad (3.5)$$

Combining (3.4) and (3.5), we get that

$$E(\min(u,\beta)) \leq E(u).$$

On the other hand, in the same way we get that $E(\max(u,\alpha)) \leq E(u)$. Therefore, we can assume that without restriction that $\alpha \leq u_n \leq \beta$.

Second, we prove that there exists $u \in S(\Omega)$ such that

$$E(u) = \min_{v \in S(\Omega)} E(v).$$

The above proof implies in particular that u_n is bounded in $L^1(\Omega)$. Moreover, by the definition of $\{u_n\}$, we get that there exists a constant C such that

$$\int_\Omega |Du_n| + \int_\Omega h(u_n) \leq C. \quad (3.6)$$

Since $\alpha \leq u_n \leq \beta$ and $h \in C[\alpha,\beta]$, we get that $h(u_n)$ is bounded. Therefore, by using (3.6), we deduce that

$$\int_\Omega |Du_n| \leq C.$$

Hence, we get that u_n is bounded in $BV(\Omega)$ and there exists u in $BV(\Omega)$ such that up to a subsequence, $u_n \to u$ in $L^1(\Omega)$-strong a.e. in Ω. Necessarily, we have $0 < \alpha \leq u \leq \beta$. This implies that $u \in S(\Omega)$. By using the lower semicontinuity of the total variation and Fatou's lemma, we get that u is a solution of problem (3.1).

It is easy to find that $h''(s) = \frac{2f^2}{s^3}$ and $h(s)$ is strictly convex as $s > 0$. Therefore, the uniqueness of the minimizer follows from the strict convexity of the energy functional in (3.1). □

4 The Associated Evolution Problem

In this section we establish the existence, uniqueness and long-time behavior of the weak solutions to the associated evolution problem (1.7)–(1.9).

First we consider the following approximation Problem $P_R^{\epsilon,\delta}$ of (1.7)–(1.9):

Problem $P_R^{\epsilon,\delta}$

$$\begin{cases} \partial_t u = \epsilon \Delta u + div\left(\frac{\nabla u}{\sqrt{|\nabla u|^2+\epsilon^2}}\right) + \lambda\left(\frac{f^2}{[u]_R^2} - 1\right), & on\ \Omega_T, \\ \frac{\partial u}{\partial n} = 0, \quad on\ \partial\Omega \times [0, T], \\ u(0) = f^\delta, \quad on\ \Omega. \end{cases} \tag{4.1}$$

Here $[\cdot]_R$ is the truncated function defined as $[\eta]_R := \max\{R, \eta\}$, R is a constant that will be determined in the latter part of this section; $f \in L^\infty(\Omega) \cap BV(\Omega)$, $f^\delta \in C^\infty(\overline{\Omega})$ such that $f^\delta \to f$ in $L^1(\Omega)$ and

$$\inf_\Omega f \le f^\delta \le \sup_\Omega f, \tag{4.2}$$

$$\int_\Omega |\nabla f^\delta| \le C_1 \int_\Omega |Df|, \tag{4.3}$$

where C_1 is a fixed constant independent of δ.

We have the following existence and uniqueness results for Problem $P_R^{\epsilon,\delta}$ (4.1).

Lemma 4.1 *Let $f \in L^\infty(\Omega) \cap BV(\Omega)$. For fixed $\epsilon, \delta, R > 0$, the approximation problem (4.1) admits a unique weak solution $u_R^{\epsilon,\delta}$ such that $u_R^{\epsilon,\delta} \in L^\infty(0, T; H^1(\Omega))$, $\partial_t u_R^{\epsilon,\delta} \in L^2(0, T; L^2(\Omega))$.*

Proof By using the Galerkin method and Lebesgue Convergence Theorem, the fact that $\frac{p}{\sqrt{p^2+\epsilon^2}}$ is a monotone operator [2] and $\frac{f^2}{[u]_R^2}$ is bounded, we get that problem (4.1) admits a unique weak solution $u_R^{\epsilon,\delta}$ such that

$$\partial_t u_R^{\epsilon,\delta} \in L^2(0, T; L^2(\Omega)),$$
$$u_R^{\epsilon,\delta} \in L^\infty(0, T; H^1(\Omega)).$$ □

In order to get the existence of weak solution for the original problem (1.7)–(1.9), we need derive some uniform estimates of the solution $\{u_R^{\epsilon,\delta}\}$ for the approximation problem (4.1).

Lemma 4.2 *Assume that $f \in L^\infty(\Omega) \cap BV(\Omega)$ with $\inf_\Omega f > 0$ and*

$$m := \inf_\Omega f, \qquad M := \sup_\Omega f. \tag{4.4}$$

Let $\{u_R^{\epsilon,\delta}\}$ be a weak solution for problem (4.1) with $0 < R \le \inf_\Omega f$. Then we have, for any $T > 0$,

$$m \le u_R^{\epsilon,\delta} \le M$$

for a.e. $(x, t) \in \Omega_T$.

Proof Since $u_R^{\epsilon,\delta}$ is a weak solution of (4.1), we see that $(u_R^{\epsilon,\delta} - M)_+(t) \in H^1(\Omega)$, where $(\cdot)_+$ is the truncated function defined as $(\zeta)_+ := \max\{0, \zeta\}$.

Multiplying the first equation in (4.1) by $(u_R^{\epsilon,\delta} - M)_+(t)$ and integrating it over Ω, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |(u_R^{\epsilon,\delta}(t) - M)_+|^2\,dx + \epsilon\int_\Omega |\nabla(u_R^{\epsilon,\delta} - M)_+|^2 \\ &\quad + \int_\Omega \frac{|\nabla(u_R^{\epsilon,\delta} - M)_+|^2}{\sqrt{|\nabla u_R^{\epsilon,\delta}|^2 + \epsilon^2}} \\ &\quad \le \lambda \int_\Omega \left(\frac{f^2}{[u_R^{\epsilon,\delta}]_R^2} - 1\right)(u_R^{\epsilon,\delta} - M)_+. \end{aligned} \tag{4.5}$$

Note that $[u]_R = \max\{R, u\} \ge M$ as $u \ge M$, and $f^2 \le M^2$. Hence, we have

$$\begin{aligned} &\int_\Omega \left(\frac{f^2}{[u_R^{\epsilon,\delta}]_R^2} - 1\right)(u_R^{\epsilon,\delta} - M)_+ \\ &\quad = \int_{\{u_R^{\epsilon,\delta} \ge M\}} \left(\frac{f^2}{[u_R^{\epsilon,\delta}]_R^2} - 1\right)(u_R^{\epsilon,\delta} - M)_+ \\ &\quad \le \int_{\{u_R^{\epsilon,\delta} \ge M\}} \left(\frac{f^2}{M^2} - 1\right)(u_R^{\epsilon,\delta} - M)_+ \\ &\quad \le 0. \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we conclude

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(u_R^{\epsilon,\delta}(t) - M)_+|^2\,dx \le 0.$$

The above inequality implies that

$$\int_\Omega |(u_R^{\epsilon,\delta}(t) - M)_+|^2\,dx \le 0$$

holds for a.e. $t \in [0.T]$, since $u_R^{\epsilon,\delta}(0) = f^\delta \le M$ from (4.2) and (4.4). Therefore, we get that

$$u_R^{\epsilon,\delta}(x, t) \le M$$

holds for a.e. $(x, t) \in \Omega_T$.

On the other hand, we see that $(u_R^{\epsilon,\delta} - m)_-(t) \in H^1(\Omega)$, where $(\cdot)_-$ is the truncated function defined as $(\zeta)_- := \min\{0, \zeta\}$.

Multiplying the first equation in (4.1) by $(u_R^{\epsilon,\delta}-m)_-(t)$ and integrating it over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(u_R^{\epsilon,\delta}(t)-m)_-|^2\,dx+\epsilon\int_\Omega|\nabla(u_R^{\epsilon,\delta}-m)_-|^2 + \int_\Omega \frac{|\nabla(u_R^{\epsilon,\delta}-m)_-|^2}{\sqrt{|\nabla u_R^{\epsilon,\delta}|^2+\epsilon^2}} \le \lambda\int_\Omega\left(\frac{f^2}{[u_R^{\epsilon,\delta}]_R^2}-1\right)(u_R^{\epsilon,\delta}-m)_-. \tag{4.7}$$

Note that if $0<R\le m$ and $u\le m$, then $0<[u]_R\le m$. Hence, we have

$$\begin{aligned}\int_\Omega\left(\frac{f^2}{[u_R^{\epsilon,\delta}]_R^2}-1\right)(u_R^{\epsilon,\delta}-m)_- &= \int_{\{u_R^{\epsilon,\delta}\le m\}}\left(\frac{f^2}{[u_R^{\epsilon,\delta}]_R^2}-1\right)(u_R^{\epsilon,\delta}-m)_-\\ &\le \int_{\{u_R^{\epsilon,\delta}\le m\}}\left(\frac{f^2}{m^2}-1\right)(u_R^{\epsilon,\delta}-m)_-\\ &\le 0.\end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we conclude

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|(u_R^{\epsilon,\delta}(t)-m)_-|^2\,dx\le 0.$$

The above inequality implies that

$$\int_\Omega|(u_R^{\epsilon,\delta}(t)-m)_-|^2\,dx\le 0$$

holds for a.e. $t\in[0.T]$, since $u_R^{\epsilon,\delta}(0)=f^\delta\ge m$ from (4.2) and (4.4). Therefore, we get that

$$u_R^{\epsilon,\delta}(x,t)\ge m$$

holds for a.e. $(x,t)\in\Omega_T$. □

Remark 2 Choosing $R=m$ in (4.1), we see that the truncated function $[\cdot]_R$ in (4.1) can be omitted. In the following, we consider the solution of Problem $\mathrm{P}_\mathrm{R}^{\epsilon,\delta}$ as $u^{\epsilon,\delta}$ depending only on ϵ,δ (not on R). From Lemma 4.1 we get that the solution $u^{\epsilon,\delta}$ satisfies $0<m\le u^{\epsilon,\delta}\le M$ a.e. in Ω_T.

Lemma 4.3 *Suppose that $f\in L^\infty(\Omega)\cap BV(\Omega)$ with $\inf_\Omega f>0$, and $\{u^{\epsilon,\delta}\}$ is a weak solution to (4.1). Then we have, for any $T>0$,*

$$\int_0^T\int_\Omega|\partial_t u^{\epsilon,\delta}|^2\,dx\,dt + \sup_{t\in[0,T]}\int_\Omega\left(\frac{\epsilon}{2}|\nabla u^{\epsilon,\delta}(t)|^2+|\nabla u^{\epsilon,\delta}(t)|\right)dx \le \frac{\epsilon}{2}\int_\Omega|\nabla f^\delta|^2\,dx+\int_\Omega|\nabla f^\delta|\,dx+|\Omega|\epsilon+C, \tag{4.9}$$

where C is a constant independent of ϵ, δ and T.

Proof Since $u^{\epsilon,\delta}$ is a weak solution of (4.1), multiplying the first equation in (4.1) by $\partial_t u^{\epsilon,\delta}$ and integrating it over Ω, we have

$$\int_\Omega|\partial_t u^{\epsilon,\delta}|^2\,dx+\frac{d}{dt}\int_\Omega\left(\frac{\epsilon}{2}|\nabla u(t)^{\epsilon,\delta}|^2 + \sqrt{|\nabla u(t)^{\epsilon,\delta}|^2+\epsilon^2}+\lambda\frac{(f-u^{\epsilon,\delta})^2}{u^{\epsilon,\delta}}\right)dx=0. \tag{4.10}$$

Note that

$$|p|\le\sqrt{|p|^2+\epsilon^2}\le|p|+\epsilon$$

holds for $p\in\mathbb{R}^n$, $u^{\epsilon,\delta}>0$ and $m\le u^{\epsilon,\delta}(0)=f^\delta\le M$. We easily obtain (4.9) by using (4.10). □

Now we are able to establish the existence and uniqueness theorem of weak solutions for problem (1.7)–(1.9).

Theorem 4.4 *Assume that $f\in L^\infty(\Omega)\cap BV(\Omega)$ with $\inf_\Omega f>0$. Then problem (1.7)–(1.9) admits a unique weak solution in the sense of Definition 2.2.*

Proof Let $u^{\epsilon,\delta}$ be the solution to (4.1). By Lemma 4.2, Remark 2 and Lemma 4.3 we get that

$$0<m\le u^{\epsilon,\delta}(x,t)\le M\quad\text{a.e. in }\Omega_\infty \tag{4.11}$$

and

$$\int_0^\infty\int_\Omega|\partial_t u^{\epsilon,\delta}|^2\,dx\,dt + \sup_{t\in[0,\infty)}\int_\Omega\left(\frac{\epsilon}{2}|\nabla u^{\epsilon,\delta}(t)|^2+|\nabla u^{\epsilon,\delta}(t)|\right)dx \le \frac{\epsilon}{2}\int_\Omega|\nabla f^\delta|^2\,dx+\int_\Omega|\nabla f^\delta|\,dx+|\Omega|\epsilon+C. \tag{4.12}$$

Hence, we know that, for fixed $\delta>0$,

$u^{\epsilon,\delta}$ is uniformly bounded in $L^\infty(0,\infty;W^{1,1}(\Omega)\cap L^\infty(\Omega))$,

$\partial_t u^{\epsilon,\delta}$ is uniformly bounded in $L^2(\Omega_\infty)$.

Therefore, combining the fact that the embedding $W^{1,1}\hookrightarrow L^1$ is compact, we conclude that for fixed $\delta>0$, there exists a subsequence of $\{u^{\epsilon,\delta}\}$ such that as $\epsilon\to 0$,

$$u^{\epsilon,\delta}(t)\to u^\delta(t)\quad\text{strongly in } L^1(\Omega)\text{ and a.e in }\Omega\text{ for each }t>0, \tag{4.13}$$

$$\partial_t u^{\epsilon,\delta}\rightharpoonup\partial_t u^\delta\quad\text{weakly in }L^2(\Omega_T). \tag{4.14}$$

Furthermore, letting $\epsilon\to 0$ in (4.11) with fixed δ and using (4.13), we have

$$m\le u^\delta(x,t)\le M\quad\text{a.e. in }\Omega_\infty. \tag{4.15}$$

Notice the fact that for each $t>0$, as $\epsilon\to 0$,

$$u^{\epsilon,\delta}(t)\to u^{\delta}(t)\quad \text{strongly in } L^2(\Omega),$$

since it follows from (4.11) and (4.15) that

$$\int_\Omega |u^{\epsilon,\delta}(t)-u^{\delta}(t)|^2\le C(M)\int_\Omega |u^{\epsilon,\delta}(t)-u^{\delta}(t)|.$$

Furthermore, $t\to u^{\epsilon,\delta}(\cdot,t)$ is equi-continuous since

$$\|u^{\epsilon,\delta}(\cdot,t)-u^{\epsilon,\delta}(\cdot,t,)\|^2_{L^2(\Omega)}\le\int_{\Omega_\infty}|\partial_t u^{\epsilon,\delta}|^2\,dx\,dt.$$

Then, by a standard argument, we can get that

$$u^{\epsilon,\delta}(t)\to u^{\delta}(t)\quad \text{strongly in } L^2(\Omega) \text{ uniformly in } t. \tag{4.16}$$

Recall that

$$\begin{cases}\sqrt{p^2+\epsilon^2}-\sqrt{q^2+\epsilon^2}\ge\frac{q}{\sqrt{q^2+\epsilon^2}}\cdot(p-q),\\ p^2-q^2\ge 2q\cdot(p-q),\\ h(s)-h(t)\ge h'(t)(s-t)\end{cases}\tag{4.17}$$

hold for $p,q\in\mathbb{R}^n$ and $s,t>0$, due to the convexity of $\sqrt{p^2+\epsilon^2}$, p^2 and $h(s)$ as $s>0$. Since $u^{\epsilon,\delta}$ is a weak solution to (4.1), multiplying the first equation in (4.1) by $v-u^{\epsilon,\delta}$, using (4.17) and integrating it over $\Omega\times[0,s]$, we conclude that

$$\begin{aligned}&\int_0^s\int_\Omega\partial_t u^{\epsilon,\delta}(v-u^{\epsilon,\delta})\,dx\,dt+\frac{\epsilon}{2}\int_0^s\int_\Omega|\nabla v|^2\,dx\,dt\\&\quad+\int_0^s\int_\Omega\sqrt{|\nabla v|^2+\epsilon^2}\,dx\,dt\\&\quad+\lambda\int_0^s\int_\Omega\frac{(f-v)^2}{v}\,dx\,dt\\&\ge\frac{\epsilon}{2}\int_0^s\int_\Omega|\nabla u^{\epsilon,\delta}|^2\,dx\,dt+\int_0^s\int_\Omega\sqrt{|\nabla u^{\epsilon,\delta}|^2+\epsilon^2}\,dx\,dt\\&\quad+\lambda\int_0^s\int_\Omega\frac{(f-u^{\epsilon,\delta})^2}{u^{\epsilon,\delta}}\,dx\,dt\\&\ge\int_0^s\int_\Omega|\nabla u^{\epsilon,\delta}|\,dx\,dt+\lambda\int_0^s\int_\Omega\frac{(f-u^{\epsilon,\delta})^2}{u^{\epsilon,\delta}}\,dx\,dt\end{aligned}\tag{4.18}$$

holds for all $v\in L^2(0,T;H^1(\Omega))$ and each $s\in[0,\infty)$. Now considering every term in (4.18), and using (4.11), (4.13)–(4.14), (4.16) and Lebesgue Convergence Theorem, we obtain that as $\epsilon\to 0$,

$$\int_0^s\int_\Omega\partial_t u^{\epsilon,\delta}(v-u^{\epsilon,\delta})\,dx\,dt\to\int_0^s\int_\Omega\partial_t u^{\delta}(v-u^{\delta})\,dx\,dt,\tag{4.19}$$

$$\frac{\epsilon}{2}\int_0^s\int_\Omega|\nabla v|^2\,dx\,dt+\int_0^s\int_\Omega\sqrt{|\nabla v|^2+\epsilon^2}\,dx\,dt\to\int_0^s\int_\Omega|\nabla v|\,dx\,dt\tag{4.20}$$

and

$$\lambda\int_0^s\int_\Omega\frac{(f-u^{\epsilon,\delta})^2}{u^{\epsilon,\delta}}\,dx\,dt\to\lambda\int_0^s\int_\Omega\frac{(f-u^{\delta})^2}{u^{\delta}}\,dx\,dt.\tag{4.21}$$

Moreover, combining (4.13) and Theorem 2.1 we have

$$\int_\Omega|Du^\delta(t)|\le\liminf_{\epsilon\to 0}\int_\Omega|\nabla u^{\epsilon,\delta}(t)|\,dx\quad\text{for each } t>0,\tag{4.22}$$

which implies

$$\int_0^s\int_\Omega|Du^\delta|\,dt\le\liminf_{\epsilon\to 0}\int_0^s\int_\Omega|\nabla u^{\epsilon,\delta}|\,dx\,dt\tag{4.23}$$

by using Fatou's Lemma. Therefore, combining (4.19)–(4.21) and (4.23), let $\epsilon\to 0$ in (4.18) to arrive at

$$\begin{aligned}&\int_0^s\int_\Omega\partial_t u^{\delta}(v-u^{\delta})\,dx\,dt+\int_0^s\int_\Omega|\nabla v|\,dx\,dt\\&\quad+\lambda\int_0^s\int_\Omega\frac{(f-v)^2}{v}\,dx\,dt\\&\ge\int_0^s\int_\Omega|Du^\delta|\,dt+\lambda\int_0^s\int_\Omega\frac{(f-u^{\delta})^2}{u^{\delta}}\,dx\,dt.\end{aligned}\tag{4.24}$$

This shows that u^δ is a weak solution to (1.7) with initial data f^δ.

Additionally using (4.3), (4.22) and letting $\epsilon\to 0$ in (4.12), we obtain

$$\begin{aligned}&\int_0^\infty\int_\Omega|\partial_t u^\delta|^2\,dx\,dt+\sup_{t\in[0,\infty)}\int_\Omega|Du^\delta(t)|\\&\le C_1\int_\Omega|Df|dx+C.\end{aligned}\tag{4.25}$$

Hence, combining (4.15) we have

$$u^\delta \text{ is uniformly bounded in } L^\infty(0,\infty;BV(\Omega)\cap L^\infty(\Omega)),$$
$$\partial_t u^\delta \text{ is uniformly bounded in } L^2(\Omega_\infty).$$

The, by the similar argument to the one for getting (4.13)–(4.14) and (4.16), we can find a subsequence of $\{u^\delta\}$ and a function $u\in L^\infty(0,\infty;BV(\Omega)\cap L^\infty)$ such that, as $\delta\to 0$,

$$\begin{aligned}&u^\delta(t)\to u(t)\quad \text{strongly in } L^1(\Omega) \text{ and a.e. in } \Omega\\&\qquad\qquad\qquad \text{for each } t>0,\\&\partial_t u^\delta\rightharpoonup\partial_t u\quad\text{weakly in } L^2(\Omega_T),\\&u^\delta(t)\to u(t)\quad\text{strongly in } L^2(\Omega) \text{ uniformly in } t.\end{aligned}\tag{4.26}$$

Therefore, we can pass to the limit as $\delta \to 0$ in (4.24) to get

$$\begin{aligned}&\int_0^s \int_\Omega (\partial_t u)(v-u)\,dx\,dt + \int_0^s \int_\Omega |Dv| \\ &\quad + \lambda \int_0^s \int_\Omega \frac{(f-v)^2}{v} \\ &\quad \geq \int_\Omega |Du| + \lambda \int_0^s \int_\Omega \frac{(f-u)^2}{u}\end{aligned} \tag{4.27}$$

for all $v \in L^2([0,T]; BV(\Omega) \cap L^2(\Omega))$ and a.e. $s \in [0,T]$. Thus we get the existence of a weak solution u to problem (1.7)–(1.9).

Furthermore, letting $\delta \to 0$ in (4.15), (4.25), we get by using (4.26) that the solution u obtained above satisfies the following estimates:

$$m \leq u(x,t) \leq M \quad \text{a.e. in } \Omega_\infty, \tag{4.28}$$

$$\int_0^\infty \int_\Omega |\partial_t u|^2\,dx\,dt + \sup_{t\in[0,\infty)} \int_\Omega |Du(t)| \leq C. \tag{4.29}$$

In the following we prove the uniqueness of weak solutions to problem (1.7)–(1.9). Let u_1, u_2 be two weak solutions to (1.7) with $u_1(0) = u_2(0) = f$. Then we have

$$\begin{aligned}&\int_0^s \int_\Omega (\partial_t u_1)(u_2-u_1)\,dx\,dt + \int_0^s E(u_2)\,dt \\ &\quad \geq \int_0^s E(u_1)\,dt\end{aligned}$$

and

$$\begin{aligned}&\int_0^s \int_\Omega (\partial_t u_2)(u_1-u_2)\,dx\,dt + \int_0^s E(u_1)\,dt \\ &\quad \geq \int_0^s E(u_2)\,dt.\end{aligned}$$

Adding the above two inequalities we get

$$\frac{1}{2}\int_0^s \frac{d}{dt}\int_\Omega |u_1-u_2|^2\,dx\,dt \leq 0.$$

This implies

$$\|u_1(\cdot,s) - u_2(\cdot,s)\|_{L^2(\Omega)} = 0$$

for a.e. $s \in [0,T]$. Therefore $u_1 = u_2$. □

At last, we show the asymptotic limit of the solution $u(\cdot,t)$ as $t \to \infty$.

Theorem 4.5 *As $t \to \infty$, the weak solution $u(x,t)$ of (1.7)–(1.9) converges strongly in $L^2(\Omega)$ and weakly in $BV(\Omega)$ to a minimizer $\hat{u}$ of the variational problem (3.1).*

Proof Take a function $v \in BV(\Omega) \cap L^2(\Omega)$ in (4.27) to get

$$\begin{aligned}&\frac{1}{s}\int_\Omega (u(x,s) - u(x,0))v(x)\,dx \\ &\quad - \frac{1}{2s}\int_\Omega (u^2(x,s) - u^2(x,0))\,dx \\ &\quad + \int_\Omega |Dv| + \lambda \int_\Omega \frac{(f-v)^2}{v}\,dx \\ &\quad \geq \frac{1}{s}\int_0^s \int_\Omega |Du| + \lambda \frac{1}{s}\int_0^s \int_\Omega \frac{(f-u)^2}{u}\,dx\,dt.\end{aligned} \tag{4.30}$$

Define

$$w(x,s) = \frac{1}{s}\int_0^s u(x,t)\,dt.$$

Then, by using the convexity of $|p|$ for $p \in \mathbb{R}^n$ and h(s) for $s > 0$, and employing the Jensey's inequality in (4.30), we obtain

$$\begin{aligned}&\frac{1}{s}\int_\Omega (u(x,s) - u(x,0))v(x)\,dx \\ &\quad - \frac{1}{2s}\int_\Omega (u^2(x,s) - u^2(x,0))\,dx \\ &\quad + \int_\Omega |Dv| + \lambda \int_\Omega \frac{(f-v)^2}{v}\,dx \\ &\quad \geq \int_\Omega |Dw(s)| + \lambda \int_\Omega \frac{(f-w(s))^2}{w(s)}\,dx.\end{aligned} \tag{4.31}$$

Furthermore, from (4.28)–(4.29), for each $s > 0$, $w(\cdot,s) \in BV(\Omega) \cap L^\infty(\Omega)$ with uniformly bounded BV and L^∞-norms. Thus there exists a subsequence $w(\cdot,s_i)$ converging strongly in $L^1(\Omega)$ and a.e. in Ω, and weakly in $BV(\Omega)$ and $L^\infty(\Omega)$ to a function $\hat{u} \in BV(\Omega) \cap L^\infty(\Omega)$ as $s_i \to \infty$. In fact, since $w(\cdot,s_i)$ have uniformly bounded L^∞-norm, the convergence of $w(\cdot,s_i)$ to $\hat{u}$ is strong in $L^2(\Omega)$.

Letting $s_i \to \infty$ in (4.31), and employing the lower semi-continuity of the total variation and Fatou's lemma, we get that

$$\begin{aligned}&\int_\Omega |Dv| + \lambda \int_\Omega \frac{(f-v)^2}{v}\,dx \\ &\quad \geq \int_\Omega |D\hat{u}| + \lambda \int_\Omega \frac{(f-\hat{u})^2}{\hat{u}}\,dx\end{aligned}$$

for all $v \in BV(\Omega) \cap L^2(\Omega)$, which implies that $\hat{u}$ is a minimizer of (3.1) □

5 Numerical Results

In this section some numerical tests on our model (1.6) are demonstrated for multiplicative noise removal. Numerically we get a solution to problem (3.1) by computing the associated (1.7) to a steady state. To discretize (1.7), the finite

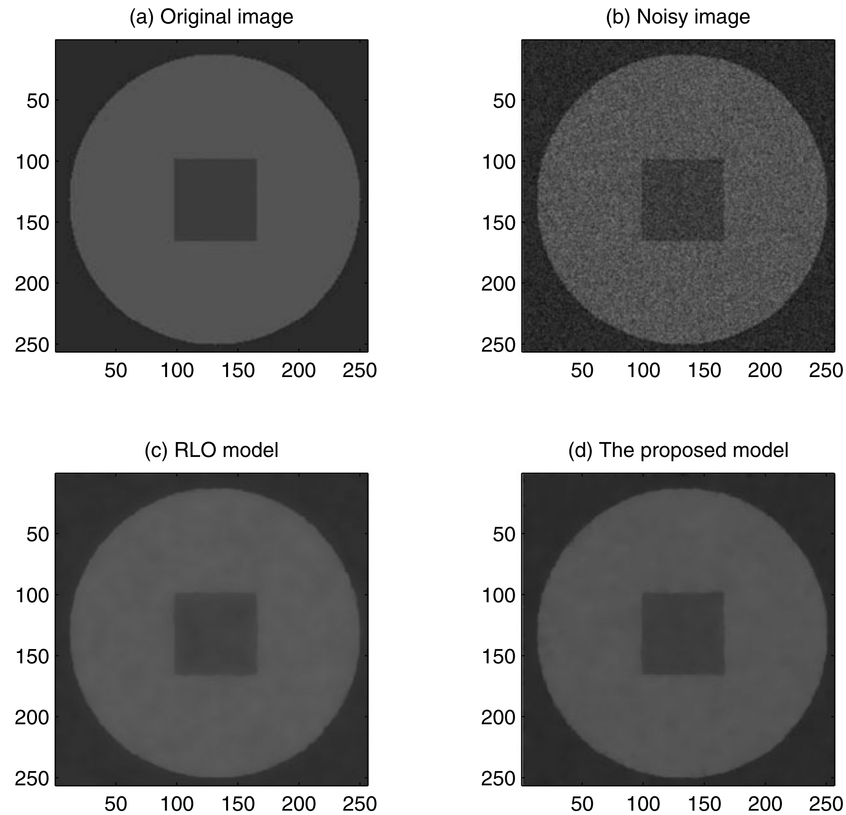

Fig. 1 (**a**) The synthetic image "SynImag1"; (**b**) the multiplicative noise is added following the model (1.4) with the standard deviation $\sigma = 2$, SNR = 3.74, ReErr = 0.23; (**c**) iteration = 140

difference scheme in [16] is used. Denote the space step by $h = 1$ and the time step by τ. Thus we have

$$D_x^{\pm}(u_{i,j}) = \pm[u_{i\pm1,j} - u_{i,j}],$$
$$D_y^{\pm}(u_{i,j}) = \pm[u_{i,j\pm1} - u_{i,j}],$$
$$|D_x(u_{i,j})| = \sqrt{(D_x^+(u_{i,j}))^2 + (m[D_y^+(u_{i,j}), D_y^-(u_{i,j})])^2 + \delta},$$
$$|D_y(u_{i,j})| = \sqrt{(D_y^+(u_{i,j}))^2 + (m[D_x^+(u_{i,j}), D_x^-(u_{i,j})])^2 + \delta},$$

where $m[a,b] = (\frac{\operatorname{sign} a + \operatorname{sign} b}{2}) \cdot \min(|a|,|b|)$ and $\delta > 0$ is the regularized parameter chosen near 0.

The numerical algorithm for (1.7) are given in the following (the subscripts i, j are omitted):

$$\frac{u^{n+1} - u^n}{\tau} = \left[D_x^-\left(\frac{D_x^+ u^n}{|D_x u^n|}\right) + D_y^-\left(\frac{D_y^+ u^n}{|D_y u^n|}\right)\right] + \lambda^n\left(\frac{f^2}{(u^n)^2} - 1\right),$$

with boundary conditions

$$u_{0,j}^n = u_{1,j}^n, \qquad u_{N,j}^n = u_{N-1,j}^n,$$
$$u_{i,0}^n = u_{i,1}^n, \qquad u_{i,N}^n = u_{i,N-1}^n$$

for $i, j = 1, \ldots, N-1$. Note that the corresponding Euler Lagrange equation to (1.6) is

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda\left(\frac{f^2}{u^2} - 1\right) = 0.$$

Multiplying the above equation by $\frac{(f-u)u}{f+u}$ and integrating it on the image domain Ω yields

$$\lambda \int_\Omega \frac{(f-u)^2}{u} = \int_\Omega \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\frac{(u-f)u}{u+f}.$$

Therefore, by using the assumption that the Gaussian variable $n = \frac{f-u}{\sqrt{u}}$ has mean 0 and variance σ^2, we obtain that the weighted parameter λ can be automatically computed by

$$\lambda^n = \frac{1}{\sigma^2|\Omega|}\sum_{i,j}\left(\left[D_x^-\left(\frac{D_x^+ u^n}{|D_x u^n|}\right) + D_y^-\left(\frac{D_y^+ u^n}{|D_y u^n|}\right)\right]\frac{(u^n - f)u^n}{u^n + f}\right).$$

The parameters are chosen like this: $\tau = 0.2$, $\delta = 0.0001$. In addition, we take $u^0 = f$ as the initial value. In the following numerical experiments, for the above algorithm it is will be stopped at the index where the variance of the recovered noise matches that of our prior knowledge.

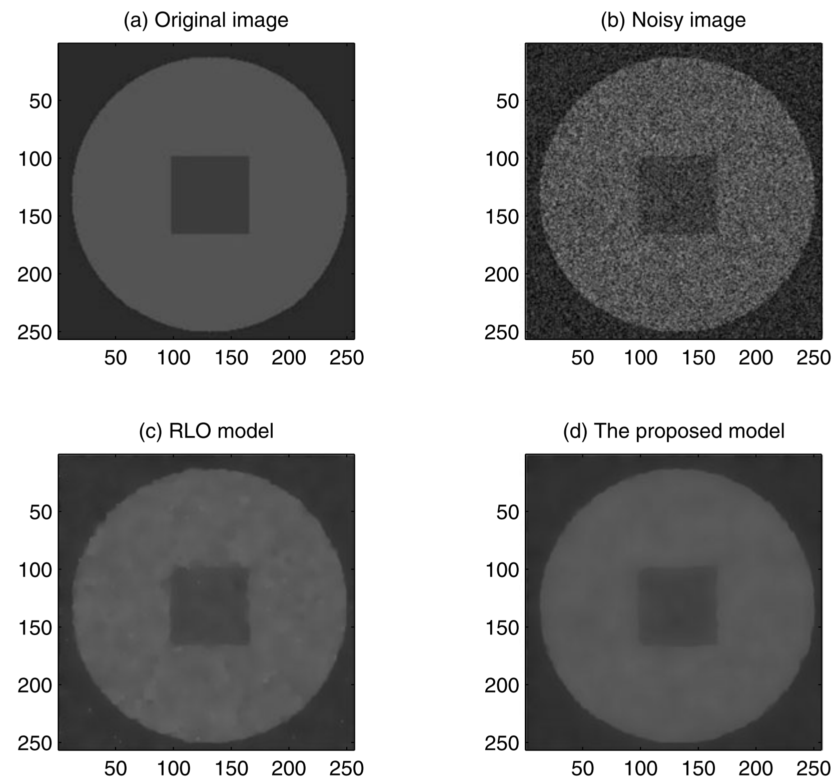

Fig. 2 (**a**) The synthetic image "SynImag1"; (**b**) the multiplicative noise is added following the model (1.4) with the standard deviation $\sigma = 4$, SNR = 1.26, ReErr = 0.46; (**c**) iteration = 180

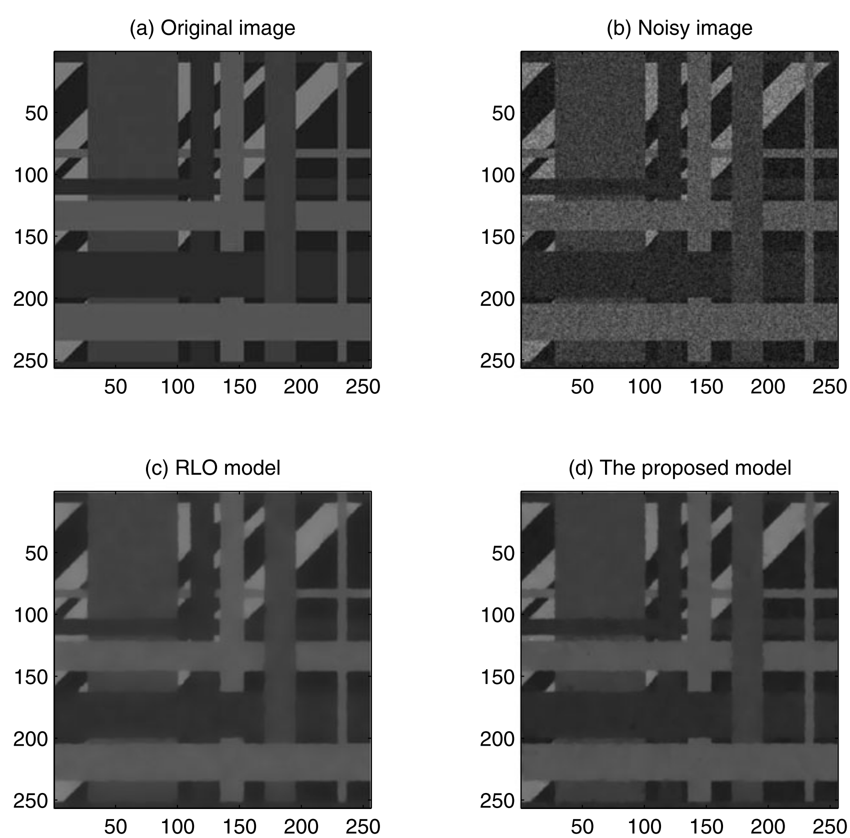

Fig. 3 (**a**) The synthetic image "SynImag2"; (**b**) the multiplicative noise is added following the model (1.4) with the standard deviation $\sigma = 2$, SNR = 5.41, ReErr = 0.23; (**c**) iteration = 140

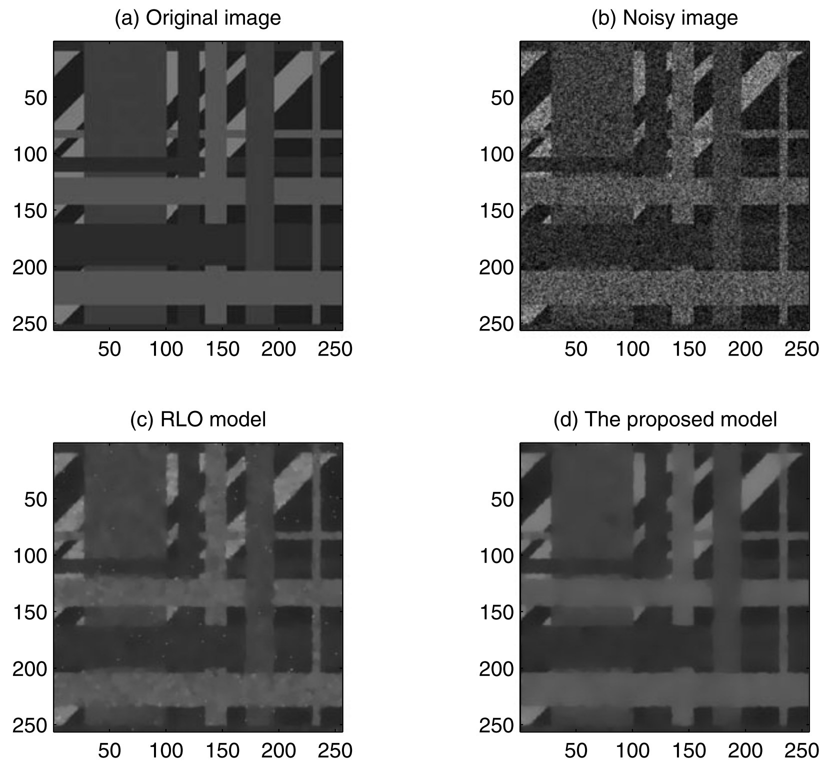

Fig. 4 (**a**) The synthetic image "SynImag2"; (**b**) the multiplicative noise is added following the model (1.4) with the standard deviation $\sigma = 4$, SNR $= 2.09$, ReErr $= 0.47$; (**c**) iteration $= 190$

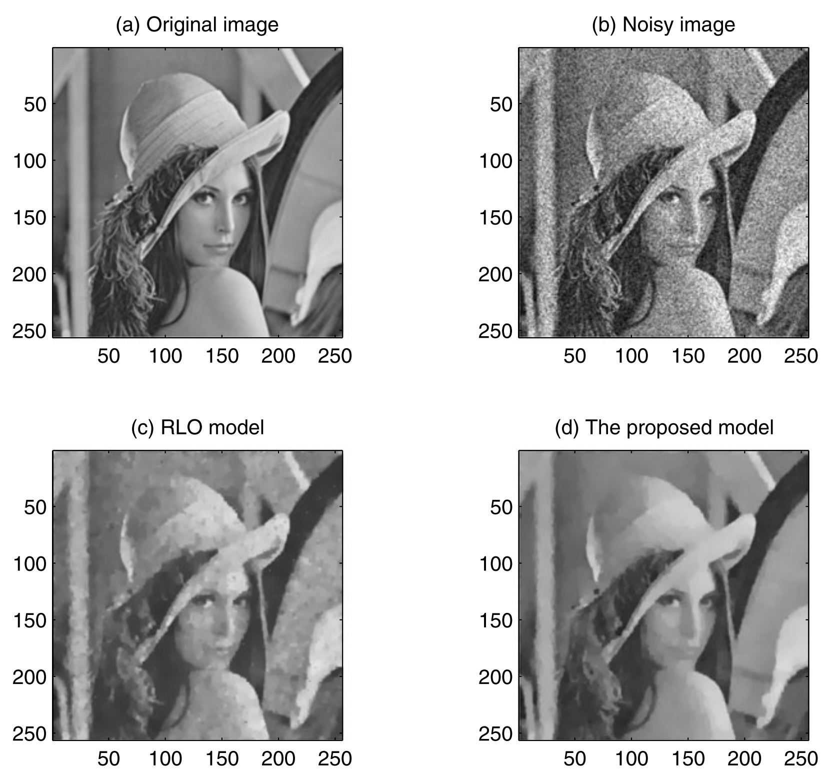

Fig. 5 (**a**) The original "Lena" image; (**b**) the multiplicative noise is added following the model (1.4) with the standard deviation $\sigma = 3$, SNR $= 4.87$, ReErr $= 0.25$; (**c**) iteration $= 190$

In the following numerical experiments, we display the denoising results obtained by our approaches, as well as with the RLO model (1.3). The solutions of the RLO model are also computed by discretizing the corresponding gradient descent flow equation

$$\partial_t u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda_1 \frac{f}{u^2} + \lambda_2 \frac{f}{u^2}\left(\frac{f}{u} - 1\right),$$

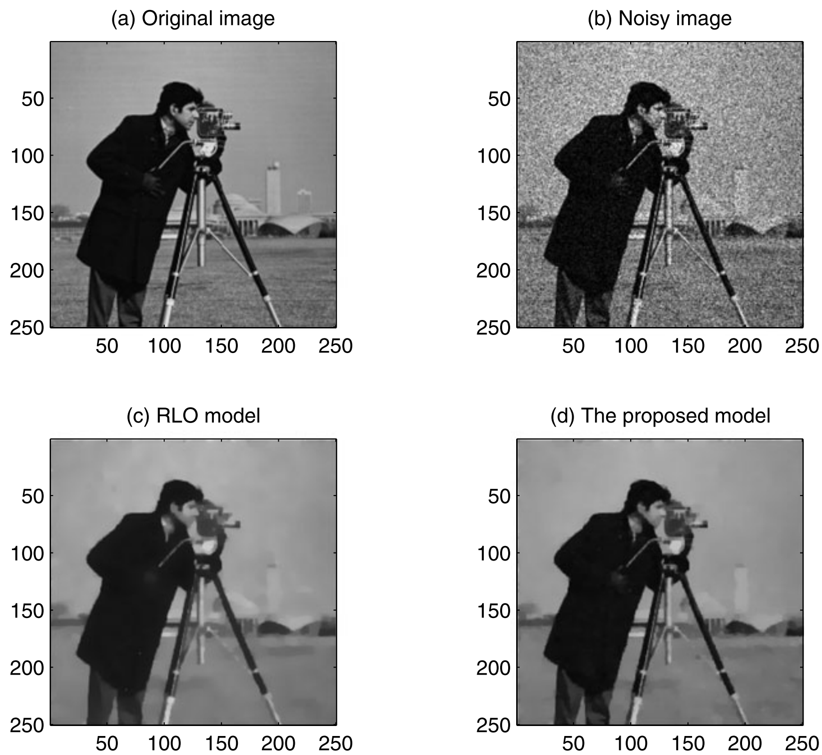

Fig. 6 (**a**) The original "Cameraman" image; (**b**) the multiplicative noise is added following the model of (1.4) with the standard deviation $\sigma = 3$, SNR $= 6.36$, ReErr $= 0.26$; (**c**) iteration $= 200$

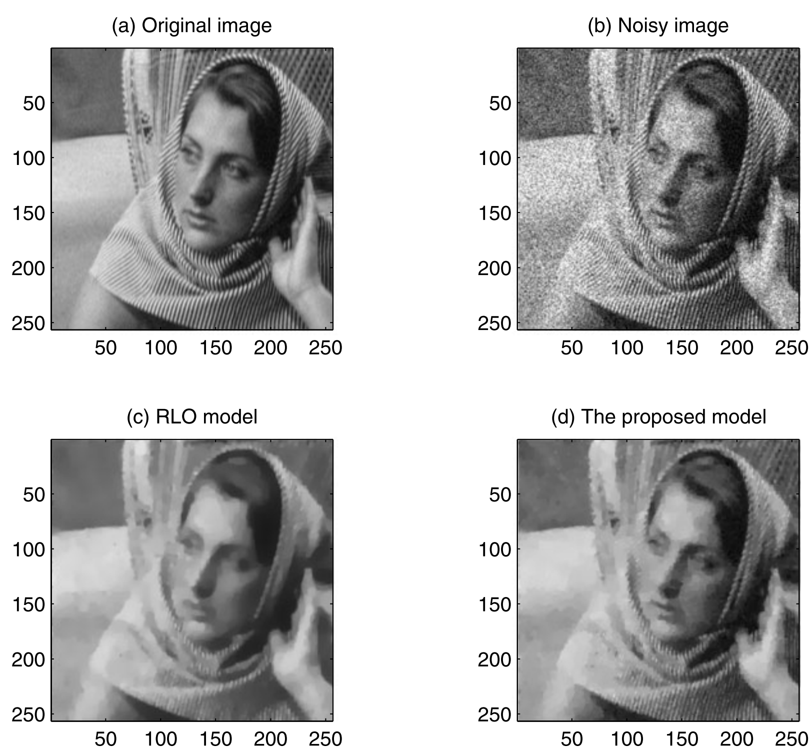

Fig. 7 (**a**) The texture image "TexImag"; (**b**) the multiplicative noise is added following the model of (1.4) with the standard deviation $\sigma = 3$, SNR $= 5.86$, ReErr $= 0.25$; (**c**) iteration $= 180$

with the above algorithm. The two weighted parameters λ_1 and λ_2 are dynamically updated as explained in [15]. Here the only parameter to tune is the number of the iterations.

For the emulated experiments, a signal to noise ratio (SNR) and a relative error (ReErr) of a restored image are used to measure the quality of the restoration. For a given clean image u and its noisy observation u_0, denote by $n =$

Table 1 The SNRs and ReErrs of the restored images

Image	RLO model	Proposed model
“SynImag1” in Fig. 1	SNR = 12.86,	SNR = 13.20,
($\sigma = 2$)	ReErr = 0.127	ReErr = 0.087
“SynImag1” in Fig. 2	SNR = 16.92,	SNR = 17.58,
($\sigma = 4$)	ReErr = 0.127	ReErr = 0.037
“SynImag1” in Fig. 3	SNR = 11.50,	SNR = 13.20,
($\sigma = 2$)	ReErr = 0.118	ReErr = 0.076
“SynImag2” in Fig. 4	SNR = 8.04,	SNR = 8.98,
($\sigma = 4$)	ReErr = 0.17	ReErr = 0.13
“Lena” in Fig. 5	SNR = 10.27,	SNR = 11.51,
($\sigma = 3$)	ReErr = 0.109	ReErr = 0.089
“Cameraman” in Fig. 6	SNR = 12.03,	SNR = 13.21,
($\sigma = 3$)	ReErr = 0.13	ReErr = 0.09
“‘TexImag” in Fig. 7	SNR = 7.64,	SNR = 9.20,
($\sigma = 3$)	ReErr = 0.16	ReErr = 0.13

$u_0 - u$. With this we define the SNR and the ReErr as follows:

$$\mathrm{SNR} = 10\log_{10}\left(\frac{\int_\Omega (u_0 - \bar{u}_0)^2\,dx\,dy}{\int_\Omega (n - \bar{n})^2\,dx\,dy}\right),$$

$$\mathrm{ReErr} = \frac{\|n\|_2^2}{\|u\|_2^2},$$

where

$$\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0\,dx\,dy, \qquad \bar{n} = \frac{1}{|\Omega|}\int_\Omega n\,dx\,dy.$$

In Figs. 1–4, two synthetic images are corrupted by the multiplicative noises following the model (1.4) with $\sigma = 2$ and $\sigma = 4$ respectively. It is clear that the restoration results by the proposed model are visually much better that those by the RLO model, especially when the noise variance is large, i.e., when σ is large (we can compare Figs. 2 and 4).

In Figs. 5–7, three real images are corrupted by the multiplicative noise following the model (1.4) with $\sigma = 3$, and the original image in Fig. 7 includes plenty of textures. We see from these figures that the proposed model gets a very good visual effect and works well for the texture image.

We see from Table 1 that our model has a higher SNR and a lower ReErr than the RLO model for all the simulated images.

In Fig. 8, we show the denoising result for a real ultrasound image. We see that our model can effectively remove the speckle noise in the ultrasound image.

6 Conclusion

In this paper, we consider a variational model to deal with the speckle noise in ultrasound images. We establish the existence and uniqueness of the minimizer for the variational problem, and derive the existence and uniqueness of weak solutions for the associated evolution equation. Then, we show that the solution of the evolution equation converges weakly in BV and strongly in L^2 to the minimizer as $t \to \infty$. Furthermore, some numerical results are presented on simulated images and a real ultrasound image by comparing the proposed model with the existing RLO model. These numerical results show the effectiveness of the proposed model for multiplicative noise removal.

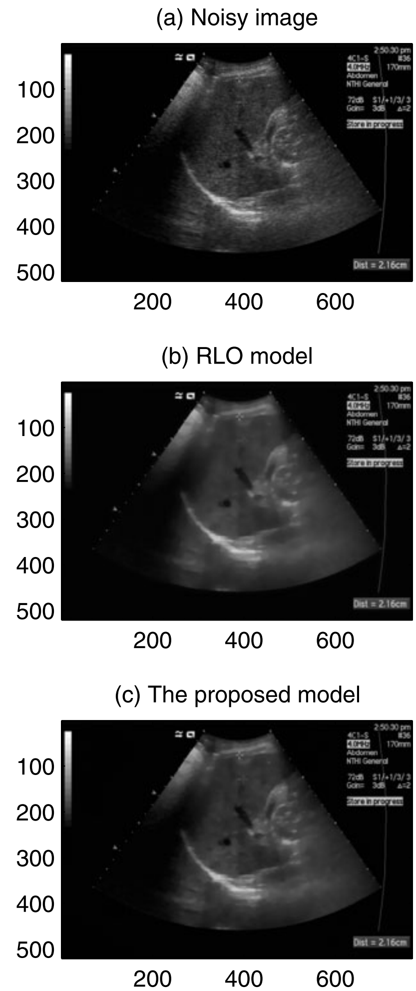

Fig. 8 (**a**) The real ultrasound image; (**b**) iteration = 140; (**c**) iteration = 120

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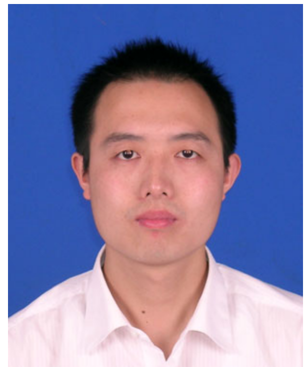
References

1. Aubert, G., Aujol, J.F.: A variational approach to removing multiplicative noise. SIAM J. Appl. Math. **68**, 925–946 (2008)
2. Brezis, H.: Operateures Maximaux Monotone. North-Holland, Amsterdam (1993)

3. Chen, Y.M., Rao, M.: Minimization problems and associated flows related to weighted p energy and total variation. SIAM J. Math. Anal. **34**(5), 1084–1104 (2002)
4. Dutt, V., Greenleaf, J.: Adaptative speckle reduction filter for log-compressed b-scan images. IEEE Trans. Med. Imag. **15**(6), 802–813 (1996)
5. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions. CRC Press, Boca Raton (1992)
6. Giusti, F.: Minimal Surfaces and Functions of Bounded Variation. Monographs in Math., vol. 80. Birkhäuser, Basel (1984)
7. Huang, Y., Ng, M., Wen, Y.: A new total variation method for multiplicative noise removal. SIAM J. Imaging Sci. **2**(1), 20–40 (2009)
8. Jin, Z.M., Yang, X.P.: Analysis of a new variation model for multiplicative noise removal. J. Math. Anal. Appl. **362**, 415–426 (2010)
9. Kaplan, D., Ma, Q.: On the statistical characteristics of the log-compressed Rayleigh signals: theoretical formulation and experimental results. J. Acoust. Soc. Am. **95**, 1396–1400 (1994)
10. Kornprobst, P., Deriche, R., Aubert, G.: Image sequence analysis via partial differential equations. J. Math. Imaging Vis. **11**(1), 5–26 (1999)
11. Krissian, K., Kikinis, R., Westin, C.F., Vosburgh, K.: Speckle-constrained filtering of ultrasound images. IEEE Comput. Vis. Pattern Recogn. 547–552 (2005)
12. Lin, F.H., Yang, X.P.: Geometric Measure Theory—An Introduction. Science Press/International Press, Beijing/Boston (2002)
13. Loupas, A.: Digital image processing for noise reduction in medical ultrasonics. PhD thesis, University of Edinburgh, UK (1988)
14. Lysaker, M., Lundervold, A., Tai, X.C.: Noise removal using fourth-order partial differential equation with application to medical magnetic resonance images in space and time. IEEE Trans. Image Process. **12**(12), 1579–1590 (2003)
15. Rudin, L., Lions, P.L., Osher, S.: Multiplicative denoising and deblurring: theory and algorithms. In: Osher, S., Paragios, N. (eds.) Geometric Level Sets in Imaging, Vision, and Graphics, pp. 103–119. Springer, Berlin (2003)
16. Rudin, L., Osher, S., Fatemi, E.: Nonlinear total variation based noise removal algorithms. Physica D **60**, 259–268 (1992)
17. Tur, M., Chin, C., Goodman, J.W.: When is speckle noise multiplicative? Appl. Opt. **21**(7), 1157–1159 (1982)
18. Tuthill, T.A., Sperry, R.H., Parker, K.J.: Deviation from Rayleigh statistics in ultrasonic speckle. Ultrason. Imag. **10**, 81–90 (1988)
19. Zhou, X.: An evolution problem for plastic antiplanar shear. Appl. Math. Optim. **25**, 263–285 (1992)



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