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On the Decomposition of Interval-Valued Fuzzy Morphological Operators

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Abstract Interval-valued fuzzy mathematical morphology is an extension of classical fuzzy mathematical morphology, which is in turn one of the extensions of binary morphology to greyscale morphology. The uncertainty that may exist concerning the grey value of a pixel due to technical limitations or bad recording circumstances, is taken into account by mapping the pixels in the image domain onto an interval to which the pixel's grey value is expected to belong instead of one specific value. Such image representation corresponds to the representation of an interval-valued fuzzy set and thus techniques from interval-valued fuzzy set theory can be applied to extend greyscale mathematical morphology. In this paper, we study the decomposition of the interval-valued fuzzy morphological operators. We investigate in which cases the $[\alpha_1, \alpha_2]$-cuts of these operators can be written or approximated in terms of the corresponding binary operators. Such conversion into binary operators results in a reduction of the computation time and is further also theoretically interesting since it provides us a link between interval-valued fuzzy and binary morphology.

Keywords Interval-valued fuzzy sets · Mathematical morphology · Decomposition

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1 Introduction

Many theories have been developed in the domain of image processing to extract specific information from images such as edges, patterns, ... One of these theories is mathematical morphology, in which an image is transformed into another image by a morphological operator, using a structuring element. The basic morphological operators are the dilation, erosion, opening and closing. The original binary morphology [1], for binary (black-white) images, was extended to greyscale images by two different approaches: (i) the threshold approach [1] and (ii) the umbra approach [2]. In the first approach, the structuring element still has to be binary; in the second approach, also greyscale structuring elements are allowed. Later, a third approach was introduced, inspired on the observation that greyscale images and fuzzy sets are modelled in the same way (i.e. as mappings from a universe $\mathcal{U}$ into the unit interval $[0, 1]$): fuzzy mathematical morphology [3–5]. Recently, also extensions of fuzzy mathematical morphology started to get attention [6–9]. In this paper, we concentrate on an extension based on interval-valued fuzzy set theory. A pixel is now mapped onto an interval of grey levels instead of one specific grey level, in this way allowing uncertainty regarding the measured grey levels.

In this paper, we investigate the relationships between the $[\alpha_1, \alpha_2]$-cuts of the interval-valued fuzzy dilation, erosion, opening and closing and the corresponding binary operators. This is first of all interesting from a theoretical point of view because it provides us a link between interval-valued fuzzy and binary morphology but secondly also because such conversion into binary operators is likely to result in a lower complexity for the calculation or approximation of the $[\alpha_1, \alpha_2]$-cuts. Moreover, the binary dilation and erosion can be further sped up by a decomposition of the structuring element.

The paper is organized as follows: in Sect. 2 we give in more detail the basic principles of interval-valued fuzzy mathematical morphology; Sect. 3 investigates the relationships between the $[\alpha_1, \alpha_2]$-cuts of the interval-valued fuzzy morphological operators and the corresponding binary operators in both a continuous and a discrete framework. The results are discussed in Sect. 4 and the paper is concluded in Sect. 5.

2 Interval-Valued Fuzzy Mathematical Morphology

2.1 Interval-Valued Fuzzy Sets

An interval-valued fuzzy set [10] is an extension of a classical fuzzy set [11] that is modelled by a mapping from a universe $\mathcal{U}$ into the unit interval [0, 1]. For a fuzzy set F in a universe $\mathcal{U}$, every element $u \in \mathcal{U}$ is mapped onto its membership degree $F(u) \in [0, 1]$ in the fuzzy set F. Interval-valued fuzzy sets now allow uncertainty about the membership degree and are given by mappings from a universe $\mathcal{U}$ into the class of closed intervals $L^I = \{[x_1, x_2] | [x_1, x_2] \subseteq [0, 1]\}$. Thus, for an interval-valued fuzzy set G in a universe $\mathcal{U}$, $G(u) = [G_1(u), G_2(u)] \subseteq [0, 1]$, $\forall u \in \mathcal{U}$. We will denote the class of interval-valued sets over the universe $\mathcal{U}$ by $\mathcal{IVFS}(\mathcal{U})$. Further, we denote the lower and upper bound of an element x of L^I by respectively x_1 and x_2: $x = [x_1, x_2]$ (Fig. 1).

For the partial ordering $\leq_{L^I}$ on L^I, defined by

$$x \leq_{L^I} y \quad \Leftrightarrow \quad x_1 \leq y_1 \text{ and } x_2 \leq y_2, \quad \forall x, y \in L^I, \tag{1}$$

the structure $(L^I, \leq_{L^I})$ forms a complete lattice [12]. The infimum and supremum of an arbitrary subset S of L^I are then respectively given by:

$$\inf S = \left[\inf_{x \in S} x_1, \inf_{x \in S} x_2\right], \tag{2}$$

$$\sup S = \left[\sup_{x \in S} x_1, \sup_{x \in S} x_2\right]. \tag{3}$$

We use the notations 0_{L^I} for $\inf L^I = [0, 0]$ and 1_{L^I} for $\sup L^I = [1, 1]$.

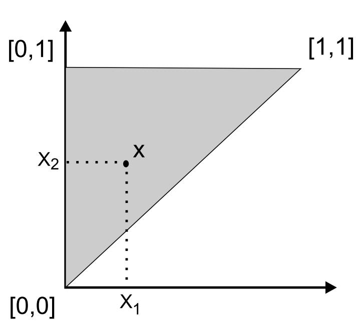

Fig. 1 Graphical representation of L^I

Related orderings on L^I that we will also use in this paper are ($\forall x, y \in L^I$):

$$x <_{L^I} y \quad \Leftrightarrow \quad x \leq_{L^I} y \text{ and } x \neq y, \tag{4}$$

$$x \ll_{L^I} y \quad \Leftrightarrow \quad x_1 < y_1 \text{ and } x_2 < y_2, \tag{5}$$

$$x \geq_{L^I} y \quad \Leftrightarrow \quad y \leq_{L^I} x, \tag{6}$$

$$x >_{L^I} y \quad \Leftrightarrow \quad y <_{L^I} x, \tag{7}$$

$$x \gg_{L^I} y \quad \Leftrightarrow \quad y \ll_{L^I} x. \tag{8}$$

Interval-valued fuzzy sets have a nice interpretation in the domain of image processing [9]. A pixel in the image domain is no longer mapped onto one specific grey value, but onto an interval of grey values to which the grey value is expected to belong. The grey levels of the pixels in a greyscale image namely can be uncertain. Firstly, in any device, the captured grey levels are rounded up or down to an element of a finite set of allowed values. Further, uncertainty may also arise when several takes of an image result in different grey levels for some of the pixels. This is sometimes the case under identical recording circumstances and can certainly be expected under variable circumstances such as illumination changes due to clouds covering the sun, … Also, the camera or an object in the scenery can slightly shift position in between takes, which might result in large differences (uncertainty) in the measured grey level of pixels. Especially pixels at the edge of an object will suffer from this. Finally, in the context of mathematical morphology, there might also exist uncertainty regarding the grey levels in the structuring element that is used. This structuring element can be chosen by the user, but in some cases he might not be completely sure how to estimate the importance or thus weight that is assigned to a pixel in this structuring element. In this case, the use of intervals to which the value is likely to belong instead of choosing one specific value, might offer a solution. Interval-valued images, where the image domain pixels are each mapped onto an interval of grey values (i.e., a closed subinterval of [0, 1]), now have the same representation as interval-valued fuzzy sets, which allows us to apply techniques from interval-valued fuzzy set theory to extend greyscale mathematical morphology.

Also in other image processing problems such as inverse halftoning [13], as well as in the context of wavelets [14], interval-valued representations occur in a natural way. They have also found to be usefull in edge detection applications [15]. Further, imprecision in grey levels is also considered in [16].

As a side note, we also mention that interval-valued fuzzy set theory is equivalent to intuitionistic fuzzy set theory [17] as shown in [12].

In the sequel, the universe $\mathcal{U}$ is restricted to $\mathbb{R}^n$, corresponding to the coordinates of an n-dimensional image.

2.2 Interval-Valued Fuzzy Mathematical Morphology

In this paper, we investigate the relationships between the $[\alpha_1, \alpha_2]$-cuts of the interval-valued fuzzy dilation, erosion, opening and closing and the corresponding binary operators. Those binary morphological operators are defined as follows (with the translation $T_y(B)$ of $B \subseteq \mathbb{R}^n$ by the vector $y \in \mathbb{R}^n$ defined as $T_y(B) = \{x \in \mathbb{R}^n | x - y \in B\}$, and the reflection $-B$ of B given by $-B = \{-b | b \in B\}$).

Definition 1 [1] Let $A, B \subseteq \mathbb{R}^n$. The binary dilation $D(A, B)$, the binary erosion $E(A, B)$, the binary closing $C(A, B)$ and the binary opening $O(A, B)$ are the sets given by:

$$D(A, B) = \{y | T_y(-B) \cap A \neq \emptyset\},$$

$$E(A, B) = \{y | T_y(B) \subseteq A\},$$

$$C(A, B) = E(D(A, B), -B),$$

$$O(A, B) = D(E(A, B), -B).$$

The notions of intersection and inclusion are clearly quite important in these definitions. Hence, to extend the binary morphological operators to interval-valued fuzzy ones, it is needed to extend the underlying Boolean conjunction and Boolean implication. Moreover, an extension of the Boolean negation is given below.

Definition 2 [18]

- A negator $\mathcal{N}$ on L^I is a decreasing $L^I - L^I$ mapping that coincides with the Boolean negation on $\{0, 1\}$ ($\mathcal{N}(0_{L^I}) = 1_{L^I}$ and $\mathcal{N}(1_{L^I}) = 0_{L^I}$).
- A negator $\mathcal{N}$ is an involutive negator on L^I if $(\forall x \in L^I)$ $(\mathcal{N}(\mathcal{N}(x)) = x)$.

The standard negator $\mathcal{N}_s$, defined by $\mathcal{N}_s([x_1, x_2]) = [1 - x_2, 1 - x_1]$, for all $x = [x_1, x_2] \in L^I$, is an example of an involutive negator on L^I.

Definition 3 [18]

- A conjunctor $\mathcal{C}$ on L^I is an increasing $(L^I)^2 - L^I$ mapping that coincides with the Boolean conjunction on $\{0, 1\}^2$, i.e., $\mathcal{C}(0_{L^I}, 0_{L^I}) = \mathcal{C}(0_{L^I}, 1_{L^I}) = \mathcal{C}(1_{L^I}, 0_{L^I}) = 0_{L^I}$ and $\mathcal{C}(1_{L^I}, 1_{L^I}) = 1_{L^I}$.
- A conjunctor $\mathcal{C}$ is a semi-norm on L^I if it satisfies $(\forall x \in L^I)(\mathcal{C}(1_{L^I}, x) = \mathcal{C}(x, 1_{L^I}) = x)$.
- A semi-norm $\mathcal{C}$ is a t-norm on L^I if it is commutative and associative.

The conjunctor $\mathcal{C}_{\min}$, that maps every $(x, y) \in (L^I)^2$ onto $\mathcal{C}_{\min}(x, y) = [\min(x_1, y_1), \min(x_2, y_2)]$, is an example of a t-norm on L^I.

Definition 4 [18]

- An implicator $\mathcal{I}$ on L^I is a hybrid monotonic $(L^I)^2 - L^I$ mapping (i.e., decreasing in the first argument and increasing in the second argument) that coincides with the Boolean implication on $\{0, 1\}^2$, i.e., $\mathcal{I}(0_{L^I}, 0_{L^I}) = \mathcal{I}(0_{L^I}, 1_{L^I}) = \mathcal{I}(1_{L^I}, 1_{L^I}) = 1_{L^I}$ and $\mathcal{I}(1_{L^I}, 0_{L^I}) = 0_{L^I}$. Every implicator $\mathcal{I}$ induces a negator $\mathcal{N}_\mathcal{I}$ defined by $\mathcal{N}_\mathcal{I}(x) = \mathcal{I}(x, 0_{L^I}), \forall x \in L^I$.
- An implicator $\mathcal{I}$ is a border implicator on L^I if it satisfies $(\forall x \in L^I)(\mathcal{I}(1_{L^I}, x) = x)$.
- A border implicator $\mathcal{I}$ is a model implicator on L^I if it is contrapositive w.r.t. its induced negator, i.e., $(\forall (x, y) \in (L^I)^2)(\mathcal{I}(x, y) = \mathcal{I}(\mathcal{N}_\mathcal{I}(y), \mathcal{N}_\mathcal{I}(x)))$, and if it fulfills the exchange principle, i.e., $(\forall (x, y, z) \in (L^I)^3)(\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(y, \mathcal{I}(x, z)))$.

The implicator $\mathcal{I}_{\min, \mathcal{N}_s}$, given by $\mathcal{I}_{\min, \mathcal{N}_s}(x, y) = [\max(1 - x_2, y_1), \max(1 - x_1, y_2)]$, for all $(x, y) \in (L^I)^2$, is an example of a model implicator on L^I.

Using the above introduced concepts, the interval-valued fuzzy morphological operators can be defined. The support d_A of an interval-valued fuzzy set A in $\mathbb{R}^n$, used in the definition, is given by $d_A = \{x | x \in \mathbb{R}^n \text{ and } A(x) \neq 0_{L^I}\}$.

Definition 5 [7] Let $\mathcal{C}$ be a conjunctor on L^I, let $\mathcal{I}$ be an implicator on L^I, and let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$. The interval-valued fuzzy dilation $D^I[\mathcal{C}](A, B)$ and erosion $E^I[\mathcal{I}](A, B)$ are the interval-valued fuzzy sets defined for all y in $\mathbb{R}^n$ by (where $T_y(d_B) = \{x \in \mathbb{R}^n | x - y \in d_B\}$ and $-d_B = \{x \in \mathbb{R}^n | -x \in d_B\}$)

$$D^I[\mathcal{C}](A, B)(y) = \sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}(B(y - x), A(x)), \tag{9}$$

and

$$E^I[\mathcal{I}](A, B)(y) = \inf_{x \in T_y(d_B)} \mathcal{I}(B(x - y), A(x)). \tag{10}$$

Remark that if $y \notin D(d_A, d_B)$, then $D^I[\mathcal{C}](A, B)(y) = 0_{L^I}$.

With the reflection $-B$ of an interval-valued fuzzy set B in $\mathbb{R}^n$ defined as $(-B)(x) = B(-x), \forall x \in \mathbb{R}^n$, the definitions of the interval-valued fuzzy closing and fuzzy opening are then given by:

Definition 6 Let $\mathcal{C}$ be a conjunctor on L^I, let $\mathcal{I}$ be an implicator on L^I, and let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$. The interval-valued fuzzy closing $C^I[\mathcal{C}, \mathcal{I}](A, B)$ and interval-valued fuzzy opening $O^I[\mathcal{C}, \mathcal{I}](A, B)$ are the interval-valued fuzzy sets in $\mathbb{R}^n$ given by:

$$C^I[\mathcal{C}, \mathcal{I}](A, B) = E^I[\mathcal{I}](D^I[\mathcal{C}](A, B), -B), \tag{11}$$

$$O^I[\mathcal{C}, \mathcal{I}](A, B) = D^I[\mathcal{C}](E^I[\mathcal{I}](A, B), -B). \tag{12}$$

In [20] it is shown that fuzzy morphology is compatible with binary morphology and that fuzzy morphology is compatible with greyscale morphology based on the threshold approach if we restrict ourselves to semi-norms and border implicators. The interval-valued fuzzy morphology is compatible with the fuzzy morphology and because we want to preserve also the compatibility with the greyscale morphology based on the threshold approach, we will restrict ourselves in the following to semi-norms and border implicators on L^I. Remark that if also stronger morphological properties such as the commutativity and the iterativity of the dilation are required, the conjunctors will need to be further restricted to t-norms [19].

2.3 Interval-Valued Fuzzy Morphological Operators in a Discrete Framework

In practice, when processing images on a computer, two technical limitations arise: (i) images are stored as matrices with a given number of rows and columns and the image domain is thus $\mathbb{Z}^n$ instead of $\mathbb{R}^n$; and (ii) also the grey levels are sampled and do not belong to the unit interval [0, 1], but to a finite subchain of it. As a consequence the greyscale intervals used in interval-valued fuzzy morphology belong to the finite subchain $L^I_{r,s}$ of L^I, with $L^I_{r,s} = \{[\frac{r-k}{r-1}, \frac{s-l}{s-1}] | k, l \in \mathbb{Z}$ and $1 \le k \le r$ and $1 \le l \le s\}$ for given integers r and s. The interval-valued fuzzy sets that correspond to the image A and the structuring element B consequently belong to $\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, i.e., the class of all interval-valued fuzzy sets in $\mathbb{Z}^n$ with membership intervals in $L^I_{r,s}$. If an interval-valued fuzzy set A belongs to $\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then, $\forall x \in \mathbb{Z}^n$, $A_1(x) \in I_r = \{\frac{r-k}{r-1} | k \in \mathbb{Z}$ and $1 \le k \le r\}$. Analogously the upper bound $A_2(x) \in I_s\{\frac{s-l}{s-1} | l \in \mathbb{Z}$ and $1 \le l \le s\}$.

Further, the definitions of negators, conjunctors and implicators on the chain $L^I_{r,s}$ are analogous to the corresponding definitions on L^I (where now $L^I_{r,s}$ takes the role of L^I). However, not every operator on L^I has a corresponding operator on $L^I_{r,s}$. The conjunctor $\mathcal{C}$, given by $\mathcal{C}(x, y) = [x_1 \cdot y_1, x_2 \cdot y_2]$ for all $x, y \in L^I$, for example, is not defined on $L^I_{r,s}$ due to the fact that the interval with as lower and upper bound the product of respectively the lower and upper bounds of two elements of $L^I_{r,s}$ does not necessarily belong to $L^I_{r,s}$.

The definitions of the discrete interval-valued fuzzy dilation and erosion can now be written as follows:

Definition 7 Let $\mathcal{C}$ be a conjunctor on $L^I_{r,s}$, let $\mathcal{I}$ be an implicator on $L^I_{r,s}$, and let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$. The discrete interval-valued fuzzy dilation $D^I[\mathcal{C}](A, B) \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ is for all $y \in \mathbb{Z}^n$ defined by:

$$\begin{aligned} D^I[\mathcal{C}](A,B)(y) &= \sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}(B(y-x), A(x)) \\ &= \Big[\max_{x \in T_y(-d_B) \cap d_A} \mathcal{C}(B(y-x), A(x))_1, \\ &\qquad \max_{x \in T_y(-d_B) \cap d_A} \mathcal{C}(B(y-x), A(x))_2 \Big]. \end{aligned}$$

(For $y \notin D(d_A, d_B)$, $D^I[\mathcal{C}](A, B)(y) = 0_{L^I}$.)
The discrete interval-valued fuzzy erosion $E_{\mathcal{I}}(A, B) \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ is for all $y \in \mathbb{Z}^n$ defined by:

$$\begin{aligned} E^I[\mathcal{I}](A,B)(y) &= \inf_{x \in T_y(d_B)} \mathcal{I}(B(x-y), A(x)) \\ &= \Big[\min_{x \in T_y(d_B)} \mathcal{I}(B(x-y), A(x))_1, \\ &\qquad \min_{x \in T_y(d_B)} \mathcal{I}(B(x-y), A(x))_2 \Big]. \end{aligned}$$

3 Decomposition of Interval-Valued Fuzzy Morphological Operators

In this section, the relationships between the $[\alpha_1, \alpha_2]$-cuts of the interval-valued fuzzy morphological operators applied on interval-valued fuzzy sets and the corresponding binary operators applied on the $[\alpha_1, \alpha_2]$-cuts of those interval-valued fuzzy sets are investigated. Therefore, we first refresh the definitions of the different $[\alpha_1, \alpha_2]$-cuts.

3.1 The Different $[\alpha_1, \alpha_2]$-Cuts

In the definitions of the different $[\alpha_1, \alpha_2]$-cuts of an interval-valued fuzzy set below, the notation U_{L^I} stands for $U_{L^I} = \{[x_1, x_2] \in L^I | x_2 = 1\}$.

Definition 8 [21] For $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$, we define the weak $[\alpha_1, \alpha_2]$-cut $A^{\alpha_2}_{\alpha_1}$ of an interval-valued fuzzy set $A \in \mathcal{IVFS}(\mathbb{R}^n)$ as:

$$\begin{aligned} A^{\alpha_2}_{\alpha_1} &= \{x | x \in \mathbb{R}^n,\ A_1(x) \ge \alpha_1 \text{ and } A_2(x) \ge \alpha_2\} \\ &= \{x | x \in \mathbb{R}^n \text{ and } A(x) \ge_{L^I} [\alpha_1, \alpha_2]\}. \end{aligned}$$

For $[\alpha_1, \alpha_2] \in L^I \backslash U_{L^I}$, the strict $[\alpha_1, \alpha_2]$-cut $A^{\overline{\alpha_2}}_{\overline{\alpha_1}}$ is given by:

$$\begin{aligned} A^{\overline{\alpha_2}}_{\overline{\alpha_1}} &= \{x | x \in \mathbb{R}^n,\ A_1(x) > \alpha_1 \text{ and } A_2(x) > \alpha_2\} \\ &= \{x | x \in \mathbb{R}^n \text{ and } A(x) \gg_{L^I} [\alpha_1, \alpha_2]\}. \end{aligned}$$

The cases $[\alpha_1, \alpha_2] = 0_{L^I}$ and $[\alpha_1, \alpha_2] \in U_{L^I}$ are excluded for respectively the weak and the strict $[\alpha_1, \alpha_2]$-cut. Since $\{x | x \in \mathbb{R}^n,\ A_1(x) \ge 0$ and $A_2(x) \ge 0\} = \mathbb{R}^n$ and $\{x | x \in \mathbb{R}^n$ and $A_2(x) > 1\} = \emptyset$, these cases don't yield new information.

Definition 9 For $\alpha_1 \in]0,1]$, the weak α_1-subcut A_{α_1} of an interval-valued fuzzy set $A \in \mathcal{IVFS}(\mathbb{R}^n)$ is given by:

$$A_{\alpha_1} = \{x | x \in \mathbb{R}^n \text{ and } A_1(x) \geq \alpha_1\}.$$

For $\alpha_2 \in]0,1]$, the weak α_2-supercut A^{α_2} is given by:

$$A^{\alpha_2} = \{x | x \in \mathbb{R}^n \text{ and } A_2(x) \geq \alpha_2\}.$$

For $\alpha_1 \in [0,1[$, the strict α_1-subcut $A_{\overline{\alpha_1}}$ is given by:

$$A_{\overline{\alpha_1}} = \{x | x \in \mathbb{R}^n \text{ and } A_1(x) > \alpha_1\}.$$

For $\alpha_2 \in [0,1[$, the strict α_2-supercut $A^{\overline{\alpha_2}}$ is given by:

$$A^{\overline{\alpha_2}} = \{x | x \in \mathbb{R}^n \text{ and } A_2(x) > \alpha_2\}.$$

The cases $\alpha_1 = 0$ and $\alpha_1 = 1$ are excluded for respectively the weak and the strict α_1-subcut. Since $\{x | x \in \mathbb{R}^n \text{ and } A_1(x) \geq 0\} = \mathbb{R}^n$ and $\{x | x \in \mathbb{R}^n \text{ and } A_1(x) > 1\} = \emptyset$, these cases don't yield new information. An analogous reasoning holds for the weak and strict α_2-supercut.

Definition 10 For $[\alpha_1, \alpha_2] \in L^I \setminus U_{L^I}$, the weak-strict $[\alpha_1, \alpha_2]$-cut $A_{\alpha_1}^{\overline{\alpha_2}}$ of an interval-valued fuzzy set $A \in \mathcal{IVFS}(\mathbb{R}^n)$ is given by:

$$A_{\alpha_1}^{\overline{\alpha_2}} = \{x | x \in \mathbb{R}^n,\ A_1(x) \geq \alpha_1 \text{ and } A_2(x) > \alpha_2\}. \tag{13}$$

For $[\alpha_1, \alpha_2] \in L^I \setminus \{1_{L^I}\}$, the strict-weak $[\alpha_1, \alpha_2]$-cut $A_{\overline{\alpha_1}}^{\alpha_2}$ is given by:

$$A_{\overline{\alpha_1}}^{\alpha_2} = \{x | x \in \mathbb{R}^n,\ A_1(x) > \alpha_1 \text{ and } A_2(x) \geq \alpha_2\}. \tag{14}$$

The cases $[\alpha_1, \alpha_2] \in U_{L^I}$ and $[\alpha_1, \alpha_2] = 1_{L^I}$ are excluded for respectively the weak-strict and strict-weak $[\alpha_1, \alpha_2]$-cut. Since $\{x | x \in \mathbb{R}^n \text{ and } A_2(x) > 1\} = \emptyset$ and $\{x | x \in \mathbb{R}^n \text{ and } A_1(x) > 1\} = \emptyset$, these cases don't yield new information.

3.2 Decomposition of the Interval-Valued Fuzzy Dilation

Lemma 1 [22] *If $\mathcal{C}$ is a semi-norm on L^I, then it holds that $\mathcal{C} \leq \mathcal{C}_{\min}$, i.e.:*

$$(\forall (x,y) \in (L^I)^2)(\mathcal{C}(x,y) \leq_{L^I} \mathcal{C}_{\min}(x,y)).$$

Note that lemma 1 does not necessarily hold if $\mathcal{C}$ is not a semi-norm on L^I.

Example 1 Let $\mathcal{C}$ be the conjunctor defined as:

$$\mathcal{C}(x,y) = \begin{cases} 0_{L^I} & \text{if } \inf(x,y) = 0_{L^I}, \\ 1_{L^I} & \text{else}, \end{cases} \quad \forall (x,y) \in (L^I)^2.$$

One easily verifies that $\mathcal{C}$ is no semi-norm on L^I since e.g. $\mathcal{C}([1,1],[1/4,1/2]) = 1_{L^I} \neq [1/4,1/2]$ and that $\mathcal{C} \not\leq \mathcal{C}_{\min}$ since e.g. $[1,1] = \mathcal{C}([1,1],[1/4,1/2]) >_{L^I} \mathcal{C}_{\min}([1,1],[1/4,1/2]) = [1/4,1/2]$.

3.2.1 Decomposition by Strict Sub- and Supercuts

Proposition 1 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for respectively all $\alpha_1 \in [0,1[$ and all $\alpha_2 \in [0,1[$ that:*

(i) $D^I[\mathcal{C}_{\min}](A,B)_{\overline{\alpha_1}} = D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}})$,

(ii) $D^I[\mathcal{C}_{\min}](A,B)^{\overline{\alpha_2}} = D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}})$.

Proof Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, and let $\alpha_1, \alpha_2 \in [0,1[$.
(i)

$$\begin{aligned}
& y \in D^I[\mathcal{C}_{\min}](A,B)_{\overline{\alpha_1}} \\
\Leftrightarrow\ & D^I[\mathcal{C}_{\min}](A,B)(y)_1 > \alpha_1 \\
\Leftrightarrow\ & \sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}_{\min}(B(y-x), A(x))_1 > \alpha_1 \\
\Leftrightarrow\ & (\exists x \in T_y(-d_B) \cap d_A) \\
& (\mathcal{C}_{\min}(B(y-x), A(x))_1 > \alpha_1) \\
\Leftrightarrow\ & (\exists x \in T_y(-d_B) \cap d_A) \\
& (\min(B_1(y-x), A_1(x)) > \alpha_1) \\
\Leftrightarrow\ & (\exists x \in T_y(-d_B) \cap d_A) \\
& (B_1(y-x) > \alpha_1 \text{ and } A_1(x) > \alpha_1) \\
\Leftrightarrow\ & (\exists x \in T_y(-d_B) \cap d_A) \\
& (x \in T_y(-B_{\overline{\alpha_1}}) \text{ and } x \in A_{\overline{\alpha_1}}) \\
\Leftrightarrow\ & T_y(-B_{\overline{\alpha_1}}) \cap A_{\overline{\alpha_1}} \neq \emptyset \\
\Leftrightarrow\ & y \in D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}}).
\end{aligned}$$

This proves that $D^I[\mathcal{C}_{\min}](A,B)_{\overline{\alpha_1}} = D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}})$.
(ii) Analogous. □

Proposition 2 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$ and let $\mathcal{C}$ be a semi-norm on L^I, then it holds for respectively all $\alpha_1 \in [0,1[$ and all $\alpha_2 \in [0,1[$ that:*

(i) $D^I[\mathcal{C}](A,B)_{\overline{\alpha_1}} \subseteq D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}})$,

(ii) $D^I[\mathcal{C}](A,B)^{\overline{\alpha_2}} \subseteq D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}})$.

Proof (i) The proof is completely analogous to the one from Proposition 1(i). We only have that due to Lemma 1

$$(\exists x \in T_y(-d_B) \cap d_A)(\mathcal{C}(B(y-x), A(x))_1 > \alpha_1)$$

$$\Downarrow$$

$$(\exists x \in T_y(-d_B) \cap d_A)(\mathcal{C}_{\min}(B(y-x), A(x))_1 > \alpha_1)$$

only holds in one direction for an arbitrary semi-norm on L^I.
(ii) Analogous. □

The reverse inclusion does not hold in general.

Example 2 Let $[\alpha_1, \alpha_2] = [1/4, 1/2]$, $\mathcal{C}(r,s) = [r_1 \cdot s_1, r_2 \cdot s_2]$ for all $r, s \in L^I$, $A(x) = [0.3, 0.6]$ for all $x \in [0,1]$, $A(x) = 0_{L^I}$ for all $x \in \mathbb{R}\backslash[0,1]$, $B(x) = [0.4, 0.7]$ for all $x \in [-1, 0]$ and $B(x) = 0_{L^I}$ for all $x \in \mathbb{R}\backslash[-1, 0]$).

Then on the one hand

$$0 \in D(A_{\overline{0.25}}, B_{\overline{0.25}}) = D(A^{\overline{0.5}}, B^{\overline{0.5}}) = [-1, 1].$$

On the other hand $D^I[\mathcal{C}](A,B)(0) = [0.12, 0.42]$ and thus $0 \notin D^I[\mathcal{C}](A,B)_{\overline{0.25}}$ and $0 \notin D^I[\mathcal{C}](A,B)^{\overline{0.5}}$. As a consequence $D^I[\mathcal{C}](A,B)_{\overline{0.25}} \not\supseteq D(A_{\overline{0.25}}, B_{\overline{0.25}})$ and $D^I[\mathcal{C}](A,B)^{\overline{0.5}} \not\supseteq D(A^{\overline{0.5}}, B^{\overline{0.5}})$.

Further, the following example illustrates that Proposition 2 is restricted to semi-norms.

Example 3 Let $\mathcal{C}$ be the conjunctor defined in Example 1 (which is not a semi-norm). Further, let $A(x) = [1/4, 1/2]$ for all $x \in [0,1]$, $A(x) = 0_{L^I}$ for all $x \in \mathbb{R}\backslash[0,1]$ and $B(x) = 1_{L^I}$ for all $x \in [-1, 0]$, $B(x) = 0_{L^I}$ for all $x \in \mathbb{R}\backslash[-1, 0]$. Then for all $y \in D(d_A, d_B) = [-1, 1]$ it holds that $D^I[\mathcal{C}](A,B)(y) = 1_{L^I}$ and thus $y \in D^I[\mathcal{C}](A,B)_{\overline{0.25}}$ and $y \in D^I[\mathcal{C}](A,B)^{\overline{0.5}}$. On the other hand, from $A_{\overline{0.25}} = A^{\overline{0.5}} = \emptyset$ it follows that $D(A_{\overline{0.25}}, B_{\overline{0.25}}) = \emptyset$ and $D(A^{\overline{0.5}}, B^{\overline{0.5}}) = \emptyset$, such that $D^I[\mathcal{C}](A,B)_{\overline{0.25}} \not\subseteq D(A_{\overline{0.25}}, B_{\overline{0.25}})$ and $D^I[\mathcal{C}](A,B)^{\overline{0.5}} \not\subseteq D(A^{\overline{0.5}}, B^{\overline{0.5}})$.

Remark that the above decomposition properties for strict sub- and supercuts remain valid in the discrete framework.

3.2.2 Decomposition by Strict $[\alpha_1, \alpha_2]$-Cuts

Proposition 3 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $[\alpha_1, \alpha_2] \in L^I \backslash U_{L^I}$ that*:

$$D^I[\mathcal{C}_{\min}](A,B)_{\overline{\alpha_1}}^{\overline{\alpha_2}} \supseteq D(A_{\overline{\alpha_1}}^{\overline{\alpha_2}}, B_{\overline{\alpha_1}}^{\overline{\alpha_2}}).$$

Proof The proof is analogous to the one from Proposition 1. Only, now we have that

$$\sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}_{\min}(B(y-x), A(x)) \gg_{L^I} [\alpha_1, \alpha_2]$$

$$\Uparrow$$

$$(\exists x \in T_y(-d_B) \cap d_A)$$

$$(\mathcal{C}_{\min}(B(y-x), A(x)) \gg_{L^I} [\alpha_1, \alpha_2])$$

only holds in one direction. □

The reverse inclusion does not hold in general.

Example 4 Let $[\alpha_1, \alpha_2] = [0.3, 0.7]$ and let

$$A(x) = \begin{cases} [0.1, 0.8], & x \in [0, 0.5[, \\ [0.5, 0.6], & x \in [0.5, 1], \\ 0_{L^I}, & \text{else} \end{cases}$$

and

$$B(x) = \begin{cases} [0.2, 0.9], & x \in]-0.5, 0], \\ [0.4, 0.5], & x \in [-1, -0.5], \\ 0_{L^I}, & \text{else.} \end{cases}$$

It then holds that $D^I[\mathcal{C}](A,B)(0) = [0.4, 0.8]$, which means that $0 \in D^I[\mathcal{C}](A,B)_{\overline{0.3}}^{\overline{0.7}}$.

On the other hand, since $A_{\overline{0.3}}^{\overline{0.7}} = \emptyset$ it holds that $D(A_{\overline{0.3}}^{\overline{0.7}}, B_{\overline{0.3}}^{\overline{0.7}}) = \emptyset$, which means that $0 \notin D(A_{\overline{0.3}}^{\overline{0.7}}, B_{\overline{0.3}}^{\overline{0.7}})$. As a consequence $D^I[\mathcal{C}](A,B)_{\overline{0.3}}^{\overline{0.7}} \not\subseteq D(A_{\overline{0.3}}^{\overline{0.7}}, B_{\overline{0.3}}^{\overline{0.7}})$.

The strict $[\alpha_1, \alpha_2]$-cut of the interval-valued fuzzy dilation based on the conjunctor $\mathcal{C}_{\min}$ can however always be constructed from binary dilations as follows.

Proposition 4 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $[\alpha_1, \alpha_2] \in L^I \backslash U_{L^I}$ that*:

$$D^I[\mathcal{C}_{\min}](A,B)_{\overline{\alpha_1}}^{\overline{\alpha_2}} = D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}}) \cap D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}}).$$

Proof Follows from Proposition 1 and the fact that $D^I[\mathcal{C}_{\min}](A,B)_{\overline{\alpha_1}}^{\overline{\alpha_2}} = D^I[\mathcal{C}_{\min}](A,B)_{\overline{\alpha_1}} \cap D^I[\mathcal{C}_{\min}](A,B)^{\overline{\alpha_2}}$. □

Due to Lemma 1, Proposition 3 is restricted to the semi-norm $\mathcal{C}_{\min}$. For an arbitrary semi-norm $\mathcal{C}$ there is no relation between the strict $[\alpha_1, \alpha_2]$-cuts $D^I[\mathcal{C}](A,B)_{\overline{\alpha_1}}^{\overline{\alpha_2}}$ and the binary dilation $D(A_{\overline{\alpha_1}}^{\overline{\alpha_2}}, B_{\overline{\alpha_1}}^{\overline{\alpha_2}})$ as the following example illustrates.

Example 5 To illustrate that, for an arbitrary semi-norm $\mathcal{C}$, it does not hold in general that $(\forall [\alpha_1, \alpha_2] \in L^I \backslash U_{L^I})$ $(D^I[\mathcal{C}](A,B)_{\overline{\alpha_1}}^{\overline{\alpha_2}} \supseteq D(A_{\overline{\alpha_1}}^{\overline{\alpha_2}}, B_{\overline{\alpha_1}}^{\overline{\alpha_2}}))$, Example 4 can be used again.

For a counterexample of the reverse inclusion we refer to Example 2, where $0 \in D(A_{\overline{0.25}}^{\overline{0.5}}, B_{\overline{0.25}}^{\overline{0.5}}) = [-1, 1]$ and $D^I[\mathcal{C}](A,B)(0) = [0.12, 0.42]$, which means that $0 \notin D^I[\mathcal{C}](A,B)_{\overline{0.25}}^{\overline{0.5}}$.

The strict $[\alpha_1, \alpha_2]$-cut of an interval-valued fuzzy dilation based on an arbitrary semi-norm $\mathcal{C}$ can however always be approximated by binary dilations.

Proposition 5 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $[\alpha_1, \alpha_2] \in L^I \backslash U_{L^I}$ that*:

$$D^I[\mathcal{C}](A, B)_{\overline{\alpha}_1}^{\overline{\alpha}_2} \subseteq D(A_{\overline{\alpha}_1}, B_{\overline{\alpha}_1}) \cap D(A^{\overline{\alpha}_2}, B^{\overline{\alpha}_2}).$$

Proof Follows from Proposition 2 and the fact that $D^I[\mathcal{C}](A, B)_{\overline{\alpha}_1}^{\overline{\alpha}_2} = D^I[\mathcal{C}](A, B)_{\overline{\alpha}_1} \cap D^I[\mathcal{C}](A, B)^{\overline{\alpha}_2}$. □

Remark that the above decomposition properties for strict $[\alpha_1, \alpha_2]$-cuts remain valid in the discrete framework.

3.2.3 Decomposition by Weak Sub- and Supercuts

In general, there is no relation between the weak sub- and supercut $D^I[\mathcal{C}](A, B)_{\alpha_1}$ and $D^I[\mathcal{C}](A, B)^{\alpha_2}$ and the binary dilations $D(A_{\alpha_1}, B_{\alpha_1})$ and $D(A^{\alpha_2}, B^{\alpha_2})$ for an arbitrary semi-norm $\mathcal{C}$.

Example 6 To illustrate that in general it does not hold that $D^I[\mathcal{C}](A, B)_{\alpha_1} \subseteq D(A_{\alpha_1}, B_{\alpha_1})$, we can use Example 4 again. However, we can now also construct a counterexample based on the fact that for a weak α_1-subcut of an interval-valued fuzzy set A, the inequality $A_1(x) \geq \alpha_1$, that needs to hold for $x \in \mathbb{R}$ to belong to A_{α_1}, is not strict. Let $[\alpha_1, \alpha_2] = [1/4, 1]$, $A(x) = [x/2, x]$ for all $x \in [0, 1[$, $A(x) = 0_{L^I}$ for all $x \in \mathbb{R}\backslash[0, 1[$, $B(x) = 1_{L^I}$ for all $x \in [-1, 0]$ and $B(x) = 0_{L^I}$ for all $x \in \mathbb{R}\backslash[-1, 0]$. Let $\mathcal{C}$ be the conjunctor defined in Example 2.

It then holds that $D^I[\mathcal{C}](A, B)(0) = [1/2, 1]$, which means that $0 \in D^I[\mathcal{C}](A, B)_{0.5}$.

On the other hand, however, since $A_{0.5} = \emptyset$ also $D(A_{0.5}, B_{0.5}) = \emptyset$ and thus $0 \notin D(A_{0.5}, B_{0.5})$. As a consequence $D^I[\mathcal{C}](A, B)_{0.5} \nsubseteq D(A_{0.5}, B_{0.5})$. Note that the above example holds for any semi-norm $\mathcal{C}$, since for any semi-norm $\mathcal{C}$ it holds in the example that $\mathcal{C}(B(x), A(x)) = \mathcal{C}(1_{L^I}, A(x)) = A(x)$ for all $x \in]0, 1[$. (An analogous example can be found for weak supercuts. The above results still hold for the weak α_2-supercut where $\alpha_2 = 1$. It then holds that $0 \in D^I[\mathcal{C}](A, B)^1$ and $D(A^1, B^1) = \emptyset$.)

In general also $D^I[\mathcal{C}](A, B)_{\alpha_1} \nsupseteq D(A_{\alpha_1}, B_{\alpha_1})$. To illustrate this, we can use Example 2 again (where the strict and weak 0.25-subcuts of A and B coincide). Adapting that example we get that $0 \in D(A_{0.25}, B_{0.25})$ and $0 \notin D^I[\mathcal{C}](A, B)_{0.25}$, which leads to $D^I[\mathcal{C}](A, B)_{0.25} \nsupseteq D(A_{0.25}, B_{0.25})$. (Analogously for weak supercuts.)

For the semi-norm $\mathcal{C} = \mathcal{C}_{\min}$, the following partial result holds.

Proposition 6 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for respectively all $\alpha_1 \in]0, 1]$ and all $\alpha_2 \in]0, 1]$ that*:

(i) $D^I[\mathcal{C}_{\min}](A, B)_{\alpha_1} \supseteq D(A_{\alpha_1}, B_{\alpha_1})$,

(ii) $D^I[\mathcal{C}_{\min}](A, B)^{\alpha_2} \supseteq D(A^{\alpha_2}, B^{\alpha_2})$.

Proof Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, and let $\alpha_1, \alpha_2 \in]0, 1]$.

(i) Analogous to the Proof of Proposition 1. Only, now it holds that:

$$(\exists x \in T_y(-d_B) \cap d_A)(\mathcal{C}_{\min}(B(y - x), A(x))_1 \geq \alpha_1)$$

$$\Downarrow$$

$$\sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}_{\min}(B(y - x), A(x))_1 \geq \alpha_1.$$

(ii) Analogous. □

To illustrate that the reverse inclusion does not hold, we refer to Example 6.

Proposition 6 remains valid in the discrete framework. Moreover, in the discrete framework, the result also holds for arbitrary semi-norms and for $\mathcal{C}_{\min}$ also the reverse inclusion holds.

Proposition 7 *Let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for respectively all $\alpha_1 \in]0, 1] \cap I_r$ and all $\alpha_2 \in]0, 1] \cap I_s$ that*:

(i) $D^I[\mathcal{C}_{\min}](A, B)_{\alpha_1} = D(A_{\alpha_1}, B_{\alpha_1})$,

(ii) $D^I[\mathcal{C}_{\min}](A, B)^{\alpha_2} = D(A^{\alpha_2}, B^{\alpha_2})$.

Proof Analogous to the proof of Proposition 6, where now in the discrete case also

$$(\exists x \in T_y(-d_B) \cap d_A)(\mathcal{C}_{\min}(B(y - x), A(x))_1 \geq \alpha_1)$$

$$\Updownarrow$$

$$\sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}_{\min}(B(y - x), A(x))_1 \geq \alpha_1.$$ □

Proposition 8 *Let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for respectively all $\alpha_1 \in]0, 1] \cap I_r$ and all $\alpha_2 \in]0, 1] \cap I_s$ that*:

(i) $D^I[\mathcal{C}](A, B)_{\alpha_1} \subseteq D(A_{\alpha_1}, B_{\alpha_1})$,

(ii) $D^I[\mathcal{C}](A, B)^{\alpha_2} \subseteq D(A^{\alpha_2}, B^{\alpha_2})$.

Proof Analogous to the proof of Proposition 7, but for an arbitrary semi-norm $\mathcal{C}$, so that

$$D^I[\mathcal{C}_{\min}](A, B)(y)_1 \geq \alpha_1$$

$$\Uparrow$$

$$D^I[\mathcal{C}](A, B)(y)_1 \geq \alpha_1$$

$$\Updownarrow$$

$$y \in D^I[\mathcal{C}](A, B)_{\alpha_1}.$$ □

3.2.4 Decomposition by Weak $[\alpha_1, \alpha_2]$-Cuts

In general, there is no relation between the weak $[\alpha_1, \alpha_2]$-cut $D^I[\mathcal{C}](A, B)_{\alpha_1}^{\alpha_2}$ and the binary dilation $D(A_{\alpha_1}^{\alpha_2}, B_{\alpha_1}^{\alpha_2})$ for an arbitrary semi-norm $\mathcal{C}$. To illustrate this, we can use Example 6 again, where the weak 0.5-supercut and weak [0.5, 1]-cut of A and B coincide and the results remain valid when using the weak [0.5, 1]-cut.

For the semi-norm $\mathcal{C} = \mathcal{C}_{\min}$, the following partial result holds.

Proposition 9 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $[\alpha_1, \alpha_2] \in L^I \setminus \{0_{L^I}\}$ that*:

$$D^I[\mathcal{C}_{\min}](A, B)^{\alpha_2}_{\alpha_1} \supseteq D(A^{\alpha_2}_{\alpha_1}, B^{\alpha_2}_{\alpha_1}).$$

Proof Analogous to the proof of Proposition 3. □

The reverse inclusion $D^I[\mathcal{C}_{\min}](A, B)^{\alpha_2}_{\alpha_1} \subseteq D(A^{\alpha_2}_{\alpha_1}, B^{\alpha_2}_{\alpha_1})$ does not hold in general. To illustrate this, we again refer to Example 6, where using the weak [0.5, 1]-cut instead of the weak 0.5-subcut doesn't affect the results.

Remark that the above decomposition properties for weak $[\alpha_1, \alpha_2]$-cuts remain valid in the discrete framework. Moreover, the weak $[\alpha_1, \alpha_2]$-cut of the discrete interval-valued fuzzy dilation based on the conjunctor $\mathcal{C}_{\min}$ (respectively semi-norm $\mathcal{C}$) can always be constructed from (respectively approximated by) binary dilations as follows.

Proposition 10 *Let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for all $[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus \{0_{L^I}\}$ and every semi-norm $\mathcal{C}$ that*:

(i) $D^I[\mathcal{C}_{\min}](A, B)^{\alpha_2}_{\alpha_1} = D(A_{\alpha_1}, B_{\alpha_1}) \cap D(A^{\alpha_2}, B^{\alpha_2})$,

(ii) $D^I[\mathcal{C}](A, B)^{\alpha_2}_{\alpha_1} \subseteq D(A_{\alpha_1}, B_{\alpha_1}) \cap D(A^{\alpha_2}, B^{\alpha_2})$.

Proof Follows from Propositions 7 and 8 and the fact that $D^I[\mathcal{C}](A, B)^{\alpha_2}_{\alpha_1} = D^I[\mathcal{C}](A, B)_{\alpha_1} \cap D^I[\mathcal{C}](A, B)^{\alpha_2}$ for every semi-norm $\mathcal{C}$. □

3.2.5 Decomposition by Strict-Weak and Weak-Strict $[\alpha_1, \alpha_2]$-Cuts

In general, for an arbitrary semi-norm $\mathcal{C}$, there is no relation between the strict-weak and weak-strict $[\alpha_1, \alpha_2]$-cuts $D^I[\mathcal{C}](A, B)^{\alpha_2}_{\overline{\alpha_1}}$ and $D^I[\mathcal{C}](A, B)^{\overline{\alpha_2}}_{\alpha_1}$ and the binary dilations $D(A^{\alpha_2}_{\overline{\alpha_1}}, B^{\alpha_2}_{\overline{\alpha_1}})$ and $D(A^{\overline{\alpha_2}}_{\alpha_1}, B^{\overline{\alpha_2}}_{\alpha_1})$.

To illustrate this, an analogous example as in Example 6 can be found.

For the semi-norm $\mathcal{C} = \mathcal{C}_{\min}$, the following partial result holds.

Proposition 11 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$. It holds for respectively all $[\alpha_1, \alpha_2] \in L^I \setminus \{1_{L^I}\}$ and for all $[\alpha_1, \alpha_2] \in L^I \setminus U_{L^I}$ that*:

(i) $D^I[\mathcal{C}_{\min}](A, B)^{\alpha_2}_{\overline{\alpha_1}} \supseteq D(A^{\alpha_2}_{\overline{\alpha_1}}, B^{\alpha_2}_{\overline{\alpha_1}})$,

(ii) $D^I[\mathcal{C}_{\min}](A, B)^{\overline{\alpha_2}}_{\alpha_1} \supseteq D(A^{\overline{\alpha_2}}_{\alpha_1}, B^{\overline{\alpha_2}}_{\alpha_1})$.

Proof Analogous to the proof of Proposition 3. □

As can be illustrated analogously as in Example 6, the reverse inclusion does not hold.

Remark that the above decomposition properties for strict-weak and weak-strict $[\alpha_1, \alpha_2]$-cuts remain valid in the discrete framework. Moreover, the weak-strict and strict-weak $[\alpha_1, \alpha_2]$-cut of the discrete interval-valued fuzzy dilation based on the conjunctor $\mathcal{C}_{\min}$ (respectively semi-norm $\mathcal{C}$) can always be constructed from (respectively approximated by) binary dilations as follows.

Proposition 12 *Let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ and let $\mathcal{C}$ be a semi-norm. For all $[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus U_{L^I}$ it holds that*:

(i) $D^I[\mathcal{C}_{\min}](A, B)^{\overline{\alpha_2}}_{\alpha_1} = D(A_{\alpha_1}, B_{\alpha_1}) \cap D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}})$,

(ii) $D^I[\mathcal{C}](A, B)^{\overline{\alpha_2}}_{\alpha_1} \subseteq D(A_{\alpha_1}, B_{\alpha_1}) \cap D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}})$.

For all $[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus \{1_{L^I}\}$ it holds that:

(i) $D^I[\mathcal{C}_{\min}](A, B)^{\alpha_2}_{\overline{\alpha_1}} = D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}}) \cap D(A^{\alpha_2}, B^{\alpha_2})$,

(ii) $D^I[\mathcal{C}](A, B)^{\alpha_2}_{\overline{\alpha_1}} \subseteq D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}}) \cap D(A^{\alpha_2}, B^{\alpha_2})$.

Proof Follows from Propositions 1, 2, 7 and 8 and the fact that $D^I[\mathcal{C}](A, B)^{\overline{\alpha_2}}_{\alpha_1} = D^I[\mathcal{C}](A, B)_{\alpha_1} \cap D^I[\mathcal{C}](A, B)^{\overline{\alpha_2}}$ and $D^I[\mathcal{C}](A, B)^{\alpha_2}_{\overline{\alpha_1}} = D^I[\mathcal{C}](A, B)_{\overline{\alpha_1}} \cap D^I[\mathcal{C}](A, B)^{\alpha_2}$ for every semi-norm $\mathcal{C}$. □

3.3 Decomposition of the Interval-Valued Fuzzy Erosion

As mentioned before, every implicator $\mathcal{I}$ induces a negator $\mathcal{N}_{\mathcal{I}}$ defined by $\mathcal{N}_{\mathcal{I}}(x) = \mathcal{I}(x, 0_{L^I})$, $\forall x \in L^I$. Based on this induced negator, the class of border implicators can be split into two subclasses.

Definition 11 [23] Let $\mathcal{I}$ be a border implicator on L^I. $\mathcal{I}$ is called an upper border implicator if $\mathcal{N}_{\mathcal{I}} \geq \mathcal{N}_s$; $\mathcal{I}$ is called a lower border implicator if $\mathcal{N}_{\mathcal{I}} \leq \mathcal{N}_s$.

Lemma 2 [23] *If $\mathcal{I}$ is an upper border implicator on L^I, then it holds that $\mathcal{I} \geq \mathcal{I}_{\min,\mathcal{N}_s}$, i.e.*:

$$\begin{aligned}(\forall (x, y) \in (L^I)^2)(\mathcal{I}(x, y) &\geq_{L^I} \mathcal{I}_{\min,\mathcal{N}_s}(x, y)\\ &= [\max(1 - x_2, y_1), \max(1 - x_1, y_2)]).\end{aligned} \tag{15}$$

The previous lemma does not necessarily hold if $\mathcal{I}$ is not an upper border implicator. Also, a lower border implicator $\mathcal{I}$ doesn't necessarily satisfy $\mathcal{I} \leq \mathcal{I}_{\min,\mathcal{N}_s}$.

Example 7 Let $\mathcal{I}$ be the implicator defined as:

$$\mathcal{I}(x, y) = \begin{cases} 1_{L^I} & \text{if } \inf(x, y) = x, \\ y & \text{else,} \end{cases} \qquad \forall (x, y) \in (L^I)^2.$$

It is easily verified that $\mathcal{I}$ is a border implicator on L^I, with induced negator $\mathcal{N}_\mathcal{I}$ given by:

$$\mathcal{N}_\mathcal{I}(x) = \mathcal{I}(x, 0_{L^I}) = \begin{cases} 1_{L^I} & \text{if } x = 0_{L^I}, \\ 0_{L^I} & \text{else,} \end{cases} \qquad \forall x \in [0,1].$$

From

$$\begin{cases} \mathcal{N}_\mathcal{I}(x) = 0_{L^I} \text{ and } 0_{L^I} \leq_{L^I} \mathcal{N}_s(x), & x \neq 0_{L^I}, \\ \mathcal{N}_\mathcal{I}(x) = 1_{L^I} = \mathcal{N}_s(x), & x = 0_{L^I}, \end{cases}$$

it follows that $\mathcal{N}_\mathcal{I} \leq \mathcal{N}_s$ and thus $\mathcal{I}$ is a lower border implicator. Further, since e.g. $\mathcal{I}([0.2, 0.3], [0.4, 0.5]) >_{L^I} \mathcal{I}_{\min,\mathcal{N}_s}([0.2, 0.3], [0.4, 0.5])$, while we also have that e.g. $\mathcal{I}([0.4, 0.5], [0.2, 0.3]) <_{L^I} \mathcal{I}_{\min,\mathcal{N}_s}([0.4, 0.5], [0.2, 0.3])$, it holds that neither $\mathcal{I} \leq \mathcal{I}_{\min,\mathcal{N}_s}$, nor $\mathcal{I} \geq \mathcal{I}_{\min,\mathcal{N}_s}$.

3.3.1 Decomposition by Weak Sub- and Supercuts

Proposition 13 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for respectively all $\alpha_1 \in]0,1]$ and all $\alpha_2 \in]0,1]$ that*:

(i) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1} = E(A_{\alpha_1}, B^{\overline{1-\alpha_1}})$,

(ii) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\alpha_2} = E(A^{\alpha_2}, B_{\overline{1-\alpha_2}})$.

Proof Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, and let $\alpha_1, \alpha_2 \in]0,1]$.

(i) It holds that:

$$\begin{aligned}
& y \in E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}) \\
\Leftrightarrow\ & T_y(B^{\overline{1-\alpha_1}}) \subseteq A_{\alpha_1} \\
\Leftrightarrow\ & (\forall x \in T_y(d_B)) \\
& (B_2(x-y) > 1-\alpha_1 \Rightarrow A_1(x) \geq \alpha_1) \\
\Leftrightarrow\ & (\forall x \in T_y(d_B)) \\
& (B_2(x-y) \leq 1-\alpha_1 \text{ or } A_1(x) \geq \alpha_1) \\
\Leftrightarrow\ & (\forall x \in T_y(d_B)) \\
& (1 - B_2(x-y) \geq \alpha_1 \text{ or } A_1(x) \geq \alpha_1) \\
\Leftrightarrow\ & (\forall x \in T_y(d_B)) \\
& (\max(1 - B_2(x-y), A_1(x)) \geq \alpha_1) \\
\Leftrightarrow\ & \inf_{x \in T_y(d_B)} \max(1 - B_2(x-y), A_1(x)) \geq \alpha_1 \\
\Leftrightarrow\ & \inf_{x \in T_y(d_B)} \mathcal{I}_{\min,\mathcal{N}_s}(B(x-y), A(x))_1 \geq \alpha_1 \\
\Leftrightarrow\ & E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_1(y) \geq \alpha_1 \\
\Leftrightarrow\ & y \in E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1}.
\end{aligned}$$

Thus $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1} = E(A_{\alpha_1}, B^{\overline{1-\alpha_1}})$.

(ii) Analogous. □

Proposition 14 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$ and let $\mathcal{I}$ be an upper border implicator on L^I, then it holds for respectively all $\alpha_1 \in]0,1]$ and all $\alpha_2 \in]0,1]$ that*:

(i) $E^I[\mathcal{I}](A, B)_{\alpha_1} \supseteq E(A_{\alpha_1}, B^{\overline{1-\alpha_1}})$,

(ii) $E^I[\mathcal{I}](A, B)^{\alpha_2} \supseteq E(A^{\alpha_2}, B_{\overline{1-\alpha_2}})$.

Proof (i) The proof is completely analogous to the one from Proposition 13(i). We only have that due to Lemma 2

$$\inf_{x \in T_y(d_B)} \mathcal{I}_{\min,\mathcal{N}_s}(B_2(x-y), A_1(x))_1 \geq \alpha_1$$

$$\Downarrow$$

$$\inf_{x \in T_y(d_B)} \mathcal{I}(B_2(x-y), A_1(x))_1 \geq \alpha_1$$

only holds in one direction for an arbitrary upper border implicator $\mathcal{I}$ on L^I.

(ii) Analogous. □

The reverse inclusion does not hold in general.

Example 8 Let $A(x) = [0.3, 0.5]$ for all $x \in [0,1]$, $B(x) = [0.5, 0.7]$ for all $x \in [0,1]$ and $A(x) = B(x) = 0_{L^I}$ for all $x \in \mathbb{R}\backslash[0,1]$. Let $\mathcal{I}$ be the following generalisation of the Łukasiewicz implicator: $\mathcal{I}_L(x, y) = [\min(1, 1 - x_2 + y_1), \min(1, 1 - x_1 + y_2)]$, $\forall (x, y) \in (L^I)^2$. It can be verified that this implicator is an upper border implicator.

It then holds that $E^I[\mathcal{I}_L](A, B)(0) = [0.6, 1]$ and thus $0 \in E^I[\mathcal{I}_L](A, B)_{0.4}$ and $0 \in E^I[\mathcal{I}_L](A, B)^{0.6}$.

On the other hand, $E(A_{0.4}, B^{\overline{0.6}}) = E(A^{0.6}, B_{\overline{0.4}}) = E(\emptyset, [0,1]) = \emptyset$ and thus $0 \notin E(A_{0.4}, B^{\overline{0.6}})$ and $0 \notin E(A^{0.6}, B_{\overline{0.4}})$, from which it follows that $E^I[\mathcal{I}](A, B)_{0.4} \not\subseteq E(A_{0.4}, B^{\overline{0.6}})$ and $E^I[\mathcal{I}](A, B)^{0.6} \not\subseteq E(A^{0.6}, B_{\overline{0.4}})$.

Further, Proposition 14 is also restricted to upper border implicator as the following example shows.

Example 9 Let $[\alpha_1, \alpha_2] = [0.3, 0.4]$, $A(x) = [0.4, 0.5]$ for all $x \in [0, 0.5]$, $A(x) = [0.2, 0.3]$ for all $x \in]0.5, 1]$, $B(x) = [0.7, 0.8]$ for all $x \in [0, 0.5]$, $B(x) = [0.4, 0.5]$ for all $x \in]0.5, 1]$ and $A(x) = B(x) = 0_{L^I}$ for all $x \in \mathbb{R}\backslash[0,1]$. Let $\mathcal{I}$ be the lower border implicator from Example 7.

It then holds that $E^I[\mathcal{I}](A, B)(0) = [0.2, 0.3]$, which means that $0 \notin E^I[\mathcal{I}](A, B)_{0.3}$ and $0 \notin E^I[\mathcal{I}](A, B)^{0.4}$.

On the other hand, $E(A_{0.3}, B^{\overline{0.7}}) = E(A^{0.4}, B_{\overline{0.6}}) = E([0, 0.5], [0, 0.5]) = \{0\}$. Consequently $E^I[\mathcal{I}](A, B)_{0.3} \not\supseteq E(A_{0.3}, B^{\overline{0.7}})$ and $E^I[\mathcal{I}](A, B)^{0.4} \not\supseteq E(A^{0.4}, B_{\overline{0.6}})$.

Remark that the above decomposition properties for weak sub- and supercuts remain valid in the discrete framework.

3.3.2 *Decomposition by Weak* $[\alpha_1, \alpha_2]$*-Cuts*

In general, for an arbitrary upper border implicator $\mathcal{I}$, there is no relation between the weak $[\alpha_1, \alpha_2]$-cut $E^I[\mathcal{I}](A, B)_{\alpha_1}^{\alpha_2}$ and the binary erosion $E(A_{\alpha_1}^{\alpha_2}, B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}})$. This is illustrated in the following example.

Example 10 To illustrate that it does not always hold that $E^I[\mathcal{I}](A, B)_{\alpha_1}^{\alpha_2} \subseteq E(A_{\alpha_1}^{\alpha_2}, B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}})$, we can use Example 8 again. For $[\alpha_1, \alpha_2] = [0.4, 0.6]$, the weak $[0.4, 0.6]$-cut and the weak 0.4-subcut and 0.6-supercut coincide and the results remain valid for the weak $[0.4, 0.6]$-cut.

In general also $E^I[\mathcal{I}](A, B)_{\alpha_1}^{\alpha_2} \not\supseteq E(A_{\alpha_1}^{\alpha_2}, B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}})$. Let $\mathcal{I}$ be $\mathcal{I}_{\min,\mathcal{N}_s}$, $[\alpha_1, \alpha_2] = [0.3, 0.4]$, $A(x) = [0.4, 0.5]$ for all $x \in [0, 0.5]$ and $A(x) = [0.2, 0.3]$ for all $x \in]0.5, 1]$, $B(x) = [0.7, 0.8]$ for all $x \in [0, 0.5]$ and $B(x) = [0.4, 0.8]$ for all $x \in]0.5, 1]$.

For the binary erosion we find that $E(A_{\alpha_1}^{\alpha_2}, B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}}) = E([0, 0.5], [0, 0.5]) = \{0\}$. Further, it also holds that $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)(0) = [0.2, 0.5] \not\geq_{L^I} [\alpha_1, \alpha_2]$. As a consequence, $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1}^{\alpha_2} \not\supseteq E(A_{\alpha_1}^{\alpha_2}, B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}})$.

For the upper border implicator $\mathcal{I} = \mathcal{I}_{\min,\mathcal{N}_s}$, the following partial result holds.

Proposition 15 *Let* $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, *then it holds for all* $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$ *that*:

$$E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1}^{\alpha_2} \subseteq E(A_{\alpha_1}^{\alpha_2}, B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}}).$$

Proof Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$ and $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$. It holds that:

$$
\begin{aligned}
& y \in E(A_{\alpha_1}^{\alpha_2}, B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}}) \\
\Leftrightarrow\quad & T_y(B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}}) \subseteq A_{\alpha_1}^{\alpha_2} \\
\Leftrightarrow\quad & (\forall x \in T_y(d_B)) \\
& ((B_1(x-y) > 1-\alpha_2 \text{ and } B_2(x-y) > 1-\alpha_1) \\
\Rightarrow\quad & (A_1(x) \geq \alpha_1 \text{ and } A_2(x) \geq \alpha_2)) \\
\Leftrightarrow\quad & (\forall x \in T_y(d_B)) \\
& ((B_1(x-y) \leq 1-\alpha_2 \text{ or } B_2(x-y) \leq 1-\alpha_1) \text{ or} \\
& (A_1(x) \geq \alpha_1 \text{ and } A_2(x) \geq \alpha_2)) \\
\Leftrightarrow\quad & (\forall x \in T_y(d_B)) \\
& ((1-B_1(x-y) \geq \alpha_2 \text{ or } 1-B_2(x-y) \geq \alpha_1) \text{ or} \\
& (A_1(x) \geq \alpha_1 \text{ and } A_2(x) \geq \alpha_2)) \\
\Leftarrow\quad & (\forall x \in T_y(d_B)) \\
& (\max(1-B_2(x-y), A_1(x)) \geq \alpha_1 \text{ and} \\
& \max(1-B_1(x-y), A_2(x)) \geq \alpha_2) \\
\Leftrightarrow\quad & (\forall x \in T_y(d_B)) \\
& (\mathcal{I}_{\min,\mathcal{N}_s}(B(x-y), A(x)) \geq_{L^I} [\alpha_1, \alpha_2]) \\
\Leftrightarrow\quad & \inf_{x \in T_y(d_B)} \mathcal{I}_{\min,\mathcal{N}_s}(B(x-y), A(x)) \geq_{L^I} [\alpha_1, \alpha_2] \\
\Leftrightarrow\quad & E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)(y) \geq_{L^I} [\alpha_1, \alpha_2] \\
\Leftrightarrow\quad & y \in E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1}^{\alpha_2}
\end{aligned}
$$

This proves that $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1}^{\alpha_2} \subseteq E(A_{\alpha_1}^{\alpha_2}, B_{\overline{1-\alpha_2}}^{\overline{1-\alpha_1}})$. □

The reverse inclusion does not hold as illustrated in Example 10.

The weak $[\alpha_1, \alpha_2]$-cut of the interval-valued fuzzy erosion based on the implicator $\mathcal{I}_{\min,\mathcal{N}_s}$ can however always be constructed by binary erosions as follows.

Proposition 16 *Let* $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, *then it holds for all* $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$ *that*:

$$
\begin{aligned}
& E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1}^{\alpha_2} \\
& = E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}) \cap E(A^{\alpha_2}, B_{\overline{1-\alpha_2}}). \qquad (16)
\end{aligned}
$$

Proof Follows from Proposition 13 and the fact that

$$
\begin{aligned}
& E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1}^{\alpha_2} \\
& = E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1} \cap E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\alpha_2}. \qquad (17)
\end{aligned}
$$

□

Analogously, an interval-valued fuzzy erosion based an upper-border implicator $\mathcal{I}$ can be approximated by binary erosions.

Proposition 17 *Let* $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, *then it holds for all* $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$ *that*:

$$E^I[\mathcal{I}](A, B)_{\alpha_1}^{\alpha_2} \supseteq E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}) \cap E(A^{\alpha_2}, B_{\overline{1-\alpha_2}}).$$

Proof Follows from Proposition 14 and the fact that $E^I[\mathcal{I}](A, B)_{\alpha_1}^{\alpha_2} = E^I[\mathcal{I}](A, B)_{\alpha_1} \cap E^I[\mathcal{I}](A, B)^{\alpha_2}$. □

Remark that the above decomposition properties for weak $[\alpha_1, \alpha_2]$-cuts remain valid in the discrete framework.

3.3.3 *Decomposition by Strict Sub- and Supercuts*

In general, there is no relation between the strict sub- and supercuts $E^I[\mathcal{I}](A, B)_{\overline{\alpha_1}}$ and $E^I[\mathcal{I}](A, B)^{\overline{\alpha_2}}$ and the binary erosions $E(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$ and $E(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$ for an arbitrary upper border implicator $\mathcal{I}$. This is illustrated in the following example.

Example 11 To show that the inclusions $E^I[\mathcal{I}](A,B)_{\overline{\alpha_1}} \subseteq E(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$ and $E^I[\mathcal{I}](A,B)^{\overline{\alpha_2}} \subseteq E(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$ do not always hold for an arbitrary upper border implicator $\mathcal{I}$, we can use Example 8 again, where working with strict sub- and supercuts instead of weak sub- and supercuts does not affect the results.

In general also $E^I[\mathcal{I}](A,B)_{\overline{\alpha_1}} \not\supseteq E(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$ and $E^I[\mathcal{I}](A,B)^{\overline{\alpha_2}} \not\supseteq E(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$. Let $\mathcal{I}$ be $\mathcal{I}_{\min,\mathcal{N}_s}$, $\alpha_1 = 0.5$, $A(x) = [\frac{2-x}{2}, 1]$ for all $x \in]0,1]$, $B(x) = [0.7, 0.8]$ for all $x \in]0,1]$ and $A(x) = B(x) = 0_{L^I}$ for all $x \in \mathbb{R} \backslash]0,1]$.

For the binary erosion we find $E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}) = E(]0,1],]0,1]) = \{0\}$.

Further, it also holds that $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A,B)(0) = [0.5, 1]$, which means that $0 \notin E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{0.5}}$.

Consequently, $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}} \not\supseteq E(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$.

An analogous example can be found for strict α_2-supercuts.

For the upper border implicator $\mathcal{I} = \mathcal{I}_{\min,\mathcal{N}_s}$, the following partial result holds.

Proposition 18 *For $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$ it holds for respectively all $\alpha_1 \in [0,1[$ and all $\alpha_2 \in [0,1[$ that*:

(i) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}} \subseteq E(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$,

(ii) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\overline{\alpha_2}} \subseteq E(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$.

Proof Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, and let $\alpha_1, \alpha_2 \in [0,1[$.

(i) Analogous to the proof of Proposition 13. However, now we only have that for all $y \in \mathbb{R}$:

$$\inf_{x \in T_y(d_B)} \max(1 - B_2(x-y), A_1(x)) > \alpha_1$$

$$\Downarrow$$

$$(\forall x \in T_y(d_B))(\max(1 - B_2(x-y), A_1(x)) > \alpha_1)$$

(ii) Analogous. □

The reverse inclusion does not hold as illustrated in Example 11.

Proposition 18 remains valid in the discrete framework. Moreover, in the discrete framework, the result also holds for arbitrary lower border implicators and for $\mathcal{I}_{\min,\mathcal{N}_s}$ also the reverse inclusion holds.

Proposition 19 *For $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ it holds for respectively all $\alpha_1 \in]0,1] \cap I_r$ and all $\alpha_2 \in]0,1] \cap I_s$ that*:

(i) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}} = E(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$,

(ii) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\overline{\alpha_2}} = E(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$.

Proof Analogous to the proof of Proposition 18, where now in the discrete case also

$$(\forall x \in T_y(d_B))(\max(1 - B_2(x-y), A_1(x)) > \alpha_1)$$

$$\Updownarrow$$

$$\inf_{x \in T_y(d_B)} \max(1 - B_2(x-y), A_1(x)) > \alpha_1.$$ □

Proposition 20 *For $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ it holds for respectively all $\alpha_1 \in]0,1] \cap I_r$ and all $\alpha_2 \in]0,1] \cap I_s$ that*:

(i) $E^I[\mathcal{I}](A,B)_{\overline{\alpha_1}} \supseteq E(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$,

(ii) $E^I[\mathcal{I}](A,B)^{\overline{\alpha_2}} \supseteq E(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$.

Proof Analogous to the proof of Proposition 19, but for an arbitrary upper border implicator $\mathcal{I}$, so that

$$E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_1(y) > \alpha_1$$

$$\Downarrow$$

$$E^I[\mathcal{I}](A,B)_1(y) > \alpha_1$$

$$\Updownarrow$$

$$y \in E^I[\mathcal{I}](A,B)_{\overline{\alpha_1}}.$$ □

3.3.4 *Decomposition by Strict $[\alpha_1,\alpha_2]$-Cuts*

In general, for an arbitrary upper border implicator $\mathcal{I}$, there is no relation between the strict $[\alpha_1,\alpha_2]$-cut $E^I[\mathcal{I}](A,B)_{\overline{\alpha_1}}^{\overline{\alpha_2}}$ and the binary erosion $E(A_{\overline{\alpha_1}}^{\overline{\alpha_2}}, B_{1-\alpha_2}^{1-\alpha_1})$. To illustrate this, we can use Examples 8 and 10 again, where working with strict $[\alpha_1,\alpha_2]$-cuts instead of respectively weak sub- and supercuts and weak $[\alpha_1,\alpha_2]$-cuts does not effect the results.

For the upper border implicator $\mathcal{I} = \mathcal{I}_{\min,\mathcal{N}_s}$, the following partial result holds.

Proposition 21 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds that for all $[\alpha_1,\alpha_2] \in L^I \backslash U_{L^I}$ that*:

$$E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}^{\overline{\alpha_2}} \subseteq E(A_{\overline{\alpha_1}}^{\overline{\alpha_2}}, B_{1-\alpha_2}^{1-\alpha_1}).$$

Proof Analogous to the proof of Proposition 15. Only, now it holds for all $y \in \mathbb{R}$ that:

$$\inf_{x \in T_y(d_B)} \mathcal{I}_{\min,\mathcal{N}_s}(B(x-y), A(x)) \gg_{L^I} [\alpha_1,\alpha_2])$$

$$\Downarrow$$

$$(\forall x \in T_y(d_B))(\mathcal{I}_{\min,\mathcal{N}_s}(B(x-y), A(x)) \gg_{L^I} [\alpha_1,\alpha_2])$$

This however does not change the result. □

To illustrate that the reverse inclusion does not hold, we refer to Example 10, where using strict $[\alpha_1,\alpha_2]$-cuts instead of the weak $[\alpha_1,\alpha_2]$-cuts does not affect the results.

Remark that the above decomposition properties for strict $[\alpha_1,\alpha_2]$-cuts remain valid in the discrete framework. Moreover, the strict $[\alpha_1,\alpha_2]$-cut of the discrete interval-valued fuzzy erosion based on the implicator $\mathcal{I}_{\min,\mathcal{N}_s}$ (respectively upper-border implicator $\mathcal{I}$) can always be constructed from (respectively approximated by) binary erosion as follows.

Proposition 22 *For $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ it holds for all $[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus U_{L^I}$ and every upper border implicator $\mathcal{I}$ that*:

(i) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\alpha_1}$

$$= E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}) \cap E(A^{\overline{\alpha_2}}, B_{1-\alpha_2}),$$

(ii) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\alpha_1}$

$$\supseteq E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}) \cap E(A^{\overline{\alpha_2}}, B_{1-\alpha_2}).$$

Proof Follows from Propositions 19 and 20. □

3.3.5 Decomposition by Weak-Strict and Strict-Weak $[\alpha_1, \alpha_2]$-Cuts

In general, for an arbitrary upper border implicator $\mathcal{I}$, there is no relation between the weak-strict and the strict-weak $[\alpha_1, \alpha_2]$-cuts $E^I[\mathcal{I}](A, B)^{\overline{\alpha_2}}_{\alpha_1}$ and $E^I[\mathcal{I}](A, B)^{\alpha_2}_{\overline{\alpha_1}}$ and the respective binary erosions $E(A^{\overline{\alpha_2}}_{\alpha_1}, B^{\overline{1-\alpha_1}}_{1-\alpha_2})$ and $E(A^{\alpha_2}_{\overline{\alpha_1}}, B^{1-\alpha_1}_{\overline{1-\alpha_2}})$ respectively. To illustrate this, we can use Examples 8 and 10 again, where working with weak-strict and strict-weak $[\alpha_1, \alpha_2]$-cuts instead of respectively weak sub- and supercuts and weak $[\alpha_1, \alpha_2]$-cuts does not effect the results.

For the upper border implicator $\mathcal{I} = \mathcal{I}_{\min,\mathcal{N}_s}$, the following partial result holds.

Proposition 23 *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for respectively all $[\alpha_1, \alpha_2] \in L^I \backslash U_{L^I}$ and all $[\alpha_1, \alpha_2] \in L^I \backslash \{1_{L^I}\}$ that*:

(i) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\alpha_1} \subseteq E(A^{\overline{\alpha_2}}_{\alpha_1}, B^{\overline{1-\alpha_1}}_{1-\alpha_2})$,

(ii) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\alpha_2}_{\overline{\alpha_1}} \subseteq E(A^{\alpha_2}_{\overline{\alpha_1}}, B^{1-\alpha_1}_{\overline{1-\alpha_2}})$.

Proof (i) Analogous to the proof of Proposition 15. Only, now it holds for all $y \in \mathbb{R}$ that:

$$\inf_{x \in T_y(d_B)} \mathcal{I}_{\min,\mathcal{N}_s}(B(x-y), A(x))_1 \geq \alpha_1 \quad \text{and}$$

$$\inf_{x \in T_y(d_B)} \mathcal{I}_{\min,\mathcal{N}_s}(B(x-y), A(x))_2 > \alpha_2$$

$$\Downarrow$$

$$(\forall x \in T_y(d_B))(\mathcal{I}_{\min,\mathcal{N}_s}(B(x-y), A(x))_1 \geq \alpha_1 \quad \text{and}$$

$$\mathcal{I}_{\min,\mathcal{N}_s}(B(x-y), A(x))_2 > \alpha_2)$$

This however does not change the result.

(ii) Analogous. □

To illustrate that the reverse inclusion does not hold, we refer to Example 10 again, where working with weak-strict or strict-weak $[\alpha_1, \alpha_2]$-cuts instead of weak $[\alpha_1, \alpha_2]$-cuts does not affect the results.

Remark that the above decomposition properties for weak-strict and strict-weak $[\alpha_1, \alpha_2]$-cuts remain valid in the discrete framework. Moreover, the weak-strict and strict-weak $[\alpha_1, \alpha_2]$-cut of the discrete interval-valued fuzzy erosion based on the implicator $\mathcal{I}_{\min,\mathcal{N}_s}$ (respectively upper-border implicator $\mathcal{I}$) can always be constructed from (respectively approximated by) binary erosion as follows.

Proposition 24 *Let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ and let $\mathcal{I}$ be an upper border implicator. For all $[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus U_{L^I}$ it holds that*:

(i) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\alpha_1}$

$$= E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}) \cap E(A^{\overline{\alpha_2}}, B_{1-\alpha_2}),$$

(ii) $E^I[\mathcal{I}](A, B)^{\overline{\alpha_2}}_{\alpha_1} \supseteq E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}) \cap E(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$.

For all $[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus 1_{L^I}$ it holds that:

(i) $E^I[\mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\alpha_2}_{\overline{\alpha_1}}$

$$= E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}) \cap E(A^{\alpha_2}, B_{\overline{1-\alpha_2}}),$$

(ii) $E^I[\mathcal{I}](A, B)^{\alpha_2}_{\overline{\alpha_1}} \supseteq E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}) \cap E(A^{\alpha_2}, B_{\overline{1-\alpha_2}})$.

Proof Follows from Propositions 13, 14, 19 and 20. □

3.4 Decomposition of the Interval-Valued Fuzzy Closing and Opening

We first prove the following lemma:

Lemma 3 *Let $A \in \mathcal{IVFS}(\mathbb{R}^n)$ and let $[\alpha_1, \alpha_2] \in L^I$, then it holds that*:

(i) $\alpha_2 \in]0, 0.5] \quad \Rightarrow \quad A^{\alpha_2} \supseteq A^{\overline{\alpha_2}} \supseteq A_{\overline{1-\alpha_2}}$,

(ii) $\alpha_1 \in [0.5, 1[\quad \Rightarrow \quad A_{\overline{\alpha_1}} \subseteq A^{\overline{1-\alpha_1}} \subseteq A^{1-\alpha_1}$,

(iii) $\alpha_2 \in [0, 0.5[\quad \Rightarrow \quad A^{\overline{\alpha_2}} \supseteq A_{1-\alpha_2}$,

(iv) $\alpha_1 \in]0.5, 1] \quad \Rightarrow \quad A_{\alpha_1} \subseteq A^{\overline{1-\alpha_1}}$.

Proof (i) $\alpha_2 \in]0, 0.5]$

$$\begin{aligned} x \in A_{\overline{1-\alpha_2}} \quad &\Leftrightarrow \quad A_1(x) > 1 - \alpha_2 \\ &\Rightarrow \quad A_2(x) \geq A_1(x) > 1 - \alpha_2 \geq \alpha_2 \\ &\qquad \text{(i.e. } x \in A^{\overline{\alpha_2}}) \\ &\Rightarrow \quad A_2(x) \geq \alpha_2 \quad \text{(i.e. } x \in A^{\alpha_2}). \end{aligned}$$

(ii) $\alpha_1 \in [0.5, 1[$

$$\begin{aligned} x \in A_{\overline{\alpha_1}} &\Leftrightarrow A_1(x) > \alpha_1 \\ &\Rightarrow A_2(x) \geq A_1(x) > \alpha_1 \geq 1-\alpha_1 \\ &\quad \text{(i.e. } x \in A^{\overline{1-\alpha_1}}) \\ &\Rightarrow A_2(x) \geq 1-\alpha_1 \quad \text{(i.e. } x \in A^{1-\alpha_1}). \end{aligned}$$

(iii) $\alpha_2 \in [0, 0.5[$

$$\begin{aligned} x \in A_{1-\alpha_2} &\Leftrightarrow A_1(x) \geq 1-\alpha_2 \\ &\Rightarrow A_2(x) \geq A_1(x) \geq 1-\alpha_2 > \alpha_2 \\ &\quad \text{(i.e. } x \in A^{\overline{\alpha_2}}). \end{aligned}$$

(iv) $\alpha_1 \in]0.5, 1]$

$$\begin{aligned} x \in A_{\alpha_1} &\Leftrightarrow A_1(x) \geq \alpha_1 \\ &\Rightarrow A_2(x) \geq A_1(x) \geq \alpha_1 > 1-\alpha_1 \\ &\quad \text{(i.e. } x \in A^{\overline{1-\alpha_1}}). \end{aligned}$$

□

3.4.1 Decomposition by Weak Sub- and Supercuts

Proposition 25 *Let $\mathcal{I}$ be an upper border implicator on L^I and let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $\alpha_1 \in]0, 1]$ that*:

(i) $C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)_{\alpha_1} \supseteq E(D(A_{\alpha_1}, B_{\alpha_1}), -B^{\overline{1-\alpha_1}})$,

(ii) $O^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)_{\alpha_1} \supseteq D(E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}), -B_{\alpha_1})$,

and for all $\alpha_2 \in]0, 1]$ that:

(iii) $C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\alpha_2} \supseteq E(D(A^{\alpha_2}, B^{\alpha_2}), -B_{\overline{1-\alpha_2}})$,

(iv) $O^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\alpha_2} \supseteq D(E(A^{\alpha_2}, B_{\overline{1-\alpha_2}}), -B^{\alpha_2})$.

Proof As an example we prove (i). Let $\mathcal{I}$ be an upper border implicator on L^I, let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$ and let $\alpha_1, \alpha_2 \in]0, 1]$. From respectively Propositions 13, 6, and because the binary erosion is increasing in its first argument, we have that:

$$\begin{aligned} C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)_{\alpha_1} &= E^I[\mathcal{I}](D^I[\mathcal{C}_{\min}](A, B), -B)_{\alpha_1} \\ &\supseteq E(D^I[\mathcal{C}_{\min}](A, B)_{\alpha_1}, -B^{\overline{1-\alpha_1}}) \\ &\supseteq E(D(A_{\alpha_1}, B_{\alpha_1}), -B^{\overline{1-\alpha_1}}). \end{aligned}$$

(ii), (iii) and (iv) follow analogously from Propositions 6, 13, and because the binary dilation and the binary erosion are increasing in their first argument. □

The previous result allows us to derive, under the restriction of $\alpha_2 \in]0, 0.5]$, a lower bound for the weak α_2-supercut of the interval-valued fuzzy closing and opening in terms of the binary closing and opening.

Proposition 26 *Let $\mathcal{I}$ be an upper border implicator on L^I and let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $\alpha_2 \in]0, 0.5]$ that*:

(i) $C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\alpha_2} \supseteq C(A^{\alpha_2}, B^{\alpha_2})$,

(ii) $C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\alpha_2} \supseteq C(A^{\alpha_2}, B_{\overline{1-\alpha_2}})$,

and:

(iii) $O^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\alpha_2} \supseteq O(A^{\alpha_2}, B^{\alpha_2})$,

(iv) $O^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\alpha_2} \supseteq O(A^{\alpha_2}, B_{\overline{1-\alpha_2}})$.

Proof As an example, we prove (i). Let $\mathcal{I}$ be an upper border implicator on L^I, let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$ and let $\alpha_2 \in]0, 0.5]$. From Proposition 25, Lemma 3 and the fact that the binary erosion is decreasing in its second argument, it follows that:

$$\begin{aligned} C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\alpha_2} &\supseteq E(D(A^{\alpha_2}, B^{\alpha_2}), -B_{\overline{1-\alpha_2}}) \\ &\supseteq E(D(A^{\alpha_2}, B^{\alpha_2}), -B^{\alpha_2}) \\ &= C(A^{\alpha_2}, B^{\alpha_2}). \end{aligned}$$

(ii), (iii) and (iv) follow in an analogous way from Proposition 25, Lemma 3 and the fact that the binary dilation is increasing in both its arguments and the binary erosion is increasing in its first argument and decreasing in its second argument. □

The above results for weak sub- and supercuts remain valid in the discrete framework. Since we had found a new relationship for the decomposition by weak sub- and supercuts of the interval-valued fuzzy dilation in the discrete framework compared to the continuous framework, also a new relationship can be found for the interval-valued fuzzy closing and opening.

Proposition 27 *Let $\mathcal{C}$ be a semi-norm on $L^I_{r,s}$ and $\mathcal{I}$ an upper border implicator on $L^I_{r,s}$ and let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for all $\alpha_1 \in]0, 1] \cap I_r$ that*:

(i) $C^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1}$

$\quad = E(D(A_{\alpha_1}, B_{\alpha_1}), -B^{\overline{1-\alpha_1}})$,

(ii) $C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)_{\alpha_1} \supseteq E(D(A_{\alpha_1}, B_{\alpha_1}), -B^{\overline{1-\alpha_1}})$,

(iii) $C^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1}$

$\quad \subseteq E(D(A_{\alpha_1}, B_{\alpha_1}), -B^{\overline{1-\alpha_1}})$,

(iv) $O^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\alpha_1}$

$\quad = D(E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}), -B_{\alpha_1})$,

(v) $O^I[\mathcal{C}_{\min},\mathcal{I}](A,B)_{\alpha_1} \supseteq D(E(A_{\alpha_1},\overline{B^{1-\alpha_1}}),-B_{\alpha_1})$,

(vi) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1}$
$$\subseteq D(E(A_{\alpha_1},\overline{B^{1-\alpha_1}}),-B_{\alpha_1}),$$

and for all $\alpha_2 \in]0,1] \cap I_s$ *that*:

(i) $C^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\alpha_2}$
$$= E(D(A^{\alpha_2},B^{\alpha_2}),-B_{\overline{1-\alpha_2}}),$$

(ii) $C^I[\mathcal{C}_{\min},\mathcal{I}](A,B)^{\alpha_2} \supseteq E(D(A^{\alpha_2},B^{\alpha_2}),-B_{\overline{1-\alpha_2}})$,

(iii) $C^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\alpha_2}$
$$\subseteq E(D(A^{\alpha_2},B^{\alpha_2}),-B_{\overline{1-\alpha_2}}),$$

(iv) $O^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\alpha_2}$
$$= D(E(A^{\alpha_2},B_{\overline{1-\alpha_2}}),-B^{\alpha_2}),$$

(v) $O^I[\mathcal{C}_{\min},\mathcal{I}](A,B)^{\alpha_2} \supseteq D(E(A^{\alpha_2},B_{\overline{1-\alpha_2}}),-B^{\alpha_2})$,

(vi) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\alpha_2}$
$$\subseteq D(E(A^{\alpha_2},B_{\overline{1-\alpha_2}}),-B^{\alpha_2}).$$

Proof Follows in an analogous way as in the proof of Proposition 25 from Propositions 7, 8, 13, 14 and the fact that the binary dilation is increasing in its first and second argument and that the binary erosion is increasing in its first argument and decreasing in its second argument. □

The previous result allows us to derive, under the restriction of $\alpha_1 \in]0.5,1] \cap I_r$, an upper bound for the weak subcut of the interval-valued fuzzy closing and opening in terms of the binary closing and opening.

Proposition 28 *Let* $\mathcal{C}$ *be a semi-norm on* $L^I_{r,s}$ *and let* $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, *then it holds for all* $\alpha_1 \in]0.5,1] \cap I_r$ *that*:

(i) $C^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1} \subseteq C(A_{\alpha_1},B_{\alpha_1})$,

(ii) $C^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1} \subseteq C(A_{\alpha_1},\overline{B^{1-\alpha_1}})$,

(iii) $C^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1} \subseteq C(A_{\alpha_1},B_{\alpha_1})$,

(iv) $C^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1} \subseteq C(A_{\alpha_1},\overline{B^{1-\alpha_1}})$,

(v) $O^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1} \subseteq O(A_{\alpha_1},B_{\alpha_1})$,

(vi) $O^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1} \subseteq O(A_{\alpha_1},\overline{B^{1-\alpha_1}})$,

(vii) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1} \subseteq O(A_{\alpha_1},B_{\alpha_1})$,

(viii) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1} \subseteq O(A_{\alpha_1},\overline{B^{1-\alpha_1}})$,

Proof Follows in an analogous way as in the proof of Proposition 26 from Proposition 27 and Lemma 3 and the fact that the binary dilation is increasing in its first and second argument and that the binary erosion is increasing in its first argument and decreasing in its second argument. □

The result also allows us to derive, under the restriction of $\alpha_2 \in]0,0.5] \cap I_s$, a lower bound for the weak supercut of the interval-valued fuzzy closing and opening in terms of the binary closing and opening.

Proposition 29 *Let* $\mathcal{I}$ *be an upper border implicator on* $L^I_{r,s}$ *and let* $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, *then it holds for all* $\alpha_2 \in]0,0.5] \cap I_s$ *that*:

(i) $C^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\alpha_2} \supseteq C(A^{\alpha_2},B^{\alpha_2})$,

(ii) $C^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\alpha_2} \supseteq C(A^{\alpha_2},B_{\overline{1-\alpha_2}})$,

(iii) $C^I[\mathcal{C}_{\min},\mathcal{I}](A,B)^{\alpha_2} \supseteq C(A^{\alpha_2},B^{\alpha_2})$,

(iv) $C^I[\mathcal{C}_{\min},\mathcal{I}](A,B)^{\alpha_2} \supseteq C(A^{\alpha_2},B_{\overline{1-\alpha_2}})$,

(v) $O^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\alpha_2} \supseteq O(A^{\alpha_2},B^{\alpha_2})$,

(vi) $O^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\alpha_2} \supseteq O(A^{\alpha_2},B_{\overline{1-\alpha_2}})$,

(vii) $O^I[\mathcal{C}_{\min},\mathcal{I}](A,B)^{\alpha_2} \supseteq O(A^{\alpha_2},B^{\alpha_2})$,

(viii) $O^I[\mathcal{C}_{\min},\mathcal{I}](A,B)^{\alpha_2} \supseteq O(A^{\alpha_2},B_{\overline{1-\alpha_2}})$.

Proof Follows in an analogous way as in the proof of Proposition 26 from Proposition 27 and Lemma 3 and the fact that the binary dilation is increasing in its first and second argument and that the binary erosion is increasing in its first argument and decreasing in its second argument. □

3.4.2 Decomposition by Weak $[\alpha_1,\alpha_2]$*-Cuts*

For the conjunctor $\mathcal{C}_{\min}$ and the implicator $\mathcal{I}_{\min,\mathcal{N}_s}$, the weak $[\alpha_1,\alpha_2]$-cuts of the discrete interval-valued fuzzy closing and opening can be found as a combination of binary dilations and erosions. For an arbitrary semi-norm $\mathcal{C}$ and an arbitrary upper border implicator $\mathcal{I}$ analogous approximations exist.

Proposition 30 *Let* $\mathcal{C}$ *be a semi-norm on* $L^I_{r,s}$ *and* $\mathcal{I}$ *an upper border implicator on* $L^I_{r,s}$ *and let* $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, *then it holds for all* $[\alpha_1,\alpha_2] \in L^I_{r,s} \setminus \{0_{L^I}\}$ *that*:

(i) $C^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\alpha_2}_{\alpha_1}$
$$= E(D(A_{\alpha_1},B_{\alpha_1}),-\overline{B^{1-\alpha_1}}) \cap E(D(A^{\alpha_2},B^{\alpha_2}),-B_{\overline{1-\alpha_2}}),$$

(ii) $C^I[\mathcal{C}_{\min},\mathcal{I}](A,B)^{\alpha_2}_{\alpha_1}$
$$\supseteq E(D(A_{\alpha_1},B_{\alpha_1}),-\overline{B^{1-\alpha_1}}) \cap E(D(A^{\alpha_2},B^{\alpha_2}),-B_{\overline{1-\alpha_2}}),$$

(iii) $C^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1}^{\alpha_2}$

$\subseteq E(D(A_{\alpha_1},B_{\alpha_1}),-B^{\overline{1-\alpha_1}})$

$\cap\, E(D(A^{\alpha_2},B^{\alpha_2}),-B_{\overline{1-\alpha_2}})$,

(iv) $O^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1}^{\alpha_2}$

$= D(E(A_{\alpha_1},B^{\overline{1-\alpha_1}}),-B_{\alpha_1})$

$\cap\, D(E(A^{\alpha_2},B_{\overline{1-\alpha_2}}),-B^{\alpha_2})$,

(v) $O^I[\mathcal{C}_{\min},\mathcal{I}](A,B)_{\alpha_1}^{\alpha_2}$

$\supseteq D(E(A_{\alpha_1},B^{\overline{1-\alpha_1}}),-B_{\alpha_1})$

$\cap\, D(E(A^{\alpha_2},B_{\overline{1-\alpha_2}}),-B^{\alpha_2})$,

(vi) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\alpha_1}^{\alpha_2}$

$\subseteq D(E(A_{\alpha_1},B^{\overline{1-\alpha_1}}),-B_{\alpha_1})$

$\cap\, D(E(A^{\alpha_2},B_{\overline{1-\alpha_2}}),-B^{\alpha_2})$.

Proof Follows from Proposition 27. □

3.4.3 Decomposition by Strict Sub- and Supercuts

Proposition 31 *Let $\mathcal{C}$ be a semi-norm on L^I and let $A,B\in\mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $\alpha_1\in[0,1[$ that*:

(i) $C^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}\subseteq E(D(A_{\overline{\alpha_1}},B_{\overline{\alpha_1}}),-B^{1-\alpha_1})$,

(ii) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}\subseteq D(E(A_{\overline{\alpha_1}},B^{1-\alpha_1}),-B_{\overline{\alpha_1}})$,

and for all $\alpha_2\in[0,1[$ that:

(iii) $C^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\overline{\alpha_2}}\subseteq E(D(A^{\overline{\alpha_2}},B^{\overline{\alpha_2}}),-B_{1-\alpha_2})$,

(iv) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\overline{\alpha_2}}\subseteq D(E(A^{\overline{\alpha_2}},B_{1-\alpha_2}),-B^{\overline{\alpha_2}})$.

Proof Follows in an analogous way as in the proof of Proposition 25 from Propositions 18 and 2 and the fact that the binary dilation is increasing in its first and second argument and that the binary erosion is increasing in its first argument and decreasing in its second argument. □

The previous result allows us to derive, under the restriction of $\alpha_1\in[0.5,1[$, an upper bound for the strict α_1-subcut of the interval-valued fuzzy closing and opening in terms of the binary closing and opening.

Proposition 32 *Let $\mathcal{C}$ be a semi-norm on L^I and let $A,B\in\mathcal{IVFS}(\mathbb{R}^n)$, then it holds for all $\alpha_1\in[0.5,1[$ that*:

(i) $C^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}\subseteq C(A_{\overline{\alpha_1}},B_{\overline{\alpha_1}})$,

(ii) $C^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}\subseteq C(A_{\overline{\alpha_1}},B^{1-\alpha_1})$,

and:

(iii) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}\subseteq O(A_{\overline{\alpha_1}},B_{\overline{\alpha_1}})$,

(iv) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}\subseteq O(A_{\overline{\alpha_1}},B^{1-\alpha_1})$.

Proof Follows in an analogous way as in the proof of Proposition 26 from Proposition 31 and Lemma 3 and the fact that the binary dilation is increasing in its first and second argument and that the binary erosion is increasing in its first argument and decreasing in its second argument. □

The above results for strict sub- and supercuts remain valid in the discrete framework. Since we had found a new relationship for the decomposition by strict sub- and supercuts of the interval-valued fuzzy erosion in the discrete framework compared to the continuous framework, also a new relationship can be found for the interval-valued fuzzy closing and opening.

Proposition 33 *Let $\mathcal{C}$ be a semi-norm on $L^I_{r,s}$ and $\mathcal{I}$ an upper border implicator on $L^I_{r,s}$ and let $A,B\in\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for all $\alpha_1\in[0,1[\,\cap I_r$ that*:

(i) $C^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}$

$= E(D(A_{\overline{\alpha_1}},B_{\overline{\alpha_1}}),-B^{1-\alpha_1})$,

(ii) $C^I[\mathcal{C}_{\min},\mathcal{I}](A,B)_{\overline{\alpha_1}}\supseteq E(D(A_{\overline{\alpha_1}},B_{\overline{\alpha_1}}),-B^{1-\alpha_1})$,

(iii) $C^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}$

$\subseteq E(D(A_{\overline{\alpha_1}},B_{\overline{\alpha_1}}),-B^{1-\alpha_1})$,

(iv) $O^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}$

$= D(E(A_{\overline{\alpha_1}},B^{1-\alpha_1}),-B_{\overline{\alpha_1}})$,

(v) $O^I[\mathcal{C}_{\min},\mathcal{I}](A,B)_{\overline{\alpha_1}}\supseteq D(E(A_{\overline{\alpha_1}},B^{1-\alpha_1}),-B_{\overline{\alpha_1}})$,

(vi) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)_{\overline{\alpha_1}}$

$\subseteq D(E(A_{\overline{\alpha_1}},B^{1-\alpha_1}),-B_{\overline{\alpha_1}})$,

and for all $\alpha_2\in[0,1[\,\cap I_s$ that:

(i) $C^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\overline{\alpha_2}}$

$= E(D(A^{\overline{\alpha_2}},B^{\overline{\alpha_2}}),-B_{1-\alpha_2})$,

(ii) $C^I[\mathcal{C}_{\min},\mathcal{I}](A,B)^{\overline{\alpha_2}}\supseteq E(D(A^{\overline{\alpha_2}},B^{\overline{\alpha_2}}),-B_{1-\alpha_2})$,

(iii) $C^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\overline{\alpha_2}}$

$\subseteq E(D(A^{\overline{\alpha_2}},B^{\overline{\alpha_2}}),-B_{1-\alpha_2})$,

(iv) $O^I[\mathcal{C}_{\min},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\overline{\alpha_2}}$

$= D(E(A^{\overline{\alpha_2}},B_{1-\alpha_2}),-B^{\overline{\alpha_2}})$,

(v) $O^I[\mathcal{C}_{\min},\mathcal{I}](A,B)^{\overline{\alpha_2}}\supseteq D(E(A^{\overline{\alpha_2}},B_{1-\alpha_2}),-B^{\overline{\alpha_2}})$,

(vi) $O^I[\mathcal{C},\mathcal{I}_{\min,\mathcal{N}_s}](A,B)^{\overline{\alpha_2}}$

$\subseteq D(E(A^{\overline{\alpha_2}},B_{1-\alpha_2}),-B^{\overline{\alpha_2}})$.

Proof Follows in an analogous way as in the proof of Proposition 25 from Propositions 1, 2, 19, 20 and the fact that the binary dilation is increasing in its first and second argument and that the binary erosion is increasing in its first argument and decreasing in its second argument. □

The previous result allows us to derive, under the restriction of $\alpha_1 \in [0.5, 1[\cap I_r$, an upper bound for the strict subcut of the interval-valued fuzzy closing and opening in terms of the binary closing and opening.

Proposition 34 *Let $\mathcal{C}$ be a semi-norm on $L^I_{r,s}$ and let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for all $\alpha_1 \in [0.5, 1[\cap I_r$ that:*

(i) $C^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\overline{\alpha_1}} \subseteq C(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}})$,

(ii) $C^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\overline{\alpha_1}} \subseteq C(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$,

(iii) $C^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\overline{\alpha_1}} \subseteq C(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}})$,

(iv) $C^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\overline{\alpha_1}} \subseteq C(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$,

(v) $O^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\overline{\alpha_1}} \subseteq O(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}})$,

(vi) $O^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\overline{\alpha_1}} \subseteq O(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$,

(vii) $O^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\overline{\alpha_1}} \subseteq O(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}})$,

(viii) $O^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)_{\overline{\alpha_1}} \subseteq O(A_{\overline{\alpha_1}}, B^{1-\alpha_1})$,

Proof Follows in an analogous way as in the proof of Proposition 26 from Proposition 33 and Lemma 3 and the fact that the binary dilation is increasing in its first and second argument and that the binary erosion is increasing in its first argument and decreasing in its second argument. □

The result also allows us to derive, under the restriction of $0 \leq \alpha_2 < 0.5$, a lower bound for the strict supercut of the interval-valued fuzzy closing and opening in terms of the binary closing and opening.

Proposition 35 *Let $\mathcal{I}$ be an upper border implicator on $L^I_{r,s}$ and let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for all $\alpha_2 \in [0, 0.5[\cap I_s$ that:*

(i) $C^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}} \supseteq C(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}})$,

(ii) $C^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}} \supseteq C(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$,

(iii) $C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\overline{\alpha_2}} \supseteq C(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}})$,

(iv) $C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\overline{\alpha_2}} \supseteq C(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$,

(v) $O^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}} \supseteq O(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}})$,

(vi) $O^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}} \supseteq O(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$,

(vii) $O^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\overline{\alpha_2}} \supseteq O(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}})$,

(viii) $O^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\overline{\alpha_2}} \supseteq O(A^{\overline{\alpha_2}}, B_{1-\alpha_2})$.

Proof Follows in an analogous way as in the proof of Proposition 26 from Proposition 33 and Lemma 3 and the fact that the binary dilation is increasing in its first and second argument and that the binary erosion is increasing in its first argument and decreasing in its second argument. □

3.4.4 Decomposition by Strict $[\alpha_1, \alpha_2]$-Cuts

For the conjunctor $\mathcal{C}_{\min}$ and the implicator $\mathcal{I}_{\min,\mathcal{N}_s}$, the strict $[\alpha_1, \alpha_2]$-cuts of the discrete interval-valued fuzzy closing and opening can be found as a combination of binary dilations and erosions. For an arbitrary semi-norm $\mathcal{C}$ and an arbitrary upper border implicator $\mathcal{I}$ analogous approximations exist.

Proposition 36 *Let $\mathcal{C}$ be a semi-norm on $L^I_{r,s}$ and $\mathcal{I}$ an upper border implicator on $L^I_{r,s}$ and let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds for all $[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus U_{L^I}$ that:*

(i) $$C^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\overline{\alpha_1}} = E(D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}}), -B^{1-\alpha_1}) \cap E(D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}}), -B_{1-\alpha_2}),$$

(ii) $$C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\overline{\alpha_2}}_{\overline{\alpha_1}} \supseteq E(D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}}), -B^{1-\alpha_1}) \cap E(D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}}), -B_{1-\alpha_2}),$$

(iii) $$C^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\overline{\alpha_1}} \subseteq E(D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}}), -B^{1-\alpha_1}) \cap E(D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}}), -B_{1-\alpha_2}),$$

(iv) $$O^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\overline{\alpha_1}} = D(E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}), -B_{\overline{\alpha_1}}) \cap D(E(A^{\overline{\alpha_2}}, B_{1-\alpha_2}), -B^{\overline{\alpha_2}}),$$

(v) $$O^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\overline{\alpha_2}}_{\overline{\alpha_1}} \supseteq D(E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}), -B_{\overline{\alpha_1}}) \cap D(E(A^{\overline{\alpha_2}}, B_{1-\alpha_2}), -B^{\overline{\alpha_2}}),$$

(vi) $$O^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\overline{\alpha_1}} \subseteq D(E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}), -B_{\overline{\alpha_1}}) \cap D(E(A^{\overline{\alpha_2}}, B_{1-\alpha_2}), -B^{\overline{\alpha_2}}).$$

Proof Follows from Proposition 33. □

3.4.5 Decomposition by Weak-Strict and Strict-Weak $[\alpha_1, \alpha_2]$-Cuts

For the conjunctor $\mathcal{C}_{\min}$ and the implicator $\mathcal{I}_{\min,\mathcal{N}_s}$, the strict $[\alpha_1, \alpha_2]$-cuts of the discrete interval-valued fuzzy closing and opening can be found as a combination of binary dilations and erosions. For an arbitrary semi-norm $\mathcal{C}$ and an arbitrary upper border implicator $\mathcal{I}$ analogous approximations exist.

Proposition 37 *Let $\mathcal{C}$ be a semi-norm on $L^I_{r,s}$ and $\mathcal{I}$ an upper border implicator on $L^I_{r,s}$ and let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$. For all $[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus U_{L^I}$ it holds that*:

(i) $$C^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\alpha_1} = E(D(A_{\alpha_1}, B_{\alpha_1}), -B^{\overline{1-\alpha_1}}) \cap E(D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}}), -B_{1-\alpha_2}),$$

(ii) $$C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\overline{\alpha_2}}_{\alpha_1} \supseteq E(D(A_{\alpha_1}, B_{\alpha_1}), -B^{\overline{1-\alpha_1}}) \cap E(D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}}), -B_{1-\alpha_2}),$$

(iii) $$C^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\alpha_1} \subseteq E(D(A_{\alpha_1}, B_{\alpha_1}), -B^{\overline{1-\alpha_1}}) \cap E(D(A^{\overline{\alpha_2}}, B^{\overline{\alpha_2}}), -B_{1-\alpha_2}),$$

(iv) $$O^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\alpha_1} = D(E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}), -B_{\alpha_1}) \cap D(E(A^{\overline{\alpha_2}}, B_{1-\alpha_2}), -B^{\overline{\alpha_2}}),$$

(v) $$O^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\overline{\alpha_2}}_{\alpha_1} \supseteq D(E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}), -B_{\alpha_1}) \cap D(E(A^{\overline{\alpha_2}}, B_{1-\alpha_2}), -B^{\overline{\alpha_2}}),$$

(vi) $$O^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\overline{\alpha_2}}_{\alpha_1} \subseteq D(E(A_{\alpha_1}, B^{\overline{1-\alpha_1}}), -B_{\alpha_1}) \cap D(E(A^{\overline{\alpha_2}}, B_{1-\alpha_2}), -B^{\overline{\alpha_2}}).$$

For all $[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus 1_{L^I}$ it holds that:

(i) $$C^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\alpha_2}_{\overline{\alpha_1}} = E(D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}}), -B^{1-\alpha_1}) \cap E(D(A^{\alpha_2}, B^{\alpha_2}), -B_{\overline{1-\alpha_2}}),$$

(ii) $$C^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\alpha_2}_{\overline{\alpha_1}} \supseteq E(D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}}), -B^{1-\alpha_1}) \cap E(D(A^{\alpha_2}, B^{\alpha_2}), -B_{\overline{1-\alpha_2}}),$$

(iii) $$C^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\alpha_2}_{\overline{\alpha_1}} \subseteq E(D(A_{\overline{\alpha_1}}, B_{\overline{\alpha_1}}), -B^{1-\alpha_1}) \cap E(D(A^{\alpha_2}, B^{\alpha_2}), -B_{\overline{1-\alpha_2}}),$$

(iv) $$O^I[\mathcal{C}_{\min}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\alpha_2}_{\overline{\alpha_1}} = D(E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}), -B_{\overline{\alpha_1}}) \cap D(E(A^{\alpha_2}, B_{\overline{1-\alpha_2}}), -B^{\alpha_2}),$$

(v) $$O^I[\mathcal{C}_{\min}, \mathcal{I}](A, B)^{\alpha_2}_{\overline{\alpha_1}} \supseteq D(E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}), -B_{\overline{\alpha_1}}) \cap D(E(A^{\alpha_2}, B_{\overline{1-\alpha_2}}), -B^{\alpha_2}),$$

(vi) $$O^I[\mathcal{C}, \mathcal{I}_{\min,\mathcal{N}_s}](A, B)^{\alpha_2}_{\overline{\alpha_1}} \subseteq D(E(A_{\overline{\alpha_1}}, B^{1-\alpha_1}), -B_{\overline{\alpha_1}}) \cap D(E(A^{\alpha_2}, B_{\overline{1-\alpha_2}}), -B^{\alpha_2}).$$

Proof Follows from Propositions 27 and 33. □

4 Discussion

The conversion of the $[\alpha_1, \alpha_2]$-cut of an interval-valued fuzzy morphological operator into binary operations on the $[\alpha_1, \alpha_2]$-cuts of the image and structuring element may result in a reduction of the time needed to compute such $[\alpha_1, \alpha_2]$-cut. For example, in the calculation of the binary dilation of a binary image A by a binary structuring element B, an element $y \in \mathbb{R}^n$ can be considered to belong to this dilation as soon as one element in $T_y(-B)$ also belongs to A. The other elements in $T_y(-B)$ don't need to be checked anymore. For the calculation of the interval-valued fuzzy dilation, all elements in $T_y(-d_B)$ need to be considered to find the supremum over those elements. Additionally, the binary dilation (respectively erosion) of an image can be further sped up by a decomposition of the structuring element [24, 25], which is especially useful for image processing systems. An analogous reasoning holds for the erosion.

As was shown in the previous sections, we only had equalities for the conjunctor $\mathcal{C}_{\min}$ and the implicator $\mathcal{I}_{\min,\mathcal{N}_s}$. For arbitrary semi-norms and upper border implicators only approximations that are not necessarily equalities could be found. As an example, we will illustrate the approximation in Proposition 10 on the camera image. In Fig. 2, three different takes of this scene are given: a cloudy, a sunny and a slightly shifted take. Due to different recording circumstances and a shift in position of the objects in the image, there is uncertainty concerning the grey values in the image. To take this uncertainty into account, an image pixel is not mapped onto one specific grey value, but onto an interval of grey values to which its grey value is expected to belong. The lower bound (respectively the upper bound) of such interval is chosen as the lowest (respectively the highest) grey level over the three takes. These lower bound and upper

Fig. 2 Three different takes on the camera image: cloudy (*upper*), sunny (*middle*) and shifted (*lower*)

Fig. 3 Lower bound image (*upper*), upper bound image (*middle*) and difference image (*lower*) of the interval-valued camera image

bound image are given in Fig. 3 together with a representation of the difference between the two. The larger this difference (more white in the difference image), the wider the corresponding interval and the larger the uncertainty at the considered pixel position. Consider e.g. the interval-valued structuring element

$$B = \begin{bmatrix} [0.6, 0.8] & [0.7, 0.9] & [0.6, 0.8] \\ [0.7, 0.9] & \underline{[1, 1]} & [0.7, 0.9] \\ [0.6, 0.8] & [0.7, 0.9] & [0.6, 0.8] \end{bmatrix}, \tag{18}$$

where the underlined element corresponds to the origin. For this structuring element, we are certain that the central pixel should get the weight 1. On the other hand, the importance of e.g. the pixel above the central pixel is thought to lie somewhere between 0.7 and 0.9, but there exists some uncertainty.

The lower bound image, the upper bound image and the difference image of the interval-valued fuzzy dilation (based on the conjunctor $\mathcal{C}(x, y) = [\max(0, x_1 + y_1 - 1), \max(0, x_2 + y_2 - 1)], \forall x, y \in \mathbb{Z}^n$) of the camera image by the above structuring element are then given in Fig. 4. The weak $[0.4, 0.6]$-cut of this dilation and the binary approximation determined in Proposition 10(ii) are finally given in Fig. 5. We see that we get a rather rough approximation.

Fig. 4 Lower bound image (*upper*), upper bound image (*middle*) and difference image (*lower*) of the dilated interval-valued camera image

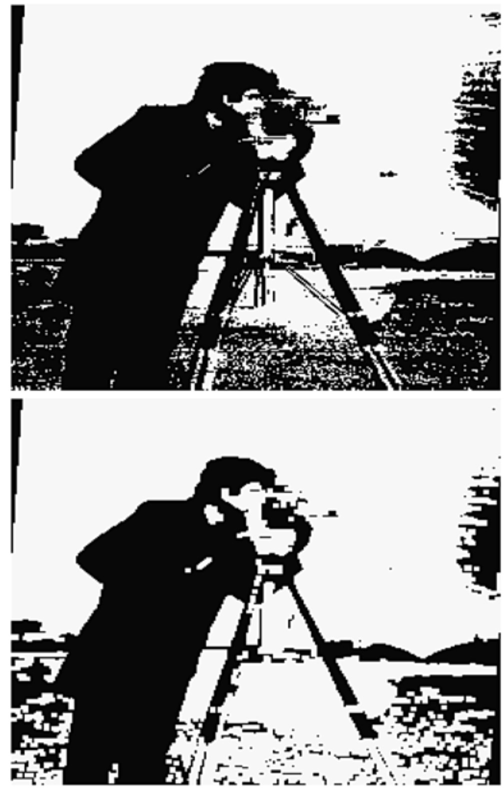

Fig. 5 Weak [0.4, 0.6]-cut of the dilated interval-valued camera image (*upper*) and binary approximation (*lower*)

5 Conclusion

In this paper we have revealed the relationships between the different $[\alpha_1, \alpha_2]$-cuts of the interval-valued fuzzy morphological operators and the corresponding binary operators. We investigated both the general continuous case and the discrete case, which is the practical case. Indeed, in practice, we deal with a sampled image domain and a sampled range of grey values, resulting in interval-valued fuzzy sets from $\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$. In the discrete case, the $[\alpha_1, \alpha_2]$-cuts of the interval-valued fuzzy dilation based on the conjunctor $\mathcal{C}_{\min}$, the erosion based on the implicator $\mathcal{I}_{\min,\mathcal{N}_s}$, and the opening and closing based on those two can always be written in terms of binary operators. For other semi-norms and upper border implicators, we found an approximation in terms of binary operators. Such conversion into binary operators provides us a link between interval-valued fuzzy and binary morphology and may be useful to reduce the computation time.

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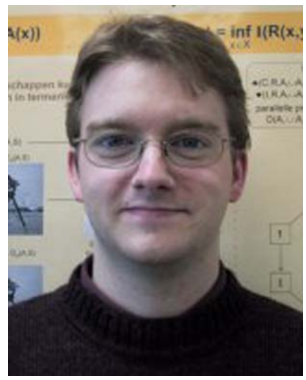

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