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Classification of Fuzzy Mathematical Morphologies Based on Concepts of Inclusion Measure and Duality

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Abstract Mathematical morphology was originally conceived as a set theoretic approach for the processing of binary images. Extensions of classical binary morphology to gray-scale morphology include approaches based on fuzzy set theory. This paper discusses and compares several well-known and new approaches towards gray-scale and fuzzy mathematical morphology. We show in particular that a certain approach to fuzzy mathematical morphology ultimately depends on the choice of a fuzzy inclusion measure and on a notion of duality. This fact gives rise to a clearly defined scheme for classifying fuzzy mathematical morphologies. The umbra and the level set approach, an extension of the threshold approach to gray-scale mathematical morphology, can also be embedded in this scheme since they can be identified with certain fuzzy approaches.

Keywords Morphological image operator · Complete lattice · Fuzzy set theory · Fuzzy mathematical morphology · Fuzzy inclusion measure · Duality · Negation · Adjunction

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1 Introduction

In the early 1960s, Matheron and Serra invented *mathematical morphology* (MM) as the part of binary image processing that is concerned with image filtering and geometric analysis by means of structuring elements [24, 32]. Relying heavily on the early work of Minkowski and Hadwiger on geometric measure theory and integral geometry [17, 25], Matheron and Serra succeeded in developing a collection of tools, called *morphological operators*, that proved to be extremely useful for the analysis of shape and structure in binary images. Traditionally, binary images are represented as subsets of $\mathbb{R}^n$ while gray-scale images can be represented as functions $\mathbb{R}^n \to [0, 1]$. Therefore, fuzzy set theory appears to be a logical choice for extensions of binary MM to gray-scale images and several researchers view fuzzy mathematical morphology as yet another approach to gray-scale mathematical morphology [15, 26] although—strictly speaking—there is a semantic difference between these theories. In contrast to gray-scale MM, fuzzy MM deals with different types of imprecision or uncertainty, for instance regarding the boundary of objects [8].

The most general mathematical framework in which mathematical morphology can be conducted is given by complete lattices [4, 18, 33]. Using this framework, Ronse formulates necessary and sufficient conditions for dilation and erosion [31]. These conditions reveal that every dilation, erosion respectively, is associated with a structuring function. Moreover, the complete lattice approach exposes and resolves certain inconsistencies that are an intrinsic part of the umbra approach in the continuous case.

The two basic morphological operators are *erosion* and *dilation* [18]. Other morphological operators can be derived from the basic ones. Erosion marks structuring element origin locations at which a structuring element fits within an image [35]. This concept can be expressed in terms of *set inclusion* or *subsethood*. Depending on the particular choice of set inclusion, we obtain different notions of *fuzzy erosion*. Examples include the approaches of De Baets [1], Bloch and Maître [6], Kaufmann and Gupta [19], Zadeh [40], Sinha and Dougherty [34–36], Kitainik [20] as well as Bandler and Kohout [2]. These approaches are compared in the excellent paper of Nachtegael and Kerre on connections between binary, gray-scale, and fuzzy mathematical morphologies [26]. This paper goes beyond the comparison of Nachtegael and Kerre by providing simple criteria for classifying approaches—including the ones mentioned above—to fuzzy and gray-scale MM. In addition, the insights gained in this paper can serve as a mathematical basis for developing new approaches to fuzzy MM.

Fuzzy dilation is usually defined as the dual of fuzzy erosion. The notion of duality that is used varies among the researchers of fuzzy MM. Many researchers—including Bloch and Maître [6], Sinha and Dougherty [36], as well as Nachtegael and Kerre [26]—introduce a duality relation based on some concept of negation. Other researchers such as Deng and Heijmans [15], Ronse [31], and Maragos [23] advocate a duality relation based on the notion of adjunction. Recently, Bloch showed that adjunction-based and negation-based approaches are generally not equivalent [7].

This paper demonstrates that both groups of approaches that we mentioned above can be embedded into our classification scheme. In contrast, the earlier survey papers of both Bloch and Maître [6] and of Nachtegael and Kerre [26] do not contain yet the adjunction-based approaches to fuzzy mathematical morphology. The survey of Bloch and Maître already appeared in 1995 and its focus was much more on comparing the main approaches to FMM, that had been published at the time, in terms of their properties than on providing a classification of these approaches. Unlike Bloch, Maître, Nachtegael, and Kerre, we also emphasize the complete lattice framework of mathematical morphology which allows for the definition of erosion, dilation, negation, and adjunction, i.e., the basic concepts that represent the foundations of our classification scheme.

The paper is organized as follows. First, we review some basic concepts of MM. Section 3 discusses some important approaches towards gray-scale MM including the level set approach which generalizes Serra's threshold approach [18, 32]. After providing some background information on fuzzy MM in Sect. 4, we proceed by classifying approaches towards fuzzy MM in terms of the fuzzy inclusion measures and relationships of duality. The paper finishes with a classification scheme for fuzzy mathematical morphologies that summarizes the main results and observations presented in this paper. Appendix summarizes the mathematical notations used in the paper.

2 A Brief Review of Mathematical Morphology

2.1 Basic Concepts on Lattice Theory

The mathematical foundations of morphology can be found in lattice theory which is concerned with algebraic structures that arise by imposing some type of ordering on a set [4, 31].

A partially ordered set X is called a *lattice* if and only if every finite, non-empty subset of X has an infimum and a supremum in X. The infimum of Y is also denoted by $\bigwedge_{j\in J} y_j$ instead of $\bigwedge Y$ if $Y = \{y_j : j \in J\}$ for some index set J. Similar notations are used to denote the supremum of Y.

Suppose that X and Y are lattices. A function $f : X \to Y$ that satisfies the following equations for all $x \in X$ and for all $y \in Y$ is called *lattice homomorphism*.

$$f(x \vee y) = f(x) \vee f(y) \quad \text{and} \quad f(x \wedge y) = f(x) \wedge f(y). \tag{1}$$

A bijective lattice homomorphism is called *lattice isomorphism* and it is called an *automorphism* when $X = Y$. An injective lattice homomorphism is called *lattice endomorphism*. An involutive bijection $\nu : X \to X$ which reverses the partial ordering is said to be a *negation* on X. Recall that a function $f : X \to X$ is called *involutive* if $f(f(x)) = x$ for all $x \in X$. We say that a function $f : X \to Y$ is *increasing* (*decreasing*) if $x \leq y$ implies $f(x) \leq f(y)$ ($f(x) \geq f(y)$) for all $x, y \in X$. We refer to f as an *extensive* (*anti-extensive*) function if $f(x) \geq x$ ($f(x) \leq x$) for all $x \in X$.

We speak of a *complete lattice* X if every non-empty (finite or infinite) subset has an infimum and a supremum in X. From now on, we will denote a complete lattice by $\mathbb{L}$.

2.2 The Complete Lattice Framework for Mathematical Morphology

In the general complete lattice setting, an operator $\varepsilon : \mathbb{L} \to \mathbb{L}$ that commutes with the infimum operation is called an *erosion* [18]. In other words, the operator ε represents an erosion if the following equality holds for every subset $Y \subseteq \mathbb{L}$:

$$\varepsilon\left(\bigwedge Y\right) = \bigwedge_{y\in Y} \varepsilon(y). \tag{2}$$

Similarly, an operator $\delta : \mathbb{L} \to \mathbb{L}$ that commutes with the supremum operation is called a *dilation*. In other words, the operator δ represents a dilation if the following equality holds for every subset $Y \subseteq \mathbb{L}$:

$$\delta\left(\bigvee Y\right) = \bigvee_{y\in Y} \delta(y). \tag{3}$$

Note that both erosion and dilation are increasing lattice operators.

An operator α is an *opening* if it is increasing, idempotent, and anti-extensive [18]. Dually, an operator β is called a *closing* if it is increasing, idempotent, and extensive.

The operators of erosion and dilation are often linked in terms of a relationship of duality. Some authors such as Maragos [23], Deng and Heijmans [15] advocate the relationship of adjunction since—among other advantages—the compositions of dilations and erosions yield openings and closings if the pairing between erosion and dilation forms an adjunction [18].

Consider two arbitrary operators $\varepsilon, \delta : \mathbb{L} \to \mathbb{L}$. We say that (ε, δ) is an *adjunction* on $(\mathbb{L}, \leq)$ if we have

$$\delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y), \quad \forall x,\ y \in \mathbb{L}. \tag{4}$$

The following proposition shows that adjunction yields a duality between erosions and dilations [18, 33].

Proposition 1 *Let $\mathbb{L}$ be a complete lattice and consider two operations $\delta, \varepsilon : \mathbb{L} \to \mathbb{L}$.*

1. *If (ε, δ) is an adjunction, then δ is a dilation and ε is an erosion.*
2. *For any dilation δ, there is a unique erosion ε such that (ε, δ) is an adjunction. The adjoint erosion is given by*

$$\varepsilon(y) = \bigvee \{x \in \mathbb{L} : \delta(x) \leq y\}, \tag{5}$$

for every $y \in \mathbb{L}$.

3. *For any erosion ε, there is a unique dilation δ such that (ε, δ) is an adjunction. The adjoint dilation is given by*

$$\delta(x) = \bigwedge \{y \in \mathbb{L} : \varepsilon(y) \geq x\}, \tag{6}$$

for every $x \in \mathbb{L}$.

A second type of duality is based on *negation*. Let Ψ be an operator mapping a complete lattice $\mathbb{L}$ into itself and let ν be a negation on $\mathbb{L}$. The operator Ψ^ν given by

$$\Psi^\nu(x) = \nu(\Psi(\nu(x))), \forall x \in \mathbb{L}, \tag{7}$$

is called the *negation* or the *dual* of Ψ with respect to ν (note that we have $(\Psi^\nu)^\nu = \Psi$). The negation of an erosion is a dilation, and vice-versa [18]. Similarly, the negation of an opening is a closing and vice-versa.

The preceding observations clarify that there is a unique erosion that can be associated with a certain dilation and vice-versa in terms of either negation or adjunction. The following proposition establishes a relationship between the notions of negation and adjunction [18].

Proposition 2 *Let $\mathbb{L}$ be a complete lattice. For a given negation ν and an adjunction (ε, δ) on $\mathbb{L}$, the pair $(\delta^\nu, \varepsilon^\nu)$ forms an adjunction on $\mathbb{L}$.*

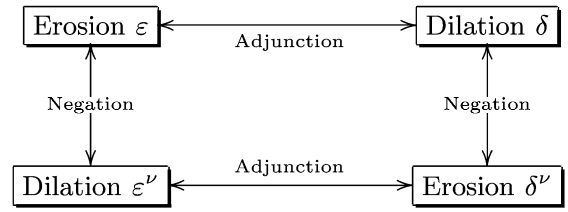

Fig. 1 Scheme to obtain a dilation from an erosion and vice-versa

Propositions 1 and 2 lead to the commutative diagram that is depicted in Fig. 1.

Apart from erosions and dilations, there are two other types of elementary morphological operators, namely anti-erosions and anti-dilations [3, 18]. An anti-erosion arises as the composition of an erosion followed by a negation and an anti-dilation arises as the composition of a dilation followed by a negation [39]. Banon and Barrera have proven that every mapping $\Psi : \mathbb{L} \to \mathbb{L}$ can be expressed as a supremum of infimums of erosions and anti-dilations or, alternatively, as an infimum of supremums of dilations and anti-erosions [3].

2.3 Binary Mathematical Morphology

Mathematical morphology was initially developed for the analysis of binary images. We identify a binary image $\mathbf{A}$ with a subset of $\mathbf{X}$, where the symbol $\mathbf{X}$ denotes the Euclidean space $\mathbb{R}^d$ or the digital space $\mathbb{Z}^d$ throughout the paper. The power set of $\mathbf{X}$ is partially ordered in terms of the operation "$\subseteq$" and forms a complete lattice. Thus, the complete lattice framework that we introduced above can be applied to binary MM.

The basic operations of binary MM are the erosion $E_\mathcal{B}$ and the dilation $D_\mathcal{B}$ that are defined below. These operations are associated with a subset $\mathbf{S}$ of $\mathbf{X}$ that is called *structuring element* (SE). We will point out below that $E_\mathcal{B}$ and $D_\mathcal{B}$ represent operations of erosion and dilation in the sense of Eqs. 2 and 3 if we fix the SE $\mathbf{S}$.

Let $\mathbf{A} \subseteq \mathbf{X}$ be a binary image and let $\mathbf{S} \subseteq \mathbf{X}$ be a binary SE. The *binary erosion* $E_\mathcal{B}(\mathbf{A}, \mathbf{S})$ and the *binary dilation* $D_\mathcal{B}(\mathbf{A}, \mathbf{S})$ of the image $\mathbf{A}$ by the SE $\mathbf{S}$ are defined in terms of translations of sets. For example, the *translation* $\mathbf{S_x}$ of $\mathbf{S}$ by $\mathbf{x} \in \mathbf{X}$ is given by $\mathbf{S_x} = \{\mathbf{s} + \mathbf{x} : \mathbf{s} \in S\}$. We have

$$E_\mathcal{B}(\mathbf{A}, \mathbf{S}) = \{\mathbf{x} \in \mathbf{X} : \mathbf{S_x} \subseteq \mathbf{A}\} = \bigcap_{\mathbf{s} \in \bar{\mathbf{S}}} \mathbf{A_s}, \tag{8}$$

$$D_\mathcal{B}(\mathbf{A}, \mathbf{S}) = \{\mathbf{x} \in \mathbf{X} : \bar{\mathbf{S}}_\mathbf{x} \cap \mathbf{A} \neq \emptyset\} = \bigcup_{\mathbf{s} \in \mathbf{S}} \mathbf{A_s} = \bigcup_{\mathbf{a} \in \mathbf{A}} S_\mathbf{a}. \tag{9}$$

Here, the symbol $\bar{\mathbf{S}}$ denotes the *reflection* of $\mathbf{S}$ around the origin. Formally, we have $\bar{\mathbf{S}} = \{-\mathbf{s} \in \mathbf{X} : \mathbf{s} \in \mathbf{S}\}$. Thus, $\bar{\mathbf{S}}_\mathbf{x}$ denotes the translation of $\bar{\mathbf{S}}$ by $\mathbf{x}$.

Note that the definition of erosion in Eq. 8 corresponds to Minkowski subtraction and that the definition of dilation in Eq. 9 corresponds to Minkowski addition. Serra slightly diverges from these definitions which are due to Sternberg by defining the dilation of $\mathbf{A}$ by the SE $\mathbf{S}$ as $\{\mathbf{x} \in \mathbf{X} : \mathbf{S_x} \cap \mathbf{A} \neq \emptyset\} = D_{\mathcal{B}}(\mathbf{A}, \bar{\mathbf{S}})$ [32, 38].

We refer to $E_{\mathcal{B}}$ as an erosion and we refer to $D_{\mathcal{B}}$ as a dilation because $E_{\mathcal{B}}(\cdot, \mathbf{S})$ satisfies Eq. 2 and $D_{\mathcal{B}}(\cdot, \mathbf{S})$ satisfies Eq. 3 for a fixed SE $\mathbf{S}$. The pair consisting of $E_{\mathcal{B}}(\cdot, \mathbf{S})$ and $D_{\mathcal{B}}(\cdot, \mathbf{S})$ forms an adjunction satisfying Eq. 4. Also note that complementation represents a negation on $\mathcal{P}(\mathbf{X})$ and that $D_{\mathcal{B}}(\cdot, \mathbf{S})$ is the dual of $E_{\mathcal{B}}(\cdot, \bar{\mathbf{S}})$ with respect to complementation.

3 Gray-Scale Mathematical Morphology

The tools of binary MM are limited and cannot be applied to gray-scale images. Serra and Sternberg have developed successful approaches to extend binary to gray-scale MM in the 1980s [32, 38]. We will refer to these approaches as the umbra and the threshold (or flat) approach. This paper also includes another approach called the level set approach that can be viewed as an extension of the threshold approach.

In gray-scale morphology, we apply the concepts of lattice theory to images, in other words, functions from some point set to a set of gray-levels that forms a complete lattice. In this paper, we restrict our attention to images $\mathbf{X} \to \mathbb{G}$ where the symbol $\mathbb{G}$ stands either for the set of extended real numbers $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ or for the set of extended integers $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$.

The symbol $\mathbb{G}^{\mathbf{X}}$ denotes the set of images $\mathbf{X} \to \mathbb{G}$. For an image $\mathbf{a} \in \mathbb{G}^{\mathbf{X}}$, the *reflection* $\bar{\mathbf{a}}$ of $\mathbf{a}$ around the origin and the *translation* $\mathbf{a_y}$ of $\mathbf{a}$ by $\mathbf{y} \in \mathbf{X}$ are defined as follows:

$$\bar{\mathbf{a}}(\mathbf{x}) = \mathbf{a}(-\mathbf{x}) \quad \text{and} \quad \mathbf{a_y}(\mathbf{x}) = \mathbf{a}(\mathbf{x} - \mathbf{y}), \quad \forall\, \mathbf{x} \in \mathbf{X}. \tag{10}$$

A negation on $\mathbb{G}^{\mathbf{X}}$ is given by the following operator $* : \mathbb{G}^{\mathbf{X}} \to \mathbb{G}^{\mathbf{X}}$:

$$\mathbf{a}^*(\mathbf{x}) = -\mathbf{a}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{G}. \tag{11}$$

The set $\mathbb{G}^{\mathbf{X}}$ inherits the complete lattice structure from $\mathbb{G}$ if $\mathbf{a} \leq \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{G}^{\mathbf{X}}$ is defined as follows:

$$\mathbf{a} \leq \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a}(\mathbf{x}) \leq \mathbf{b}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbf{X}. \tag{12}$$

The basic operators of gray-scale MM are erosions and dilations on the complete lattice $\mathbb{G}^{\mathbf{X}}$. These operations consist of probing a given image with a SE, i.e. another image, in order to extract some relevant information on the shape and form of objects.

In this context, we speak of an *erosion* as an operator $E : \mathbb{G}^{\mathbf{X}} \times \mathbb{G}^{\mathbf{X}} \to \mathbb{G}^{\mathbf{X}}$ that commutes with the infimum in the first argument. For an image $\mathbf{a} \in \mathbb{G}^{\mathbf{X}}$ and a SE $\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$, $E(\mathbf{a}, \mathbf{s})$ is said to be the erosion of the image $\mathbf{a}$ by the SE $\mathbf{s}$. Similarly, we speak of a *dilation* as an operator $D : \mathbb{G}^{\mathbf{X}} \times \mathbb{G}^{\mathbf{X}} \to \mathbb{G}^{\mathbf{X}}$ that commutes with the supremum in the first argument and $D(\mathbf{a}, \mathbf{s})$ is said to be the dilation of the image $\mathbf{a}$ by the SE $\mathbf{s}$.

For simplicity, we say that (E, D) forms an *adjunction* if and only $(E(\cdot, \mathbf{s}), D(\cdot, \mathbf{s}))$ forms an adjunction for every SE $\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$. Negation represents another duality relationship that plays an important role in MM. Note that the negation ν on $\mathbb{G}$ induces a negation $\boldsymbol{\nu}$ on $\mathbb{G}^{\mathbf{X}}$ that arises by applying ν pointwise. We say that D is *the dual* of E *with respect to the negation* ν, in symbols $D = E^{\nu}$ if and only if we have that $D(\cdot, \mathbf{s})$ is the negation of $E(\cdot, \bar{\mathbf{s}})$ with respect to $\boldsymbol{\nu}$ for all $\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$. Note that D is *the dual* of E with respect to ν if and only if E is *the dual* of D with respect to ν, that is $E(\cdot, \mathbf{s})$ is the negation of $D(\cdot, \bar{\mathbf{s}})$ for all $\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$. In this case, we write $E = D^{\nu}$.

Now, we are ready to discuss some particular approaches towards gray-scale MM.

3.1 The Threshold or Flat Approach

Although the threshold approach dates back to Serra's work [32], we will permit ourselves to slightly adapt Serra's original definitions in order to streamline them with the other definitions of dilation and erosion in this paper. In other words, we choose to follow Sternberg's line of reasoning as far as the definitions of dilation and erosion are concerned.

An image $\mathbf{a} \in \mathbb{G}^{\mathbf{X}}$ can be decomposed into its *threshold or level sets* $\mathcal{S}_t(\mathbf{a})$ given by [18]:

$$\mathcal{S}_t(\mathbf{a}) = \{\mathbf{x} \in \mathbf{X} : \mathbf{a}(\mathbf{x}) \geq t\}. \tag{13}$$

In other words, $S_t(\mathbf{a}) \subseteq \mathbf{X}$ is the set of all points $\mathbf{x} \in \mathbf{X}$ at which the image $\mathbf{a}$ exceeds the threshold t. Note that $S_t(\mathbf{a}) \subseteq S_r(\mathbf{a})$ for every $r, t \in \mathbb{G}$ such that $r \leq t$.

The level sets of an image $\mathbf{a}$ can be combined to yield the following mapping $\mathcal{S} : \mathbb{G}^{\mathbf{X}} \to \hat{\mathcal{P}}(\mathbf{X})^{\mathbb{G}}$, where $\hat{\mathcal{P}}(\mathbf{X})^{\mathbb{G}}$ is the set of decreasing functions from $\mathbb{G}$ into $\mathcal{P}(\mathbf{X})$.

$$\mathcal{S}(\mathbf{a}) = (\mathcal{S}_t(\mathbf{a}))_{t \in \mathbb{G}}. \tag{14}$$

The mapping $\mathcal{S}$ constitutes a lattice isomorphism between $\mathbb{G}^{\mathbf{X}}$ and $\hat{\mathcal{P}}(\mathbf{X})^{\mathbb{G}}$. The inverse of this isomorphism is defined as follows. Let $\mathbf{p}$ be a decreasing function from $\mathbb{G}$ into $\mathcal{P}(\mathbf{X})$. If $\mathbf{a} = \mathcal{S}^{-1}(\mathbf{p})$ then $\mathbf{a}$ is given by

$$\mathbf{a}(\mathbf{x}) = \bigvee \{t \in \mathbb{G} : \mathbf{x} \in \mathbf{p}(t)\}. \tag{15}$$

This result yields a systematic approach to building operators on the lattice $\mathbb{G}^{\mathbf{X}}$ from operators on $\mathcal{P}(\mathbf{X})$.

Let $\mathbf{a} \in \mathbb{G}^{\mathbf{X}}$ be a gray-scale image and let $\mathbf{S} \subseteq \mathbf{X}$ be a SE. The increasing property on the first argument of the binary erosion implies that $\mathbf{p}_E(t) = E_{\mathcal{B}}(S_t(\mathbf{a}), \mathbf{S})$ is a decreasing operator from $\mathbb{G}$ into $\mathcal{P}(\mathbf{X})$. Similarly, the function $\mathbf{p}_D(t) =$

$D_{\mathcal{B}}(S_t(\mathbf{a}), \mathbf{S})$ is a decreasing function from $\mathbb{G}$ into $\mathcal{P}(\mathbf{X})$. The $\mathcal{T}$*-erosion* $E_{\mathcal{T}}(\mathbf{a}, \mathbf{S})$ and the $\mathcal{T}$*-dilation* $D_{\mathcal{T}}(\mathbf{a}, \mathbf{S})$ are defined as follows [18]:

$$E_{\mathcal{T}}(\mathbf{a}, \mathbf{S}) = \mathcal{S}^{-1}(\mathbf{p}_E) \quad \text{and}$$
$$D_{\mathcal{T}}(\mathbf{a}, \mathbf{S}) = \mathcal{S}^{-1}(\mathbf{p}_D). \tag{16}$$

The following proposition provides a characterization of the $\mathcal{T}$-erosion and the $\mathcal{T}$-dilation.

Proposition 3 *Let* $\mathbf{a} \in \mathbb{G}^{\mathbf{X}}$ *be a gray-scale image and let* $\mathbf{S} \subseteq \mathbf{X}$ *be a SE. The* $\mathcal{T}$*-erosion and the* $\mathcal{T}$*-dilation are given by*

$$E_{\mathcal{T}}(\mathbf{a}, \mathbf{S})(\mathbf{x}) = \bigwedge_{\mathbf{y} \in \mathbf{S}_{\mathbf{x}}} \mathbf{a}(\mathbf{y}) \quad \textit{and}$$
$$D_{\mathcal{T}}(\mathbf{a}, \mathbf{S})(\mathbf{x}) = \bigvee_{\mathbf{y} \in \bar{\mathbf{S}}_{\mathbf{x}}} \mathbf{a}(\mathbf{y}), \tag{17}$$

for every $\mathbf{x} \in \mathbf{X}$.

The following theorem concerns the duality relationships between the $\mathcal{T}$-erosion and the $\mathcal{T}$-dilation.

Theorem 1 *The erosion* $E_{\mathcal{T}}$ *and the dilation* $D_{\mathcal{T}}$ *are dual operators with respect to adjunction and with respect to the negation* $*$ *given in Eq.* 11.

We provide the proof of this theorem in the next subsections.

3.2 The Level Set Approach

Allowing for SEs in $\mathbb{G}^{\mathbf{X}}$, the level set approach can be viewed as an extension of the threshold approach. Given a gray-scale image $\mathbf{a} \in \mathbb{G}^{\mathbf{X}}$ and a SE $\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$, the underlying idea of the level set approach is decompose the image $\mathbf{a}$ and the SE $\mathbf{s}$ into its level sets and apply the binary morphological operators to $S_t(\mathbf{a})$ and $S_t(\mathbf{s})$ for every $t \in \mathbb{G}$. The $\mathcal{L}$-erosion $E_{\mathcal{L}}(\mathbf{a}, \mathbf{s})$ and $\mathcal{L}$-dilation $D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})$ are obtained by means of the inverse of the isomorphism $\mathcal{S}$. The following formalizes this idea.

Given a family of set operators $\{\psi_t : t \in \mathbb{G}\}$, the operator $\Psi : \mathbb{G}^{\mathbf{X}} \to \mathcal{P}(\mathbf{X})^{\mathbb{G}}$ is defined as follows:

$$\Psi(\mathbf{a})(\mathbf{x}) = \bigvee \{t \in \mathbb{G} : \mathbf{x} \in \psi_t(\mathcal{S}_t(\mathbf{a}))\}. \tag{18}$$

The operator Ψ is called a *semi-flat operator* generated by the family $\{\psi_t : t \in \mathbb{G}\}$. If this family contains only one operator ψ, then Ψ is called a *flat operator* generated by ψ [18].

Suppose that the family $\{\psi_t : t \in \mathbb{G}\}$ in Eq. 18 is a decreasing family of increasing operators on $\mathcal{P}(\mathbf{X})$. For this case, Heijmans showed that if every ψ_t is a dilation, then Ψ is a dilation as well [18]. Similarly, if every ψ_t is an erosion, then Ψ is also an erosion.

Note that, in particular, every operator $\psi_t : \mathcal{P}(\mathbf{X}) \to \mathcal{P}(\mathbf{X})$ given by $\psi_t(A) = D_{\mathcal{B}}(A, \mathcal{S}_t(\mathbf{s}))$ is increasing for an arbitrary, fixed function $\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$. Moreover, we recognize that the family $\{\psi_t = D_{\mathcal{B}}(\cdot, \mathcal{S}_t(\mathbf{s})) : t \in \mathbb{G}\}$ is decreasing. Therefore, the following definitions derived from Eq. 18 yield a dilation (Eq. 19) and an erosion (Eq. 20) for fixed $\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$:

$$D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})(\mathbf{x}) = \bigvee \{t \in \mathbb{G} : \mathbf{x} \in D_{\mathcal{B}}(\mathcal{S}_t(\mathbf{a}), \mathcal{S}_t(\mathbf{s}))\}, \tag{19}$$

$$E_{\mathcal{L}}(\mathbf{a}, \mathbf{s})(\mathbf{x}) = \bigvee \{t \in \mathbb{G} : \mathbf{x} \in E_{\mathcal{B}}(\mathcal{S}_t(\mathbf{a}), \mathcal{S}_t(\mathbf{s}))\}. \tag{20}$$

Note that the operators $D_{\mathcal{L}}(\cdot, \mathbf{s})$ and $E_{\mathcal{L}}(\cdot, \mathbf{s})$ represent semi-flat operators on $\mathcal{P}(\mathbf{X})$ in the sense of Heijmans [18].

The $\mathcal{T}$-dilation $D_{\mathcal{T}}(\mathbf{a}, \mathbf{S})$ can be expressed as an $\mathcal{L}$-dilation as follows. Given a binary SE $\mathbf{S} \subseteq \mathbf{X}$, let $\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$ be defined as follows:

$$\mathbf{s}(\mathbf{x}) = \begin{cases} \infty & \text{if } \mathbf{x} \in \mathbf{S}, \\ -\infty & \text{if } \mathbf{x} \notin \mathbf{S}. \end{cases} \tag{21}$$

Consequently,

$$\begin{aligned} D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})(\mathbf{x}) &= \bigvee \{t \in \mathbb{G} : \mathbf{x} \in D_{\mathcal{B}}(\mathcal{S}_t(\mathbf{a}), \mathcal{S}_t(\mathbf{s}))\} \\ &= \bigvee \{t \in \mathbb{G} : \mathbf{x} \in D_{\mathcal{B}}(\mathcal{S}_t(\mathbf{a}), \mathbf{S})\}. \end{aligned} \tag{22}$$

From these equations, we also infer that the operator Ψ given by $\Psi(\mathbf{a}) = D_{\mathcal{T}}(\mathbf{a}, \mathbf{S})$ is flat in Heijman's sense. Similar arguments show that $E_{\mathcal{T}}(\mathbf{a}, \mathbf{S}) = E_{\mathcal{L}}(\mathbf{a}, \mathbf{s})$. Moreover, we have a flat operator Ψ given by $\Psi(\mathbf{a}) = E_{\mathcal{T}}(\mathbf{a}, \mathbf{S})$.

The $\mathcal{L}$-dilation and the $\mathcal{L}$-erosion form an adjunction since for every gray-scale images $\mathbf{a}, \mathbf{b} \in \mathbb{G}^{\mathbf{X}}$ and every SE $\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$, we have

$$\begin{aligned} D_{\mathcal{L}}(\mathbf{a}, \mathbf{s}) \leq \mathbf{b} \quad &\Leftrightarrow \quad \mathcal{S}_t(D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})) \subseteq \mathcal{S}_t(\mathbf{b}), \quad \forall t \in \mathbb{G} \\ &\Leftrightarrow \quad D_{\mathcal{B}}(\mathcal{S}_t(\mathbf{a}), \mathcal{S}_t(\mathbf{s})) \subseteq \mathcal{S}_t(\mathbf{b}), \quad \forall t \in \mathbb{G} \\ &\Leftrightarrow \quad \mathcal{S}_t(\mathbf{a}) \subseteq E_{\mathcal{B}}(\mathcal{S}_t(\mathbf{b}), \mathcal{S}_t(\mathbf{s})), \quad \forall t \in \mathbb{G} \\ &\Leftrightarrow \quad \mathcal{S}_t(\mathbf{a}) \subseteq \mathcal{S}_t(E_{\mathcal{L}}(\mathbf{b}, \mathbf{s})), \quad \forall t \in \mathbb{G} \\ &\Leftrightarrow \quad \mathbf{a} \leq E_{\mathcal{L}}(\mathbf{b}, \mathbf{s}). \end{aligned} \tag{23}$$

Here we used the fact that the binary dilation and erosion represent adjoint operators on $(\mathcal{P}(\mathbf{X}), \subseteq)$.

Theorem 2 *The pairs* $(E_{\mathcal{L}}, D_{\mathcal{L}})$ *and* $(E_{\mathcal{T}}, D_{\mathcal{T}})$ *form adjunctions on* $\mathbb{G}^{\mathbf{X}}$.

The following theorem provides a characterization of the $\mathcal{L}$-dilation and the $\mathcal{L}$-erosion.

Theorem 3 *Let* $\mathbf{a} \in \mathbb{G}^{\mathbf{X}}$ *be a gray-scale image and let* $\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$ *be a SE. The* $\mathcal{L}$*-erosion is given by*

$$E_{\mathcal{L}}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \bigwedge_{\mathbf{y}\in\mathbf{X}} f\left(\mathbf{s}_{\mathbf{x}}(\mathbf{y}),\mathbf{a}(\mathbf{y})\right), \tag{24}$$

where $\mathbf{s}_{\mathbf{x}}$ *is the translation of* $\mathbf{s}$ *by* $\mathbf{x}$ *and* $f : \mathbb{G}\times\mathbb{G} \to \mathbb{G}$ *is such that*

$$f(x,y) = \begin{cases} +\infty, & x \le y, \\ y, & x > y. \end{cases} \tag{25}$$

The $\mathcal{L}$*-dilation is given by*

$$D_{\mathcal{L}}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \bigvee_{\mathbf{y}\in\mathbf{X}} \left(\bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y}) \wedge \mathbf{a}(\mathbf{y})\right), \tag{26}$$

where $\bar{\mathbf{s}} \in \mathbb{G}^{\mathbf{X}}$ *denotes the reflection of* $\mathbf{s}$ *around the origin, i.e.* $\bar{\mathbf{s}}(\mathbf{x}) = \mathbf{s}(-\mathbf{x})$ *for every* $\mathbf{x} \in \mathbf{X}$.

Proof We first prove Eq. 26. Let $\mathbf{a},\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$. For all $\mathbf{x} \in \mathbf{X}$, we have

$$\begin{aligned} \mathbf{x} \in D_{\mathcal{B}}(\mathcal{S}_t(\mathbf{a}),\mathcal{S}_t(\mathbf{s})) \quad &\Leftrightarrow \quad \overline{[\mathcal{S}_t(\mathbf{s})]}_{\mathbf{x}} \cap \mathcal{S}_t(\mathbf{a}) \neq \emptyset \\ &\Leftrightarrow \quad \exists\, \mathbf{y} \in \mathbf{X} : \mathbf{y} \in \overline{[\mathcal{S}_t(\mathbf{s})]}_{\mathbf{x}} \\ &\qquad \text{and} \quad \mathbf{y} \in \mathcal{S}_t(\mathbf{a}) \\ &\Leftrightarrow \quad \exists\, \mathbf{y} \in \mathbf{X} : \mathbf{s}(\mathbf{x}-\mathbf{y}) \ge t \\ &\qquad \text{and} \quad \mathbf{a}(\mathbf{y}) \ge t \\ &\Leftrightarrow \quad \bigvee_{\mathbf{y}\in\mathbf{X}} (\mathbf{s}(\mathbf{x}-\mathbf{y}) \wedge \mathbf{a}(\mathbf{y})) \ge t. \end{aligned} \tag{27}$$

The following sequence of equations concludes the proof of Eq. 26.

$$\begin{aligned} D_{\mathcal{L}}(\mathbf{a},\mathbf{s})(\mathbf{x}) &= \bigvee \{t \in \mathbb{G} : \mathbf{x} \in D_{\mathcal{B}}(\mathcal{S}_t(\mathbf{a}),\mathcal{S}_t(\mathbf{s}))\} \\ &= \bigvee \left\{ t \in \mathbb{G} : \bigvee_{\mathbf{y}\in\mathbf{X}} (\mathbf{s}(\mathbf{x}-\mathbf{y}) \wedge \mathbf{a}(\mathbf{y})) \ge t \right\} \\ &= \bigvee_{\mathbf{y}\in\mathbf{X}} (\mathbf{s}(\mathbf{x}-\mathbf{y}) \wedge \mathbf{a}(\mathbf{y})). \end{aligned} \tag{28}$$

The proof of the second part follows from Theorem 2 and statement 2 of Proposition 1.

Consider the following sequence of inequalities for $\mathbf{a},\mathbf{b},\mathbf{s} \in \mathbb{G}^{\mathbf{X}}$.

$$\begin{aligned} & D_{\mathcal{L}}(\mathbf{a},\mathbf{s})(\mathbf{y}) \le \mathbf{b}(\mathbf{y}), \quad \forall\, \mathbf{y} \in \mathbf{X} \\ \Leftrightarrow\quad & \bigvee_{\mathbf{x}\in\mathbf{X}} \bar{\mathbf{s}}_{\mathbf{y}}(\mathbf{x}) \wedge \mathbf{a}(\mathbf{x}) \le \mathbf{b}(\mathbf{y}), \quad \forall\, \mathbf{y} \in \mathbf{X} \\ \Leftrightarrow\quad & \mathbf{s}(\mathbf{y}-\mathbf{x}) \wedge \mathbf{a}(\mathbf{x}) \le \mathbf{b}(\mathbf{y}), \quad \forall\, \mathbf{x},\mathbf{y} \in \mathbf{X} \\ \Leftrightarrow\quad & \mathbf{a}(\mathbf{x}) \le f\left(\mathbf{s}(\mathbf{y}-\mathbf{x}),\mathbf{b}(\mathbf{y})\right), \quad \forall\, \mathbf{x},\mathbf{y} \in \mathbf{X} \\ \Leftrightarrow\quad & \mathbf{a}(\mathbf{x}) \le \bigwedge_{\mathbf{y}\in\mathbf{X}} f\left(\mathbf{s}_{\mathbf{x}}(\mathbf{y}),\mathbf{b}(\mathbf{y})\right), \quad \forall\, \mathbf{x} \in \mathbf{X}. \end{aligned} \tag{29}$$

By Proposition 1 and Theorem 2, $E_{\mathcal{L}}$ is the unique erosion that forms an adjunction together with $D_{\mathcal{L}}$. Therefore, Eqs. 29 through 29 imply Eq. 24. □

Note that Proposition 3 follows from Theorem 3 considering an SE given by means of Eq. 21.

Recall that we can obtain an erosion from a dilation and vice-versa by means of a negation. For example, consider the negation $*$ of Eq. 11. Straightforward computation shows that the erosion $D^*_{\mathcal{L}}$ and the dilation $E^*_{\mathcal{L}}$ have the following representations:

$$D^*_{\mathcal{L}}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \bigwedge_{\mathbf{y}\in\mathbf{X}} (-\mathbf{s}_{\mathbf{x}}(\mathbf{y})) \vee \mathbf{a}(\mathbf{y}), \tag{30}$$

$$E^*_{\mathcal{L}}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \bigvee_{\mathbf{y}\in\mathbf{X}} g\left(\bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y}),\mathbf{a}(\mathbf{y})\right), \tag{31}$$

where $g : \mathbb{G}\times\mathbb{G} \to \mathbb{G}$ is defined as follows:

$$g(x,y) = -f(x,-y) = \begin{cases} -\infty, & x \le -y, \\ y, & x > -y. \end{cases} \tag{32}$$

3.3 The Umbra Approach

Sternberg introduced the umbra approach to gray-scale MM [37, 38] which is based on the observation that the points on and below the graph of an image $\mathbf{a} : \mathbf{X} \to \mathbb{G}$ yield a subset of the abelian group $\mathbf{X}\times\mathbb{G}$ and thus the tools of binary MM can be applied. The resulting subsets are of a special form and are called umbras. More precisely, a subset U of $\mathbf{X}\times\mathbb{G}$ is called an *umbra* if—for every $\mathbf{x} \in \mathbf{X}$ and for every $t \in \mathbb{G}$—$(\mathbf{x},t) \in U$ implies that $(\mathbf{x},s) \in U$ for every $s < t$.

Let $\mathcal{U}_{\mathbf{X}\times\mathbb{G}}$ denote the set of all umbras contained in $\mathbf{X}\times\mathbb{G}$. Note that $\mathcal{U}_{\mathbf{X}\times\mathbb{G}}$ represents a complete sublattice of $\mathcal{P}(\mathbf{X}\times\mathbb{G})$. We have a lattice endomorphism $\mathcal{U} : \mathbb{G}^{\mathbf{X}} \to \mathcal{U}_{\mathbf{X}\times\mathbb{G}}$ that associates every gray-scale image $\mathbf{a} \in \mathbb{G}^{\mathbf{X}}$ with its umbra $\mathcal{U}(\mathbf{a})$ where $\mathcal{U}(\mathbf{a})$ is defined as follows [31]:

$$\mathcal{U}(\mathbf{a}) = \left\{(\mathbf{x},t) \in \mathbf{X}\times\mathbb{G}' : t \le \mathbf{a}(\mathbf{x})\right\}. \tag{33}$$

The symbol $\mathbb{G}'$ denotes $\mathbb{G} \setminus \{-\infty,\infty\}$.

Using this definition of $\mathcal{U}(\mathbf{a})$, the lattice homomorphism $\mathcal{U} : \mathbb{G}^{\mathbf{X}} \to \mathcal{U}_{\mathbf{X}\times\mathbb{G}}$ becomes a lattice isomorphism if the set of gray-levels $\mathbb{G}$ is discrete, in other words if $\mathbb{G} = \bar{\mathbb{Z}}$. The inverse of this isomorphism is given by $\mathcal{T} : \mathcal{U}_{\mathbf{X}\times\mathbb{G}} \to \mathbb{G}^{\mathbf{X}}$ where $\mathcal{T}(U)$ is defined as follows:

$$(\mathcal{T}(U))(\mathbf{x}) = \bigvee \{t \in \mathbb{G} : (\mathbf{x},t) \in U\}, \quad \forall\, \mathbf{x} \in \mathbf{X}. \tag{34}$$

We refer to $\mathcal{T}(U)$ as the *top of the umbra* U. In the case of continuous gray-levels, i.e. $\mathbb{G} = \bar{\mathbb{R}}$, the homomorphism $\mathcal{U}$

is also injective but fails to be surjective. For both discrete as well as continuous gray-levels, the $\mathcal{U}$-dilation $D_{\mathcal{U}}(\mathbf{a},\mathbf{s})$ and the $\mathcal{U}$-erosion $E_{\mathcal{U}}(\mathbf{a},\mathbf{s})$ are defined as follows for every $\mathbf{a},\mathbf{s}\in\mathbb{G}^{\mathbf{X}}$:

$$\begin{aligned} D_{\mathcal{U}}(\mathbf{a},\mathbf{s}) &= \mathcal{T}\left(D_{\mathcal{B}}\left(\mathcal{U}(\mathbf{a}),\mathcal{U}(\mathbf{s})\right)\right) \quad \text{and} \\ E_{\mathcal{U}}(\mathbf{a},\mathbf{s}) &= \mathcal{T}\left(E_{\mathcal{B}}\left(\mathcal{U}(\mathbf{a}),\mathcal{U}(\mathbf{s})\right)\right). \end{aligned} \tag{35}$$

The definitions in Eqs. 35 lead to the following alternative formulations of $\mathcal{U}$-dilation and $\mathcal{U}$-erosion:

$$\begin{aligned} E_{\mathcal{U}}(\mathbf{a},\mathbf{s})(\mathbf{x}) &= \bigwedge_{\mathbf{y}\in\mathbf{X}} \left(\mathbf{a}(\mathbf{y}) +' (-\mathbf{s}_{\mathbf{x}}(\mathbf{y}))\right) \quad \text{and} \\ D_{\mathcal{U}}(\mathbf{a},\mathbf{s})(\mathbf{x}) &= \bigvee_{\mathbf{y}\in\mathbf{X}} \left(\mathbf{a}(\mathbf{y}) + \bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y})\right). \end{aligned} \tag{36}$$

The operations "$+$" and "$+'$" differ with respect to the sum of ∞ and $-\infty$ as described in Eq. 38 below. Otherwise, these operations behave as one would expect.

$$\begin{aligned} &\infty + (-\infty) = (-\infty) + \infty = -\infty \quad \text{and} \\ &\infty +' (-\infty) = (-\infty) +' \infty = \infty. \end{aligned} \tag{37}$$

Proposition 4 *The erosion $E_{\mathcal{U}}$ and the dilation $D_{\mathcal{U}}$ are dual operators with respect to adjunction and with respect to the negation $*$ given in Eq. 11 [18].*

For a given binary SE $\mathbf{S}\subseteq\mathbf{X}$, let us construct a gray-scale SE $\mathbf{s}\in\mathbb{G}^{\mathbf{X}}$ as follows:

$$\mathbf{s}(\mathbf{x}) = \begin{cases} 0, & \mathbf{x}\in\mathbf{S}, \\ -\infty, & \mathbf{x}\notin\mathbf{S}. \end{cases} \tag{38}$$

For every image $\mathbf{a}\in\mathbb{G}^{\mathbf{X}}$, we can compute the $\mathcal{U}$-dilation $D_{\mathcal{U}}(\mathbf{a},\mathbf{s})(\mathbf{x})$ of $\mathbf{a}$ by this SE $\mathbf{s}$ as follows

$$\begin{aligned} D_{\mathcal{U}}(\mathbf{a},\mathbf{s})(\mathbf{x}) &= \bigvee_{\mathbf{y}\in\mathbf{X}} \left(\mathbf{a}(\mathbf{x}-\mathbf{y}) + \mathbf{s}(\mathbf{y})\right) \\ &= \left[\bigvee_{\mathbf{y}\in\mathbf{S}} (\mathbf{a}(\mathbf{x}-\mathbf{y}))\right] = \left[\bigvee_{\mathbf{y}\in\bar{\mathbf{S}}_{\mathbf{x}}} (\mathbf{a}(\mathbf{y}))\right]. \end{aligned} \tag{39}$$

In a similar vein, we recognize that

$$E_{\mathcal{U}}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \bigwedge_{\mathbf{y}\in\mathbf{S}_{\mathbf{x}}} \mathbf{a}(\mathbf{y}). \tag{40}$$

In view of the definitions of $\mathcal{T}$-dilation and $\mathcal{T}$-erosion that we presented in Eq. 18, these observations yield the equalities $D_{\mathcal{T}}(\mathbf{a},\mathbf{S}) = D_{\mathcal{U}}(\mathbf{a},\mathbf{s})$ and $E(\mathbf{a},\mathbf{S}) = E_{\mathcal{U}}(\mathbf{a},\mathbf{s})$ where $\mathbf{s}$ is given by Eq. 38.

4 Fuzzy Mathematical Morphology

4.1 Basic Concepts of Fuzzy Set Theory

Fuzzy set theory extends conventional (crisp) set theory. Lotfi Zadeh introduced this mathematical theory as a tool to model the vagueness and ambiguity in complex systems [40]. A *fuzzy set* is formally defined as a function $\mathbf{a}$ from a set $\mathbf{X}$ to $[0,1]$. The function $\mathbf{a}$ is also called *membership function* and the value $\mathbf{a}(\mathbf{x})$ is the *degree of membership* of $\mathbf{x}$ in the fuzzy set $\mathbf{a}$. In particular, we have that $\mathbf{a}(\mathbf{x}) = 0$ represents complete exclusion and that $\mathbf{a}(\mathbf{x}) = 1$ represents complete membership. The class of fuzzy sets in $\mathbf{X}$ will be denoted by $\mathcal{F}(\mathbf{X}) = [0,1]^{\mathbf{X}}$. Note that fuzzy set theory can be used for the design of image operators since an image $\mathbf{a}:\mathbf{X}\to[0,1]$ can be interpreted as a fuzzy set of $\mathbf{X}$. From now on, an image $\mathbf{a}\in\mathcal{F}(\mathbf{X})$ will be called *fuzzy image*. We identify every crisp set $\mathbf{A}\in\mathcal{P}(\mathbf{X})$ with a fuzzy set $\mathbf{a}\in\mathcal{F}(\mathbf{X})$ via the relationship

$$\mathbf{a}(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x}\in\mathbf{A}, \\ 0, & \text{else.} \end{cases} \tag{41}$$

A *fuzzy conjunction* is an increasing mapping $C_{\mathcal{F}}: [0,1]\times[0,1]\longrightarrow[0,1]$ that satisfies $C_{\mathcal{F}}(0,0) = C_{\mathcal{F}}(0,1) = C_{\mathcal{F}}(1,0) = 0$ and $C_{\mathcal{F}}(1,1) = 1$. The minimum operator obviously yields a simple example. Some other particular choices of fuzzy conjunction are due to Lukasiewicz and to Kleene and Dienes [15]:

$$C_M(x,y) = x\wedge y, \tag{42}$$

$$C_L(x,y) = 0\vee(x+y-1), \tag{43}$$

$$C_K(x,y) = \begin{cases} 0, & y\le 1-x, \\ y, & y>1-x. \end{cases} \tag{44}$$

In particular, a commutative, associative, and non-decreasing fuzzy conjunction $T:[0,1]\times[0,1]\to[0,1]$ that satisfies $T(x,1) = x$ for every $x\in[0,1]$ is called *triangular norm* or simply *t-norm* [21, 27]. The fuzzy conjunctions C_M and C_L are examples of t-norms. An increasing, commutative, and associative mapping $S:[0,1]\times[0,1]\to[0,1]$ that satisfies $S(0,x) = x$ for every $x\in[0,1]$ is called *triangular co-norm*, for short *s-norm*.

An operator $I_{\mathcal{F}}:[0,1]\times[0,1]\longrightarrow[0,1]$ that is decreasing in the first argument and that is increasing in the second argument is called a *fuzzy implication* if $I_{\mathcal{F}}$ extends the usual crisp implication on $\{0,1\}\times\{0,1\}$, i.e. $I_{\mathcal{F}}(0,0) = I_{\mathcal{F}}(0,1) = I_{\mathcal{F}}(1,1) = 1$ and $I_{\mathcal{F}}(1,0) = 0$. Some particular fuzzy implications, that were introduced by Gödel, Lukasiewicz, and by Kleene and Dienes, can be found below [15].

$$I_G(x,y) = \begin{cases} 1, & x\le y, \\ y, & x>y, \end{cases} \tag{45}$$

$$I_L(x,y) = 1 \wedge (y - x + 1), \quad (46)$$

$$I_K(x,y) = (1-x) \vee y. \quad (47)$$

The Lukasiewicz implication I_L can be generalized to a class of fuzzy implications. Let λ be a strictly decreasing mapping $[0,1] \to [0,1]$ satisfying

$$\lambda(1) = 0, \lambda(0) = 1 \quad \text{and} \quad (48)$$

$$x \leq y \quad \Leftrightarrow \quad \lambda(x) + \lambda(1-y) \geq 1 \quad \forall x, y \in [0,1]. \quad (49)$$

A fuzzy implication $I_{GL} : [0,1] \times [0,1] \to [0,1]$ is called a *generalized Lukasiewicz implication* if I_{GL} has the following representation [10].

$$I_{GL}(x,y) = 1 \wedge [\lambda(x) + \lambda(1-y)], \quad \forall x, y \in [0,1]. \quad (50)$$

A negation on the unit interval $[0,1]$ is also called a *fuzzy negation*. The following unary operators represent examples of fuzzy negations.

$$N_S(x) = 1 - x, \quad (51)$$

$$N_D(x) = \frac{1-x}{1+px}, \quad p > -1, \quad (52)$$

$$N_R(x) = \sqrt[p]{1 - x^p}, \quad p \in (0, \infty). \quad (53)$$

Note that a fuzzy negation $N_{\mathcal{F}}$ on $[0,1]$ induces a negation $\mathbf{N}_{\mathcal{F}}$ on $\mathcal{F}(\mathbf{X})$ that is given by applying $N_{\mathcal{F}}$ pointwise, i.e. $\mathbf{N}_{\mathcal{F}}(\mathbf{a})(\mathbf{x}) = N_{\mathcal{F}}(\mathbf{a}(\mathbf{x}))$. Here, we use a bold symbol $\mathbf{N}$ for a negation to indicate that the negation is vector-valued. A subscript of the bold symbol $\mathbf{N}$ indicates the corresponding negation on $[0,1]$.

Let $N_{\mathcal{F}}$ be an arbitrary fuzzy negation. A fuzzy implication $I_{\mathcal{F}}$ satisfying $I_{\mathcal{F}}(x,y) = I_{\mathcal{F}}(N_{\mathcal{F}}(y), N_{\mathcal{F}}(x))$ for all $x, y \in [0,1]$ is called *contrapositive* with respect to $N_{\mathcal{F}}$. For example, every generalized Lukasiewicz implication I_{GL} is contrapositive with respect to the standard negation N_S.

Let $N_{\mathcal{F}}$ be a fuzzy negation. We construct a fuzzy implication $I_{\mathcal{F}}$ from a fuzzy conjunction $C_{\mathcal{F}}$ by setting $I_{\mathcal{F}}(x,\cdot) = (C_{\mathcal{F}}(x,\cdot))^{N_{\mathcal{F}}}$ for all $x \in [0,1]$. In a similar fashion, we derive a fuzzy conjunction $C_{\mathcal{F}}$ from a fuzzy implication $I_{\mathcal{F}}$ by defining $C_{\mathcal{F}}(x,\cdot)$ as the dual of $I_{\mathcal{F}}(x,\cdot)$ with respect to $N_{\mathcal{F}}$ for all $x \in [0,1]$. Simplifying our terminology, we say that a fuzzy conjunction $C_{\mathcal{F}}$ and a fuzzy negation $I_{\mathcal{F}}$ are *dual with respect to a fuzzy negation* $N_{\mathcal{F}}$ if and only if $C_{\mathcal{F}}(x,\cdot)$ and $I_{\mathcal{F}}(x,\cdot)$ are dual with respect to $N_{\mathcal{F}}$ for all $x \in [0,1]$.

Similarly, we say that a fuzzy conjunction $C_{\mathcal{F}}$ and a fuzzy implication $I_{\mathcal{F}}$ form an *adjunction* if and only if $C_{\mathcal{F}}(x,\cdot)$ and $I_{\mathcal{F}}(x,\cdot)$ form an adjunction for every $x \in [0,1]$. In this case, we call $C_{\mathcal{F}}$ and $I_{\mathcal{F}}$ *adjoint* operators. For example, the pairs (C_M, I_G), (C_L, I_L), and (C_K, I_K) represent adjunctions.

Duality with respect to negation and duality with respect to adjunction are two distinct concepts. For example, on one hand we have that the conjunction C_M and the implication I_K constitute dual operations with respect to the standard negation N_S but fail to be adjoint. On the other hand, the conjunction C_M is adjoint to the implication I_G but they are not dual with respect to any fuzzy negation for the following reason. Suppose that there exists a fuzzy negation $N_{\mathcal{F}} : [0,1] \to [0,1]$ that satisfies the equation $C_M(x,y) = N_{\mathcal{F}}(I_G(x, N_{\mathcal{F}}(y)))$ for all $x, y \in [0,1]$. The existence of such a fuzzy negation $N_{\mathcal{F}}$ would lead to the following contradiction if x is such that $1 > x > 0$.

$$\begin{aligned} 1 > x = x \wedge 1 = C_M(x,1) &= N_{\mathcal{F}}(I_G(x, N_{\mathcal{F}}(1))) \\ &= N_{\mathcal{F}}(I_G(x,0)) = N_{\mathcal{F}}(0) = 1. \end{aligned} \quad (54)$$

Let $C_{\mathcal{F}}$ be a fuzzy conjunction and let $I_{\mathcal{F}}$ be a fuzzy implication such that $C_{\mathcal{F}}$ and $I_{\mathcal{F}}$ are adjoint. Proposition 1 implies that $I_{\mathcal{F}}(a,\cdot)$ is an erosion and $C_{\mathcal{F}}(a,\cdot)$ is a dilation on $[0,1]$ in the sense of Eqs. 2 and 3 for every $a \in [0,1]$. As another consequence of Proposition 1, we have that for every fuzzy implication $I_{\mathcal{F}}$ there is at most one fuzzy conjunction $C_{\mathcal{F}}$ and vice-versa such that $I_{\mathcal{F}}$ and $C_{\mathcal{F}}$ are adjoint. We would like to point out, however, that there are fuzzy implications that do not yield erosions and there are fuzzy conjunctions that do not yield dilations. Consider for example the fuzzy implication I_{CE} given by

$$I_{CE}(x,y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y = 0, \\ 1, & \text{otherwise.} \end{cases} \quad (55)$$

Note that $I_{CE}(1,\cdot)$ does not satisfy Eq. 2 since

$$I_{CE}\left(1, \bigwedge_{n\in\mathbb{N}} \frac{1}{n}\right) = I_{CE}(1,0) = 0, \quad (56)$$

whereas

$$\bigwedge_{n\in\mathbb{N}} I_{CE}\left(1, \frac{1}{n}\right) = \bigwedge_{n\in\mathbb{N}} 1 = 1. \quad (57)$$

4.2 Fuzzy Mathematical Morphology Based on Fuzzy Inclusion and Intersection Measures

In Eq. 8, the binary erosion of a set $\mathbf{A}$ by a SE $\mathbf{S}$ was defined as the set of all points $\mathbf{x}$ such that the translated structuring element $\mathbf{S_x}$ is contained in $\mathbf{A}$. Formally, we obtain the following equivalent definition of $E_{\mathcal{B}}(\mathbf{A},\mathbf{S})$.

$$E_{\mathcal{B}}(\mathbf{A},\mathbf{S}) = \{\mathbf{x} \in \mathbf{X} : \text{Inc}(\mathbf{S_x}, \mathbf{A}) = 1\}, \quad (58)$$

where $\text{Inc} : \mathcal{P}(\mathbf{X}) \times \mathcal{P}(\mathbf{X}) \to \{0,1\}$ represents the set inclusion for crisp sets, i.e. $\text{Inc}(\mathbf{S_x}, \mathbf{A}) = 1$ if and only if $\mathbf{S_x} \subseteq \mathbf{A}$. We also defined the binary dilation of $\mathbf{A}$ by $\mathbf{S}$ as the set of all $\mathbf{x}$ such that the reflection of $\mathbf{S_x}$ hits $\mathbf{A}$. This notion can be expressed as follows in terms of the intersection $\mathrm{Sec}: \mathcal{P}(\mathbf{X}) \times \mathcal{P}(\mathbf{X}) \to \{0,1\}$ of crisp sets.

$$D_{\mathcal{B}}(\mathbf{A},\mathbf{S}) = \{\mathbf{x} \in \mathbf{X} : \mathrm{Sec}(\bar{\mathbf{S}}_{\mathbf{x}}, \mathbf{A}) = 1\}. \tag{59}$$

A consistent fuzzy morphology should be based on definitions of fuzzy erosion and fuzzy dilation that extend the definitions of binary erosion and dilation to the fuzzy domain. This goal can be achieved as follows.

In this paper, we adhere to the definition that a *fuzzy inclusion measure* or *fuzzified set inclusion* $\mathrm{Inc}_{\mathcal{F}}$ is a $\mathcal{F}(\mathbf{X}) \times \mathcal{F}(\mathbf{X}) \to [0,1]$ mapping whose restriction to $\mathcal{P}(\mathbf{X}) \times \mathcal{P}(\mathbf{X})$ coincides with the set inclusion for crisp sets [26]. Formally, we have the following implications for all $\mathbf{A},\mathbf{B} \in \mathcal{P}(\mathbf{X})$ and their corresponding fuzzy sets $\mathbf{a},\mathbf{b} \in \mathcal{F}(\mathbf{X})$.

$$\begin{aligned} &\mathbf{A} \subseteq \mathbf{B} \Rightarrow \mathrm{Inc}_{\mathcal{F}}(\mathbf{a},\mathbf{b}) = \mathrm{Inc}(\mathbf{A},\mathbf{B}) = 1 \quad \text{and} \\ &\mathbf{A} \not\subseteq \mathbf{B} \Rightarrow \mathrm{Inc}_{\mathcal{F}}(\mathbf{a},\mathbf{b}) = \mathrm{Inc}(\mathbf{A},\mathbf{B}) = 0. \end{aligned} \tag{60}$$

The value $\mathrm{Inc}_{\mathcal{F}}(\mathbf{a},\mathbf{b})$ is interpreted as the *degree of subsethood or inclusion* of the fuzzy set $\mathbf{a}$ in the fuzzy set $\mathbf{b}$.

Various researchers have set out to define fuzzy inclusion measures. Among these definitions are the inclusion measures of Zadeh, Bandler and Kohout, Kitainik, and Sinha and Dougherty [2, 20, 34, 35, 40]. Straightforward verification shows that all of these measures fuzzify the notion of crisp set inclusion. The subsethood measure S_K that is described below (for a such that $\mathbf{a}(\mathbf{x}) > 0$ for some $\mathbf{x} \in \mathbf{X}$) violates the definition of fuzzy inclusion measure. Kosko has proposed this measure of subsethood for a finite universe $\mathbf{X}$ [22]

$$S_K(\mathbf{a},\mathbf{b}) = 1 - \frac{1}{\sum_{\mathbf{x}} \mathbf{a}(\mathbf{x})} \sum_{\mathbf{x}\in\mathbf{X}} 0 \vee (\mathbf{a}(\mathbf{x}) - \mathbf{b}(\mathbf{x})). \tag{61}$$

A fuzzy erosion $E_{\mathcal{F}} : \mathcal{F}(\mathbf{X}) \times \mathcal{F}(\mathbf{X}) \to \mathcal{F}(\mathbf{X})$ based on a certain fuzzy inclusion measure $\mathrm{Inc}_{\mathcal{F}}$ arises via the following definition [26]:

$$E_{\mathcal{F}}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \mathrm{Inc}_{\mathcal{F}}(\mathbf{s_x},\mathbf{a}). \tag{62}$$

Note that $E_{\mathcal{F}}$ extends the binary erosion $E_{\mathcal{B}} : \mathcal{P}(\mathbf{X}) \times \mathcal{P}(\mathbf{X}) \to \mathcal{P}(\mathbf{X})$ to the fuzzy domain.

We would like to clarify a fact that Nachtegael and Kerre have failed to mention [26]. Strictly speaking, we may only refer to $E_{\mathcal{F}}$ as a *fuzzy erosion* if the operators $\mathrm{Inc}_{\mathcal{F}}(\mathbf{s},\cdot)$ commute with the infimum operation for all $\mathbf{s} \in \mathcal{F}(\mathbf{X})$. Otherwise, the operator $E_{\mathcal{F}}(\cdot,\mathbf{s})$ does not represent an erosion by the SE $\mathbf{s}$.

In analogy to the measure $\mathrm{Inc}_{\mathcal{F}}$, we define a *fuzzy intersection measure* or *fuzzified set intersection* $\mathrm{Sec}_{\mathcal{F}}$ as a $\mathcal{F}(\mathbf{X}) \times \mathcal{F}(\mathbf{X}) \to [0,1]$ mapping whose restriction to $\mathcal{P}(\mathbf{X}) \times \mathcal{P}(\mathbf{X})$ coincides with the set intersection for crisp sets. We interpret the value $\mathrm{Sec}_{\mathcal{F}}(\mathbf{a},\mathbf{b})$ as the degree of intersection of the fuzzy sets $\mathbf{a}$ and $\mathbf{b}$ or the degree of the fuzzy set $\mathbf{a}$ hitting the fuzzy set $\mathbf{b}$. Given a *fuzzified set intersection* $\mathrm{Sec}_{\mathcal{F}}$ such that $\mathrm{Sec}_{\mathcal{F}}(\mathbf{s},\cdot)$ commutes with the supremum operation, we obtain a function $D_{\mathcal{F}} : \mathcal{F}(\mathbf{X}) \times \mathcal{F}(\mathbf{X}) \to \mathcal{F}(\mathbf{X})$ by setting

$$D_{\mathcal{F}}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \mathrm{Sec}_{\mathcal{F}}(\bar{\mathbf{s}}_{\mathbf{x}},\mathbf{a}), \tag{63}$$

Note that $D_{\mathcal{F}}$ coincides with $D_{\mathcal{B}}$ on $\mathcal{P}(\mathbf{X}) \times \mathcal{P}(\mathbf{X})$. We refer to $D_{\mathcal{F}}$ using the terminology *fuzzy dilation* if $D_{\mathcal{F}}(\cdot,\mathbf{s})$ commutes with the supremum operator for every $\mathbf{s} \in \mathcal{F}(\mathbf{X})$.

In our opinion, almost all approaches towards fuzzy mathematical morphology are based on a certain fuzzy inclusion measure and employ Eq. 62 in order to define the concept of fuzzy erosion. The constructive approach of Ronse that we will discuss in Sect. 5.3.3 represents an exception to this rule. After defining a fuzzy erosion according to Eq. 62, researchers in fuzzy mathematical morphology have chosen to define fuzzy dilation as the dual operation of fuzzy erosion with respect to either adjunction or (a particular operation of) negation.

In analogy to the gray-scale case, we say that a pair $(E_{\mathcal{F}}, D_{\mathcal{F}})$ consisting of a fuzzy erosion $E_{\mathcal{F}}$ and a fuzzy dilation $D_{\mathcal{F}}$ forms an *adjunction* if and only if $(E_{\mathcal{F}}(\cdot,\mathbf{s}), D_{\mathcal{F}}(\cdot,\mathbf{s}))$ forms an adjunction for every SE $\mathbf{s} \in \mathcal{F}(\mathbf{X})$. We say that a fuzzy erosion $E_{\mathcal{F}}$ and a fuzzy dilation $D_{\mathcal{F}}$ are *dual operators with respect to a fuzzy negation* $N_{\mathcal{F}}$ if and only if $E_{\mathcal{F}}(\cdot,\mathbf{s})$ and $D_{\mathcal{F}}(\cdot,\bar{\mathbf{s}})$ are dual operators with respect to $\mathbf{N}_{\mathcal{F}}$ for all $\mathbf{s} \in \mathcal{F}(\mathbf{X})$.

4.3 Fuzzy Inf-I Inclusion and the Sup-C Intersection Measures

In the previous section, we have underlined the importance of fuzzy inclusion and intersection measures in the definition of fuzzy erosions and fuzzy dilations. This section explains how a fuzzy inclusion measure can be derived from the crisp inclusion measure.

Consider arbitrary crisp sets $\mathbf{A},\mathbf{B} \subseteq \mathbf{X}$. Obviously, we have $\mathbf{A} \subseteq \mathbf{B}$ if and only if $\mathbf{x} \in \mathbf{A}$ implies that $\mathbf{x} \in \mathbf{B}$ for all $\mathbf{x} \in \mathbf{X}$. If $\mathbf{a},\mathbf{b} : \mathbf{X} \to \{0,1\}$ denote the corresponding crisp membership functions then this statement can be reformulated as follows:

$$\mathrm{Inc}(\mathbf{A},\mathbf{B}) = \bigwedge_{\mathbf{x}\in\mathbf{X}} I\,(\mathbf{a}(\mathbf{x}),\mathbf{b}(\mathbf{x}))\,. \tag{64}$$

Now consider arbitrary fuzzy sets $\mathbf{a},\mathbf{b} \in \mathcal{F}(\mathbf{X})$. A straightforward fuzzification of Eq. 64 leads to the following fuzzy inclusion measure $\mathrm{Inc}_{\mathcal{F}}$ [2]:

$$\mathrm{Inc}_{\mathcal{F}}(\mathbf{a},\mathbf{b}) = \bigwedge_{\mathbf{x}\in\mathbf{X}} I_{\mathcal{F}}(\mathbf{a}(\mathbf{x}),\mathbf{b}(\mathbf{x}))\,. \tag{65}$$

A fuzzy operation $\mathrm{Inc}_{\mathcal{F}}$ of this form will be called *fuzzy Inf-I inclusion measure* or *Bandler-Kohout inclusion measure* [2].

We will speak of Inf-$I_{\mathcal{F}}$ inclusion measures if we want to refer to a specific implication in Eq. 65. Clearly, the restriction of $\text{Inc}_{\mathcal{F}}$ to $\mathcal{P}(\mathbf{X}) \times \mathcal{P}(\mathbf{X})$ is given by the crisp set inclusion Inc since I represents the restriction of $I_{\mathcal{F}}$ to $\{0, 1\}$.

Following a similar line of reasoning, we derive a fuzzy intersection measure $\text{Sec}_{\mathcal{F}}$ by means of the following equation.

$$\text{Sec}_{\mathcal{F}}(\mathbf{a}, \mathbf{b}) = \bigvee_{\mathbf{x} \in \mathbf{X}} C_{\mathcal{F}}(\mathbf{a}(\mathbf{x}), \mathbf{b}(\mathbf{x}))\,. \tag{66}$$

We will call a fuzzy operation $\text{Sec}_{\mathcal{F}}$ of this form *fuzzy Sup-C intersection measure*. Note that $\text{Sec}_{\mathcal{F}}$ fuzzifies the crisp set intersection measure.

Particular choices of fuzzy implications, conjunctions respectively, yield particular fuzzy inclusion measures, intersection measures respectively. Given a particular fuzzy Inf-I inclusion measure, we will indicate the underlying type of fuzzy implication by means of a subscript. For example, Inc_L will denote the fuzzy Inf-I inclusion measure that is based on the Lukasiewicz implication.

4.3.1 Kitainik Inclusion Measure

Kitainik developed an axiomatic approach to fuzzy inclusion measures [20]. A *Kitainik inclusion measure* Inc_{KT} is an $\mathcal{F}(\mathbf{X}) \times \mathcal{F}(\mathbf{X}) \to [0, 1]$ mapping that satisfies a set of four axioms. For details, we refer the reader to [10].

The following proposition by Fordor and Yager provides an exact characterization of the Kitainik inclusion measures in terms of fuzzy Inf-I inclusion measures [10, 16]:

Proposition 5 *A $\mathcal{F}(\mathbf{X}) \times \mathcal{F}(\mathbf{X}) \to [0, 1]$ mapping $\text{Inc}_{\mathcal{F}}$ is a Kitainik inclusion measure if and only if $\text{Inc}_{\mathcal{F}}$ is an Inf-$I_{\mathcal{F}}$ inclusion measure for some fuzzy implication $I_{\mathcal{F}}$ that is contrapositive with respect to the standard fuzzy negation N_S.*

For example, the Inf-I inclusion measure Inc_K and Inc_{GL} represent Kitainik inclusion measures because the implication of Kleene and Dienes I_K as well as every generalized Lukasiewicz implication I_{GL} are contrapositive with respect to N_S.

4.3.2 Sinha-Dougherty Inclusion Measure

According to Sinha and Dougherty, a fuzzy inclusion measure should satisfy the seven axioms that are enumerated in [34, 36]. Specifically, Sinha and Dougherty focus their attention on inclusion measures of a certain form, namely Inf-I_λ inclusion measures where I_λ is given in terms of the right hand side of Eq. 50 for some $\lambda : [0, 1] \to [0, 1]$.

Burillo *et al.* proved that an Inf-I_λ inclusion measure satisfies the seven axioms of Sinha-Dougherty if and only if I_λ belongs to the generalized Lukasiewicz implication class [9].

From now on, we will refer to a Sinha-Dougherty inclusion measure as an Inf-I inclusion measure of the form Inc_{GL}. In other words, we set

$$\text{Inc}_{SD}(\mathbf{a}, \mathbf{b}) = \text{Inc}_{GL}(\mathbf{a}, \mathbf{b}) = \bigwedge_{\mathbf{x} \in \mathbf{X}} I_{GL}(\mathbf{a}(\mathbf{x}), \mathbf{b}(\mathbf{x}))\,, \tag{67}$$

where I_{GL} is a generalized Lukasiewicz implication. The symbols Inc_{SD} and Inc_{GL} will be used interchangeably. Note that the class of Sinha-Dougherty inclusion measures is contained in the class of Kitainik inclusion measures. Equality does not hold. For example, the inclusion measure Inc_K is a Kitainik inclusion measure but fails to constitute a Sinha-Dougherty inclusion measure since I_K cannot be written as a generalized Lukasiewicz implication [10].

4.3.3 Zadeh Inclusion Measure

Zadeh defined an inclusion measure Inc_Z [40] in terms of the following implication I_Z, called the Zadeh implication.

$$I_Z(x, y) = \begin{cases} 1, & x \leq y, \\ 0, & \text{otherwise.} \end{cases} \tag{68}$$

Note that I_Z is contrapositive with respect to the standard negation N_S and therefore the *Zadeh inclusion measure* Inc_Z belongs to the class of Kitainik inclusion measures. If $\mathbf{a}$ and $\mathbf{b}$ are fuzzy sets in an universe $\mathbf{X}$ then Inc_Z satisfies the following equation:

$$\text{Inc}_{ZD}(\mathbf{a}, \mathbf{b}) = \begin{cases} 1, & \mathbf{a}(\mathbf{x}) \leq \mathbf{b}(\mathbf{x})\ \forall \mathbf{x} \in \mathbf{X}, \\ 0, & \text{otherwise.} \end{cases} \tag{69}$$

5 Classification of Some Specific Approaches to Fuzzy Mathematical Morphology

In this section, we provide a classification of some important particular approaches towards fuzzy MM. Each of these approaches depends on a particular choice of fuzzy erosion and fuzzy dilation.

5.1 The General Inf-I/Sup-C Approach of De Baets

De Baets' definitions of fuzzy erosion and dilation [14] yield the most general approach towards fuzzy MM among the ones presented in this section.

If, for all $\mathbf{a} \in \mathcal{F}(\mathbf{X})$, the symbol $d_{\mathbf{a}}$ denotes the set of points $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{a}(\mathbf{x}) > 0$ then De Baets defines the fuzzy "erosion" $E_{DB}(\mathbf{a}, \mathbf{s})$ as follows:

$$E_{DB}(\mathbf{a}, \mathbf{s})(\mathbf{x}) = \bigwedge_{\mathbf{y} \in (d_{\mathbf{s}})_{\mathbf{x}}} I_{\mathcal{F}}(\mathbf{s}_{\mathbf{x}}(\mathbf{y}), \mathbf{a}(\mathbf{y}))\,. \tag{70}$$

Here, the symbol $I_{\mathcal{F}}$ represents an arbitrary fuzzy implication. Using the fact that $I_{\mathcal{F}}(0,x)=1$ for all $x \in [0,1]$, we realize that

$$E_{DB}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \bigwedge_{\mathbf{y}\in\mathbf{X}} I_{\mathcal{F}}(\mathbf{s}_{\mathbf{x}}(\mathbf{y}),\mathbf{a}(\mathbf{y}))\,. \tag{71}$$

If $\mathrm{Inc}_{\mathcal{F}}$ represents a fuzzy Inf-I inclusion measure given by Eq. 65, we obtain

$$E_{DB}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \mathrm{Inc}_{\mathcal{F}}(\mathbf{s}_{\mathbf{x}},\mathbf{a})(\mathbf{x}) = E_{\mathcal{F}}(\mathbf{a},\mathbf{s})(\mathbf{x})\,. \tag{72}$$

Similarly, we will see that De Baets' definition of fuzzy "dilation" D_{DB} amounts to defining D_{DB} as the operator $D_{\mathcal{F}}:\mathcal{F}(\mathbf{X})\times\mathcal{F}(\mathbf{X})\to\mathcal{F}(\mathbf{X})$ of Eq. 63 with $\mathrm{Sec}_{\mathcal{F}}$ given by a Sup-C intersection measure. Formally, we have

$$D_{DB}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \bigvee_{\mathbf{y}\in\mathbf{X}} C_{\mathcal{F}}(\bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y}),\mathbf{a}(\mathbf{y}))\,. \tag{73}$$

Here we slightly adapted De Baets's original definition according to Sternberg by reflecting the SE $\mathbf{s}$ around the origin.

Thus, De Baets' approach is based on fuzzy "erosions" and "dilations" that are defined via Eqs. 62 and 63, the only restriction being that $\mathrm{Inc}_{\mathcal{F}}$ must be an Inf-I inclusion measure and that $\mathrm{Sec}_{\mathcal{F}}$ must be a Sup-C intersection measure. Different choices of fuzzy implications and conjunctions in Eqs. 71 and 73 lead to different fuzzy "erosions" and "dilations" in the sense of De Baets. From now on, a subscript of the symbol E will indicate the type of implication that is used in Eq. 71. For example, the symbol E_G stands for the fuzzy erosion that is given by the equation $E_G(\mathbf{a},\mathbf{s})(\mathbf{x}) = \bigwedge_{\mathbf{y}\in\mathbf{X}} I_G(\mathbf{s}_{\mathbf{x}}(\mathbf{y}),\mathbf{a}(\mathbf{y}))$. Similarly, a subscript of the symbol D will indicate the type of conjunction that is used in Eq. 73.

Recall that the operators $E_{\mathcal{F}}(\cdot,\mathbf{s}):\mathcal{F}(\mathbf{X})\to\mathcal{F}(\mathbf{X})$ given by Eq. 62 represent erosions for all $\mathbf{s}\in\mathcal{F}(\mathbf{X})$ if and only if $\mathrm{Inc}(\mathbf{s},\cdot)$ commutes with the infimum operation for all $\mathbf{s}\in\mathcal{F}(\mathbf{X})$. The latter statement is certainly true for Inf-$I_{\mathcal{F}}$ inclusion measures such that $I_{\mathcal{F}}(s,\cdot)$ is an erosion for all $s\in[0,1]$.

For example, the implications I_G, I_L, and I_K commute with the infimum operation in the second argument and thus E_G, E_L, and E_K yield erosions. If I_{CE} that is defined as in Eq. 55 then E_{CE} does not lead to an erosion as we shall demonstrate below.

Example 1 Let $\mathbf{s}$ denote the constant SE $\mathbf{1}$, i.e. $\mathbf{s}(\mathbf{x})=1$ for all $\mathbf{x}\in\mathbf{X}$. For all $n\in\mathbb{N}$, let $\mathbf{a}_n$ denote the constant fuzzy image whose value $\mathbf{a}(\mathbf{x})$ equals $1/n$ for all $\mathbf{x}\in\mathbf{X}$. On one hand, we infer the following sequence of equations from Eqs. 71 and 56.

$$E_{CE}\left(\bigwedge_{n\in\mathbb{N}}\mathbf{a}_n,\mathbf{s}\right)(\mathbf{x}) = \bigwedge_{\mathbf{y}\in\mathbf{X}} I_{CE}\left(1,\bigwedge_{n\in\mathbb{N}}\frac{1}{n}\right) = 0\,. \tag{74}$$

On the other hand, Eq. 57 implies that

$$\bigwedge_{n\in\mathbb{N}} E_{CE}(\mathbf{a}_n,\mathbf{s})(\mathbf{x}) = \bigwedge_{n\in\mathbb{N}}\left[\bigwedge_{\mathbf{y}\in\mathbf{X}} I_{CE}\left(1,\frac{1}{n}\right)\right] = 1\,. \tag{75}$$

Thus, there exists a SE $\mathbf{s}$ such that the operator $E_{CE}(\cdot,\mathbf{s})$ does not satisfy the definition of an erosion.

Similar observations can be made for the operator $D_{DB}:\mathcal{F}(\mathbf{X})\times\mathcal{F}(\mathbf{X})\to\mathcal{F}(\mathbf{X})$. If $C(s,\cdot)$ is a dilation in $[0,1]$ for every $s\in[0,1]$ then the operator $D_{DB}(\cdot,\mathbf{s})$ is a dilation for every $\mathbf{s}\in\mathcal{F}(\mathbf{X})$. In this case, we may speak of the fuzzy dilation of the image $\mathbf{a}$ by the SE $\mathbf{s}$. In a recent paper [7], I. Bloch shows that a few additional conditions are enough to guarantee that a fuzzy dilation $\delta:\mathcal{F}(\mathbf{X})\to\mathcal{F}(\mathbf{X})$ is of the form $D_{DB}(\cdot,\mathbf{s})$ where $D_{DB}(\cdot,\mathbf{s})$ is given by Eq. 73.

5.2 Inf-I/Sup-C Approaches Based on (Fuzzy) Negations

The approaches that we will discuss in this section are based on the following step-wise procedure. First, one constructs either a fuzzy erosion or a fuzzy dilation satisfying the general framework of De Baets. Then, one generates the dual operator of the previously defined fuzzy erosion or dilation. In contrast to the Deng-Heijmans approach of Sect. 5.3 that employs the duality relationship of adjunction, the subsequent approaches derive fuzzy dilations from fuzzy erosions (or vice-versa) by means of the duality relationship of negation.

5.2.1 Approaches of Nachtegael and Kerre Based on Various Inclusion Measures

Several approaches toward fuzzy MM that comply with the general framework of De Baets were discussed in [26]. Each of these approaches is based on a different fuzzy Inf-I inclusion measure. Therefore, Nachtegael and Kerre have named them accordingly: *Bandler-Kohout*, *Kitainik*, and *Zadeh approach* towards fuzzy MM. All three approaches follow the same scheme:

1. Given a certain fuzzy inclusion measure, one generates a fuzzy erosion according to Eq. 71.
2. One obtains a corresponding fuzzy dilation as the dual of this fuzzy erosion with respect to the standard fuzzy negation N_S.

The Bandler-Kohout approach constitutes the most general one among the three respective approaches since the Bandler-Kohout inclusion measure represents an Inf-I inclusion measure in its most general form.

By Proposition 5, Kitainik's approach is more restrictive, considering only fuzzy Inf-I_{KT} inclusion measures, where the fuzzy implication I_{KT} is contrapositive with respect to N_S, in order to construct fuzzy erosions E_{KT}. Kitainik's fuzzy dilation D_{KT}, given by the N_S-dual of E_{KT}, involves the N_S-dual of the contrapositive fuzzy implication I_{KT} which yields a commutative fuzzy conjunction C_{KT}. Similarly, calculating the N_S-dual of a commutative fuzzy conjunction C_{KT} yields a contrapositive fuzzy implication I_{KT}. Thus, Kitainik's fuzzy dilation D_{KT} is given by Eq. 73 provided that the conjunction is commutative.

We pointed out in Sect. 4.3.3 that Zadeh inclusion measures Inc_{ZD} belong to the class of Kitainik inclusion measure. This observation clarifies that the Kitainik approach generalizes the Zadeh approach to fuzzy MM.

5.2.2 *The approach of Bloch and Maître*

Bloch and Maître's definition of fuzzy dilation relies on Sup-C intersection measure with a t-norm instead of a general fuzzy conjunction [6]. Thus, the Bloch-Maître fuzzy dilation represents a special case of the De Baets fuzzy dilation that is given in Eq. 73.

Given a fuzzy dilation, Bloch and Maître derive a corresponding fuzzy erosion as the dual operator with respect to a fuzzy negation $N_{\mathcal{F}}$. Note that this strategy differs from the one that we described in Sect. 5.2.1 by allowing for an arbitrary fuzzy negation instead of N_S when forming the dual operator. Consequently, the Bloch-Maître fuzzy erosion is based on Inf-$I_{\mathcal{F}}$ inclusion measure where $I_{\mathcal{F}}(x,\cdot) = T(x,\cdot)^{N_{\mathcal{F}}}$ for all $x \in [0,1]$.

As we explained above, there is no direct relationship between the Bloch-Maître approach and the Nachtegael and Kerre's approaches of Sect. 5.2.1 due to fact that Bloch and Maître apply general fuzzy negations $N_{\mathcal{F}}$. The special case of the Bloch-Maître approach that is associated with the standard fuzzy negation N_S, however, complies with Kitainik's scheme since every t-norm represents a commutative fuzzy conjunction.

5.2.3 *Sinha and Dougherty Approach*

Recall that Sinha and Dougherty's approach towards fuzzy inclusion measures revolves around seven axioms [34, 36]. We pointed out that the seven axioms hold in particular for Inf-I_{GL} where I_{GL} denotes a generalized Lukasiewicz implication. In fact, Sinha and Dougherty focus on this type of inclusion measure Inc_{GL}—also denoted using the symbol Inc_{SD} in this paper—and their approach to fuzzy MM consists of the following.

For $\mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X})$, the SD-erosion $E_{SD}(\mathbf{a},\mathbf{s})$ of the image $\mathbf{a}$ by the SE is defined as a special case of Eqs. 62 and 72.

$$E_{SD}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \text{Inc}_{SD}(\mathbf{s_x},\mathbf{a}), \quad \forall \mathbf{x} \in \mathbf{X}. \tag{76}$$

The SD-dilation immediately arises as the dual operator with respect to the negation N_S. The following sequence of equalities reveals that the SD-dilation can be expressed in terms of a Sup-C intersection measure.

$$\begin{aligned} D_{SD}(\mathbf{a},\mathbf{s})(\mathbf{x}) &= 1 - \left[\bigwedge_{\mathbf{y}\in\mathbf{X}} I_{GL}\left(\bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y}), 1-\mathbf{a}(\mathbf{x})\right)\right] \\ &= 1 - \left\{\bigwedge_{\mathbf{y}\in\mathbf{X}} 1 \wedge \left[\lambda(\bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y})) + \lambda(1-\mathbf{a}(\mathbf{x}))\right]\right\} \\ &= \bigvee_{\mathbf{y}\in\mathbf{X}} 0 \vee \left[1 - \lambda(\bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y})) - \lambda(1-\mathbf{a}(\mathbf{x}))\right] \\ &= \bigvee_{\mathbf{y}\in\mathbf{X}} C_{GL}\left(\bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y}), \mathbf{a}(\mathbf{x})\right). \end{aligned} \tag{77}$$

Here C_{GL} is the fuzzy conjunction given by

$$C_{GL}(x,y) = 0 \vee \left[1 - \lambda(x) - \lambda(y)\right], \quad \forall x,y \in [0,1]. \tag{78}$$

To conclude this section, we explain how the Sinha-Dougherty approach to fuzzy MM fits into Kitainik's framework. Note that Sinha and Dougherty as well as Kitainik derive fuzzy erosions from fuzzy inclusion measures according to Eq. 62. The corresponding fuzzy dilation are generated by applying the relationship of duality with respect to the standard fuzzy negation N_S. The two respective approaches only differ regarding the fuzzy inclusion measures Inc_{SD} and Inc_{KT}. Since Inc_{SD} is an Inf-I_{GL} inclusion measure and I_{GL} is contrapositive, Proposition 5 implies that the Sinha-Dougherty approach can be viewed as a special case of Kitainik's approach to fuzzy MM.

5.2.4 *The Approach of Minkowski Addition*

Recall that the binary dilation of a set $\mathbf{A}$ by a structuring element $\mathbf{S}$ corresponds to the Minkowski addition of $\mathbf{A}$ and $\mathbf{S}$. Therefore, extending the Minkowski addition to fuzzy sets seems to provide a natural way to derive a fuzzy dilation [1, 19]. Then, a fuzzy erosion arises as the N_S-dual of this fuzzy dilation. This approach yields the particular case where $C_{\mathcal{F}} = C_M$ of De Baets' fuzzy dilation given in Eq. 73. Since the implication I_K represents the N_S-dual of the conjunction C_M, the corresponding fuzzy erosion is given in terms of the Inf-I_K inclusion measure as a special case of Eq. 71.

The approach of Minkowski addition can be embedded into the framework of Bloch and Maître because the minimum conjunction C_M represents a t-norm. In contrast to the Bloch-Maître approach, the former employs duality with respect to the standard fuzzy negation N_S only.

5.3 Inf-I/Sup-C Approaches Based on Adjunction: The Deng-Heijmans Approach

Deng and Heijmans, as well as other prominent researchers including Maragos, continue to emphasize the central role of the concept of *adjunction* in MM [15, 18, 23]. Therefore, the Deng-Heijmans approach to fuzzy MM differs from the general De Baets approach in one crucial aspect: Deng and Heijmans generate a fuzzy MM based on a pair (E_{DB}, D_{DB}) that is required to form an adjunction. According to the following proposition, an equivalent condition is the adjointness of $I_{\mathcal{F}}$ and $C_{\mathcal{F}}$, the operators appearing in Eqs. 71 and 73 [15].

Proposition 6 *Let $I_{\mathcal{F}}$ be a fuzzy implication and $C_{\mathcal{F}}$ be a fuzzy conjunction. The pair $(I_{\mathcal{F}}, C_{\mathcal{F}})$ forms an adjunction on* $[0,1]$ *if and only if the pair (E_{DB}, D_{DB}) given by Eqs.* 71 *and* 73 *is an adjunction on $\mathcal{F}(\mathbf{X})$.*

Unlike the pair (I_K, C_M), the pairs (I_L, C_L) and (I_G, C_M) form adjunctions on $[0,1]$. I. Bloch succeeded in relating the Inf-I/Sup-C approaches based on adjunction and the Inf-I/Sup-C approaches based on negation as follows [7]. A t-norm T and a fuzzy implication $I_{\mathcal{F}}$ are adjoint if and only if we have that for every fuzzy SE $\mathbf{s}$ the compositions of the corresponding fuzzy erosion $E_{DB}(\cdot, \mathbf{s})$ and the fuzzy dilation $D_{DB}(\cdot, \mathbf{s})$ that is the dual of $E_{DB}(\cdot, \mathbf{s})$ with respect to a fuzzy negation $N_{\mathcal{F}}$ yield (idempotent) openings and closings. Thus, openings and closings can only be derived from a fuzzy erosion E_{DB} and a fuzzy dilation D_{DB} based on a t-norm T that is the dual of $I_{\mathcal{F}}$ with respect to a fuzzy negation $N_{\mathcal{F}}$ if T and $I_{\mathcal{F}}$ are adjoint.

5.3.1 Approach of Maragos

Recently, Maragos presented a theory that is geared at unifying MM and lattice algebraic systems such as image algebra and minimax algebra [11, 23, 28, 29]. Ritter et al. had previously established image algebra [30], a heterogeneous or many-valued algebra in the sense of Birkhoff and Lipson [5] that provides a mathematical background for image processing and computer vision. Davidson proved that the lattice algebra known as minimax algebra can be embedded into image algebra [13]. The theory of minimax algebra arose from problems in operations research and machine scheduling [12]. Minimax algebra and MM are closely related [13] despite the fact that these theories were developed for completely different purposes.

In contrast to MM that focuses on the complete lattice structure of the sets of images $\mathbb{G}^{\mathbf{X}}$ and $\mathcal{F}(\mathbf{X})$, image algebra and minimax algebra investigate the interactions between the lattice sup/inf structure and the group structure of real addition or multiplication. However, image algebra and minimax algebra fail to exploit the lattice structure to the level that MM has and these theories have neglected important concepts of MM such as adjunctions.

According to the image algebra point of view, MM can be conducted in a minimax algebra structure that Cuninghame-Green named *blog*, which stands for *bounded lattice ordered group* [11]. Typical examples for blogs include $(\mathbb{G}, \vee, \wedge, +, +')$ where $\mathbb{G}$ equals $\bar{\mathbb{R}}$ or $\bar{\mathbb{Z}}$ as before and where $+$ and $+'$ denote the addition and the dual addition that we introduced in Sect. 3.3. As Maragos pointed out in [23], the blog structure does not capture several important aspects of MM, in particular fuzzy MM. Therefore, Maragos defines the less restrictive notion of *clodum* or *complete lattice-ordered double monoid*. A clodum $(\mathbb{V}, \vee, \wedge, \star, \star')$ consists of the following:

(C1) A complete infinitely-distributive lattice $(\mathbb{V}, \vee, \wedge)$;
(C2) A commutative monoid $(\mathbb{V}, \star)$ such that $\star$ is a dilation;
(C3) A commutative monoid $(\mathbb{V}, \star')$ such that $\star'$ is an erosion.

For example, we have that $[0,1]$ together with $\vee$, $\wedge$, a continuous t-norm T, and a continuous s-norm S represents a clodum but does not represent a blog. Note that $([0,1], \vee, \wedge, C_{\mathcal{F}}, I_{\mathcal{F}})$, the underlying algebraic structure of the more general approaches of De Baets and of Deng and Heijmans, does not constitute a clodum because the implication I is not commutative by definition. The clodum $([0,1], \vee, \wedge, T, S)$ lies at the root of Maragos' approach. Observe that the set of images $\mathcal{F}(\mathbf{X})$ inherits the clodum structure of $[0,1]$. Maragos also takes the concepts of adjunction and (dual) translation invariance into account. The latter concept refers the following operators of translation $\tau_{\mathbf{h},v}$ and dual translation $\tau'_{\mathbf{h},v}$ where $\mathbf{h} \in \mathbf{X}$ and $v \in [0,1]$.

$$\tau_{\mathbf{h},v}(\mathbf{a})(\mathbf{x}) = T(\mathbf{a}(\mathbf{x}-\mathbf{h}), v) \quad \text{and} \\ \tau'_{\mathbf{h},v}(\mathbf{a})(\mathbf{x}) = S(\mathbf{a}(\mathbf{x}-\mathbf{h}), v). \tag{79}$$

An operator on $\mathcal{F}(\mathbf{X})$ that commutes with the translations $\tau_{h,v}$ is called *translation invariant*. Similarly, an operator on $\mathcal{F}(\mathbf{X})$ that commutes with the dual translations $\tau'_{h,v}$ is called *dual-translation invariant*.

Maragos defines a fuzzy dilation $D_{MT}(\mathbf{a}, \mathbf{s})$ of the image $\mathbf{a}$ by the SE $\mathbf{s}$ as a Sup-T convolution of $\mathbf{a}$ and $\mathbf{s}$—in other words, he requires the fuzzy conjunction in Eq. 73 to be a continuous t-norm. Maragos proved that every translation-invariant dilation on the clodum $\mathcal{F}(\mathbf{X})$ is of this form. The adjoint fuzzy erosion E_{MT} can be expressed in terms of an Inf-I_T convolution of $\mathbf{a}$ and $\mathbf{s}$ where I_T is such that the pair (T, I_T) forms an adjunction. The implication I_T has the following representation [21, 23].

$$I_T(x, y) = \bigvee \{z \in [0,1] : T(x, z) \leq y\}. \tag{80}$$

Maragos also proposes a second, alternative approach to fuzzy MM. First, a fuzzy erosion E_{MS} is defined as an Inf-S convolution of the image $\mathbf{a}$ with the SE $\mathbf{s}$:

$$E_{MS}(\mathbf{a}, \mathbf{s})(\mathbf{x}) = \bigwedge_{\mathbf{y} \in \mathbf{X}} S(\mathbf{s}_{\mathbf{x}}(\mathbf{y}), \mathbf{a}(\mathbf{y})), \tag{81}$$

The function E_{MS} is a dual-translation invariant erosion. In fact, every dual-translation invariant erosion can be written as in Eq. 81 [23]. If D_{MS} denotes the adjoint fuzzy dilation, then $D_{MS}(\mathbf{a},\mathbf{s})$ is given by a Sup-J_S convolution of $\mathbf{a}$ with $\mathbf{s}$ where J_S is the adjoint operator of S that is defined as follows for every $x, y \in [0,1]$:

$$J_S(x,y) = \bigwedge\{z \in [0,1] : S(x,z) \geq y\}. \quad (82)$$

Note that $J_S(x,y) \leq z$ if and only if $y \leq S(x,z)$. Thus, $J_S(x,\cdot)$ and $S(x,\cdot)$ form an adjunction for every $x \in [0,1]$. Finally, we observe that J_S is not a fuzzy conjunction since $J_S(1,1) = \bigwedge\{z \in [0,1] : S(1,z) \geq 1\} = 0$.

5.3.2 The Approach of Lattice Isomorphism

An approach to fuzzy MM can be deduced from gray-scale MM on $\bar{\mathbb{R}}^{\mathbf{X}}$ by taking advantage of the fact that the complete lattices $[0,1]$ and $\bar{\mathbb{R}}$ are isomorphic [15].

Let $\mathbf{X}$ be an arbitrary point set and let $\theta : [0,1] \to \bar{\mathbb{R}}$ be a continuous lattice isomorphism such as $\theta(t) = tan(\pi(t-0.5))$. Note that θ and its inverse θ^{-1} induce continuous lattice isomorphisms $\boldsymbol{\theta} : \mathcal{F}(\mathbf{X}) \to \bar{\mathbb{R}}^{\mathbf{X}}$ and $\boldsymbol{\theta}^{-1} : \bar{\mathbb{R}}^{\mathbf{X}} \to \mathcal{F}(\mathbf{X})$ which establish a one-to-one relationship between fuzzy images in $\mathcal{F}(\mathbf{X})$ and gray-scale images in $\bar{\mathbb{R}}^{\mathbf{X}}$.

The following strategy can be employed to derive a fuzzy erosion from a gray-scale erosion. In a similar way, a fuzzy dilation can be obtained from a gray-scale dilation. Consider a fuzzy image $\mathbf{a} \in \mathcal{F}(\mathbf{X})$ and a fuzzy SE $\mathbf{s} \in \mathcal{F}(\mathbf{X})$. An application of the lattice isomorphism $\boldsymbol{\theta}$ converts $\mathbf{a}$ and $\mathbf{s}$ into a gray-scale image $\boldsymbol{\theta}(\mathbf{a})$ and a gray-scale SE $\boldsymbol{\theta}(\mathbf{s})$. After computing the gray-scale erosion of the image $\boldsymbol{\theta}(\mathbf{a})$ by the SE $\boldsymbol{\theta}(\mathbf{s})$, one transforms the resulting gray-scale image $\mathbf{b}$ into a fuzzy image $\boldsymbol{\theta}^{-1}(\mathbf{b})$.

An application of this strategy to the level set erosion and dilation as well as the umbra erosion and dilation results in the $\theta\mathcal{L}$-erosion $E_{\theta\mathcal{L}}$, the $\theta\mathcal{L}$-dilation $D_{\theta\mathcal{L}}$, the $\theta\mathcal{U}$-erosion $E_{\theta\mathcal{U}}$ and the $\theta\mathcal{U}$-dilation $D_{\theta\mathcal{U}}$:

$$\begin{aligned} E_{\theta\mathcal{L}}(\mathbf{a},\mathbf{s}) &= \boldsymbol{\theta}^{-1}\left[E_{\mathcal{L}}(\boldsymbol{\theta}(\mathbf{a}),\boldsymbol{\theta}(\mathbf{s}))\right] \quad \text{and} \\ D_{\theta\mathcal{L}}(\mathbf{a},\mathbf{s}) &= \boldsymbol{\theta}^{-1}\left[D_{\mathcal{L}}(\boldsymbol{\theta}(\mathbf{a}),\boldsymbol{\theta}(\mathbf{s}))\right], \end{aligned} \quad (83)$$

$$\begin{aligned} E_{\theta\mathcal{U}}(\mathbf{a},\mathbf{s}) &= \boldsymbol{\theta}^{-1}\left[E_{\mathcal{U}}(\boldsymbol{\theta}(\mathbf{a}),\boldsymbol{\theta}(\mathbf{s}))\right] \quad \text{and} \\ D_{\theta\mathcal{U}}(\mathbf{a},\mathbf{s}) &= \boldsymbol{\theta}^{-1}\left[D_{\mathcal{U}}(\boldsymbol{\theta}(\mathbf{a}),\boldsymbol{\theta}(\mathbf{s}))\right]. \end{aligned} \quad (84)$$

The following theorem formulates these fuzzy erosions and dilations in terms of Inf-I inclusion and Sup-C intersection measures.

Theorem 4 *Suppose that $\theta : [0,1] \to \bar{\mathbb{R}}$ is a continuous lattice isomorphism. The $\theta\mathcal{L}$-erosion $E_{\theta\mathcal{L}}$ coincides with E_G and the $\theta\mathcal{L}$-dilation coincides with D_M.*

Moreover, if $I_\theta, C_\theta : [0,1] \times [0,1] \to [0,1]$ denote the fuzzy operators that are defined below then I_θ represents a fuzzy implication and C_θ represents a fuzzy conjunction.

$$\begin{aligned} I_\theta(x,y) &= \theta^{-1}\left(\theta(y) +' (-\theta(x))\right) \quad \text{and} \\ C_\theta(x,y) &= \theta^{-1}\left(\theta(y) + \theta(x)\right). \end{aligned} \quad (85)$$

In addition, we have

$$\begin{aligned} E_{\theta\mathcal{U}}(\mathbf{a},\mathbf{s})(\mathbf{x}) &= \bigwedge_{\mathbf{y}\in\mathbf{X}} I_\theta\left(\mathbf{s}_{\mathbf{x}}(\mathbf{y}),\mathbf{a}(\mathbf{y})\right) \quad \text{and} \\ D_{\theta\mathcal{U}}(\mathbf{a},\mathbf{s})(\mathbf{x}) &= \bigvee_{\mathbf{y}\in\mathbf{X}} C_\theta\left(\bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y}),\mathbf{a}(\mathbf{y})\right). \end{aligned} \quad (86)$$

Proof We only need to prove the first claim since the proof of the second part of the theorem can be found in [15]. In fact, we will only show that $\theta\mathcal{L}$-dilation coincides with D_M. The equality $E_{\theta\mathcal{L}}(\mathbf{a},\mathbf{s})(\mathbf{x}) = \bigwedge_{\mathbf{y}\in\mathbf{X}} I_G(\mathbf{s}_{\mathbf{x}}(\mathbf{y}),\mathbf{a}(\mathbf{y}))$ can be obtained in a similar fashion.

The following equations follow from the definition of the $\theta\mathcal{L}$-dilation and from the fact that θ is a continuous lattice isomorphism. The latter implies that θ commutes with the operations of supremum and infimum.

$$\begin{aligned} D_{\theta\mathcal{L}}(\mathbf{a},\mathbf{s})(\mathbf{x}) &= \theta^{-1}\left\{\bigvee_{\mathbf{y}\in\mathbf{X}} \theta\left(\mathbf{s}_{\mathbf{x}}(\mathbf{y})\right) \wedge \theta\left(\mathbf{a}(\mathbf{y})\right)\right\} \\ &= \bigvee_{\mathbf{y}\in\mathbf{X}} \theta^{-1}\left[\theta\left(\mathbf{s}(\mathbf{x}-\mathbf{y})\right) \wedge \theta\left(\mathbf{a}(\mathbf{y})\right)\right] \\ &= \bigvee_{\mathbf{y}\in\mathbf{X}} \mathbf{s}(\mathbf{x}-\mathbf{y}) \wedge \mathbf{a}(\mathbf{y}) = \bigvee_{\mathbf{y}\in\mathbf{X}} \bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y}) \wedge \mathbf{a}(\mathbf{y}) \\ &= D_M(\mathbf{a},\mathbf{s})(\mathbf{x}). \end{aligned} \quad (87)$$

□

Recall that (I_G, C_M) forms an adjunction. As Deng and Heijmans have pointed out, I_θ and C_θ are adjoint as well. Therefore, Proposition 6 induces the following corollary.

Corollary 1 *The pairs $(E_{\theta\mathcal{L}}, D_{\theta\mathcal{L}})$ and $(E_{\theta\mathcal{U}}, D_{\theta\mathcal{U}})$ form adjunctions on $\mathcal{F}(\mathbf{X})$.*

The approaches $(E_{\theta\mathcal{L}}, D_{\theta\mathcal{L}})$ and $(E_{\theta\mathcal{U}}, D_{\theta\mathcal{U}})$ can not only be embedded into the Deng-Heijmans framework as stated in Corollary 1 but also into the Maragos framework in view of the following theorem.

Theorem 5 *Let $\theta : [0,1] \to \bar{\mathbb{R}}$ be a continuous lattice isomorphism. Consider C_θ given by Eq. 85 and define $C'_\theta : [0,1] \times [0,1] \to [0,1]$ as follows for every $x, y \in [0,1]$:*

$$C'_\theta(x,y) = \theta^{-1}\left(\theta(y) +' \theta(x)\right). \quad (88)$$

Then, the fuzzy interval $[0, 1]$ *equipped with* C_θ *and* C'_θ *represents a clodum.*

Proof Clearly, the structure $([0, 1], \vee, \wedge)$ constitutes a complete infinitely-distributive lattice. The following shows that C_θ is a dilation and that the structure $([0, 1], C_\theta)$ is a commutative monoid. Analogously, one can show that C'_θ is an erosion and that $([0, 1], C'_\theta)$ represents a commutative monoid.

First, note that the continuity of θ and "+" implies that C_θ is continuous. Consequently, C_θ represents a dilation. Now, let $x, y, z \in [0, 1]$. The following sequence of equations shows that C_θ is an associative operation.

$$\begin{aligned} C_\theta(x, C_\theta(y, z)) &= \theta^{-1}\left(\theta(x) + \theta(\theta^{-1}(\theta(y) + \theta(z)))\right) \\ &= \theta^{-1}(\theta(x) + \theta(y) + \theta(z)) \\ &= \theta^{-1}\left(\theta(\theta^{-1}(\theta(x) + \theta(y))) + \theta(z)\right) \\ &= C_\theta(C_\theta(x, y), z). \end{aligned} \tag{89}$$

The commutativity and the monotonicity properties of C_θ follow from the fact that θ is a lattice isomorphism and the fact that "+" a is commutative and monotonic binary operation. Finally, the following equalities reveal that the element $e = \theta^{-1}(0) \in [0, 1]$ is an identity of C_θ.

$$\begin{aligned} C_\theta(x, e) &= \theta^{-1}(\theta(x) + \theta(e)) \\ &= \theta^{-1}(\theta(x) + 0) = x \quad \forall x \in [0, 1]. \end{aligned} \tag{90}$$

Thus, the structure $([0, 1], C_\theta)$ represents a commutative monoid. □

Since $E_\mathcal{U}$ and $D_\mathcal{U}$ constitute dual operators with respect to the negation $*$ according to Proposition 5, it seems reasonable to assume that $E_{\theta\mathcal{U}}$ and $D_{\theta\mathcal{U}}$ are dual operators with respect to some negation on $\mathcal{F}(\mathbf{X})$. In fact, we have the following theorem:

Theorem 6 *Let* $\theta : [0, 1] \to \bar{\mathbb{R}}$ *be a continuous lattice isomorphism. The operator* $N_\theta : [0, 1] \to [0, 1]$ *given by* $N_\theta(\mathbf{x}) = \theta^{-1}(-\theta(\mathbf{x}))$ *represents a fuzzy negation and the fuzzy morphological operators* $E_{\theta\mathcal{U}}$ *and* $D_{\theta\mathcal{U}}$ *are dual with respect to this fuzzy negation.*

Proof Clearly, we have $N_\theta(1) = 0$ and $N_\theta(0) = 1$. Moreover, straightforward computation reveals that $N_\theta(N_\theta(x)) = x$ for every $x \in [0, 1]$. Therefore, N_θ represents a fuzzy negation and we obtain the following sequence of equalities for all $x, y \in [0, 1]$:

$$\begin{aligned} &N_\theta\left(C_\theta\left(x, N_\theta\left(y\right)\right)\right) \\ &\quad = \theta^{-1}\left[-\theta\left(\theta^{-1}\left(\theta\left(\theta^{-1}\left(-\theta(y)\right)\right) + \theta\left(x\right)\right)\right)\right] \\ &\quad = \theta^{-1}\left[-\left(\left(-\theta(y)\right) + \theta\left(x\right)\right)\right] \\ &\quad = \theta^{-1}\left[\theta(y) +' \left(-\theta\left(x\right)\right)\right] = I_\theta(x, y). \end{aligned} \tag{91}$$

We conclude the proof of the theorem as follows. For all $\mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X})$ and for all $\mathbf{x} \in \mathbf{X}$, we have

$$\begin{aligned} &\mathbf{N}_\theta\left[D_{\theta\mathcal{U}}\left(\mathbf{N}_\theta(\mathbf{a}), \bar{\mathbf{s}}\right)\right](\mathbf{x}) \\ &\quad = N_\theta\left[\bigvee_{\mathbf{y}\in\mathbf{X}} C_\theta\left(\mathbf{s}_\mathbf{x}(\mathbf{y}), N_\theta(\mathbf{a}(\mathbf{y}))\right)\right] \\ &\quad = \bigwedge_{\mathbf{y}\in\mathbf{X}} N_\theta\left[C_\theta\left(\mathbf{s}_\mathbf{x}(\mathbf{y}), N_\theta(\mathbf{a}(\mathbf{y}))\right)\right] \\ &\quad = \bigwedge_{\mathbf{y}\in\mathbf{X}} I_\theta(\mathbf{s}_\mathbf{x}(\mathbf{y}), \mathbf{a}(\mathbf{y})) = E_{\theta\mathcal{U}}\left(\mathbf{a}, \mathbf{s}\right). \end{aligned} \tag{92}$$

□

As a consequence of Theorem 6, the pair $(E_{\theta\mathcal{U}}, D_{\theta\mathcal{U}})$ also fits into a general approach based on negation.

Finally, note that the lattice isomorphism θ and its inverse θ^{-1} also provide a tool for constructing gray-scale operators from fuzzy operators. Let $E_\mathcal{F}$ and $D_\mathcal{F}$ be a fuzzy erosion and a fuzzy dilation. A gray-scale erosion can be defined as follows: Given a gray-scale image $\mathbf{a} \in \bar{\mathbb{R}}^\mathbf{X}$ and a gray-scale SE $\mathbf{s} \in \bar{\mathbb{R}}^\mathbf{X}$, the $\theta^{-1}\mathcal{F}$-erosion $E_{\theta^{-1}\mathcal{F}}$ is given by

$$E_{\theta^{-1}\mathcal{F}}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\theta}\left[E_\mathcal{F}\left(\boldsymbol{\theta}^{-1}(\mathbf{a}), \boldsymbol{\theta}^{-1}(\mathbf{s})\right)\right]. \tag{93}$$

The $\theta^{-1}\mathcal{F}$-dilation $D_{\theta^{-1}\mathcal{F}}$ is defined in a similar fashion. Straightforward computation shows that the following equations hold:

$$\begin{aligned} &E_\mathcal{U}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\theta}[E_{\theta\mathcal{U}}(\boldsymbol{\theta}^{-1}(\mathbf{a}), \boldsymbol{\theta}^{-1}(\mathbf{s}))] \quad \text{and} \\ &D_\mathcal{U}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\theta}[D_{\theta\mathcal{U}}(\boldsymbol{\theta}^{-1}(\mathbf{a}), \boldsymbol{\theta}^{-1}(\mathbf{s}))], \end{aligned} \tag{94}$$

$$\begin{aligned} &E_\mathcal{L}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\theta}[E_G(\boldsymbol{\theta}^{-1}(\mathbf{a}), \boldsymbol{\theta}^{-1}(\mathbf{s}))] \quad \text{and} \\ &D_\mathcal{L}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\theta}[D_M(\boldsymbol{\theta}^{-1}(\mathbf{a}), \boldsymbol{\theta}^{-1}(\mathbf{s}))]. \end{aligned} \tag{95}$$

Thus, we can identify the umbra approach and the level set approach with certain particular approaches to fuzzy MM [15].

5.3.3 The Approach of Ronse

In view of the fact that $\mathcal{F}(\mathbf{X}) \subset \bar{\mathbb{R}}^\mathbf{X}$, truncating the gray-levels outside the unit interval yields a simple method for transforming a gray-scale image into a fuzzy image while preserving the order of the gray-levels. In particular, this transformation can be performed after an application of a gray-scale morphological operator to a fuzzy image $\mathbf{a} \in \mathcal{F}(\mathbf{X})$. Thus, the sequence consisting of the gray-scale morphological operator followed by truncation yields a fuzzy morphological operator.

For example, let us consider the fuzzy morphological operators $\boldsymbol{\sigma}(E_{\mathcal{L}}(\cdot, \mathbf{s}))$ and $\boldsymbol{\sigma}(D_{\mathcal{L}}(\cdot, \mathbf{s}))$, where $\mathbf{s} \in \mathcal{F}(\mathbf{X})$ and where $\boldsymbol{\sigma}$ denotes the transformation of truncation that is formally defined as follows:

$$\boldsymbol{\sigma}(\mathbf{b})(\mathbf{x}) = 1 \wedge [0 \vee \mathbf{b}(\mathbf{x})], \quad \forall \mathbf{b} \in \bar{\mathbb{R}}^{\mathbf{X}}, \ \mathbf{x} \in \mathbf{X}. \tag{96}$$

Let us recall the formula for the level set dilation $D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})$ that we presented in Eq. 26. We realize that the image $D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})$ belongs to $\mathcal{F}(\mathbf{X})$ if the image $\mathbf{a}$ and the SE $\mathbf{s}$ are fuzzy. Thus, truncating $D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})$ becomes obsolete and we have $\boldsymbol{\sigma}(D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})) = D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})$ for all $\mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X})$. Furthermore, a comparison of Eq. 26 with Eq. 73 reveals that $\boldsymbol{\sigma}(D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})) = D_M(\mathbf{a}, \mathbf{s})$.

Now let us consider the level set erosion $E_{\mathcal{L}}$. After a short glance at Eq. 24, it becomes apparent that $E_{\mathcal{L}}(\mathbf{a}, \mathbf{s})(\mathbf{x}) \geq 0$ for all $\mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X})$ and for all $\mathbf{x} \in \mathbf{X}$. Therefore, we have $\boldsymbol{\sigma}(E_{\mathcal{L}}(\mathbf{a}, \mathbf{s}))(\mathbf{x}) = E_{\mathcal{L}}(\mathbf{a}, \mathbf{s})(\mathbf{x}) \wedge 1$ for all $\mathbf{x} \in \mathbf{X}$. This identity shows that truncating $E_{\mathcal{L}}$ coincides with a special case of De Baets' fuzzy erosion given by Eq. 72, namely the Gödel fuzzy erosion E_G. The latter is defined in terms of the Gödel implication I_G that was introduced in Eq. 45.

We summarize the preceding observations in the following theorem:

Theorem 7 *The following equations hold for all* $\mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X})$.

$$\begin{aligned} &\boldsymbol{\sigma}(E_{\mathcal{L}}(\mathbf{a}, \mathbf{s})) = E_G(\mathbf{a}, \mathbf{s}) \quad \textit{and} \\ &\boldsymbol{\sigma}(D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})) = D_{\mathcal{L}}(\mathbf{a}, \mathbf{s}) = D_M(\mathbf{a}, \mathbf{s}). \end{aligned} \tag{97}$$

As mentioned in Sect. 5.3, the pair (I_G, C_M) forms an adjunction which implies that E_G and D_M are adjoint as well. By Proposition 1, this fact guarantees that $\boldsymbol{\sigma}(E_{\mathcal{L}}(\cdot, \mathbf{s}))$ is an erosion and that $\boldsymbol{\sigma}(D_{\mathcal{L}}(\cdot, \mathbf{s}))$ is a dilation for every $\mathbf{s} \in \mathcal{F}(\mathbf{X})$.

We would like to point out, however, that the pair consisting of gray-scale erosion followed by truncation and the corresponding gray-scale dilation followed by truncation does not always form an adjunction. This phenomenon may even occur in the case where the underlying gray-scale operators are adjoint. An excellent example for this situation is given by $\boldsymbol{\sigma}(E_{\mathcal{U}}(\cdot, \mathbf{s}))$ and $\boldsymbol{\sigma}(D_{\mathcal{U}}(\cdot, \mathbf{s}))$, whose composition $\boldsymbol{\sigma}(D_{\mathcal{U}}(\cdot, \mathbf{s})) \circ \boldsymbol{\sigma}(E_{\mathcal{U}}(\cdot, \mathbf{s}))$ does not lead to an opening [31]. In fact, $\boldsymbol{\sigma}(E_{\mathcal{U}}(\cdot, \mathbf{s}))$ does not even represent an erosion and $\boldsymbol{\sigma}(D_{\mathcal{U}}(\cdot, \mathbf{s}))$ does not even represent a dilation for every $\mathbf{s} \in \mathcal{F}(\mathbf{X})$. Furthermore, as an additional problem, truncating the gray-level outside the interval [0, 1] may entail a considerable loss of information.

In the umbra case, the problems that we outlined above can be circumvented by translating the gray-levels before truncating them. Specifically, if $\mathbf{1} : \mathbf{X} \to \bar{\mathbb{R}}$ denotes the constant gray-scale image such that $\mathbf{1}(\mathbf{x}) = 1$ for every $\mathbf{x} \in \mathbf{X}$, then we define $E_{\mathcal{U}1}$ and $D_{\mathcal{U}1}$ as follows for every gray-scale image $\mathbf{a} \in \bar{\mathbb{R}}^{\mathbf{X}}$ and for every SE $\mathbf{s} \in \bar{\mathbb{R}}^{\mathbf{X}}$:

$$\begin{aligned} &E_{\mathcal{U}1}(\mathbf{a}, \mathbf{s}) = E_{\mathcal{U}}(\mathbf{a}, \mathbf{s}) + \mathbf{1} \quad \text{and} \\ &D_{\mathcal{U}1}(\mathbf{a}, \mathbf{s}) = D_{\mathcal{U}}(\mathbf{a}, \mathbf{s}) - \mathbf{1}. \end{aligned} \tag{98}$$

The argumentation below shows that the pair $(E_{\mathcal{U}1}, D_{\mathcal{U}1})$ forms an adjunction. Therefore, $E_{\mathcal{U}1}$ represents an erosion and $D_{\mathcal{U}1}$ represents a dilation by Proposition 1. Let $\mathbf{a}, \mathbf{b} \in \bar{\mathbb{R}}^{\mathbf{X}}$ be gray-scale images and let $\mathbf{s} \in \bar{\mathbb{R}}^{\mathbf{X}}$ be a gray-scale SE.

$$\begin{aligned} &D_{\mathcal{U}1}(\mathbf{a}, \mathbf{s}) \leq \mathbf{b} \\ &\Leftrightarrow \quad D_{\mathcal{U}}(\mathbf{a}, \mathbf{s}) - \mathbf{1} \leq \mathbf{b} \\ &\Leftrightarrow \quad D_{\mathcal{U}}(\mathbf{a}, \mathbf{s}) \leq \mathbf{b} + \mathbf{1} \\ &\Leftrightarrow \quad \mathbf{a} \leq E_{\mathcal{U}}(\mathbf{b} + \mathbf{1}, \mathbf{s}) = E_{\mathcal{U}}(\mathbf{b}, \mathbf{s}) + \mathbf{1} = E_{\mathcal{U}1}(\mathbf{a}, \mathbf{s}). \end{aligned} \tag{99}$$

The following theorem demonstrates that truncating the adjoint operators $E_{\mathcal{U}1}$ and $D_{\mathcal{U}1}$ results in the adjoint operators E_L and D_L, i.e. the Lukasiewicz erosion and the Lukasiewicz dilation.

Theorem 8 *The following equations hold for all* $\mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X})$.

$$\begin{aligned} &\boldsymbol{\sigma}(E_{\mathcal{U}1}(\mathbf{a}, \mathbf{s})) = E_L(\mathbf{a}, \mathbf{s}) \quad \textit{and} \\ &\boldsymbol{\sigma}(D_{\mathcal{U}1}(\mathbf{a}, \mathbf{s})) = D_L(\mathbf{a}, \mathbf{s}). \end{aligned} \tag{100}$$

Proof From the definitions of the fuzzy erosion E_L, we infer that

$$\begin{aligned} E_L(\mathbf{a}, \mathbf{s})(\mathbf{x}) &= \bigwedge_{\mathbf{y} \in \mathbf{X}} I_L(\mathbf{s}_{\mathbf{x}}(\mathbf{y}), \mathbf{a}(\mathbf{y})) \\ &= \bigwedge_{\mathbf{y} \in \mathbf{X}} \left[1 \wedge (\mathbf{a}(\mathbf{y}) - \mathbf{s}_{\mathbf{x}}(\mathbf{y}) + 1)\right] \\ &= 1 \wedge \left[\bigwedge_{\mathbf{y} \in \mathbf{X}} (\mathbf{a}(\mathbf{y}) - \mathbf{s}_{\mathbf{x}}(\mathbf{y})) + 1\right] \\ &= 1 \wedge \left(0 \vee \left[\bigwedge_{\mathbf{y} \in \mathbf{X}} (\mathbf{a}(\mathbf{y}) - \mathbf{s}_{\mathbf{x}}(\mathbf{y})) + 1\right]\right) \\ &= 1 \wedge [0 \vee (E_{\mathcal{U}}(\mathbf{a}, \mathbf{s})(\mathbf{x}) + 1)] \\ &= \boldsymbol{\sigma}(E_{\mathcal{U}1}(\mathbf{a}, \mathbf{s}))(\mathbf{x}), \end{aligned} \tag{101}$$

for all $\mathbf{x} \in \mathbf{X}$ and for all $\mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X})$.

In a similar fashion, we obtain $D_L(\mathbf{a}, \mathbf{s})(\mathbf{x}) = \boldsymbol{\sigma}(D_{\mathcal{U}1}(\mathbf{a}, \mathbf{s}))(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{X}$ and for all $\mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X})$. □

As Ronse pointed out in [31], the problems regarding $\boldsymbol{\sigma}(E_{\mathcal{U}}(\cdot, \mathbf{s}))$ and $\boldsymbol{\sigma}(D_{\mathcal{U}}(\cdot, \mathbf{s}))$ are due to conceptual differences between the universal bounds of $\bar{\mathbb{R}}$ and [0, 1]. Seeking a general solution to these problems for all approaches based on truncation, Ronse introduced the following functions $\theta_0, \theta_1 : [0,1] \to \bar{\mathbb{R}}$.

$$\theta_0(x) = \begin{cases} x, & \text{if } x > 0, \\ -\infty, & \text{if } x = 0, \end{cases} \qquad \theta_1(x) = \begin{cases} x, & \text{if } x < 1, \\ +\infty, & \text{if } x = 1. \end{cases} \tag{102}$$

Note that θ_0 and θ_1 effect transformations between the universal bounds.

Let $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$ be the transformations obtained applying θ_0 and θ_1 pointwise. A general approach to fuzzy MM arises from an adjunction consisting of an erosion and a dilation in the gray-scale domain and by transforming these operators using $\boldsymbol{\theta}_0$ or $\boldsymbol{\theta}_1$ as well as the truncation $\boldsymbol{\sigma}$. Ronse's approach preserves the adjointness of erosion and dilation. More precisely, the following fundamental proposition forms the backbone of Ronse's approach to fuzzy MM.

Proposition 7 *If the pair* (ε, δ) *forms an adjunction on* $\mathbb{G}^{\mathbf{X}}$ *then the pair* $(\boldsymbol{\sigma}\, \varepsilon\, \boldsymbol{\theta}_1, \boldsymbol{\sigma}\, \delta\, \boldsymbol{\theta}_0)$ *forms an adjunction on* $\mathcal{F}(\mathbf{X})$.

In some cases, the operators $\boldsymbol{\sigma}\, \varepsilon\, \boldsymbol{\theta}_1$ and $\boldsymbol{\sigma}\, \delta\, \boldsymbol{\theta}_0$ can be simplified as follows.

Proposition 8 *Let* (ε, δ) *be an adjunction on* $\mathbb{G}^{\mathbf{X}}$. *The following statements hold true for every* $\mathbf{a} \in \mathcal{F}(\mathbf{X})$.

1. *If* $\varepsilon(\mathbf{0}) \geq \mathbf{0}$ *then* $\boldsymbol{\sigma}\varepsilon\boldsymbol{\theta}_1(\mathbf{a}) = \boldsymbol{\sigma}\varepsilon(\mathbf{a})$.
2. *If* $\delta(\mathbf{1}) \leq \mathbf{1}$ *then* $\boldsymbol{\sigma}\delta\boldsymbol{\theta}_0(\mathbf{a}) = \boldsymbol{\sigma}\delta(\mathbf{a})$.
3. *If* $\varepsilon(\mathbf{1}) = \mathbf{1}$ *and* $\delta(\mathbf{0}) = \mathbf{0}$ *then* $\boldsymbol{\sigma}\varepsilon\boldsymbol{\theta}_1(\mathbf{a}) = \varepsilon(\mathbf{a})$ *and* $\boldsymbol{\sigma}\delta\boldsymbol{\theta}_0(\mathbf{a}) = \delta(\mathbf{a})$.

As an example for this simplified situation, let us apply Ronse's approach to the $\mathcal{L}$-erosion $E_{\mathcal{L}}(\cdot, \mathbf{s})$ and the $\mathcal{L}$-dilation $D_{\mathcal{L}}(\cdot, \mathbf{s})$, where $\mathbf{s} \in \mathcal{F}(\mathbf{X})$. We will use the following notations:

$$E_{R\mathcal{L}}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\sigma}(E_{\mathcal{L}}(\boldsymbol{\theta}_1(\mathbf{a}), \mathbf{s})) \quad \text{and} \quad D_{R\mathcal{L}}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\sigma}(D_{\mathcal{L}}(\boldsymbol{\theta}_0(\mathbf{a}), \mathbf{s})) \quad \forall \mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X}). \tag{103}$$

As we have mentioned before, $E_{\mathcal{L}}(\mathbf{a}, \mathbf{s}) \geq \mathbf{0}$ and that $D_{\mathcal{L}}(\mathbf{a}, \mathbf{s}) \leq \mathbf{1}$ for all fuzzy $\mathbf{a}$ and $\mathbf{s}$. Therefore, combining Proposition 8 with Theorem 7 yields the following:

Theorem 9 *For all* $\mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X})$, *we have*:

$$E_{R\mathcal{L}}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\sigma}(E_{\mathcal{L}}(\mathbf{a}, \mathbf{s})) = E_G(\mathbf{a}, \mathbf{s}) \quad \text{and} \quad D_{R\mathcal{L}}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\sigma}(D_{\mathcal{L}}(\mathbf{a}, \mathbf{s})) = D_M(\mathbf{a}, \mathbf{s}). \tag{104}$$

This theorem also shows that the fuzzy MM based on $E_{R\mathcal{L}}$ and $D_{R\mathcal{L}}$ belongs to the category of Inf-I/Sup-C approaches. We would like to clarify however that there are adjunctions ε, δ on $\mathbb{G}^{\mathbf{X}}$ such that $\boldsymbol{\sigma}\varepsilon\boldsymbol{\theta}_1$ cannot be described in terms of Eq. 71 or $\boldsymbol{\sigma}\delta\boldsymbol{\theta}_0$ cannot be described in terms of Eq. 73.

Let us apply Ronse's approach to the umbra erosion and dilation. The symbols $E_{R\mathcal{U}}$ and $D_{R\mathcal{U}}$ denote the following operators.

$$E_{R\mathcal{U}}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\sigma}(E_{\mathcal{U}}(\boldsymbol{\theta}_1(\mathbf{a}), \mathbf{s})) \quad \text{and} \quad D_{R\mathcal{U}}(\mathbf{a}, \mathbf{s}) = \boldsymbol{\sigma}(D_{\mathcal{U}}(\boldsymbol{\theta}_0(\mathbf{a}), \mathbf{s})), \quad \forall \mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X}). \tag{105}$$

We compute $E_{R\mathcal{U}}(\mathbf{a}, \mathbf{s})(\mathbf{x})$ as follows for every $\mathbf{a}, \mathbf{s} \in \mathcal{F}(\mathbf{X})$ and for every $\mathbf{x} \in \mathbf{X}$:

$$\begin{aligned} E_{R\mathcal{U}}(\mathbf{a}, \mathbf{s})(\mathbf{x}) &= \sigma\left(E_{\mathcal{U}}(\boldsymbol{\theta}_1(\mathbf{a}), \mathbf{s})(\mathbf{x})\right) \\ &= 1 \wedge \left[0 \vee \left(\bigwedge_{\mathbf{y} \in \mathbf{X}} \theta_1(\mathbf{a}(\mathbf{y})) - \mathbf{s}_{\mathbf{x}}(\mathbf{y})\right)\right] \\ &= \bigwedge_{\mathbf{y} \in \mathbf{X}} 1 \wedge \left[0 \vee (\theta_1(\mathbf{a}(\mathbf{y})) - \mathbf{s}_{\mathbf{x}}(\mathbf{y}))\right] \\ &= \bigwedge_{\mathbf{y} \in \mathbf{X}} \tilde{I}_{R\mathcal{U}}(\mathbf{s}_{\mathbf{x}}(\mathbf{y}), \mathbf{a}(\mathbf{y})), \end{aligned} \tag{106}$$

where $\tilde{I}_{R\mathcal{U}} : [0,1] \times [0,1] \to [0,1]$ is defined as follows:

$$\tilde{I}_{R\mathcal{U}}(x, y) = \sigma(\theta_1(y) - x). \tag{107}$$

Here, $\sigma : \bar{\mathbb{R}} \to [0,1]$ is given by $\sigma(x) = 1 \wedge (0 \vee x)$ for every $x \in \bar{\mathbb{R}}$. Note that the function $\tilde{I}_{R\mathcal{U}}$ is decreasing in the first argument and increasing in the second argument. However, $\tilde{I}_{R\mathcal{U}}$ satisfies the boundary conditions $\tilde{I}_{R\mathcal{U}}(0,1) = \tilde{I}_{R\mathcal{U}}(1,1) = 1$ and $\tilde{I}_{R\mathcal{U}}(0,0) = \tilde{I}_{R\mathcal{U}}(1,0) = 0$. Therefore, $\tilde{I}_{R\mathcal{U}}$ does not represent a fuzzy implication. Similarly, we derive the following representation for $D_{R\mathcal{U}}(\mathbf{a}, \mathbf{s})(\mathbf{x})$.

$$\begin{aligned} D_{R\mathcal{U}}(\mathbf{a}, \mathbf{s})(\mathbf{x}) &= \sigma\left(D_{\mathcal{U}}(\boldsymbol{\theta}_0(\mathbf{a}), \mathbf{s})(\mathbf{x})\right) \\ &= 1 \wedge \left[0 \vee \left(\bigvee_{\mathbf{y} \in \mathbf{X}} \theta_0(\mathbf{a}(\mathbf{y})) + \mathbf{s}(\mathbf{x} - \mathbf{y})\right)\right] \\ &= \bigvee_{\mathbf{y} \in \mathbf{X}} 1 \wedge \left[0 \vee (\bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y}) + \theta_0(\mathbf{a}(\mathbf{y})))\right] \\ &= \bigvee_{\mathbf{y} \in \mathbf{X}} \tilde{C}_{R\mathcal{U}}(\bar{\mathbf{s}}_{\mathbf{x}}(\mathbf{y}), \mathbf{a}(\mathbf{y})), \end{aligned} \tag{108}$$

where $\tilde{C}_{R\mathcal{U}} : [0,1] \times [0,1] \to [0,1]$ is defined as follows:

$$\tilde{C}_{R\mathcal{U}}(x, y) = \sigma(x + \theta_0(y)). \tag{109}$$

The function $\tilde{C}_{R\mathcal{U}}$ does not constitute a fuzzy conjunction since $\tilde{C}_{R\mathcal{U}}(0,1) = 1 \neq 0$.

Furthermore, we have that $D_{R\mathcal{U}}(\mathbf{a}, \mathbf{s})$ cannot be written in terms of a supremum of conjunctions as in Eq. 71. To explain this fact, let us consider an image $\mathbf{a}$ such that $\mathbf{a}(\mathbf{x}_0) = 1$ and $\mathbf{a}(\mathbf{x}) = 0$ for all $\mathbf{x} \neq \mathbf{x}_0$. Also assume that $\mathbf{s}_{\mathbf{x}_0}(\mathbf{x}_0) = 0$. Computing $D_{R\mathcal{U}}(\mathbf{a}, \mathbf{s})(\mathbf{x}_0)$ should yield $C_{\mathcal{F}}(\mathbf{s}_{\mathbf{x}_0}(\mathbf{x}_0), \mathbf{a}(\mathbf{x}_0)) = C_{\mathcal{F}}(0, 1)$ for some fuzzy conjunction $C_{\mathcal{F}}$. This expression equals 0 by the definition of a fuzzy conjunction, contradicting the fact that $D_{R\mathcal{U}}(\mathbf{a}, \mathbf{s})(\mathbf{x}_0) = \tilde{C}_{R\mathcal{U}}(0, 1) = 1$.

These comments reveal that $(E_{R\mathcal{U}}, D_{R\mathcal{U}})$ does not fit into the Inf-I/Sup-C framework although $(E_{R\mathcal{U}}, D_{R\mathcal{U}})$ form an adjunction and $E_{R\mathcal{U}}$ can be expressed as an Inf-$I_{R\mathcal{U}}$ erosion. This situation occurs because there is no fuzzy conjunction $C_{R\mathcal{U}}$ such that $I_{R\mathcal{U}}$ and $C_{R\mathcal{U}}$ are adjoint.

A slight modification of the $(E_{R\mathcal{U}}, D_{R\mathcal{U}})$-approach yields an Inf-I/Sup-C approach. Consider the translated umbra operators $E_{\mathcal{U}1}$ and $D_{\mathcal{U}1}$. As mentioned before, these operators are adjoint. An application of Ronse's approach to the pair $(E_{\mathcal{U}1}, D_{\mathcal{U}1})$ yields the adjunction $(E_{R\mathcal{U}1}, D_{R\mathcal{U}1})$. Straightforward calculations show that $E_{R\mathcal{U}1}$ can be written as an Inf-$I_{R\mathcal{U}1}$ erosion where $I_{R\mathcal{U}1}$ is a fuzzy implication given by

$$I_{R\mathcal{U}1}(x, y) = \sigma(\theta_1(y) - x + 1)\,. \tag{110}$$

Similarly, we have that $D_{R\mathcal{U}1}$ can be written as a Sup-$C_{R\mathcal{U}1}$ dilation where $C_{R\mathcal{U}1}$ is a fuzzy conjunction given by

$$C_{R\mathcal{U}1}(x, y) = \sigma(x + \theta_0(y) - 1)\,. \tag{111}$$

6 Conclusions

To our knowledge, this paper is the first to classify the main approaches to fuzzy (and gray-scale) MM in terms of two simple criteria: the underlying notions of inclusion measure and duality. The paper includes new theorems and observations that are relevant for our classification scheme and that clarify a number of important facts concerning fuzzy MM. Figure 2 exhibits the resulting classification. We believe that the results of this paper will be useful for developing new approaches and for choosing an approach to fuzzy MM that is suited for a given application.

The definition of a fuzzy erosion and a fuzzy dilation lies at the root of an approach to fuzzy MM. A fuzzy erosion is determined by the choice of a fuzzy inclusion measure and a fuzzy dilation is determined by the choice of a fuzzy intersection measure. Almost all approaches to fuzzy MM comply with the general framework of De Baets, i.e. they use an infimum of implications to generate an inclusion measure and a supremum of conjunctions to generate an intersection measure. Strictly speaking, we may not refer to the resulting operators $E_{\mathcal{F}}$ and $D_{\mathcal{F}}$ as fuzzy erosion and fuzzy dilation, since an operator of the form given by Eq. 71 does not necessarily commute with the infimum and since an operator of the form given by Eq. 73 does not necessarily commute with the supremum operation.

Apart from Eqs. 71 and 73, De Baets does not impose any restrictions on fuzzy erosions and dilations. Other researchers, however, hold the firm conviction that a fuzzy erosion should be linked to a fuzzy dilation in terms of a relationship of duality, that can be either adjunction or negation. The duality of a fuzzy erosion and dilation is induced by the duality of the underlying fuzzy implication and conjunction. Thus, we distinguish between approaches that are based on adjunction and approaches that are based on negation. The former approaches have the advantage that an adjoint pair of operators is guaranteed to consist of an erosion and a dilation. Moreover, compositions of adjoint erosions and dilations yield openings and closings. Similar observations cannot be made concerning a pair of operators that are dual with respect to negation. On the other hand, for every negation we have that a dual conjunction, a dual implication respectively, can be easily constructed from a given implication, a conjunction respectively, whereas a conjunction that is adjoint to a given implication or vice-versa does not always exist.

Deng and Heijmans proposed the Inf-I/Sup-C approach based on adjunction in its' most general form. This class comprises the approach of Maragos and the restriction of Ronse's approach to pairs of Inf-I erosions and Sup-C dilations such as $(E_{R\mathcal{L}}, D_{R\mathcal{L}})$ and $(E_{R\mathcal{U}1}, D_{R\mathcal{U}1})$. In Sect. 5.3.2, we have embedded the approaches of lattice isomorphism $(E_{\theta\mathcal{L}}, D_{\theta\mathcal{L}})$ and $(E_{\theta\mathcal{U}}, D_{\theta\mathcal{U}})$ into the Maragos framework. The dashed arrows below the approaches on lattice isomorphism indicate one-to-one correspondences between $(E_{\theta\mathcal{L}}, D_{\theta\mathcal{L}})$ and $(E_{\mathcal{L}}, D_{\mathcal{L}})$ as well as between $(E_{\theta\mathcal{U}}, D_{\theta\mathcal{U}})$ and $(E_{\mathcal{U}}, D_{\mathcal{U}})$. Note that the dashed lines relate fuzzy approaches with gray-scale approaches. Since $(E_{\theta\mathcal{L}}, D_{\theta L}) = (E_G, D_M)$, we can identify the level set approach with (E_G, E_M). By Theorem 5, the adjunction $(E_{\theta\mathcal{U}}, D_{\theta\mathcal{U}})$ also forms a dual pair with respect to the fuzzy negation N_θ. Thus, the fuzzy approach $(E_{\theta\mathcal{U}}, D_{\theta\mathcal{U}})$ corresponding to the umbra approach also fits into the general framework based on negation. As mentioned in Sect. 3, the threshold or flat approach can be viewed as a special case of both the level set approach as well as the umbra approach. The level set approach can be adapted using certain transformations to obtain Ronse's $(E_{R\mathcal{L}}, D_{R_{\mathcal{L}}})$ approach and the umbra-approach can be adapted to obtain Ronse's $(E_{R\mathcal{U}1}, D_{R\mathcal{U}1})$ approach. We showed that both $(E_{R\mathcal{L}}, D_{R\mathcal{L}})$ and $(E_{R\mathcal{U}1}, D_{R\mathcal{U}1})$ fit into the Inf-I/Sup-C framework. Recall that $E_{\mathcal{U}1}$ and $D_{\mathcal{U}1}$ denote translations of the $\mathcal{U}$-erosion and $\mathcal{U}$-dilation. A pointed arrow indicates the relationship $(\sigma(E_{\mathcal{U}1}), \sigma(D_{\mathcal{U}1})) = (E_L, D_L)$.

Next to Heijmans' approach in Fig. 2, the reader encounters an approach that not associated with any particular researcher: the General Inf-I/Sup-C approach based on negation. Nachtegael and Kerre have investigated several restrictions of this approach that are named after the corresponding inclusion measures of Bandler & Kohout, Kitainik, and

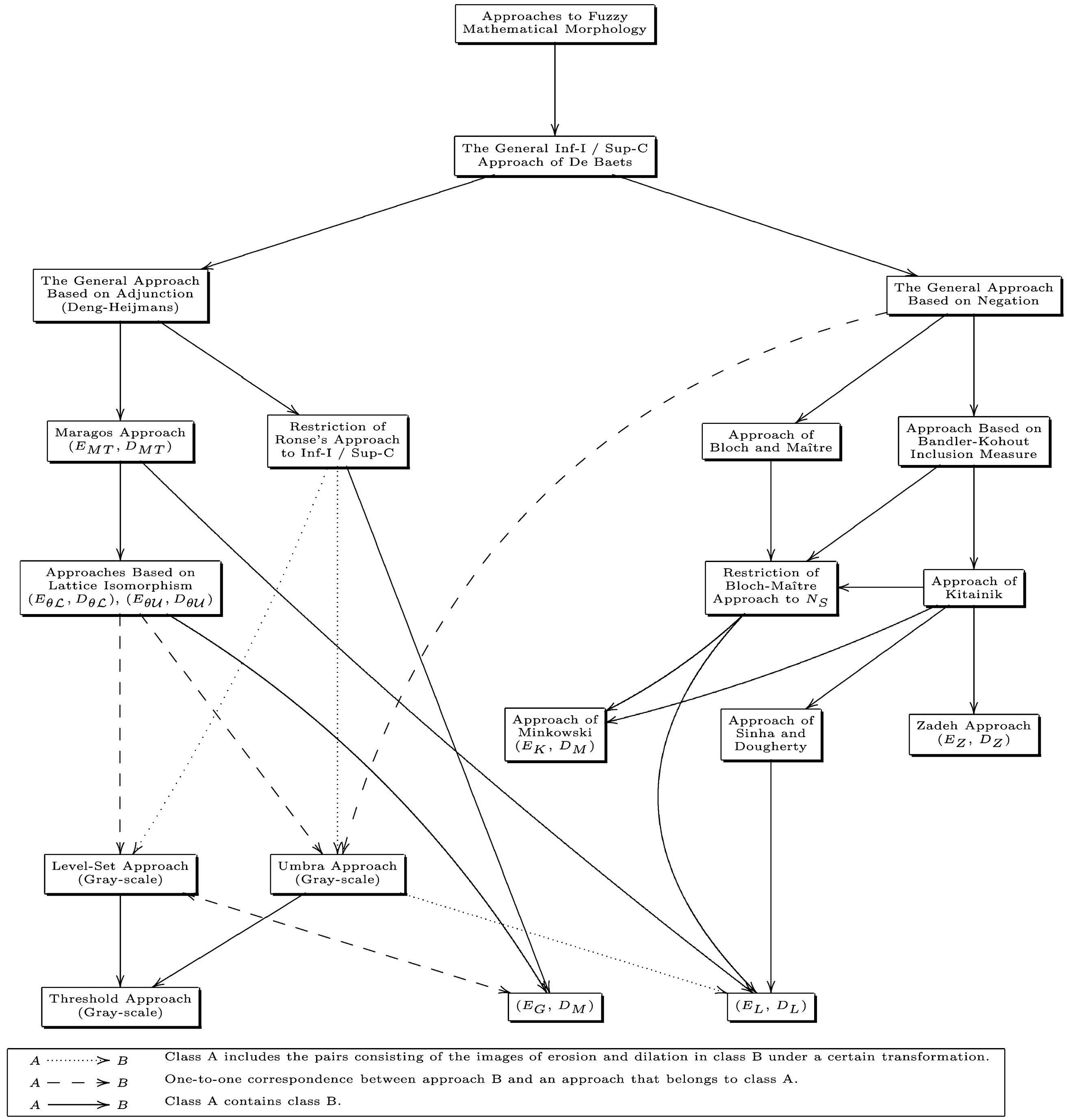

Fig. 2 Classification of fuzzy mathematical morphologies

Zadeh. All three approaches are limited to the application of the standard fuzzy negation in order to generate a fuzzy dilation from a given fuzzy erosion. Figure 2 displays these approaches from top to bottom, arranging them in the order from the most general one to the most specific one. Bandler and Kohout allow for any type of Inf-I inclusion measure. Kitainik only considers Inf-I_{KT} inclusion measures where I_{KT} is contrapositive with respect to N_S. Zadeh's inclusion measure I_Z belongs to the class of Kitainik inclusion measures.

The approach of Bloch and Maître also represents a special case of the general approach based on negation. Bloch and Maître first define a fuzzy Sup-$C_{\mathcal{F}}$ dilation in terms of Eq. 73 subject to the condition that $C_{\mathcal{F}}$ is a t-norm. The corresponding fuzzy erosion is then defined as the $N_{\mathcal{F}}$-dual of the fuzzy dilation where $N_{\mathcal{F}}$ is an arbitrary fuzzy negation.

Obviously, we have a particular case of Bandler & Kohout's approach if we impose $N_{\mathcal{F}} = N_S$ and this subset of Bandler & Kohout's approaches belongs to the class of Kitainik approaches because every t-norm represents a fuzzy conjunction. The restriction of Bloch and Maître's approach to N_S includes (E_L, D_L) as well as the approach of Minkowski addition that coincides with (E_K, D_M) because I_L is the N_S-dual of the t-norm C_L and I_K is the N_S-dual of the t-norm C_M.

The approach of Sinha and Dougherty employs an Inf-I_{GL} inclusion measure to construct a fuzzy erosion. Since I_{GL} is contra-positive, this approach can be embedded into Kitainik's framework. The Sinha-Dougherty approach comprises (E_L, D_L) since I_L is contrapositive and C_L is the N_S-dual of I_L.

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Appendix: Some Mathematical Notations

$\mathbb{L}$	A general complete lattice
ν	A general negation
Ψ^ν	The negation or the dual of Ψ with respect to ν
ε and δ	General erosions and dilations on complete lattices
$E_{\mathcal{B}}$ and $D_{\mathcal{B}}$	Binary erosion and dilation
$\mathbb{G}$	Either $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ or $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$
$\mathbb{G}^{\mathbf{X}}$	The set of images $\mathbf{X} \to \mathbb{G}$
$\bar{\mathbf{a}}$	The reflection of **a** around the origin
$\mathbf{a}_{\mathbf{y}}$	The translation of **a** by $\mathbf{y} \in \mathbf{X}$
$E_{\mathcal{T}}$ and $D_{\mathcal{T}}$	Threshold erosion and dilation
$E_{\mathcal{L}}$ and $D_{\mathcal{L}}$	Level set erosion and dilation
$E_{\mathcal{U}}$ and $D_{\mathcal{U}}$	Umbra erosion and dilation
$\mathcal{F}(\mathbf{X}) = [0, 1]^{\mathbf{X}}$	The class of fuzzy sets in **X**
$C_{\mathcal{F}}$ and $I_{\mathcal{F}}$	Fuzzy conjunctions and fuzzy implications
$E_{\mathcal{F}}$	Fuzzy erosion based on a fuzzy inclusion measure. Given a particular fuzzy Inf-I erosion, we indicate the underlying type of implication by means of a subscript
$D_{\mathcal{F}}$	Fuzzy dilation based on a fuzzy intersection measure. Given a particular fuzzy Sup-C dilation, we indicate the underlying type of conjunction by means of a subscript
E_{DB} and D_{DB}	De Baets' fuzzy erosion and dilation
E_{SD} and D_{SD}	Sinha and Dougerty approach of fuzzy erosion and dilation
$E_{\theta\mathcal{L}}$ and $D_{\theta\mathcal{L}}$	Fuzzy erosion and dilation isomorphic to $E_{\mathcal{L}}$ and $D_{\mathcal{L}}$
$E_{\theta\mathcal{U}}$ and $D_{\theta\mathcal{U}}$	Fuzzy erosion and dilation isomorphic to $E_{\mathcal{U}}$ and $D_{\mathcal{U}}$
$E_{R\mathcal{L}}$ and $D_{R\mathcal{L}}$	Ronse's approach to fuzzy erosion and dilation based on $E_{\mathcal{L}}$ and $D_{\mathcal{L}}$
$E_{R\mathcal{U}}$ and $D_{R\mathcal{U}}$	Ronse's approach to fuzzy erosion and dilation based on $E_{\mathcal{U}}$ and $D_{\mathcal{U}}$
$E_{R\mathcal{U}1}$ and $D_{R\mathcal{U}1}$	Modified versions of $E_{R\mathcal{U}}$ and $D_{R\mathcal{U}}$

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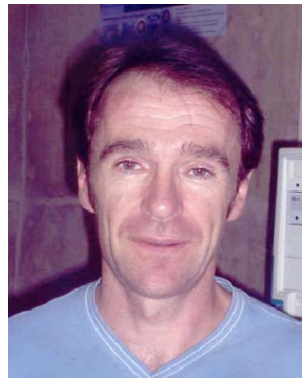

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