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Partial Partitions, Partial Connections and Connective Segmentation

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Abstract In *connective segmentation* (Serra in J. Math. Imaging Vis. 24(1):83–130, 2006), each image determines subsets of the space on which it is "homogeneous", in such a way that this family of subsets always constitutes a connection (connectivity class); then the segmentation of the image is the partition of space into its connected components according to that connection.

Several concrete examples of connective segmentations or of connections on sets, indicate that the space covering requirement of the partition should be relaxed. Furthermore, morphological operations on partitions require the consideration of wider framework.

We study thus *partial partitions* (families of mutually disjoint non-void subsets of the space) and *partial connections* (where connected components of a set are mutually disjoint but do not necessarily cover the set). We describe some methods for generating partial connections. We investigate the links between the two lattices of partial connections and of partial partitions. We generalize Serra's characterization of connective segmentation and discuss its relevance. Finally we give some ideas on how the theory of partial connections could lead to improved segmentation algorithms.

Keywords Mathematical morphology · Connective segmentation · Connections · Partitions · Complete lattice

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1 Motivation

The algebraic formalization of the concept of connectivity was given by Serra [17]. Given a space E, a *connection on* $\mathcal{P}(E)$ is a family $\mathcal{C} \subseteq \mathcal{P}(E)$ such that (a) $\emptyset \in \mathcal{C}$, (b) $\forall p \in E$, $\{p\} \in \mathcal{C}$, and (c) $\forall \mathcal{B} \subseteq \mathcal{C}$, $\bigcap \mathcal{B} \neq \emptyset \Rightarrow \bigcup \mathcal{B} \in \mathcal{C}$. The elements of $\mathcal{C}$ are then said to be *connected*. Several authors [4, 9, 12] call $\mathcal{C}$ a *connectivity class*. The family of topologically connected subsets of a topological space, the one of arc-connected subsets of $\mathbf{R}^n$, and the one of connected subsets of a graph, are connections. Hence Serra's definition unifies previous notions given in topology and graph theory.

Given a connection $\mathcal{C}$ on $\mathcal{P}(E)$, a non-empty subset X of E is partitioned into its *connected components* according to $\mathcal{C}$, that is, the maximal subsets of X that belong to $\mathcal{C}$. For the usual topological or graph-theoretical connectivity, this represents an elementary form of segmentation of X into its constituent parts. However more elaborate connections have been defined in [9, 12, 17], which give more meaningful segmentations of a shape. Let us give an example.

Let $E = \mathbf{R}^n$ or $\mathbf{Z}^n$, provided with a standard connection $\mathcal{C}_0$ (the topological or arc connectivity for $\mathbf{R}^n$, the digital connectivity based on the $(2n)$ or $(3^n - 1)$-adjacency for $\mathbf{Z}^n$). Choose a non-void structuring element $B \in \mathcal{C}_0$. Let the family $\mathcal{C}_B \subseteq \mathcal{P}(E)$ contain all $Z \in \mathcal{C}_0$ such that $Z \circ B = Z$ (i.e., all connected unions of translates of B), plus all singletons in E. Then $\mathcal{C}_B$ is a connection on $\mathcal{P}(E)$ [12]. For any $X \in \mathcal{P}(E)$, the connected components of X according to $\mathcal{C}_B$ are the connected components of $X \circ B$ according to $\mathcal{C}_0$, and the singletons of $X \setminus (X \circ B)$. We illustrate this in Fig. 1 for the two-dimensional case.

The formalism of connections was extended to complete lattices in [18], leading to an extensive analysis in [3, 14]. This new framework led to further approaches to segmentation, in particular in [19], where the jump segmentation was

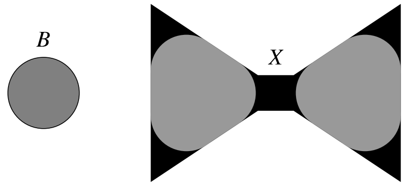

Fig. 1 *Left*: the structuring element $B \in \mathcal{C}_0$ is a disk. *Right*: the connected components of the bowtie X according to $\mathcal{C}_B$ are the two connected components (in $\mathcal{C}_0$) of its opening $X \circ B$ (shown in *grey*) and the singletons in the residual $X \setminus (X \circ B)$ (shown in *black*)

introduced. Then [23] introduced the theory of connective segmentation. Consider a set V of values (that can be numerical grey-levels, multivalued vectors, etc.). We assume a "homogeneity" criterion that is modeled by a Boolean predicate σ, associating to every function $F : E \to V$ and every subset $A \subseteq E$ a value $\sigma[F, A]$ that can be 1 (if the criterion is satisfied by F on A) or 0 (if the criterion is invalidated by F on A). The criterion σ is said to be *connective* if for any $F : E \to V$, the set of all $A \in \mathcal{P}(E)$ such that $\sigma[F, A] = 1$ constitutes a connection $\mathcal{C}_\sigma^F$; then the segmentation of F is given by the partition of E into its connected components according to $\mathcal{C}_\sigma^F$.

For example, assume that E is provided with a standard connection $\mathcal{C}_0$. Define σ by $\sigma[F, A] = 1$ iff $A \in \mathcal{C}_0$ and F has a constant value on A. Then the criterion σ is connective, and the segmentation of F according to σ is the partition of E into the maximal connected sets (in $\mathcal{C}_0$) on which F has a constant value, in other words the *flat zones* of F.

Several connective segmentation algorithms are based on first generating some subsets of E called *seeds*, then agglomerating neighbouring or overlapping seeds into mutually non-adjacent connected regions. This contrasts with watershed segmentation [24], where the seeds are the markers, which are progressively grown into connected regions, but where two neighbouring regions originating from different markers must be separated by a watershed line. Without a good choice of markers, the watershed tends to produce over-segmentation, a problem that can be avoided by connective segmentation.

Let us give some examples of connective segmentations, and describe the associated connective criteria. We assume a space E provided with a standard connection $\mathcal{C}_0$.

In the simplest methods, every $F : E \to V$ determines a subset S_F of E comprising all points where F satisfies some property; then the connected components of S_F (according to $\mathcal{C}_0$) will be the regions of the segmentation. Now the residual $R_F = E \setminus S_F$ will make the boundaries separating the regions. Thus we have a criterion σ where $\sigma[F, A] = 1$ if either $A \in \mathcal{C}_0$ and $A \subseteq S_F$, or A is a singleton. Then this criterion is connective, and the connected components of E according to $\mathcal{C}_\sigma^F$ are indeed the connected components of S_F (i.e., the regions) and the singletons included in R_F (i.e., the boundary singletons). Here we have no seeds, or rather all singletons in S_F constitute seeds, that are aggregated by connectivity in $\mathcal{C}_0$.

We describe two examples for S_F. In *thresholding*, one selects an interval $U \subset V$; then for $F : E \to V$, let $S_F = \{p \in E \mid F(p) \in U\}$ be the threshold set of F. In the *regional Lipschitz segmentation* [23], to every point $p \in E$ one associates a neighbourhood $B(p)$ (for example, in $\mathbf{R}^n$, the open ball of radius r about p); then for $F : E \to V$, let S_F be the set of $p \in E$ such that F is Lipschitz on $B(p)$.

There are more elaborated approaches where one defines non-singleton seeds. The best example is the *jump segmentation* introduced in [19] and analysed in [23]. We follow the precise description given in [15]. This method assumes that the set V of values is an interval in $\mathbf{Z}$, and requires the choice of an integer parameter $k > 0$, called the *jump constant*. Recall that a *regional minimum of* F is some $M \in \mathcal{C}_0$ such that for some $m \in V$, all $p \in M$ satisfy $F(p) = m$, but for any $N \in \mathcal{C}_0$ strictly greater than M ($N \supset M$), there is some $q \in N$ with $F(q) > m$; then m is the *level of* M. For each $m \in V$, let $\mathbf{M}_m$ be the set of regional minima of level m, and let $B(m)$ be the set of points having their level between m and $m + k - 1$: $B(m) = \{p \in E \mid m \leq F(p) < m + k\}$. For $M \in \mathbf{M}_m$, we have $M \in \mathcal{C}_0$ and $M \subseteq B(m)$; let $S(M)$ be the connected component (according to $\mathcal{C}_0$) of $B(m)$ that contains M. Then the $S(M)$ ($M \in \mathbf{M}_m$, $m \in V$) are the seeds of the segmentation, and the final regions are the maximal connected unions of seeds, in other words the connected components (according to $\mathcal{C}_0$) of $S_F = \bigcup\{S(M) \mid M \in \mathbf{M}_m, m \in V\}$. The residual set $R_F = E \setminus S_F$ separates the regions, it can be considered as a boundary, but it can be thick.

Let us describe the corresponding connection. For any $m \in V$ and $M \in \mathbf{M}_m$, we will consider as seed not only $S(M)$, but also every $A \in \mathcal{C}_0$ such that $A \subseteq B(m)$ and $A \cap M \neq \emptyset$; note that $A \subseteq S(M)$ because $S(M)$ is the connected component of $B(m)$ containing M. Then the family $\mathcal{J}_F^k$ comprising all connected unions of seeds and all singletons of E, is a connection called the k*-jump connection*. It corresponds to the connective criterion of the jump segmentation: indeed, the connected components of E according to $\mathcal{J}_F^k$ are the connected components of S_F (according to $\mathcal{C}_0$) and the singletons of R_F, in other words the regions and the boundary singletons. We illustrate $\mathcal{J}_F^k$ in Fig. 2.

We remark that in the above examples of segmentations (flat zones, thresholding, regional Lipschitz and jump), we determined not only the regions (the connected components according to the connective criterion), but the whole connection corresponding to the connective criterion. This becomes useful if one wants to segment an image by a Boolean conjunction $\sigma_1 \wedge \cdots \wedge \sigma_n$ of connective criteria $\sigma_1, \ldots, \sigma_n$, for example: a jump for both the function and its negative, a jump limited to points with Lipschitz neighbourhoods, etc. Indeed this conjunctive criterion will be connective, since

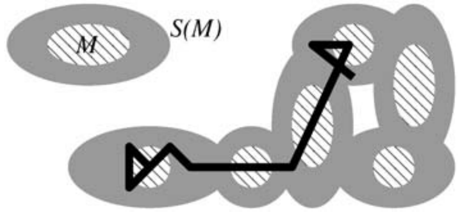

Fig. 2 *Top left*: we represent a minimum M as a hatched ellipse, and the corresponding $S(M)$ as a *grey ellipse*. *Bottom right*: several overlapping $S(M)$. The *thick black line* is a connected union of connected segments, each one included in some $S(M)$ and hitting M; therefore it belongs to $\mathcal{J}_F^k$

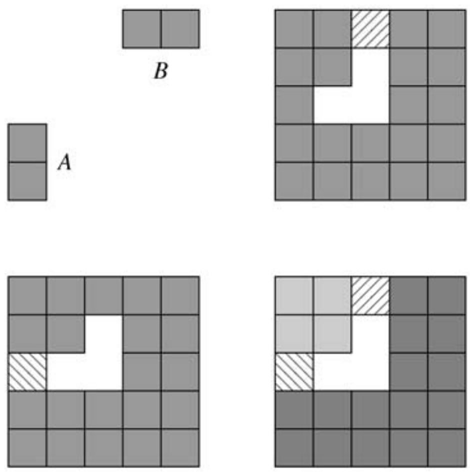

Fig. 3 Let $E = \mathbf{Z}^2$ with $\mathcal{C}_0$ being given by the 4-connectivity. *Top left*: the vertical segment A and the horizontal segment B. Let $\mathcal{C}_A$ be the connection consisting of all singletons and all 4-connected unions of translates of A, and define similarly $\mathcal{C}_B$. *Top right*: the partition of a set $X \subseteq E$ into its 2 connected components according to $\mathcal{C}_A$. *Bottom left*: the partition of X into its 2 connected components according to $\mathcal{C}_B$. *Bottom right*: the partition of X into its 4 connected components according to $\mathcal{C}_A \cap \mathcal{C}_B$; the two big components (shown in *dark* and *light grey*) belong to the same connected component according to $\mathcal{C}_A$ and to $\mathcal{C}_B$, hence their separation is not determined by the two partitions, but by the two connections

$\mathcal{C}^F_{\sigma_1 \wedge \cdots \wedge \sigma_n} = \mathcal{C}^F_{\sigma_1} \cap \cdots \cap \mathcal{C}^F_{\sigma_n}$ is an intersection of connections, hence a connection [12]. Now the partition of E into its connected components according to $\mathcal{C}^F_{\sigma_1} \cap \cdots \cap \mathcal{C}^F_{\sigma_n}$ *is not determined* by its partitions into its connected components according to $\mathcal{C}^F_{\sigma_1}, \ldots, \mathcal{C}^F_{\sigma_n}$ respectively, it requires the knowledge of the connections $\mathcal{C}^F_{\sigma_1}, \ldots, \mathcal{C}^F_{\sigma_n}$ [23], see Fig. 3.

It is also possible to combine disjunctively connective criteria $\sigma_1, \ldots, \sigma_n$. However the Boolean disjunction $\sigma_1 \vee \cdots \vee \sigma_n$ will usually not be connective, because $\mathcal{C}^F_{\sigma_1 \vee \cdots \vee \sigma_n} = \mathcal{C}^F_{\sigma_1} \cup \cdots \cup \mathcal{C}^F_{\sigma_n}$ is in general not a connection. However $\mathcal{C}^F_{\sigma_1} \cup \cdots \cup \mathcal{C}^F_{\sigma_n}$ generates a connection $\mathrm{Con}(\mathcal{C}^F_{\sigma_1} \cup \cdots \cup \mathcal{C}^F_{\sigma_n})$, namely the least connection containing it [12], and it corresponds thus to the least connective criterion majorating $\sigma_1 \vee \cdots \vee \sigma_n$, that we write $\mathrm{con}(\sigma_1 \vee \cdots \vee \sigma_n)$. Here the segmentation according to $\mathrm{con}(\sigma_1 \vee \cdots \vee \sigma_n)$ is easily obtained from those according to $\sigma_1, \ldots, \sigma_n$, we have only to agglomerate the corresponding classes whenever they overlap. More precisely, the partition of E into its connected components according to $\mathrm{Con}(\mathcal{C}^F_{\sigma_1} \cup \cdots \cup \mathcal{C}^F_{\sigma_n})$ is the supremum (in the lattice of partitions) of its partitions into its connected components according to $\mathcal{C}^F_{\sigma_1}, \ldots, \mathcal{C}^F_{\sigma_n}$ respectively (see Sect. 2.4).

In our first example (the segmentation by flat zones), the regions constitute a partition of the space E, there is no need of boundaries to separate them. In all other examples (thresholding, regional Lipschitz and jump), we constructed a subset S_F of E by agglomerating either points with a required property (in thresholding and regional Lipschitz segmentations), or seeds (in jump segmentation), and the regions were the connected components (according to the initial connection $\mathcal{C}_0$) of S_F. We had thus a residual $R_F = E \setminus S_F$ separating the regions; this set is necessary, otherwise our construction would fuse neighbouring regions. In the figures of [23], S_F was shown in white and R_F in black, illustrating the notion that R_F consists of a boundary (sometimes thick) between regions. Note that R_F can also contain edges that are not closed and do not separate regions (e.g., a crack in the middle of a wall). Now in the corresponding connection, we take not only some connected subsets of S_F, but also all singletons in R_F. The inclusion of singletons is done in order to satisfy the axiom that a connection contains all singletons. But then this induces a loss of information, namely, it becomes impossible to distinguish a region reduced to a single point (in S_F) from a singleton in the residual R_F.

We can thus broaden the notion of a connection by removing the axiom that all singletons are connected. We define thus a *partial connection* as a family $\mathcal{C} \subseteq \mathcal{P}(E)$ such that $\emptyset \in \mathcal{C}$ and $\forall \mathcal{B} \subseteq \mathcal{C}$, $\bigcap \mathcal{B} \neq \emptyset \Rightarrow \bigcup \mathcal{B} \in \mathcal{C}$. Then the connected components (according to $\mathcal{C}$) of a set X will still be non-empty and mutually disjoint, but they will not necessarily cover X; in other words they constitute what we call a *partial partition* of X. A non-empty set X can even have no connected component at all. This idea was first proposed in [21], under the name of *quasi-connection*. Thus in the thresholding and regional Lipschitz segmentations, the partial connection would be $\mathcal{C}_0 \cap \mathcal{P}(S_F)$ (i.e., consisting of all subsets of S_F that are connected for $\mathcal{C}_0$), while in the jump segmentation, the partial connection would consist of all unions of seeds that are connected for $\mathcal{C}_0$. One can make the same reasoning with the connection illustrated in Fig. 1 (see also Fig. 3): the partial connection $\mathcal{C}^*_B$ consists of all $Z \in \mathcal{C}_0$ such that $Z \circ B = Z$ (i.e., all connected unions of translates of B), so the connected components of a set X according to $\mathcal{C}^*_B$ are the connected components of $X \circ B$ according to $\mathcal{C}_0$, while $X \setminus (X \circ B)$ will be the residual.

We can thus consider a *partial segmentation* of a function, that gives a partial partition of E according to the partial connection associated to a *partially connective criterion*.

This can be useful in practice. Indeed, in region-based segmentation methods like the watershed, the only edges that are preserved are those that separate distinct regions; in particular, edges that are not closed will usually disappear. One might want to preserve unclosed edges, so that they might be closed with some post-processing. Furthermore, there is no guarantee that the watershed will always follow the most salient edges; thus one might decide to constrain the watershed not only by initial markers for the regions (as customary [24]), but also by markers for the edges (V. Agnus, personal communication).

From a theoretical point of view, partial partitions model the progressive building of a segmentation, for instance by region growing: until all regions have grown into the final segmentation classes, we have only a partial partition.

In [21], Serra applied the idea of a partial connection for the sequential partitioning of a set (or for segmentation of a function). Given $A \in \mathcal{P}(E)$,

1. construct the partial partition $\{C_i \mid i \in I\}$ of connected components of A according to a partial connection $\mathcal{C}$;
2. make a partition $\{D_j \mid j \in J\}$ of the residual $\rho(A) = A \setminus \bigcup_{i\in I} C_i$; for example the D_j $(j \in J)$ can be the connected components of $\rho(A)$ according to a connection $\mathcal{D}$.

This gives a final partition $\{C_i \mid i \in I\} \cup \{D_j \mid j \in J\}$ of A. Note that step 2 can be obtained by the connected components of a connection, but also by a recursive application of the sequence. Thus we can make a partial partition of A by the connected components according to a partial connection $\mathcal{C}_0$, leading to a residual $\rho_1(A)$; then the connected components of $\rho_1(A)$ according to a partial connection $\mathcal{C}_1$ will make a partial partition of $\rho_1(A)$, leading to a second residual $\rho_2(A)$, and so on; finally, the n-th residual $\rho_n(A)$ is partitioned into its connected component according to a connection (not partial) $\mathcal{C}_n$.

In the example of Fig. 1, the connected components of X according to the partial connection $\mathcal{C}_B^*$ (consisting of all connected unions of translates of B), are the connected components (according to $\mathcal{C}_0$) of $X \circ B$, this makes the first partial partition; then the residual $X \setminus (X \circ B)$ is partitioned into its connected component according to $\mathcal{C}_0$. As shown in Fig. 4, the classes (or blocks [11]) of the resulting partition can be regrouped by building the influence zones and the SKIZ [24] in their adjacency graph, with markers being the blocks of the first partial partition.

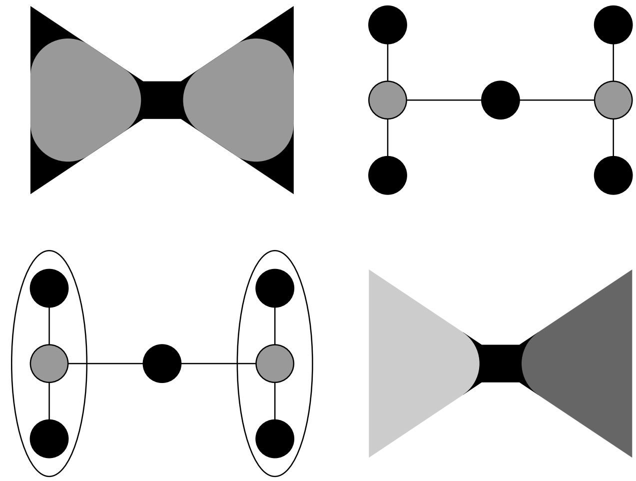

Fig. 4 *Top left*: the bowtie X is partitioned into its 2 connected components according to $\mathcal{C}_B^*$ (in *grey*), and the 5 connected components (according to $\mathcal{C}_0$) of the residual $X \setminus (X \circ B)$. *Top right*: the adjacency graph of the 7 blocks of the partition. *Bottom left*: in this graph, choose the 2 marker nodes (in *grey*) corresponding to the 2 blocks of the partial partition induced by $\mathcal{C}_B^*$; the ellipses give the set of nodes in the influence zones of these markers, and the remaining node is the SKIZ. *Bottom right*: fusing the blocks in the respective influence zones gives a final segmentation into 3 blocks

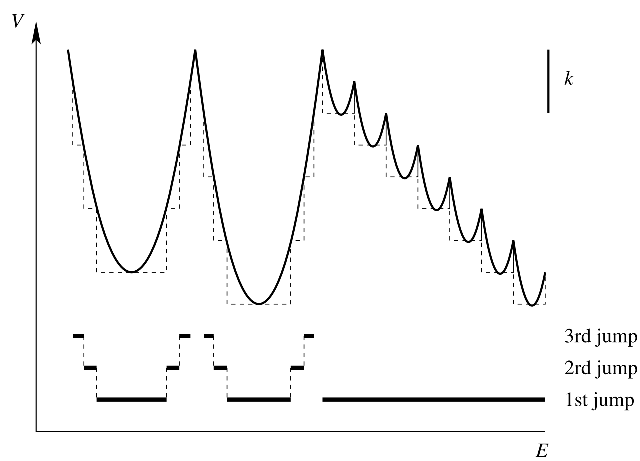

Fig. 5 Three successive jumps of a function $\mathbf{Z} \to \mathbf{Z}$, and the associated regions. The neighbouring regions in the 3 successive jumps are linked by *vertical dotted lines*, they could be aggregated

Since the jump segmentation gives often a thick residual, it is generally used sequentially as explained above. The first jump gives the set S_F whose connected components are the first regions. Then jump segmentation is applied to the space $E_1 = E \setminus S_F$ and the function F_1 that is the restriction of F to E_1, leading to the set S_{F_1} whose connected components are the second regions. Next take $E_2 = E_1 \setminus S_{F_1}$ and F_2 the restriction of F to E_2, etc. Finally we obtain a partial partition made of the connected components of $S_F, S_{F_1}, S_{F_2}, \ldots$ separately, but we can also take the connected components of $S_F \cup S_{F_1} \cup S_{F_2} \cdots$, or build influence zones of the components in S_F, as in Fig. 4. We show in Fig. 5 the construction of S_F, S_{F_1}, S_{F_2} for a one-dimensional function.

The above discussion shows that connections and partial connections are useful tools for segmentation. Now segmentation can also be improved by applying morphological operators to partitions. Indeed, segmentation is generally seen as the final step of low-level image analysis; thus when segmentation gives an unsatisfactory result, one usually returns to the previous image processing stage, where a better filtering is performed, before applying the same segmentation algorithm. An alternative approach is to consider the segmentation partition as a pictorial object that can itself be processed and filtered. In a work on the segmentation of colour images [20], Serra derived from an erosion ε on $\mathcal{P}(E)$ (such that $\varepsilon(\emptyset)=\emptyset$) an erosion ε' on partitions: in a partition, erode all blocks by ε, keep the non-void eroded blocks, and constitute into singleton blocks $\{p\}$ all points $p \in E$ which do not belong to an eroded block. In [13], the lower adjoint dilation δ' of ε' was defined: in a partition, apply the lower adjoint δ of ε to every non-singleton block, then recursively fuse all overlapping dilated blocks. The correct understanding of these two operators relies on the framework of partial partitions. Therefore the study of morphology on partial partitions is a promising topic in segmentation.

This paper is devoted to the study of the more abstract aspect of connective segmentation, namely the theory of partial partitions, partial connections, the lattices they make, and the relations between the two. The rest of this section recalls some mathematical prerequisites. Section 2 presents the various formalisms for partial partitions and partial connections, and it describes the two lattices of partial partitions and of partial connections. Section 3 describes several methods for constructing a partial connection from another partial connection and an operator. Section 4 generalizes the main theorem of [23] and discusses its relevance to segmentation. Finally Sect. 5 concludes. We give as appendix a table of our notation.

Some results from Sects. 2 and 4 have been stated without proof in [15].

The study of morphological and geodesic operators on partial partitions will be the topic of further papers by the author. Furthermore, a collaborative research will be undertaken on connective segmentation criteria and algorithms, and on image filtering adapted to this framework, cf. [25].

1.1 Mathematical Prerequisites

We assume that the reader has a basic knowledge of the lattice-theoretic framework for morphological operations, in particular increasing operators, adjunctions, dilations, erosions, openings and closings, see for example [2, 8]. Standard references in lattice theory are [1, 6, 7].

Given a poset (partially ordered set P), a *directed subset* of P is a non-empty $D \subseteq P$ such that every *finite* subset X of D has an upper bound (is majorated) in D; equivalently, every pair $\{p,q\} \subseteq D$ has an upper bound in D: $\forall p,q \in D$, $\exists r \in D$, $p,q \leq r$ [6].

Let L be a complete lattice with least and greatest elements $\mathbf{0}$ and $\mathbf{1}$. A *sup-generating family* of L is a subset S of L such that every element of L is the supremum of some elements of S; in fact, $\forall x \in L$, $x = \bigvee\{s \in S \mid s \leq x\}$. An element a of L is called an *atom* if $\mathbf{0} < a$ but there is no $b \in L$ with $\mathbf{0} < b < a$. We say that L is *atomic* if the family of atoms of L is sup-generating.

A *lower set* in L is a subset A of L such that for $x \in A$ and $y \in L$ with $y \leq x$, we must have $y \in A$; an *upper set* B satisfies the dual condition: for $x \in B$ and $y \in L$ with $y \geq x$, we must have $y \in B$. In particular, for a fixed $x_0 \in L$, $\{y \in L \mid y \geq x_0\}$ is an upper set, and also a complete lattice for the order $\leq$ of L, having the same non-empty supremum and infimum operations as in L, as well as the same greatest element (or empty infimum) $\mathbf{1}$ as in L; however its least element (or empty supremum) will be x_0 instead of $\mathbf{0}$.

Given two sets A and B, we will say that $(\alpha,\beta) : A \rightleftharpoons B$ if α is a mapping $A \to B$ and β is mapping $B \to A$. In particular, given two complete lattices L and M, we will consider adjunctions $(\varepsilon,\delta) : L \rightleftharpoons M$, that is for $\varepsilon : L \to M$ and $\delta : M \to L$.

The *invariance domain* of an operator $\psi : L \to L$ is the set $\mathrm{Inv}(\psi) = \{x \in L \mid \psi(x) = x\}$.

A *Moore family* [2] is a subset M of L that is closed under arbitrary infima: $\forall X \subseteq M$, $\bigwedge X \in M$; in particular, $\mathbf{1} = \bigwedge \emptyset \in M$; equivalently, it is the invariance domain of a closing. A *dual Moore family* is closed under arbitrary suprema, in particular it contains $\mathbf{0}$; it is the invariance domain of an opening. A Moore family or dual Moore family is itself a complete lattice for the order $\leq$.

A classic example of dual Moore family is in $\mathcal{P}(\mathbf{R}^n)$ the family $\mathcal{B}_r$ of all invariants of the opening by a disk of radius $r > 0$, in other words all subsets of $\mathbf{R}^n$ that are unions of such disks. This family $\mathcal{B}_r$ decreases with r, and the union of all $\mathcal{B}_r$ for $r > 0$ is the family of open sets, also a dual Moore family.

Given a lower set A, $A \cup \{\mathbf{1}\}$ is a Moore family, and given an upper set B, $B \cup \{\mathbf{0}\}$ is a dual Moore family.

An intersection of Moore families is a Moore family. Thus the set of Moore families of L is a Moore family of $\mathcal{P}(L)$. Similarly, the family of dual Moore families of L is a Moore family of $\mathcal{P}(L)$. They constitute thus complete lattices for the inclusion order.

The set of openings on L is a dual Moore family of the lattice of operators on L, it constitutes thus a complete lattice. The map $\gamma \mapsto \mathrm{Inv}(\gamma)$ is an isomorphism between the lattice of openings and the lattice of dual Moore families.

Throughout the paper, E will designate our "space" on which we will consider partial partitions and connections, and its elements will be called "points"; in fact E is an arbitrary set of size at least 2, although in practice E will be the Euclidean space $\mathbf{R}^n$, the digital space $\mathbf{Z}^n$, or a bounded interval in such spaces.

2 Partial Partitions and Connections

Partitions can be formalized in 3 ways: in terms of an equivalence relation, of a family of blocks, or of the map associating to each point its class. Connections are given by 2 equivalent set of axioms [17]: those for the family of connected sets and those for the openings giving the connected components. We extend these formalisms to partial partitions and partial connections. We describe also the lattices that they make, and exhibit a link between the supremum operations for partial connections and for partial partitions.

2.1 Partial Partitions

An *equivalence* on a set E is a binary relation R on E that is reflexive ($\forall p \in E,\ pRp$), symmetric ($\forall p,q \in E,\ pRq \Leftrightarrow qRp$) and transitive ($\forall p,q,r \in E,\ [pRq\ ,\ qRr] \Rightarrow pRr$). A *partition* of E is a family π of subsets of E that are non-empty ($\emptyset \notin \pi$), mutually disjoint ($\forall X,Y \in \pi,\ X \neq Y \Rightarrow X \cap Y = \emptyset$), and whose union covers E ($\bigcup \pi = E$). Equivalently, every point of E belongs to exactly one member of π. In order to generalize these notions to their partial versions, we need to introduce the *support*:

Definition 1 The *support*

1. of a binary relation R on E is the subset $\mathrm{supp}(R)$ of E comprising all $p \in E$ such that there is some $q \in E$ with pRq or qRp;
2. of a family $\mathcal{B}$ of subsets of E is the subset $\mathrm{supp}(\mathcal{B})$ of E comprising all points covered by at least one element of $\mathcal{B}$, in other words $\mathrm{supp}(\mathcal{B}) = \bigcup \mathcal{B}$.

In order to define a partial equivalence, we drop the axiom of reflexivity of the relation, and for the partial partition, we drop the axiom of the covering of the set by the blocks:

Definition 2

1. A *partial equivalence* on E is a binary relation on E that is symmetric and transitive.
2. A *partial partition* of E is a family π of subsets of E that are non-empty and mutually disjoint. Equivalently, π is a family of subsets of E such that every point of E belongs to at most one member of π. Every member of a partial partition is called a *block* [11]; given a point belonging to a block, that block is called the *class* of that point.

Now partiality links with the support:

Lemma 3

1. *A binary relation on E is a partial equivalence iff it forms an equivalence on its support.*
2. *A family of subsets of E is a partial partition iff it constitutes a partition of its support.*

Proof Item 2 is trivial, we prove only item 1. Since the symmetry and transitivity conditions are void for points outside the support, a relation R is symmetric and transitive on E iff it is symmetric and transitive on $\mathrm{supp}(R)$. Let R be symmetric and transitive. For any $p \in \mathrm{supp}(R)$, there is some $q \in E$ with pRq or qRp; by symmetry we have then both pRq and qRp, and by transitivity we deduce that pRp; hence R is reflexive on its support. □

Then the well-known bijection between equivalence relations and partitions extends to their partial counterparts:

Proposition 4 *A one-to-one correspondence exists between partial partitions of E and partial equivalences on E, associating to each partial partition π of E the partial equivalence* $\mathrm{PE}(\pi)$ *on E given by*

$$\forall p,q \in E, \qquad p\,\mathrm{PE}(\pi)\,q \iff \exists C \in \pi,\ p,q \in C. \quad (1)$$

Equivalently, π consists of the equivalence classes of the equivalence induced by $\mathrm{PE}(\pi)$ *on its support. We have then* $\mathrm{supp}(\mathrm{PE}(\pi)) = \mathrm{supp}(\pi)$.

We now turn to the third formalism for a partial partition, in terms of the map associating to each point its class. Consider a map $\mathrm{cl} : E \to \mathcal{P}(E)$, and the following properties that it can satisfy:

(P1a) For any $p \in E$, $p \in \mathrm{cl}(p)$.
(P1b) For any $p \in E$, $\mathrm{cl}(p) = \emptyset$ or $p \in \mathrm{cl}(p)$.
(P2a) For any $p,q \in E$, $q \in \mathrm{cl}(p) \Rightarrow \mathrm{cl}(p) = \mathrm{cl}(q)$.
(P2b) For any $p,q \in E$, $\mathrm{cl}(p) \cap \mathrm{cl}(q) \neq \emptyset \Rightarrow \mathrm{cl}(p) = \mathrm{cl}(q)$.

Lemma 5 (P1a) *implies* (P1b), (P2a) *implies* (P2b), *and if* (P1a) *holds, then* (P2a) *and* (P2b) *are equivalent.*

Proof It is obvious that (P1a) implies (P1b). Suppose that (P2a) holds, and let $p,q \in E$ with $\mathrm{cl}(p) \cap \mathrm{cl}(q) \neq \emptyset$; for $r \in \mathrm{cl}(p) \cap \mathrm{cl}(q)$, (P2a) gives $\mathrm{cl}(p) = \mathrm{cl}(r)$ and $\mathrm{cl}(q) = \mathrm{cl}(r)$, so $\mathrm{cl}(p) = \mathrm{cl}(q)$; thus (P2b) holds. Suppose finally that (P1a) and (P2b) hold, let us show that (P2a) follows: given $p,q \in E$ with $q \in \mathrm{cl}(p)$, by (P1a) we have $q \in \mathrm{cl}(q)$, so $q \in \mathrm{cl}(p) \cap \mathrm{cl}(q)$, thus $\mathrm{cl}(p) \cap \mathrm{cl}(q) \neq \emptyset$, hence by (P2b) we deduce that $\mathrm{cl}(p) = \mathrm{cl}(q)$. □

Definition 6 A map $\mathrm{cl} : E \to \mathcal{P}(E)$ is called

1. a *partial partition class map* on E if it satisfies (P1b) and (P2a);
2. a *partition class map* on E if it satisfies (P1a) and (P2a) (or equivalently: (P1a) and (P2b)).

Theorem 7 *There is a one-to-one correspondence between partial partitions on E and partial partition class maps on E, under which*:

- *To every partial partition π is associated the partial partition class map Cl_π given by*

$$\forall p \in E, \quad \mathrm{Cl}_\pi(p) = \begin{cases} \emptyset & \text{if } p \notin \mathrm{supp}(\pi); \\ C & \text{for } p \in C \in \pi, \\ & \text{if } p \in \mathrm{supp}(\pi); \end{cases} \tag{2}$$

  this C being unique.
- *To every partial partition class map* cl *is associated the partial partition*

$$\mathrm{PP}(\mathrm{cl}) = \{\mathrm{cl}(p) \mid p \in E,\ \mathrm{cl}(p) \neq \emptyset\}. \tag{3}$$

Furthermore, it induces a one-to-one correspondence between partitions on E and partition class maps on E.

Proof We have to show that:

- for a partial partition π: (a) Cl_π is a partial partition class map, (b) $\mathrm{PP}(\mathrm{Cl}_\pi) = \pi$, and (c) if π is a partition, then Cl_π is a partition class map;
- for a partial partition class map cl: (d) PP(cl) is a partial partition, (e) $\mathrm{Cl}_{\mathrm{PP(cl)}} = \mathrm{cl}$, and (f) if cl is a partition class map, then PP(cl) is a partition.

(a) By (2) we have $\mathrm{Cl}_\pi(p) = \emptyset$ for $p \notin \mathrm{supp}(\pi)$, and $p \in C = \mathrm{Cl}_\pi(p)$ for $p \in \mathrm{supp}(\pi)$; thus Cl_π satisfies (P1b). Let $p, q \in E$ with $q \in \mathrm{Cl}_\pi(p)$; then $\mathrm{Cl}_\pi(p) \neq \emptyset$, so by (2) we have $\mathrm{Cl}_\pi(p) = C$ for the unique C such that $p \in C \in \pi$; but then $q \in C \in \pi$ and (2) again gives $\mathrm{Cl}_\pi(q) = C$, hence $\mathrm{Cl}_\pi(p) = \mathrm{Cl}_\pi(q)$; so Cl_π satisfies (P2a). Therefore Cl_π is a partial partition class map.

(b) Here $\mathrm{PP}(\mathrm{Cl}_\pi) = \{\mathrm{Cl}_\pi(p) \mid p \in E,\ \mathrm{Cl}_\pi(p) \neq \emptyset\}$. By (2), $\mathrm{Cl}_\pi(p) \neq \emptyset$ means that $\mathrm{Cl}_\pi(p) = C$ with $C \in \pi$; thus $\mathrm{PP}(\mathrm{Cl}_\pi) \subseteq \pi$. Conversely, for any $C \in \pi$, $C \neq \emptyset$, and taking $p \in C$, (2) again gives $\mathrm{Cl}_\pi(p) = C$ with $\mathrm{Cl}_\pi(p) \neq \emptyset$; thus $\pi \subseteq \mathrm{PP}(\mathrm{Cl}_\pi)$. The equality $\mathrm{PP}(\mathrm{Cl}_\pi) = \pi$ follows.

(c) If π is a partition, then $\mathrm{supp}(\pi) = E$, so in (2) the case $p \notin \mathrm{supp}(\pi)$ may not occur, hence we always have $\mathrm{Cl}_\pi(p) = C$ for $p \in C$; thus Cl_π satisfies (P1a) and is a partition class map.

(d) As cl satisfies (P2a), by Lemma 5 it satisfies (P2b): $\mathrm{cl}(p) \cap \mathrm{cl}(q) \neq \emptyset \Rightarrow \mathrm{cl}(p) = \mathrm{cl}(q)$. Thus the elements of $\mathrm{PP}(\mathrm{cl}) = \{\mathrm{cl}(p) \mid p \in E,\ \mathrm{cl}(p) \neq \emptyset\}$ are disjoint; they are non-empty by definition, so PP(cl) is a partial partition.

(e) By (3), $\mathrm{supp}(\mathrm{PP(cl)}) = \bigcup\{\mathrm{cl}(p) \mid p \in E,\ \mathrm{cl}(p) \neq \emptyset\}$, so $q \in \mathrm{supp}(\mathrm{PP(cl)})$ iff $\exists\, p \in E$ with $q \in \mathrm{cl}(p)$. Then (2,3) gives

$$\forall q \in E, \quad \mathrm{Cl}_{\mathrm{PP(cl)}}(q) = \begin{cases} \emptyset & \text{if } \forall p \in E, q \notin \mathrm{cl}(p), \\ \mathrm{cl}(p) & \text{for } q \in \mathrm{cl}(p),\ p \in E, \end{cases}$$

this $\mathrm{cl}(p)$ being unique. In the first case where $\forall p \in E$, $q \notin \mathrm{cl}(p)$, we have $q \notin \mathrm{cl}(q)$, so by (P1b) we have $\mathrm{cl}(q) = \emptyset$. In the second case where $q \in \mathrm{cl}(p)$ for some $p \in E$, (P2a) implies that $\mathrm{cl}(p) = \mathrm{cl}(q)$. Thus in both cases $\mathrm{Cl}_{\mathrm{PP(cl)}}(q) = \mathrm{cl}(q)$ for all $q \in E$, so $\mathrm{Cl}_{\mathrm{PP(cl)}} = \mathrm{cl}$.

(f) If cl is a partition class map, then by (P1a) for every $p \in E$ we have $p \in \mathrm{cl}(p)$. By (3), $\mathrm{supp}(\mathrm{PP(cl)}) = \bigcup\{\mathrm{cl}(p) \mid p \in E,\ \mathrm{cl}(p) \neq \emptyset\} = E$, so PP(cl) covers E and is a partition. □

Cl_π is called the *class map of* π. Combining (2) with (1), we get for a partial partition π and the corresponding partial equivalence relation $\mathrm{PE}(\pi)$:

$$\forall p, q \in E, \qquad p\, \mathrm{PE}(\pi)\, q \iff q \in \mathrm{Cl}_\pi(p). \tag{4}$$

Remark 8 It is customary to use axioms (P1a) and (P2b) for the class map of a partition; indeed (P1a) means that the non-void blocks cover E, and (P2b) that they are mutually disjoint. However, in conjunction with the weaker (P1b), (P2b) becomes insufficient, (P2a) is required for a partial partition, as shows the following example.

Let E be a finite subset of $\mathbf{Z}$, of size > 3. To a partition π of E we associate the map cl defined as follows:

$$\forall C \in \pi, \quad \mathrm{cl}(\min C) = \mathrm{cl}(\max C) = C,$$

$$\mathrm{cl}(p) = \emptyset \quad \text{for } p \in C \text{ with } \min C < p < \max C.$$

In other words, for $p \in E$, $\mathrm{cl}(p) = C \neq \emptyset$ iff $C \in \pi$ and p is either the leftmost or the rightmost point of C. Then cl satisfies (P1b) and (P2b), but not (P2a), and (3) gives $\pi = \{\mathrm{cl}(p) \mid p \in E,\ \mathrm{cl}(p) \neq \emptyset\}$. However $\mathrm{cl} \neq \mathrm{Cl}_\pi$, so cl is not a partial partition class map.

2.2 The Lattice of Partial Partitions

We will describe here the complete lattice of partial partitions according to the 3 views given above: partial equivalences, families of disjoint non-void sets, and class maps.

Every binary relation R on E can be identified with the set of ordered pairs $(x, y) \in E^2$ such that $x R y$. Thus the family of binary relations becomes the complete lattice $\mathcal{P}(E^2)$, ordered by inclusion and with the supremum and infimum operations given by union and intersection:

$$R \subseteq S : pRq \Longrightarrow p\, S\, q,$$

$$R = \bigcup_{i \in I} R_i : pRq \iff \exists i \in I, p\, R_i\, q,$$

$$R = \bigcap_{i \in I} R_i : pRq \iff \forall i \in I, p\, R_i\, q.$$

Let us write $\mathcal{E}^*(E)$ for the set of partial equivalences on E.

Proposition 9 *The set $\mathcal{E}^*(E)$ of partial equivalences on E is a Moore family of $\mathcal{P}(E^2)$. It is thus a complete lattice for the inclusion order, where the infimum and supremum of a family of partial equivalences is given respectively by their intersection and the transitive closure of their union*:

$$\bigwedge_{i\in I} R_i = \bigcap_{i\in I} R_i \quad \text{and} \quad \bigvee_{i\in I} R_i = \bigcup_{n=1}^{\infty}\Big(\bigcup_{i\in I} R_i\Big)^n, \tag{5}$$

with

$$\operatorname{supp}\Big(\bigwedge_{i\in I} R_i\Big) = \bigcap_{i\in I} \operatorname{supp}(R_i)$$

and (6)

$$\operatorname{supp}\Big(\bigvee_{i\in I} R_i\Big) = \bigcup_{i\in I} \operatorname{supp}(R_i).$$

The least and greatest partial equivalences are $\emptyset$ and E^2, with $\operatorname{supp}(\emptyset)=\emptyset$ and $\operatorname{supp}(E^2)=E$.

Proof Clearly the operation of intersection of relations preserves symmetry and transitivity; moreover the greatest relation E^2 and the least one $\emptyset$ are both symmetric and transitive. Hence partial equivalences constitute a Moore family with $\emptyset$ and E^2 as universal bounds. They form thus a complete lattice where the infimum is given by the intersection. Now the supremum of a family $\{R_i \mid i \in I\}$ of partial equivalences is the least partial equivalence containing each R_i, in other words the partial equivalence generated by their union $\bigcup_{i\in I} R_i$. But the operations of union and transitive closure $R \mapsto \bigcup_{n=1}^{\infty} R^n$ both preserve symmetry, so the transitive closure $\bigcup_{n=1}^{\infty}(\bigcup_{i\in I} R_i)^n$ of the union $\bigcup_{i\in I} R_i$ will be symmetric, it is thus the least partial equivalence (symmetric and transitive relation) containing all R_i, in other words their supremum.

By item 1 of Lemma 3, a partial equivalence R satisfies $\operatorname{supp}(R) = \{p \in E \mid (p,p) \in R\}$. Thus $p \in \operatorname{supp}(\bigwedge_{i\in I} R_i)$ iff $(p,p) \in \bigwedge_{i\in I} R_i = \bigcap_{i\in I} R_i$, iff for all $i \in I$ we have $(p,p) \in R_i$, that is $p \in \operatorname{supp}(R_i)$; hence $\operatorname{supp}(\bigwedge_{i\in I} R_i) = \bigcap_{i\in I} \operatorname{supp}(R_i)$.

The support is compatible with the operations of union of relations and n-th power of a relation: $\operatorname{supp}(\bigcup_{i\in I} R_i) = \bigcup_{i\in I} \operatorname{supp}(R_i)$ and $\operatorname{supp}(R^n) = \operatorname{supp}(R)$. Hence

$$\begin{aligned}\operatorname{supp}\Big(\bigvee_{i\in I} R_i\Big) &= \operatorname{supp}\Big(\bigcup_{n=1}^{\infty}\Big(\bigcup_{i\in I} R_i\Big)^n\Big)\\ &= \bigcup_{n=1}^{\infty} \operatorname{supp}\Big(\Big(\bigcup_{i\in I} R_i\Big)^n\Big) = \bigcup_{n=1}^{\infty} \operatorname{supp}\Big(\bigcup_{i\in I} R_i\Big)\\ &= \operatorname{supp}\Big(\bigcup_{i\in I} R_i\Big) = \bigcup_{i\in I} \operatorname{supp}(R_i).\end{aligned}$$

It is also obvious that $\operatorname{supp}(\emptyset)=\emptyset$ and $\operatorname{supp}(E^2)=E$. □

Let us now turn to partial partitions. Write $\Pi(E)$ for the set of all partitions of E, and $\Pi^*(E)$ for the set of all partial partitions of E. Since a partial partition is a partition of its support, we have $\Pi^*(E) = \bigcup_{A\in\mathcal{P}(E)} \Pi(A)$.

$\Pi(\emptyset)$ has a unique element, the empty partition having no block. Formally there is a unique empty set in mathematics, however we will distinguish its two roles as set of points and as partial partition. Hence we write $\emptyset$ for the empty subset of E (thus $\emptyset \in \mathcal{P}(E)$), and $\boldsymbol{\emptyset}$ for the empty partial partition on E, thus $\boldsymbol{\emptyset} \in \Pi^*(E)$ and $\Pi(\emptyset) = \{\boldsymbol{\emptyset}\}$.

For $A \in \mathcal{P}(E)$ with $A \neq \emptyset$, let $\mathbf{0}_A$ be the partition of A into its singletons, and $\mathbf{1}_A$ the partition of A into a single block:

$$\mathbf{0}_A = \big\{\{p\} \mid p \in A\big\} \quad \text{and} \quad \mathbf{1}_A = \{A\}. \tag{7}$$

Following [11], we call $\mathbf{0}_A$ the *identity partition* of A and $\mathbf{1}_A$ the *universal partition* of A. By extension, for $A=\emptyset$, we set $\mathbf{0}_\emptyset = \mathbf{1}_\emptyset = \boldsymbol{\emptyset}$.

The family of partitions of E is known to be ordered by *refinement* and to constitute a complete lattice [11, 16, 19, 23]. We can extend this order to partial partitions:

Definition 10 Given $\pi_1, \pi_2 \in \Pi^*(E)$, we say that π_1 is *finer* than π_2, or that π_2 is *coarser* than π_1, and write $\pi_1 \leq \pi_2$ (or $\pi_2 \geq \pi_1$), iff every block of π_1 is included in a block of π_2:

$$\pi_1 \leq \pi_2 \quad \iff \quad \forall C_1 \in \pi_1,\ \exists C_2 \in \pi_2,\ C_1 \subseteq C_2.$$

This relation on $\Pi^*(E)$ is called *refinement*.

Theorem 11 *By the bijection between $\mathcal{E}^*(E)$ and $\Pi^*(E)$ given in Proposition 4, the refinement relation on $\Pi^*(E)$ corresponds to the inclusion order on $\mathcal{E}^*(E)$*:

$$\forall \pi_1, \pi_2 \in \Pi^*(E), \quad \pi_1 \leq \pi_2 \iff \operatorname{PE}(\pi_1) \subseteq \operatorname{PE}(\pi_2). \tag{8}$$

Therefore $(\Pi^*(E), \leq)$ is a complete lattice isomorphic to $(\mathcal{E}^*(E), \subseteq)$. This order corresponds to the inclusion of class maps:

$$\forall \pi_1, \pi_2 \in \Pi^*(E),$$
$$\pi_1 \leq \pi_2 \quad \iff \quad \forall p \in E,\ \operatorname{Cl}_{\pi_1}(p) \subseteq \operatorname{Cl}_{\pi_2}(p). \tag{9}$$

Given a family $\{\pi_i \mid i \in I\}$ of partial partitions, the class map of their infimum $\bigwedge_{i\in I} \pi_i$ is given by intersection of the respective class maps:

$$\forall p \in E, \quad \operatorname{Cl}_{\bigwedge_{i\in I}\pi_i}(p) = \bigcap_{i\in I} \operatorname{Cl}_{\pi_i}(p). \tag{10}$$

The class map of their supremum $\bigvee_{i\in I} \pi_i$ is given by chaining *class maps*: *for $p, q \in E$, $q \in \operatorname{Cl}_{\bigvee_{i\in I}\pi_i}(p)$ iff there is* *some integer $n \geq 1$ and a sequence $x_0, \ldots, x_n$ in E with $x_0 = p$ and $x_n = q$, such that for each $t = 1, \ldots, n$ there is some $i(t) \in I$ with $x_t \in \mathrm{Cl}_{\pi_{i(t)}}(x_{t-1})$. Furthermore,*

$$\mathrm{supp}\Big(\bigwedge_{i\in I} \pi_i\Big) = \bigcap_{i\in I} \mathrm{supp}(\pi_i)$$

and (11)

$$\mathrm{supp}\Big(\bigvee_{i\in I} \pi_i\Big) = \bigcup_{i\in I} \mathrm{supp}(\pi_i).$$

The least and greatest partial partitions are $\mathbf{Ø}$ and $\mathbf{1}_E$, with $\mathrm{supp}(\mathbf{Ø}) = \emptyset$ *and* $\mathrm{supp}(\mathbf{1}_E) = E$.

Proof For $\pi_1, \pi_2 \in \Pi^*(E)$, $\mathrm{PE}(\pi_1) \subseteq \mathrm{PE}(\pi_2)$ means that for any $p, q \in E$, $p\ \mathrm{PE}(\pi_1)\ q \Rightarrow p\ \mathrm{PE}(\pi_2)\ q$; by (4), this is equivalent to $q \in \mathrm{Cl}_{\pi_1}(p) \Rightarrow q \in \mathrm{Cl}_{\pi_1}(p)$; in other words, $\mathrm{PE}(\pi_1) \subseteq \mathrm{PE}(\pi_2)$ iff $\forall p \in E$ we have $\mathrm{Cl}_{\pi_1}(p) \subseteq \mathrm{Cl}_{\pi_2}(p)$. By Theorem 7, $\pi_1 = \{\mathrm{Cl}_{\pi_1}(p) \mid p \in E,\ \mathrm{Cl}_{\pi_1}(p) \neq \emptyset\}$, and similarly for π_2. If for all $p \in E$ we have $\mathrm{Cl}_{\pi_1}(p) \subseteq \mathrm{Cl}_{\pi_2}(p)$, then every non-void $\mathrm{Cl}_{\pi_1}(p)$ is included in a non-void $\mathrm{Cl}_{\pi_2}(p)$, so $\pi_1 \leq \pi_2$. Conversely, suppose that $\pi_1 \leq \pi_2$ and let $p \in E$; if $\mathrm{Cl}_{\pi_1}(p) = \emptyset$, then obviously $\mathrm{Cl}_{\pi_1}(p) \subseteq \mathrm{Cl}_{\pi_2}(p)$; if $\mathrm{Cl}_{\pi_1}(p) \neq \emptyset$, then $\mathrm{Cl}_{\pi_1}(p)$ is a block of π_1, so it is contained in a block C_2 of π_2, hence $p \in C_2$ and we have $C_2 = \mathrm{Cl}_{\pi_2}(p)$, thus $\mathrm{Cl}_{\pi_1}(p) \subseteq \mathrm{Cl}_{\pi_2}(p)$. Thus we have shown that

$$\begin{aligned}\mathrm{PE}(\pi_1) \subseteq \mathrm{PE}(\pi_2) &\iff (\forall p \in E,\ \mathrm{Cl}_{\pi_1}(p) \subseteq \mathrm{Cl}_{\pi_2}(p)) \\ &\iff \pi_1 \leq \pi_2,\end{aligned}$$

and (8,9) hold. Therefore the two posets $(\Pi^*(E), \leq)$ and $(\mathcal{E}^*(E), \subseteq)$ are isomorphic, and as the latter is a complete lattice, the same holds for the former.

Consider a family $\{\pi_i \mid i \in I\} \subseteq \Pi^*(E)$. By the isomorphism and (5), $\mathrm{PE}(\bigwedge_{i\in I} \pi_i) = \bigwedge_{i\in I} \mathrm{PE}(\pi_i) = \bigcap_{i\in I} \mathrm{PE}(\pi_i)$. We apply (4): for $p, q \in E$, $q \in \mathrm{Cl}_{\bigwedge_{i\in I}\pi_i}(p)$ iff $p\ \mathrm{PE}(\bigwedge_{i\in I} \pi_i)\ q$, that is, $p\ [\bigcap_{i\in I} \mathrm{PE}(\pi_i)]\ q$, in other words, $\forall i \in I$, $p\ \mathrm{PE}(\pi_i)\ q$, that is, $q \in \mathrm{Cl}_{\pi_i}(p)$. Thus $q \in \mathrm{Cl}_{\bigwedge_{i\in I}\pi_i}(p)$ iff $q \in \bigcap_{i\in I} \mathrm{Cl}_{\pi_i}(p)$, and (10) holds. (Alternately, we can show that the map $p \mapsto \bigcap_{i\in I} \mathrm{Cl}_{\pi_i}(p)$ is a partial partition class map.) Similarly,

$$\mathrm{PE}\Big(\bigvee_{i\in I} \pi_i\Big) = \bigvee_{i\in I} \mathrm{PE}(\pi_i) = \bigcup_{n=1}^{\infty}\Big(\bigcup_{i\in I} \mathrm{PE}(\pi_i)\Big)^n.$$

Then by (4) $q \in \mathrm{Cl}_{\bigvee_{i\in I}\pi_i}(p)$ iff $p\ \mathrm{PE}(\bigvee_{i\in I} \pi_i)\ q$, which means that $p\ [\bigcup_{n=1}^{\infty}(\bigcup_{i\in I} \mathrm{PE}(\pi_i))^n]\ q$, equivalently there is some $n \geq 1$ with $p(\bigcup_{i\in I} \mathrm{PE}(\pi_i))^n q$, in other words there is a sequence $p = x_0, \ldots, x_n = q$ with $x_{t-1}\ (\bigcup_{i\in I} \mathrm{PE}(\pi_i))\ x_t$ for $t = 1, \ldots, n$, that is, for each $t = 1, \ldots, n$ there is some $i(t) \in I$ with $x_{t-1}\ \mathrm{PE}(\pi_{i(t)})\ x_t$, equivalently $x_t \in \mathrm{Cl}_{\pi_{i(t)}}(x_{t-1})$.

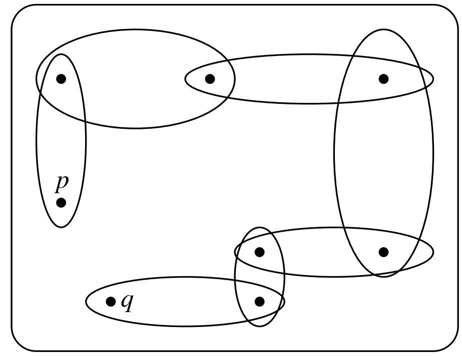

Fig. 6 A block (shown as a *rounded rectangle*) of the supremum of a family of partial partition is obtained by chaining blocks (shown as *ellipses*) of these partitions

Now (11) follows by combining (6) with the identity $\mathrm{supp}(\mathrm{PE}(\pi)) = \mathrm{supp}(\pi)$ in Proposition 4. Finally, it is obvious that every $\pi \in \Pi^*(E)$ satisfies $\mathbf{Ø} \leq \pi \leq \mathbf{1}_E$, and that $\mathrm{supp}(\mathbf{Ø}) = \emptyset$ and $\mathrm{supp}(\mathbf{1}_E) = E$. □

Let us describe the infimum and supremum of a family $\{\pi_i \mid i \in I\}$ of partial partitions, in terms of blocks. A block of $\bigwedge_{i\in I} \pi_i$ is of the form $\bigcap_{i\in I} \beta(i)$, where β is a choice map associating to each $i \in I$ a block $\beta(i) \in \pi_i$, provided that $\bigcap_{i\in I} \beta(i) \neq \emptyset$. A block of $\bigvee_{i\in I} \pi_i$ is any $A \in \mathcal{P}(E)$ such that for any $p, q \in A$, there exist $B_0, \ldots, B_n \in \bigcup_{i\in I} \pi_i$ ($n \geq 0$) with $p \in B_0$, $q \in B_n$ and $B_{t-1} \cap B_t \neq \emptyset$ for all $t = 1, \ldots, n$, see Fig. 6, but no such sequence $B_0, \ldots, B_n$ exists for $p \in A$ and $q \notin A$. The operation of agglomerating successively overlapping blocks $B_0, \ldots, B_n$, is called *chaining* [11]. The construction of a supremum of partitions by chaining the blocks from all of them was given in [11], and we simply extended it to partial partitions.

Note that when the partial partitions have pairwise disjoint supports ($i \neq j \Rightarrow \mathrm{supp}(\pi_i) \cap \mathrm{supp}(\pi_j) = \emptyset$), their supremum is their union: $\bigvee_{i\in I} \pi_i = \bigcup_{i\in I} \pi_i$. In particular, for any $\pi \in \Pi^*(E)$ we have $\bigvee_{C\in\pi} \mathbf{1}_C = \bigcup_{C\in\pi} \mathbf{1}_C = \pi$.

One can chain blocks in a family $\mathcal{B}$ of non-empty subsets of E, and this amounts to taking the supremum of the partial partitions $\mathbf{1}_B$, $B \in \mathcal{B}$. Hence we will use the following terminology:

Definition 12 Let $\mathcal{B}$ be a family of non-empty subsets of E.

1. The *partial partition spanned by* $\mathcal{B}$ is $\bigvee_{B\in\mathcal{B}} \mathbf{1}_B$, in other words the least partial partition such that every $B \in \mathcal{B}$ is included in one block of it (in fact, $\bigvee_{B\in\mathcal{B}} \mathbf{1}_B$ is a partition of $\mathrm{supp}(\mathcal{B})$).
2. For $p, q \in E$, p and q are said to be *chained by* $\mathcal{B}$ if p and q belong both to one block of $\bigvee_{B\in\mathcal{B}} \mathbf{1}_B$, in other words if there are $B_0, \ldots, B_n \in \mathcal{B}$ ($n \geq 0$) such that $p \in B_0$, $q \in B_n$ and $B_{t-1} \cap B_t \neq \emptyset$ for all $t = 1, \ldots, n$.

3. Let $A \in \mathcal{P}(E)$ such that $\mathcal{B} \subseteq \mathcal{P}(A)$. Then A is *chained by* $\mathcal{B}$ if any two points of A are chained by $\mathcal{B}$, in other words $\bigvee_{B \in \mathcal{B}} \mathbf{1}_B = \mathbf{1}_A$ (in particular, $A = \mathrm{supp}(\mathcal{B})$).

Note that any partial partition is spanned by its blocks: $\forall \pi \in \Pi^*(E)$, $\pi = \bigvee_{C \in \pi} \mathbf{1}_C$. The partial partition spanned by an empty family is $\boldsymbol{\emptyset}$. A supremum $\bigvee_{i \in I} \pi_i$ of partition is spanned by their union $\bigcup_{i \in I} \pi_i$.

There is a situation where the class maps of the supremum of partial partitions can be obtained by union instead of chaining:

Proposition 13 *Let $\{\pi_i \mid i \in I\}$ be a subset of $\Pi^*(E)$ such that for every $p \in E$, the set $\{\mathrm{Cl}_{\pi_i}(p) \mid i \in I\}$ is directed. Then*

$$\forall p \in E, \quad \mathrm{Cl}_{\bigvee_{i \in I} \pi_i}(p) = \bigcup_{i \in I} \mathrm{Cl}_{\pi_i}(p).$$

This holds in particular if the set $\{\pi_i \mid i \in I\}$ is directed.

Proof The fact that for every $p \in E$, $\{\mathrm{Cl}_{\pi_i}(p) \mid i \in I\}$ is directed means: $\forall p \in E$, $\forall i, j \in I$, $\exists k \in I$, $\mathrm{Cl}_{\pi_i}(p) \cup \mathrm{Cl}_{\pi_j}(p) \subseteq \mathrm{Cl}_{\pi_k}(p)$ (k depends on p, i, j). Let us show that for every $n \geq 1$, given a sequence $x_0, \dots, x_n$ in E such that for each $t = 1, \dots, n$ there is $i(t) \in I$ with $x_t \in \mathrm{Cl}_{\pi_{i(t)}}(x_{t-1})$, then for some $k \in I$ we have $x_n \in \mathrm{Cl}_{\pi_k}(x_0)$. We use induction on n. The result is obvious for $n = 1$. Suppose that it is true for n, and let us deduce it for $n + 1$. Given $x_0, \dots, x_{n+1}$ with $x_t \in \mathrm{Cl}_{\pi_{i(t)}}(x_{t-1})$ for $t = 1, \dots, n + 1$, the induction hypothesis gives us $x_n \in \mathrm{Cl}_{\pi_j}(x_0)$ for some $j \in I$. By (P2a), $\mathrm{Cl}_{\pi_j}(x_n) = \mathrm{Cl}_{\pi_j}(x_0)$, and by (P1b), $x_0 \in \mathrm{Cl}_{\pi_j}(x_n)$. As $\{\mathrm{Cl}_{\pi_i}(x_n) \mid i \in I\}$ is directed, there is some $k \in I$ with $\mathrm{Cl}_{\pi_j}(x_n) \cup \mathrm{Cl}_{\pi_{i(n+1)}}(x_n) \subseteq \mathrm{Cl}_{\pi_k}(x_n)$. Then $x_0, x_{n+1} \in \mathrm{Cl}_{\pi_k}(x_n)$, and by (P2a) we have $\mathrm{Cl}_{\pi_k}(x_0) = \mathrm{Cl}_{\pi_k}(x_n)$. Therefore $x_{n+1} \in \mathrm{Cl}_{\pi_k}(x_0)$.

Let $p, q \in E$; by Theorem 11, $q \in \mathrm{Cl}_{\bigvee_{i \in I} \pi_i}(p)$ iff we have a sequence $p = x_0, \dots, x_n = q$ such that for each $t = 1, \dots, n$ there is $i(t) \in I$ with $x_t \in \mathrm{Cl}_{\pi_{i(t)}}(x_{t-1})$. By the above, this implies that there is some $k \in I$ with $q \in \mathrm{Cl}_{\pi_k}(p)$; conversely, if $q \in \mathrm{Cl}_{\pi_k}(p)$ for some $k \in I$, then setting $n = 1$, $i(1) = k$, and taking the sequence $p = x_0$, $x_1 = q$, we have $q \in \mathrm{Cl}_{\bigvee_{i \in I} \pi_i}(p)$. Thus $q \in \mathrm{Cl}_{\bigvee_{i \in I} \pi_i}(p)$ iff there is $k \in I$ with $q \in \mathrm{Cl}_{\pi_k}(p)$, and the equality $\mathrm{Cl}_{\bigvee_{i \in I} \pi_i}(p) = \bigcup_{i \in I} \mathrm{Cl}_{\pi_i}(p)$ follows.

The fact that $\{\pi_i \mid i \in I\}$ is directed means that $\forall i, j \in I$, $\exists k \in I$, $\pi_i, \pi_j \leq \pi_k$; by (9) this is equivalent to: $\forall i, j \in I$, $\exists k \in I$, $\forall p \in E$, $\mathrm{Cl}_{\pi_i}(p) \cup \mathrm{Cl}_{\pi_j}(p) \subseteq \mathrm{Cl}_{\pi_k}(p)$ (k depends on i, j, not on p). By the above, this implies that for every $p \in E$, the set $\{\mathrm{Cl}_{\pi_i}(p) \mid i \in I\}$ is directed. □

An alternate proof shows that the map $p \mapsto \bigcup_{i \in I} \mathrm{Cl}_{\pi_i}(p)$ is a partial partition class map. An example of application of this result is to show that for any $B \in \mathcal{P}(E)$ and $\pi \in \Pi^*(E)$, we have

$$\pi \vee \mathbf{0}_B = \pi \cup \mathbf{0}_{B \setminus \mathrm{supp}(\pi)}. \tag{12}$$

Indeed, for $p \in \mathrm{supp}(\pi)$ we have $\mathrm{Cl}_{\mathbf{0}_B}(p) \subseteq \{p\} \subseteq \mathrm{Cl}_\pi(p)$, while for $p \notin \mathrm{supp}(\pi)$ we have $\mathrm{Cl}_\pi(p) = \emptyset \subseteq \mathrm{Cl}_{\mathbf{0}_B}(p)$; by Proposition 13, $\mathrm{Cl}_{\pi \vee \mathbf{0}_B}(p)$ will be $\mathrm{Cl}_\pi(p)$ for $p \in \mathrm{supp}(\pi)$ and $\mathrm{Cl}_{\mathbf{0}_B}(p)$ for $p \notin \mathrm{supp}(\pi)$, so (12) follows.

Given $A \in \mathcal{P}(E)$ and blocks $B_0, \dots, B_n \in \mathcal{P}(A)$, the fact that $B_0, \dots, B_n$ are chained means the same in A and in E. Hence:

Proposition 14 *Let $A \in \mathcal{P}(E)$. Then the non-empty supremum and infimum operations in $\Pi^*(A)$ are those of $\Pi^*(E)$; $\Pi^*(A)$ shares with $\Pi^*(E)$ the same least element (or empty supremum) $\boldsymbol{\emptyset}$; however its greatest element (or empty infimum) is $\mathbf{1}_A$ (instead of $\mathbf{1}_E$ in $\Pi^*(E)$).*

The structure of the lattice of partial partitions determines that of the lattice of partitions. The following result is straightforward:

Proposition 15 *A partial partition on E is a partition iff it majorates $\mathbf{0}_E$:*

$$\Pi(E) = \{\pi \in \Pi^*(E) \mid \pi \geq \mathbf{0}_E\}.$$

Then $(\Pi(E), \leq)$ is a complete lattice whose non-empty supremum and infimum operations are those of $\Pi^*(E)$; $\Pi(E)$ shares with $\Pi^*(E)$ the same greatest element (or empty infimum) $\mathbf{1}_E$; however its least element (or empty supremum) is $\mathbf{0}_E$ (instead of $\boldsymbol{\emptyset}$ in $\Pi^*(E)$).

Combining these two propositions, for any $A \in \mathcal{P}(E)$, a non-empty supremum or infimum of partitions of A is the same in $\Pi(A)$, $\Pi^*(A)$ or $\Pi^*(E)$.

By (11), the map $\Pi^*(E) \to \mathcal{P}(E) : \pi \mapsto \mathrm{supp}(\pi)$ is both a dilation and an erosion. We can thus find its upper and lower adjoints:

Proposition 16 *The support map*

$$\mathrm{supp} : \Pi^*(E) \to \mathcal{P}(E) : \pi \mapsto \mathrm{supp}(\pi)$$

has as upper adjoint the erosion

$$\mathbf{1}_\bullet : \mathcal{P}(E) \to \Pi^*(E) : A \mapsto \mathbf{1}_A,$$

and as lower adjoint the dilation

$$\mathbf{0}_\bullet : \mathcal{P}(E) \to \Pi^*(E) : A \mapsto \mathbf{0}_A,$$

in other words

$$\forall \pi \in \Pi^*(E), \forall A \in \mathcal{P}(E), \quad \mathrm{supp}(\pi) \subseteq A \iff \pi \leq \mathbf{1}_A$$
$$\textit{and} \quad A \subseteq \mathrm{supp}(\pi) \iff \mathbf{0}_A \leq \pi. \tag{13}$$

For all $A \in \mathcal{P}(E)$, $\mathrm{supp}(\mathbf{1}_A) = \mathrm{supp}(\mathbf{0}_A) = A$, *the map* $\mathrm{supp} : \pi \mapsto \mathrm{supp}(\pi)$ *is surjective*, *while the maps* $\mathbf{1}_\bullet : A \mapsto \mathbf{1}_A$ *and* $\mathbf{0}_\bullet : A \mapsto \mathbf{0}_A$ *are injective.*

Proof Let $A \in \mathcal{P}(E)$. For $A \neq \emptyset$, $\mathrm{supp}(\pi) \subseteq A$ and $\pi \leq \mathbf{1}_A$ both mean that every block of π is included in A; on the other hand, $\mathbf{0}_A \leq \pi$ and $A \subseteq \mathrm{supp}(\pi)$ both mean that every point of A belongs to a block of π. For $A = \emptyset$, $\mathbf{1}_\emptyset = \mathbf{0}_\emptyset = \boldsymbol{\emptyset}$; thus $\mathrm{supp}(\pi) \subseteq \emptyset$ and $\pi \leq \mathbf{1}_\emptyset$ both mean that $\pi = \boldsymbol{\emptyset}$; on the other hand, $\mathbf{0}_\emptyset \leq \pi$ and $\emptyset \subseteq \mathrm{supp}(\pi)$ are both always true. Therefore the two adjunctions (13) hold, and it follows that $A \mapsto \mathbf{1}_A$ is an erosion and $A \mapsto \mathbf{0}_A$ is a dilation. Obviously $\mathrm{supp}(\mathbf{1}_A) = \mathrm{supp}(\mathbf{0}_A) = A$, from which we deduce the surjectivity of $\pi \mapsto \mathrm{supp}(\pi)$ and the injectivity of $A \mapsto \mathbf{1}_A$ and $A \mapsto \mathbf{0}_A$. □

The fact that $\mathbf{0}_\bullet : A \mapsto \mathbf{0}_A$ is a dilation and $\mathbf{1}_\bullet : A \mapsto \mathbf{1}_A$ is an erosion means:

$$\forall \mathcal{B} \subseteq \mathcal{P}(E), \quad \mathbf{0}_{\bigcup \mathcal{B}} = \bigvee_{B \in \mathcal{B}} \mathbf{0}_B$$
$$\text{and} \quad \mathbf{1}_{\bigcap \mathcal{B}} = \bigwedge_{B \in \mathcal{B}} \mathbf{1}_B. \tag{14}$$

Then $\{\mathbf{0}_A \mid A \in \mathcal{P}(E)\}$ is a dual Moore family and $\{\mathbf{1}_A \mid A \in \mathcal{P}(E)\}$ is a Moore family. The maps $A \mapsto \mathbf{0}_A$ and $A \mapsto \mathbf{1}_A$ are *order-embeddings* of $\mathcal{P}(E)$ into $\Pi^*(E)$; this means that they are injective and that each one induces an order-isomorphism between $\mathcal{P}(E)$ and its image: $A \subseteq B \Leftrightarrow \mathbf{0}_A \leq \mathbf{0}_B \Leftrightarrow \mathbf{1}_A \leq \mathbf{1}_B$.

The map $\mathbf{0}_\bullet : A \mapsto \mathbf{0}_A$ is not an erosion, because it is not compatible with the greatest element or empty supremum, which is E in $\mathcal{P}(E)$ and $\mathbf{1}_E$ in $\Pi^*(E)$, with $\mathbf{0}_E < \mathbf{1}_E$. However, for a *non-void* family $\mathcal{B} \subseteq \mathcal{P}(E)$ we have $\mathbf{0}_{\bigcap \mathcal{B}} = \bigwedge_{B \in \mathcal{B}} \mathbf{0}_B$.

The map $\mathbf{1}_\bullet : A \mapsto \mathbf{1}_A$ is not a dilation: for two non-void and disjoint A, B we have $\mathbf{1}_A \vee \mathbf{1}_B = \{A, B\} < \mathbf{1}_{A \cup B}$. However it is compatible with the least element or empty infimum, which is $\emptyset$ in $\mathcal{P}(E)$ and $\boldsymbol{\emptyset} = \mathbf{1}_\emptyset$ in $\Pi^*(E)$. Moreover,

$$\forall \mathcal{B} \subseteq \mathcal{P}(E), \quad \left(\mathcal{B} \neq \emptyset, \bigcap \mathcal{B} \neq \emptyset\right)$$
$$\Longrightarrow \mathbf{1}_{\bigcup \mathcal{B}} = \bigvee_{B \in \mathcal{B}} \mathbf{1}_B. \tag{15}$$

From the two adjunctions $(\mathbf{1}_\bullet, \mathrm{supp})$ and $(\mathrm{supp}, \mathbf{0}_\bullet)$, we deduce two operators on $\Pi^*(E)$:

– the *block blending* closing **blend** : $\pi \mapsto \mathbf{1}_{\mathrm{supp}(\pi)}$, where all blocks of π are merged;
– the *block grinding* opening **grind** : $\pi \mapsto \mathbf{0}_{\mathrm{supp}(\pi)}$, where each block of π is pulverized into its singletons.

Combining these two adjunctions, we obtain

$$\forall \pi, \pi' \in \Pi^*(E), \quad \mathbf{0}_{\mathrm{supp}(\pi)} \leq \pi'$$
$$\Longleftrightarrow \mathrm{supp}(\pi) \subseteq \mathrm{supp}(\pi') \Longleftrightarrow \pi \leq \mathbf{1}_{\mathrm{supp}(\pi')},$$

in other words, (**blend**, **grind**) is an adjunction on $\Pi^*(E)$.

2.3 Partial Connections

Connections were first defined in [17]. In [21] the axiomatics was relaxed, introducing thus under the name of *quasi-connection* what we call a partial connection. We will analyse partial connections and the corresponding partial partitions of connected components.

Definition 17 A *partial connection* on $\mathcal{P}(E)$ is a family $\mathcal{C} \subseteq \mathcal{P}(E)$ such that

1. $\emptyset \in \mathcal{C}$, and
2. for any $\mathcal{B} \subseteq \mathcal{C}$ such that $\bigcap \mathcal{B} \neq \emptyset$, we have $\bigcup \mathcal{B} \in \mathcal{C}$.

We call the partial connection $\mathcal{C}$ a *connection* on $\mathcal{P}(E)$ if it satisfies the following third condition:

3. for all $p \in E$, $\{p\} \in \mathcal{C}$.

Note that in condition 2 we did not require $\mathcal{B}$ to be non-empty: for $\mathcal{B} = \emptyset$, $\bigcap \mathcal{B} = E \neq \emptyset$ and $\bigcup \mathcal{B} = \emptyset \in \mathcal{C}$ thanks to condition 1. In other words, condition 1 represents the limiting case of an empty family in condition 2. In particular, $\mathcal{C}$ is a partial connection iff it satisfies condition 1 and the restriction of condition 2 to a non-empty family $\mathcal{B}$.

For any $X \subseteq E$, let us write $\mathcal{S}(X)$ for the family of all singletons in X: $\mathcal{S}(X) = \{\{p\} \mid p \in X\}$. Formally, $\mathcal{S}(X)$ is the same set as $\mathbf{0}_X$, however we will use the notation $\mathbf{0}_X$ in the case of (partial) partitions, and $\mathcal{S}(X)$ in relation to (partial) connections.

Proposition 18 *A family* $\mathcal{C} \subseteq \mathcal{P}(E)$ *is a partial connection iff* $\mathcal{C} \cup \mathcal{S}(E)$ *is a connection*; *then* $\mathcal{C} \cup \mathcal{S}(E)$ *is the least connection containing* $\mathcal{C}$.

Proof Clearly $\mathcal{C} \cup \mathcal{S}(E)$ satisfies condition 3. As for condition 1, $\emptyset \in \mathcal{C}$ iff $\emptyset \in \mathcal{C} \cup \mathcal{S}(E)$. If $\mathcal{C} \cup \mathcal{S}(E)$ satisfies condition 2, then by restriction this condition holds also in $\mathcal{C}$. Conversely, suppose that $\mathcal{C}$ satisfies condition 2, and let us show that it holds then in $\mathcal{C} \cup \mathcal{S}(E)$. Let $\mathcal{B} \subseteq \mathcal{C} \cup \mathcal{S}(E)$ such that $\bigcap \mathcal{B} \neq \emptyset$, and let $p \in \bigcap \mathcal{B}$; we have 3 cases:

– $\mathcal{B} \subseteq \mathcal{C}$; then $\bigcup \mathcal{B} \in \mathcal{C}$ by hypothesis, so $\bigcup \mathcal{B} \in \mathcal{C} \cup \mathcal{S}(E)$.
– $\mathcal{B} = \{\{p\}\}$; then $\bigcup \mathcal{B} = \{p\} \in \mathcal{S}(E)$, so $\bigcup \mathcal{B} \in \mathcal{C} \cup \mathcal{S}(E)$.
– $\mathcal{B} = \mathcal{B}' \cup \{\{p\}\}$ for a non-void $\mathcal{B}' \subseteq \mathcal{C}$; then $p \in \bigcap \mathcal{B}'$ and $\bigcup \mathcal{B}' \in \mathcal{C}$ by hypothesis; as $\mathcal{B}'$ is non-void, we must have $p \in \bigcup \mathcal{B}'$, so $\bigcup \mathcal{B} = \bigcup \mathcal{B}' \cup \{p\} = \bigcup \mathcal{B}' \in \mathcal{C}$, hence $\bigcup \mathcal{B} \in \mathcal{C} \cup \mathcal{S}(E)$.

Therefore $\mathcal{C} \cup \mathcal{S}(E)$ satisfies also condition 2.

Every connection contains $\mathcal{S}(E)$, thus any connection containing $\mathcal{C}$ contains $\mathcal{C} \cup \mathcal{S}(E)$, so this is the least connection containing $\mathcal{C}$. □

Connections have an alternate description in terms of the operation that associates to a set and a point the connected component of the set marked by the point [17]. Let us extend it to partial connections. Suppose that to every point $p \in E$ is associated an opening γ_p on $\mathcal{P}(E)$. Consider then the following properties that they may satisfy:

(C0a) For any $p \in E$, $\gamma_p(\{p\}) = \{p\}$.
(C0b) For any $p \in E$ and $X \in \mathcal{P}(E)$, $p \in X \Rightarrow p \in \gamma_p(X)$.
(C1a) For any $p \in E$ and $X \in \mathcal{P}(E)$, $p \in X$ or $\gamma_p(X) = \emptyset$.
(C1b) For any $p \in E$ and $X \in \mathcal{P}(E)$, $p \in \gamma_p(X)$ or $\gamma_p(X) = \emptyset$.
(C2a) For any $p, q \in E$ and $X \in \mathcal{P}(E)$, $q \in \gamma_p(X) \Rightarrow \gamma_p(X) = \gamma_q(X)$.
(C2b) For any $p, q \in E$ and $X \in \mathcal{P}(E)$, $\gamma_p(X) \cap \gamma_q(X) \neq \emptyset \Rightarrow \gamma_p(X) = \gamma_q(X)$.

In order to characterize connections in terms of connected components, one usually takes the axioms (C0a), (C1a) (under the form $p \notin X \Rightarrow \gamma_p(X) = \emptyset$) and (C2b) [17]. We have the following counterpart of Lemma 5:

Lemma 19 *Let γ_p be an opening on $\mathcal{P}(E)$ for every $p \in E$. Then*

1. (C0a) *and* (C0b) *are equivalent*;
2. (C1a) *and* (C1b) *are equivalent*;
3. (C2a) *implies* (C2b), *and if* (C0b) *holds*, *then* (C2a) *and* (C2b) *are equivalent*.

Proof 1. For $p \in X$ we have $\{p\} \subseteq X$, and as γ_p is increasing, $\gamma_p(\{p\}) \subseteq \gamma_p(X)$; so if (C0a) holds, we have $p \in \gamma_p(\{p\})$, hence $p \in \gamma_p(X)$, that is (C0b). As γ_p is anti-extensive, $\gamma_p(\{p\}) \subseteq \{p\}$; applying (C0b) with $p \in \{p\}$, we get $p \in \gamma_p(\{p\})$; hence $\gamma_p(\{p\}) = \{p\}$, that is (C0a).

2. Applying (C1a) with $\gamma_p(X)$ in place of X gives $p \in \gamma_p(X)$ or $\gamma_p(\gamma_p(X)) = \emptyset$, but as γ_p is idempotent, $\gamma_p(\gamma_p(X)) = \gamma_p(X)$, hence we get (C1b). Since γ_p is anti-extensive, $\gamma_p(X) \subseteq X$, so $p \in \gamma_p(X) \Rightarrow p \in X$; hence (C1b) implies (C1a).

3. Since γ_p is anti-extensive, $\gamma_p(X) \subseteq X$. Thus we can apply the proof of Lemma 5 with $\gamma_p(X), \gamma_q(X)$ in place of $\mathrm{cl}(p), \mathrm{cl}(q)$. □

Definition 20 A *system of partial connection openings* on $\mathcal{P}(E)$ associates to each $p \in E$ an opening γ_p on $\mathcal{P}(E)$, and satisfies (C1a) (or equivalently (C1b)), and (C2a). If it satisfies also (C0a) (or equivalently (C0b)), it is a *system of connection openings* on $\mathcal{P}(E)$.

Note that in a system of connection openings, by (C0a) / (C0b) we can replace (C2a) by (C2b); however this is in general not possible for a system of partial connection openings, as we will see later.

The following result, which parallels Theorem 7, generalizes the well-known characterization [17] of connections by systems of connection openings:

Theorem 21 *There exists a one-to-one correspondence between partial connections on $\mathcal{P}(E)$ and systems of partial connection openings on $\mathcal{P}(E)$, under which*:

– *To every partial connection $\mathcal{C}$ corresponds the system of partial connection openings $(\gamma_p, p \in E)$ given by*

$$\forall p \in E, \forall X \in \mathcal{P}(E),\quad \gamma_p(X) = \bigcup \{C \in \mathcal{C} \mid p \in C, C \subseteq X\}, \tag{16}$$

and in fact

$$\begin{aligned}&\forall p \in E, \forall X \in \mathcal{P}(E),\\ &\quad \textit{either}\ \{C \in \mathcal{C} \mid p \in C, C \subseteq X\} = \emptyset\\ &\qquad \textit{and}\ \gamma_p(X) = \emptyset,\\ &\quad \textit{or}\ \gamma_p(X)\ \textit{is the greatest element of}\\ &\qquad \{C \in \mathcal{C} \mid p \in C,\ C \subseteq X\}.\end{aligned} \tag{17}$$

– *To every system of partial connection openings $(\gamma_p, p \in E)$ corresponds the partial connection $\mathcal{C}$ given by*

$$\mathcal{C} = \{\gamma_p(X) \mid p \in E, X \in \mathcal{P}(E)\}. \tag{18}$$

Furthermore, it induces a one-to-one correspondence between connections on $\mathcal{P}(E)$ and systems of connection openings on $\mathcal{P}(E)$.

Proof We have to show that:

– for a partial connection $\mathcal{C}$: (a) (16) gives a system of partial connection openings satisfying (17), (b) for which (18) gives again $\mathcal{C}$, and (c) if $\mathcal{C}$ is a connection, then (16) gives a system of connection openings.
– for a system of partial connection openings $(\gamma_p, p \in E)$: (d) (18) gives a partial connection, (e) for which (16) gives again $(\gamma_p, p \in E)$, and (f) if $(\gamma_p, p \in E)$ is a system of connection openings, then (18) gives a connection.

(a) Let $\mathcal{C}$ be a partial connection, and let $(\gamma_p, p \in E)$ be given by (16). For $p \in E$ and $X \in \mathcal{P}(E)$, let $\mathcal{B}(p, X) = \{C \in \mathcal{C} \mid p \in C \subseteq X\}$. If $\mathcal{B}(p, X) = \emptyset$, then $\gamma_p(X) = \bigcup \mathcal{B}(p, X) = \emptyset$. If $\mathcal{B}(p, X) \neq \emptyset$, $\mathcal{B}(p, X) \subseteq \mathcal{C}$ and $p \in \bigcap \mathcal{B}(p, X)$, so that $\gamma_p(X) = \bigcup \mathcal{B}(p, X) \in \mathcal{C}$; but then $p \in \gamma_p(X) \subseteq X$, so $\gamma_p(X)$ is the greatest element of $\mathcal{B}(p, X)$. Hence (17) holds. By construction, $\gamma_p(X) \subseteq X$, so γ_p is anti-extensive. For

$X \subseteq Y$, $\mathcal{B}(p, X) \subseteq \mathcal{B}(p, Y)$, so $\gamma_p(X) = \bigcup \mathcal{B}(p, X) \subseteq \bigcup \mathcal{B}(p, Y) = \gamma_p(Y)$; thus γ_p is increasing. If $\gamma_p(X) = \emptyset$, by anti-extensivity we have $\gamma_p(\gamma_p(X)) = \emptyset$; otherwise, by (17) $\gamma_p(X) \in \mathcal{C}$ and $p \in \gamma_p(X)$, but then $\gamma_p(X)$ is the greatest $C \in \mathcal{C}$ such that $p \in C \subseteq \gamma_p(X)$, so $\gamma_p(X) = \bigcup \mathcal{B}(p, \gamma_p(X)) = \gamma_p(\gamma_p(X))$. Hence γ_p is idempotent, so it is an opening. By construction, $p \notin X$ gives $\mathcal{B}(p, X) = \emptyset$, hence $\gamma_p(X) = \emptyset$; thus (C1a) holds. For $q \in \gamma_p(X)$, $\gamma_p(X) \neq \emptyset$, so by (17) $p \in \gamma_p(X)$, $\gamma_p(X) \subseteq X$ and $\gamma_p(X) \in \mathcal{C}$; but then $\gamma_p(X) \in \mathcal{B}(q, X)$, so $\gamma_p(X) \subseteq \gamma_q(X)$; thus $p \in \gamma_q(X)$, and we deduce similarly that $\gamma_q(X) \subseteq \gamma_p(X)$, and the equality follows, in other words we have (C2a). Therefore $(\gamma_p, p \in E)$ is a system of partial connection openings.

(b) By (17), either $\gamma_p(X) = \emptyset \in \mathcal{C}$, or $\gamma_p(X) \in \mathcal{C}$. Thus $\{\gamma_p(X) \mid p \in E,\ X \in \mathcal{P}(E)\} \subseteq \mathcal{C}$. Now $\gamma_p(\emptyset) = \emptyset$ for any $p \in E$, while for $C \in \mathcal{C}$ such that $C \neq \emptyset$, by (17) we have $\gamma_p(C) = C$ for $p \in C$. Thus $\{\gamma_p(X) \mid p \in E,\ X \in \mathcal{P}(E)\} \supseteq \mathcal{C}$, and we deduce the equality. Hence (18) gives again $\mathcal{C}$.

(c) If C is a connection, then for every $p \in E$, $\{p\} \in \mathcal{C}$, so (17) gives $\gamma_p(\{p\}) = \{p\}$, that is (C0a). Therefore $(\gamma_p, p \in E)$ (already a system of partial connection openings by (a)) is a system of connection openings.

(d) Let $(\gamma_p, p \in E)$ be a system of partial connection openings, and let $\mathcal{C}$ be given by (18). By anti-extensivity, $\gamma_p(\emptyset) = \emptyset$, so $\emptyset \in \mathcal{C}$. Let $\mathcal{B} \subseteq \mathcal{C}$ with $\bigcap \mathcal{B} \neq \emptyset$, and let $q \in \bigcap \mathcal{B}$. Every element of B is of the form $\gamma_p(X)$ ($p \in E$, $X \in \mathcal{P}(E)$), and then $q \in \gamma_p(X)$; by (C2a), $\gamma_p(X) = \gamma_q(X)$. Thus $\mathcal{B} \subseteq \{\gamma_q(Z) \mid Z \in \mathcal{P}(E)\} = \mathrm{Inv}(\gamma_q)$, and as the invariance domain of an opening is a dual Moore family, $\bigcup \mathcal{B} \in \mathrm{Inv}(\gamma_q)$, in other words $\bigcup \mathcal{B} = \gamma_q(Z)$ for some $Z \in \mathcal{P}(E)$, hence $\bigcup \mathcal{B} \in \mathcal{C}$. Thus $\mathcal{C}$ is a partial connection.

(e) Let $p \in E$ and $X \in \mathcal{P}(E)$. Suppose first that there exists $C \in \mathcal{C}$ such that $p \in C \subseteq X$. By (18) we have $C = \gamma_q(Y)$ for some $q \in E$ and $Y \in \mathcal{P}(E)$, and as $p \in C$, (C2a) gives $C = \gamma_p(Y)$; as γ_p is an opening, $C = \gamma_p(Y) = \gamma_p(\gamma_p(Y)) = \gamma_p(C) \subseteq \gamma_p(X)$. As $C \subseteq \gamma_p(X)$ and $p \in C$, we have $p \in \gamma_p(X)$; now $\gamma_p(X) \in \mathcal{C}$ by (18). We have thus shown that if $\{C \in \mathcal{C} \mid p \in C \subseteq X\} \neq \emptyset$, then $\gamma_p(X)$ is the greatest element of this set, so $\gamma_p(X) = \bigcup\{C \in \mathcal{C} \mid p \in C \subseteq X\}$. If $\{C \in \mathcal{C} \mid p \in C \subseteq X\} = \emptyset$, then as $\gamma_p(X) \subseteq X$ and $\gamma_p(X) \in \mathcal{C}$, we must have $p \notin \gamma_p(X)$, so (C1b) give $\gamma_p(X) = \emptyset$; thus $\bigcup\{C \in \mathcal{C} \mid p \in C \subseteq X\} = \bigcup \emptyset = \emptyset = \gamma_p(X)$. We have thus shown that (16) gives again $(\gamma_p, p \in E)$.

(f) If $(\gamma_p, p \in E)$ is a system of connection openings, then by (C0a), for every $p \in E$, we have $\gamma_p(\{p\}) = \{p\}$; by (18), this means that $\{p\} \in \mathcal{C}$. Therefore $\mathcal{C}$ (already a partial connection by (d)) is a connection. □

From (17) we deduce that

$$\forall p \in E, \quad \mathrm{Inv}(\gamma_p) = \{\emptyset\} \cup \{C \in \mathcal{C} \mid p \in C\}. \tag{19}$$

Definition 22 Let $\mathcal{C}$ be a partial connection on $\mathcal{P}(E)$. For any $X \in \mathcal{P}(E)$, a *$\mathcal{C}$-component of X*, or *connected component of X according to $\mathcal{C}$*, is any $C \in \mathcal{C}$ with $C \neq \emptyset$ and $C \subseteq X$, which is maximal for inclusion: $\forall C' \in \mathcal{C}$, $C \subseteq C' \subseteq X \Rightarrow C' = C$.

Proposition 23 *Let $\mathcal{C}$ be a partial connection on $\mathcal{P}(E)$. For any $X \in \mathcal{P}(E)$, the map $X \to \mathcal{P}(X) : p \mapsto \gamma_p(X)$ is a partial partition class map on X; when $\mathcal{C}$ is a connection, this map is a partition class map on X. The corresponding partial partition is*

$$\mathrm{PC}^{\mathcal{C}}(X) = \{\gamma_p(X) \mid p \in X,\ \gamma_p(X) \neq \emptyset\}; \tag{20}$$

it is the set of all $\mathcal{C}$-components of X. Considered as a partial partition of E, $\mathrm{PC}^{\mathcal{C}}(X)$ is increasing in X: $X \subseteq Y \Rightarrow \mathrm{PC}^{\mathcal{C}}(X) \leq \mathrm{PC}^{\mathcal{C}}(Y)$.

Proof As γ_p is anti-extensive, the map $\mathrm{cl}_X : p \mapsto \gamma_p(X)$ is indeed $X \to \mathcal{P}(X)$. Now (C1b) and (C2a) restricted to $p \in X$ mean that this map satisfies (P1b) and (P2a), in other words cl_X is a partial partition class map on X. When $\mathcal{C}$ is a connection, the openings γ_p satisfy also (C0b), which means that cl_X satisfies (P1a), so that it is a partition class map. The corresponding partial partition is given by (3): $\{\mathrm{cl}_X(p) \mid p \in E,\ \mathrm{cl}_X(p) \neq \emptyset\}$, that is $\mathrm{PC}^{\mathcal{C}}(X)$.

If $\gamma_p(X) \neq \emptyset$, then by (17) $\gamma_p(X)$ is the greatest $C \in \mathcal{C}$ such that $p \in C$ and $C \subseteq X$. Thus $\gamma_p(X)$ cannot be included in a larger $C \in \mathcal{C}$ with $C \subseteq X$, in other words $\gamma_p(X)$ is a $\mathcal{C}$-component of X. Conversely, given a $\mathcal{C}$-component C of X, $C \neq \emptyset$; taking $p \in C$, (17) implies that $C \subseteq \gamma_p(X) \subseteq X$ and $\gamma_p(X) \in \mathcal{C}$; by the maximality of C, we deduce that $\gamma_p(X) = C$. Thus $\mathrm{PC}^{\mathcal{C}}(X)$ is the set of $\mathcal{C}$-components of X.

Any partial partition of $X \subseteq E$ is a partial partition of E, so $\mathrm{PC}^{\mathcal{C}}(X) \in \Pi^*(E)$. If $X \subseteq Y$, for every $p \in X$ we have $\gamma_p(X) \subseteq \gamma_p(Y)$, in other words every block of $\mathrm{PC}^{\mathcal{C}}(X)$ is included in a block of $\mathrm{PC}^{\mathcal{C}}(Y)$, that is $\mathrm{PC}^{\mathcal{C}}(X) \leq \mathrm{PC}^{\mathcal{C}}(Y)$. □

Note that $\mathrm{PC}^{\mathcal{C}}(X)$ is the partial partition spanned by $\mathcal{C} \cap \mathcal{P}(X)$, cf. Definition 12. It is also the greatest partial partition of X whose blocks belong to $\mathcal{C}$.

In view of Proposition 18, we have the following result, which is a straightforward consequence of (16):

Proposition 24 *Let $\mathcal{C}$ be a partial connection on $\mathcal{P}(E)$ with system of partial connection openings $(\gamma_p, p \in E)$. Then the connection $\mathcal{C} \cup \mathcal{S}(E)$ has its system of connection openings $(\gamma'_p, p \in E)$ given by setting for all $p \in E$ and $X \in \mathcal{P}(E)$:*

$$\gamma'_p(X) = \gamma_p(X) \cup \big(\{p\} \cap X\big)$$

$$= \begin{cases} \gamma_p(X) & \text{if } p \notin X \text{ or } \gamma_p(X) \neq \emptyset, \\ \{p\} & \text{if } p \in X \text{ and } \gamma_p(X) = \emptyset. \end{cases}$$

For any $X \in \mathcal{P}(E)$, $\mathrm{PC}^{\mathcal{C}\cup\mathcal{S}(E)}(X)$ *is obtained from* $\mathrm{PC}^{\mathcal{C}}(X)$ *by adding the singletons outside its support*:

$$\mathrm{PC}^{\mathcal{C}\cup\mathcal{S}(E)}(X) = \mathrm{PC}^{\mathcal{C}}(X) \cup \mathbf{0}_{X\setminus \mathrm{supp}(\mathrm{PC}^{\mathcal{C}}(X))}.$$

Equivalently, $\mathrm{PC}^{\mathcal{C}\cup\mathcal{S}(E)}(X) = \mathrm{PC}^{\mathcal{C}}(X) \vee \mathbf{0}_X$, *cf.* (12).

Remark 25 A counterexample similar to that of Remark 8, but this time with the partition of $\mathcal{C}$-components, shows that in a system of partial connection openings we cannot replace (C2a) by (C2b).

Let E be a finite subset of $\mathbf{Z}$, of size > 3. Choose $A \in \mathcal{P}(E)$ of size ≥ 3, and let $\mathcal{C} = \{\emptyset, A\}$. Then $\mathcal{C}$ is a partial connection whose system of partial connection openings $(\gamma_p, p \in E)$ is given by

$$\forall p \in E, \forall X \in \mathcal{P}(E),$$
$$\gamma_p(X) = \begin{cases} A & \text{if } p \in A \text{ and } A \subseteq X, \\ \emptyset & \text{otherwise.} \end{cases}$$

Now let $a = \min A$ and $b = \max A$, and define $(\beta_p, p \in E)$ by

$$\forall p \in E, \forall X \in \mathcal{P}(E),$$
$$\beta_p(X) = \begin{cases} A & \text{if} A \subseteq X \text{ and } p = a \text{ or } p = b, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then we can check that the β_p, $p \in E$, are openings satisfying (C1b) and (C2b), but not (C2a), and that similarly to (18) we have

$$\{\beta_p(X) \mid p \in E,\ X \in \mathcal{P}(E)\} = \{\emptyset, A\} = \mathcal{C}.$$

However for $p \in A \setminus \{a, b\}$ we have $\beta_p(A) = \emptyset$ while $\gamma_p(A) = A$, hence $\beta_p \neq \gamma_p$, so $(\beta_p, p \in E)$ is not a system of partial connection openings.

Let us now give some examples of partial connections, with their systems of partial connection openings. Several connections were described in [9, 12, 17]; in each of them, if we remove the requirement that the singletons belong to the connection, we obtain a partial connection. In Sect. 3, we will describe some methods for constructing partial connections from other ones.

In $\mathbf{Z}^2$, the family of horizontal lines, half-lines or line segments, in other words, of all sets of the form $\{a\} \times C$, where $a \in \mathbf{Z}$ and C is a connected (equivalently, convex) subset of $\mathbf{Z}$, is a connection. We call it the *horizontal connection* and write it $\mathcal{C}^h$. For $p \in X$, $\gamma_p^h(X)$ is the connected component of the horizontal cross-section of X containing p. See Fig. 7.

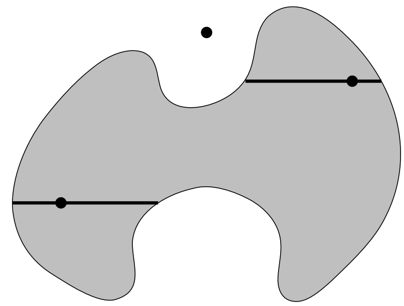

Fig. 7 Consider the horizontal connection $\mathcal{C}^h$ on $\mathcal{P}(\mathbf{Z}^2)$, consisting of all connected subsets of horizontal lines. We show a set, 3 points, and the connected components marked by the 2 points inside the set (the point outside the set marks $\emptyset$)

Given $A \in \mathcal{P}(E)$, every partial connection on $\mathcal{P}(A)$ (in particular, every connection on $\mathcal{P}(A)$) is a partial connection on $\mathcal{P}(E)$; if we write $(\gamma_p, p \in A)$ for the system of partial connection openings on $\mathcal{P}(A)$ and $(\gamma'_p, p \in E)$ for the one on $\mathcal{P}(E)$, then we have $\gamma'_p(X) = \gamma_p(X \cap A)$ for $p \in A$, and $\emptyset$ for $p \notin A$. Note that a partial connection on $\mathcal{P}(E)$ is a connection on $\mathcal{P}(A)$ for some $A \in \mathcal{P}(E)$, iff it contains all singletons in its support: $\forall C \in \mathcal{C}$, $\forall p \in C$, $\{p\} \in \mathcal{C}$.

Given $\pi \in \Pi^*(E)$, let $\mathbf{subbl}(\pi)$ be the family of all subsets of all blocks of π:

$$\mathbf{subbl}(\pi) = \bigcup_{C\in\pi} \mathcal{P}(C) = \{X \in \mathcal{P}(E) \mid \exists C \in \pi,\ X \subseteq C\}. \tag{21}$$

Then $\mathbf{subbl}(\pi)$ is a connection on $\mathcal{P}(\mathrm{supp}(\pi))$ [17, 23], in particular a partial connection on $\mathcal{P}(E)$. The corresponding partial connection openings are given by $\gamma_p(X) = X \cap \mathrm{Cl}_\pi(p)$ for all $p \in E$. In particular, the blocks of π are the connected components of E.

A wide family of partial connections is provided by dual Moore families:

Proposition 26 *Let* $\mathcal{M} \subseteq \mathcal{P}(E)$. *Then* $\mathcal{M}$ *is a dual Moore family of* $\mathcal{P}(E)$ *iff it is a partial connection such that every set has at most one* $\mathcal{M}$*-component. Given the opening* α *corresponding to* $\mathcal{M}$ (*that is*, $\mathrm{Inv}(\alpha) = \mathcal{M}$), *for every* $X \in \mathcal{P}(E)$, *the unique* $\mathcal{M}$*-component of* X *is* $\alpha(X)$ *if* $\alpha(X) \neq \emptyset$, *while there is none if* $\alpha(X) = \emptyset$. *The corresponding system of partial connection openings* $(\alpha_p, p \in E)$ *is given by*

$$\forall p \in E, \forall X \in \mathcal{P}(E),$$
$$\alpha_p(X) = \begin{cases} \alpha(X) & \text{if } p \in \alpha(X), \\ \emptyset & \text{if } p \notin \alpha(X). \end{cases} \tag{22}$$

Proof If $\mathcal{M}$ is a dual Moore family, then it contains $\emptyset$ and it is closed under arbitrary union, so it satisfies the two axioms of a partial connection. For any $X \in \mathcal{P}(E)$, $\alpha(X)$ is the greatest $Y \in \mathcal{M}$ such that $Y \subseteq X$, thus either $\alpha(X) = \emptyset$ or $\alpha(X)$ is the unique $\mathcal{M}$-component of X. Conversely, if $\mathcal{M}$ is a partial connection such that every set has at most one $\mathcal{M}$-component, let $\mathcal{B} \subseteq \mathcal{M}$. If $\mathcal{B}$ is empty or $\mathcal{B} = \{\emptyset\}$, then $\bigcup \mathcal{B} = \emptyset \in \mathcal{M}$, so we can suppose that $\mathcal{B}$ has at least one non-void member. Now every non-void $B \in \mathcal{B}$ is included in the unique $\mathcal{M}$-component of $\bigcup \mathcal{B}$, so this $\mathcal{M}$-component must be $\bigcup \mathcal{B}$, in other words $\bigcup \mathcal{B} \in \mathcal{M}$, hence $\mathcal{M}$ is a dual Moore family. Now (22) follows from (17) and the fact that $\alpha(X)$ is the greatest $C \in \mathrm{Inv}(\alpha) = \mathcal{M}$ such that $C \subseteq X$. □

We give 2 simple examples of dual Moore families:

- The least and greatest dual Moore families, namely $\{\emptyset\}$ and $\mathcal{P}(E)$, are the least and greatest partial connections (and $\mathcal{P}(E)$ is a connection).
- $\{\emptyset, A\}$ for $A \in \mathcal{P}(E) \setminus \{\emptyset\}$. We already considered such a partial connection in Remark 25. We have $\gamma_p(X) = A$ if $p \in A$ and $A \subseteq X$, and $\emptyset$ otherwise.

It is straightforward that an increasing operator α on $\mathcal{P}(E)$, such that for every $X \in \mathcal{P}(E)$, $\alpha(X) \in \{\emptyset, X\}$, is an opening. It is called a *trivial opening*. We have the following easily proved result:

Proposition 27 *Let $\mathcal{U} \subseteq \mathcal{P}(E)$, and $\mathcal{V} = \mathcal{U} \cup \{\emptyset\}$. Then $\mathcal{U}$ is an upper set iff $\mathcal{V} = \mathrm{Inv}(\alpha)$ for a trivial opening α, iff $\mathcal{V}$ is a partial connection on $\mathcal{P}(E)$ such that every set $X \notin \mathcal{V}$ has no $\mathcal{V}$-component. In particular, $\mathcal{V}$ is a dual Moore family. The corresponding system of partial connection openings $(\alpha_p, p \in E)$ is given by*

$$\forall p \in E, \forall X \in \mathcal{P}(E),$$
$$\alpha_p(X) = \begin{cases} X & \text{if } p \in X \text{ and } X \in \mathcal{U}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let us give some examples of upper sets; in each, adding $\emptyset$ yields a partial connection:

- For an integer $n > 1$, the set $\mathcal{P}_{\geq n}(E)$ of all subsets X of E such that $|X| \geq n$. Here $\alpha_p(X) = X$ if $p \in X$ and $|X| \geq n$, and $\emptyset$ otherwise.
- $\{X \in \mathcal{P}(E) \mid X \supseteq A\}$ and $\{X \in \mathcal{P}(E) \mid X \not\subseteq A\}$, for a fixed $A \in \mathcal{P}(E)$.
- $\{X \in \mathcal{P}(E) \mid \psi(X) \neq \emptyset\}$ for an increasing operator ψ on $\mathcal{P}(E)$.
- Given a metric on E, the family of all subsets of E whose diameter exceeds some fixed value.

As a partial generalization of Proposition 27: if $\mathcal{C}$ is a partial connection and $\mathcal{U}$ is an upper set, then $\mathcal{C} \cup \mathcal{U}$ is a partial connection; furthermore, if $\mathcal{C}$ is a dual Moore family, then $\mathcal{C} \cup \mathcal{U}$ is a dual Moore family. For $X \notin \mathcal{U}$, the $\mathcal{C} \cup \mathcal{U}$-components of X are its $\mathcal{C}$-components, while for $X \in \mathcal{U}$, X is its unique $\mathcal{C} \cup \mathcal{U}$-component.

If $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are partial connections such that for $1 \leq i < j \leq n$, all $X \in \mathcal{C}_i \setminus \mathcal{C}_j$ and $Y \in \mathcal{C}_j \setminus \mathcal{C}_i$ satisfy $X \cap Y = \emptyset$, then $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$ is a partial connection. A $\mathcal{C}_i$-component and a $\mathcal{C}_j$-component of a set are either equal or disjoint, so the $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$-components of that set are its $\mathcal{C}_t$-components for $t = 1, \ldots, n$.

We can also take a union $\bigcup_{i \in I} \mathcal{C}_i$ of partial connections $\mathcal{C}_i$, where for $i, j \in I$ such that $i \neq j$, every $X \in \mathcal{C}_i$ and $Y \in \mathcal{C}_i$ satisfy $X \cap Y = \emptyset$. For instance we can take each $\mathcal{C}_i$ to be a partial connection on $\mathcal{P}(A_i)$, where the A_i $(i \in I)$ are mutually disjoint subsets of E. The example of Fig. 7 belongs to that category (where the $\mathcal{C}_i$'s are the connections on the individual lines $\{a\} \times \mathbf{Z}$).

2.4 The Lattice of Partial Connections

We will generalize some results of [12]. Write $\Gamma(E)$ for the set of all connections on $\mathcal{P}(E)$, and $\Gamma^*(E)$ for the set of all partial connections on $\mathcal{P}(E)$ (in [12] we wrote $\mathrm{ConCl}(E)$ for $\Gamma(E)$).

It is easily seen that the conditions in Definition 17 are preserved by intersecting families $\mathcal{C} \subseteq \mathcal{P}(E)$. Hence:

Proposition 28 *An intersection of connections on $\mathcal{P}(E)$ is a connection on $\mathcal{P}(E)$; an intersection of partial connections on $\mathcal{P}(E)$ is a partial connection on $\mathcal{P}(E)$; $\mathcal{P}(E)$ is a connection on $\mathcal{P}(E)$. Thus $\Gamma(E)$ and $\Gamma^*(E)$ are Moore families of $\mathcal{P}(\mathcal{P}(E))$.*

We can relate the structure of the Moore family of partial connections to that of the systems of partial connection openings:

Proposition 29 *Given two partial connections $\mathcal{C}$ and $\mathcal{C}'$ with systems of partial connection openings $(\gamma_p, p \in E)$ and $(\gamma'_p, p \in E)$,*

$$[\mathcal{C} \subseteq \mathcal{C}'] \iff [\forall p \in E, \gamma_p \leq \gamma'_p]$$
$$\iff [\forall p \in E, \mathrm{Inv}(\gamma_p) \subseteq \mathrm{Inv}(\gamma'_p)].$$

Given a non-void *family $\mathcal{C}_i$ $(i \in I, I \neq \emptyset)$ of partial connections with systems of partial connection openings $(\gamma^i_p, p \in E)$, then $\bigcap_{i \in I} \mathcal{C}_i$ has the system of partial connection openings $(\gamma_p, p \in E)$ such that for every $p \in E$, γ_p is the greatest opening on $\mathcal{P}(E)$ that is $\leq \gamma^i_p$ for all $i \in I$, in other words*

$$\forall p \in E, \quad \mathrm{Inv}(\gamma_p) = \bigcap_{i \in I} \mathrm{Inv}(\gamma^i_p).$$

The system of connection openings of $\mathcal{P}(E)$ is $(\gamma^\top_p, p \in E)$ given by

$$\forall p \in E, \ \forall X \in \mathcal{P}(E), \quad \gamma^\top_p(X) = \begin{cases} X & \text{if } p \in X, \\ \emptyset & \text{if } p \notin X. \end{cases}$$

Proof By (19), $\mathcal{C} \subseteq \mathcal{C}'$ implies that for every $p \in E$ we have

$$\begin{aligned}\mathrm{Inv}(\gamma_p) &= \{\emptyset\} \cup \{C \in \mathcal{C} \mid p \in C\}\\ &\subseteq \{\emptyset\} \cup \{C \in \mathcal{C}' \mid p \in C\} = \mathrm{Inv}(\gamma'_p).\end{aligned}$$

Conversely, suppose that $\mathrm{Inv}(\gamma_p) \subseteq \mathrm{Inv}(\gamma'_p)$ for all $p \in E$. By definition, $\emptyset \in \mathcal{C}'$. Now for $C \in \mathcal{C}$ such that $C \neq \emptyset$, take $p \in C$; by (19), $C \in \mathrm{Inv}(\gamma_p)$, hence $C \in \mathrm{Inv}(\gamma'_p)$, so by (19) again we deduce that $C \in \mathcal{C}'$. Thus $\mathcal{C} \subseteq \mathcal{C}'$. We have shown that $\mathcal{C} \subseteq \mathcal{C}'$ iff $\mathrm{Inv}(\gamma_p) \subseteq \mathrm{Inv}(\gamma'_p)$ for all $p \in E$. By the isomorphism between openings and dual Moore families, $\gamma_p \leq \gamma'_p \Leftrightarrow \mathrm{Inv}(\gamma_p) \subseteq \mathrm{Inv}(\gamma'_p)$.

Take now a non-void family ($I \neq \emptyset$) of partial connections $\mathcal{C}_i$ ($i \in I$) with systems of partial connection openings (γ^i_p, $p \in E$). For $p \in E$ and $X \in \mathcal{P}(E)$, we have $X \in \bigcap_{i\in I} \mathrm{Inv}(\gamma^i_p)$ iff $\forall i \in I$, $X \in \mathrm{Inv}(\gamma^i_p)$. By (19) this means that $\forall i \in I$, $X = \emptyset$ or $p \in X$ and $X \in \mathcal{C}_i$. Since $I \neq \emptyset$, we can rewrite this as $X = \emptyset$ or $p \in X$ and $\forall i \in I$, $X \in \mathcal{C}_i$, in other words $X = \emptyset$ or $p \in X$ and $X \in \bigcap_{i\in I} \mathcal{C}_i$. By (19) again, this is equivalent to $X \in \mathrm{Inv}(\gamma_p)$ for the system of partial connection openings (γ_p, $p \in E$) corresponding to $\bigcap_{i\in I} \mathcal{C}_i$. Thus $\mathrm{Inv}(\gamma_p) = \bigcap_{i\in I} \mathrm{Inv}(\gamma^i_p)$ for all $p \in E$. By the isomorphism between openings and dual Moore families, this means that for all $p \in E$, γ_p is the infimum, *in the lattice of openings on* $\mathcal{P}(E)$, of the γ^i_p, $i \in I$.

The form taken by ($\gamma^\top_p$, $p \in E$), the system of connection openings of $\mathcal{P}(E)$, follows from (17). □

Another way of interpreting this result is that the family of systems (γ_p, $p \in E$) of increasing and anti-extensive operators γ_p satisfying (C1a) and (C2a) is closed under "pointwise" non-void infimum

$$(\gamma^i_p, p \in E)\ (i \in I,\ I \neq \emptyset) \mapsto \left(\bigwedge_{i\in I} \gamma^i_p, p \in E\right)$$

and "pointwise" composition

$$(\gamma_p, p \in E), (\gamma'_p, p \in E) \mapsto (\gamma_p\gamma'_p, p \in E).$$

Then the family closed under these two operations generated by the (γ^i_p, $p \in E$) ($i \in I$, $I \neq \emptyset$) has "pointwise" infimum (γ_p, $p \in E$), where for each $p \in E$, γ_p is the infimum, in the lattice of openings on $\mathcal{P}(E)$, of the γ^i_p, $i \in I$.

Having dealt with the intersection of partial connections, we will now be in position to exhibit the complete lattice made by partial connections, and in particular consider a supremum of partial connections.

Definition 30 Given a family $\mathcal{B}$ of subsets of E:

- The *connection generated by* $\mathcal{B}$ is the least connection containing $\mathcal{B}$, it is written $\mathrm{Con}(\mathcal{B})$.
- The *partial connection generated by* $\mathcal{B}$ is the least partial connection containing $\mathcal{B}$, it is written $\mathrm{Con}^*(\mathcal{B})$.

By Proposition 18, it is obvious that

$$\mathrm{Con}(\mathcal{B}) = \mathrm{Con}(\mathrm{Con}^*(\mathcal{B})) = \mathrm{Con}^*(\mathcal{B}) \cup \mathcal{S}(E). \tag{23}$$

For any $B \in \mathcal{P}(E)$, let us write $\mathrm{Con}^*(B)$ and $\mathrm{Con}(B)$ for $\mathrm{Con}^*(\{B\})$ and $\mathrm{Con}(\{B\})$ respectively. Thus

$$\mathrm{Con}^*(B) = \{\emptyset, B\} \quad \text{and} \quad \mathrm{Con}(B) = \{\emptyset, B\} \cup \mathcal{S}(E).$$

Proposition 31 *$\Gamma(E)$ and $\Gamma^*(E)$, ordered by inclusion, are atomic complete lattices, where the infimum operation is the intersection; they share the same greatest element $\mathcal{P}(E)$. Otherwise*:

- *In $\Gamma(E)$, the least element is $\{\emptyset\} \cup \mathcal{S}(E)$, and the supremum of a family $\mathcal{C}_i$ ($i \in I$) is $\mathrm{Con}(\bigcup_{i\in I} \mathcal{C}_i)$; for a non-void family ($I \neq \emptyset$), $\mathrm{Con}(\bigcup_{i\in I} \mathcal{C}_i) = \mathrm{Con}^*(\bigcup_{i\in I} \mathcal{C}_i)$. The atoms are $\{\emptyset\} \cup \{A\} \cup \mathcal{S}(E)$ for all $A \in \mathcal{P}(E)$ that have at least two elements.*
- *In $\Gamma^*(E)$, the least element is $\{\emptyset\}$, and the supremum of a family $\mathcal{C}_i$ ($i \in I$) is $\mathrm{Con}^*(\bigcup_{i\in I} \mathcal{C}_i)$. The atoms are $\{\emptyset\} \cup \{A\}$ for all non-void $A \in \mathcal{P}(E)$.*

Proof Since $\Gamma(E)$ and $\Gamma^*(E)$ are Moore families, they are complete lattices with $\mathcal{P}(E)$ as greatest element. The supremum of a family of partial connections $\mathcal{C}_i$ ($i \in I$) is the least partial connection containing each $\mathcal{C}_i$, that is, containing their union $\bigcup_{i\in I} \mathcal{C}_i$; by definition, this is $\mathrm{Con}^*(\bigcup_{i\in I} \mathcal{C}_i)$. Similarly, the supremum of a family $\mathcal{C}_i$ ($i \in I$) of connections is $\mathrm{Con}(\bigcup_{i\in I} \mathcal{C}_i)$; when the family is non-void, each $\mathcal{C}_i$ contains $\mathcal{S}(E)$, so $\mathcal{S}(E) \subseteq \bigcup_{i\in I} \mathcal{C}_i \subseteq \mathrm{Con}^*(\bigcup_{i\in I} \mathcal{C}_i)$, hence (23) gives $\mathrm{Con}(\bigcup_{i\in I} \mathcal{C}_i) = \mathrm{Con}^*(\bigcup_{i\in I} \mathcal{C}_i)$. Obviously the least partial connection is $\{\emptyset\}$, so the least connection is $\{\emptyset\} \cup \mathcal{S}(E)$ (cf. Proposition 18).

We know that for $A \neq \emptyset$, $\{\emptyset\} \cup \{A\}$ is a dual Moore family, hence a partial connection. Since it has exactly one more element than the least partial connection $\{\emptyset\}$, it is an atom in $\Gamma^*(E)$. Now for every partial connection $\mathcal{C}$ we have

$$\mathcal{C} = \bigcup_{A\in\mathcal{C}\setminus\{\emptyset\}} (\{\emptyset\} \cup \{A\}) = \mathrm{Con}^*\Bigg(\bigcup_{A\in\mathcal{C}\setminus\{\emptyset\}} (\{\emptyset\} \cup \{A\})\Bigg),$$

so $\Gamma^*(E)$ is atomic.

For $|A| > 1$, $A \notin \mathcal{S}(E)$, so $\{\emptyset\} \cup \{A\} \cup \mathcal{S}(E)$ is a connection (cf. Proposition 18), and it has exactly one more element than the least connection $\{\emptyset\} \cup \mathcal{S}(E)$, hence it is an atom in $\Gamma(E)$. Now for every connection $\mathcal{C}$ we have for $\mathcal{C}' = \mathcal{C} \setminus (\mathcal{S}(E) \cup \{\emptyset\})$:

$$\begin{aligned}\mathcal{C} &= \bigcup_{A\in\mathcal{C}'} (\{\emptyset\} \cup \{A\} \cup \mathcal{S}(E))\\ &= \mathrm{Con}\Bigg(\bigcup_{A\in\mathcal{C}'} (\{\emptyset\} \cup \{A\} \cup \mathcal{S}(E))\Bigg),\end{aligned}$$

so $\Gamma(E)$ is atomic. □

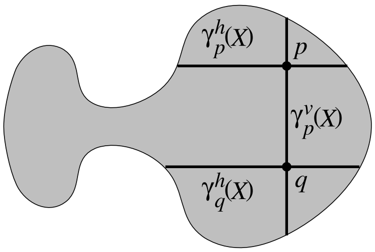

Fig. 8 Consider the horizontal connection $\mathcal{C}^h$ of Fig. 7, and the similar vertical connection $\mathcal{C}^v$. For $q \in \gamma^v_p(X)$, we generally have $\gamma^h_q(X) \not\subseteq \gamma^h_p(X) \cup \gamma^v_p(X)$, so the openings $\gamma^h_p \vee \gamma^v_p$ do not satisfy (C2a)

Proposition 29 does not extend to a supremum of partial connections. Given a non-void family $\mathcal{C}_i$ ($i \in I$, $I \neq \emptyset$) of partial connections with systems of partial connection openings (γ^i_p, $p \in E$), $\bigvee_{i \in I} \gamma^i_p$ will be an opening for every $p \in E$, but the system ($\bigvee_{i \in I} \gamma^i_p$, $p \in E$) will generally not satisfy (C2a), as shows Fig. 8.

In fact, a supremum of partial connections can be characterized in terms of the partial partition of connected components, cf. (20):

Proposition 32 *For any $A \in \mathcal{P}(E)$, the map*

$$\Gamma^*(E) \to \Pi^*(A) : \mathcal{C} \mapsto \mathrm{PC}^{\mathcal{C}}(A)$$

is a dilation. In other words, for a family $\mathcal{C}_i$ ($i \in I$) of partial connections, and for $\mathcal{C} = \mathrm{Con}^*(\bigcup_{i \in I} \mathcal{C}_i)$, we have $\mathrm{PC}^{\mathcal{C}}(A) = \bigvee_{i \in I} \mathrm{PC}^{\mathcal{C}_i}(A)$, and for I empty, $\mathrm{PC}^{\{\emptyset\}}(A) = \mathbf{\emptyset}$.

The restriction to $\Gamma(E)$ of the map $\mathcal{C} \mapsto \mathrm{PC}^{\mathcal{C}}(A)$ is a dilation $\Gamma(E) \to \Pi(A)$.

Proof For any $\pi \in \Pi^*(A)$, let

$$\mathcal{F}(\pi) = \left[\bigcup_{D \in \pi} \mathcal{P}(D)\right] \cup \left[\mathcal{P}(E) \setminus \mathcal{P}(A)\right] \cup \{\emptyset\},$$

that is the family of subsets of E that are either included in a block of π, or not included in A, plus $\emptyset$ (for the case where $\pi = \mathbf{\emptyset}$). Clearly $\emptyset \in \mathcal{F}(\pi)$. Let $\mathcal{B} \subseteq \mathcal{F}(\pi)$ such that $\bigcap \mathcal{B} \neq \emptyset$; we can assume that $\mathcal{B} \neq \emptyset$. If there is $B \in \mathcal{B}$ such that $B \not\subseteq A$, then $\bigcup \mathcal{B} \not\subseteq A$; otherwise each $B \in \mathcal{B}$ is included in a block of π, but as $\bigcap \mathcal{B} \neq \emptyset$, all $B \in \mathcal{B}$ are included in the same block D of π, hence $\bigcup \mathcal{B} \subseteq D$; thus $\bigcup \mathcal{B} \in \mathcal{F}(\pi)$ in any case. Hence $\mathcal{F}(\pi)$ is a partial connection. Thus $\mathcal{F}$ is $\Pi^*(A) \to \Gamma^*(E)$, while we have $\mathcal{G} : \Gamma^*(E) \to \Pi^*(A) : \mathcal{C} \mapsto \mathrm{PC}^{\mathcal{C}}(A)$.

For any $\mathcal{C} \in \Gamma^*(E)$ and $\pi \in \Pi^*(A)$, $\mathcal{G}(\mathcal{C}) \leq \pi$ means $\mathrm{PC}^{\mathcal{C}}(A) \leq \pi$, in other words (cf. Proposition 23) every $\mathcal{C}$-component of A is included in a block of π. By Definition 22, the $\mathcal{C}$-components of A are maximal (for inclusion) among the $C \in \mathcal{C} \setminus \{\emptyset\}$ such that $C \subseteq A$. Hence $\mathcal{G}(\mathcal{C}) \leq \pi$ iff every $C \in \mathcal{C} \setminus \{\emptyset\}$ such that $C \subseteq A$ must be included in a block of π, in other words for every $C \in \mathcal{C}$, either $C = \emptyset$, C is included in a block of π, or $C \not\subseteq A$. This means exactly that $\mathcal{C} \subseteq \mathcal{F}(\pi)$. Therefore $\mathcal{G}(\mathcal{C}) \leq \pi \Leftrightarrow \mathcal{C} \subseteq \mathcal{F}(\pi)$, in other words $(\mathcal{F}, \mathcal{G})$ is an adjunction $\Pi^*(A) \rightleftharpoons \Gamma^*(E)$. As $\mathcal{G}$ is the lower adjoint in an adjunction, it is a dilation, in other words it transforms the supremum in $\Gamma^*(E)$ into the supremum in $\Pi^*(A)$:

$$\mathcal{G}\left(\mathrm{Con}^*\left(\bigcup_{i \in I} \mathcal{C}_i\right)\right) = \bigvee_{i \in I} \mathcal{G}(\mathcal{C}_i).$$

Note that for $\pi \in \Pi(A)$, $\mathcal{F}(\pi) \in \Gamma(E)$, while for $\mathcal{C} \in \Gamma(E)$, $\mathcal{G}(\mathcal{C}) \in \Pi(A)$. Thus $(\mathcal{F}, \mathcal{G})$ is also an adjunction $\Pi(A) \rightleftharpoons \Gamma(E)$. □

In [12] we showed that in the connection generated by a family, the connected sets are obtained by chaining the elements of the family. This is in fact a consequence of the above result:

Corollary 33 *Let $\mathcal{B} \subseteq \mathcal{P}(E) \setminus \{\emptyset\}$ be non-void. Then for any $X \in \mathcal{P}(E)$, $X \in \mathrm{Con}^*(\mathcal{B})$ iff X is chained by $\mathcal{B} \cap \mathcal{P}(X)$.*

Proof We have $\mathrm{Con}^*(\mathcal{B}) = \mathrm{Con}^*(\bigcup_{B \in \mathcal{B}} \mathrm{Con}^*(B))$; Proposition 32 gives then $\mathrm{PC}^{\mathrm{Con}^*(\mathcal{B})}(X) = \bigvee_{B \in \mathcal{B}} \mathrm{PC}^{\mathrm{Con}^*(B)}$. For $B \in \mathcal{B}$, $\mathrm{Con}^*(B) = \{\emptyset, B\}$. Now $\mathrm{PC}^{\{\emptyset, B\}}(X) = \{B\}$ if $B \subseteq X$, and is empty otherwise. Thus

$$\begin{aligned}\mathrm{PC}^{\mathrm{Con}^*(\mathcal{B})}(X) &= \bigvee_{B \in \mathcal{B}} \mathrm{PC}^{\mathrm{Con}^*(B)}(X) \\ &= \bigvee \{\mathbf{1}_B \mid B \in \mathcal{B},\ B \subseteq X\}.\end{aligned}$$

Now $X \in \mathrm{Con}^*(\mathcal{B})$ iff $\mathrm{PC}^{\mathrm{Con}^*(\mathcal{B})}(X) = \{X\}$, in other words $\mathbf{1}_X = \bigvee \{\mathbf{1}_B \mid B \in \mathcal{B},\ B \subseteq X\}$, which means that X is chained by $\mathcal{B} \cap \mathcal{P}(X)$. □

This result can also be shown directly (as we did in [12]): given a partial connection $\mathcal{C}$ and blocks $B_0, \ldots, B_n \in \mathcal{C}$ such that $B_{t-1} \cap B_t \neq \emptyset$ for $t = 1, \ldots, n$, it is easily seen that $B_0 \cup \cdots \cup B_n \in \mathcal{C}$; then in a set B chained by $\mathcal{C}$, any two points belong to the same $\mathcal{C}$-component of B, hence $B \in \mathcal{C}$.

We have thus an explanation of what appeared before as a coincidence: that the same operation, chaining, is used for constructing suprema both for connections and for partitions.

As shown in Fig. 3, Proposition 32 does not extend to the infimum operation. For a family $\mathcal{C}_i$ ($i \in I$) of partial connections, and for $\mathcal{C} = \bigcap_{i \in I} \mathcal{C}_i$, we have only $\mathrm{PC}^{\mathcal{C}}(A) \leq \bigwedge_{i \in I} \mathrm{PC}^{\mathcal{C}_i}(A)$, and the inequality is often sharp. Indeed, in $\mathrm{PC}^{\mathcal{C}}(A)$, the class of a point p (of E) is $\gamma_p(A)$, while in $\bigwedge_{i \in I} \mathrm{PC}^{\mathcal{C}_i}(A)$, by (10) the class of p is $\bigcap_{i \in I} \gamma^i_p(A)$, the intersection of the classes $\gamma^i_p(A)$ in $\mathrm{PC}^{\mathcal{C}_i}(A)$. Now $\bigwedge_{i \in I} \gamma^i_p : A \mapsto \bigcap_{i \in I} \gamma^i_p(A)$ is the infimum of the γ^i_p ($i \in I$) *in the* *lattice of operators*, so it is not necessarily an opening, and by Proposition 29, γ_p is the greatest opening $\leq \bigwedge_{i\in I} \gamma_p^i$, so $\gamma_p(A) \subseteq \bigcap_{i\in I} \gamma_p^i(A)$.

3 Second-Generation Partial Connections

We will describe here methods for constructing a new partial connection from an existing partial connection and an operator on sets. We will in fact extend two well-known aproaches:

- In [12] a connection $\mathcal{C}$ was restricted to its elements that are invariant under an opening satisfying some properties (for example in $\mathbf{R}^n$ of $\mathbf{Z}^n$: the opening by a structuring element in $\mathcal{C}$), plus the singletons. In Sect. 3.1 we will study the partial connection $\mathcal{C} \cap \mathrm{Inv}(\alpha)$, where $\mathcal{C}$ is a partial connection on $\mathcal{P}(E)$ and α is an opening on $\mathcal{P}(E)$.
- Serra [17] showed that for a connection $\mathcal{C}$ on $\mathcal{P}(E)$ and an extensive dilation δ on $\mathcal{P}(E)$ such that $\delta(\mathcal{C}) \subseteq \mathcal{C}$, then $\delta^{-1}(\mathcal{C})$ is a connection containing $\mathcal{C}$. For example in $\mathbf{R}^n$ of $\mathbf{Z}^n$, δ can be the dilation by a structuring element in $\mathcal{C}$ containing the origin, and this new connection can be used to cluster neighbouring grains. This construction was modified [19] to the use of a closing instead of a dilation. A more general formulation in terms of an extensive operator on $\mathcal{P}(E)$ was given by Heijmans in [9]. In Sect. 3.2 we generalize these results to partial connections, and, given arbitrary spaces E_1 and E_2, describe new connections on $\mathcal{P}(E_1)$ that can be built from a partial connection on $\mathcal{P}(E_2)$ and a dilation $\mathcal{P}(E_1) \to \mathcal{P}(E_2)$.

3.1 Partial Connections by Restriction

As seen above, intersection gives a general method for constructing a partial connection from pre-existing partial connections. In [12] we showed that for an opening α on $\mathcal{P}(E)$ and a connection $\mathcal{C}$ on $\mathcal{P}(E)$, $(\mathcal{C} \cap \mathrm{Inv}(\alpha)) \cup \mathcal{S}(E)$ is a connection. More generally, let $\mathcal{C}$ be a partial connection; as $\mathrm{Inv}(\alpha)$ is a Moore family, it is a partial connection, so the intersection $\mathcal{C} \cap \mathrm{Inv}(\alpha)$ is a partial connection, and by Proposition 18, $(\mathcal{C} \cap \mathrm{Inv}(\alpha)) \cup \mathcal{S}(E)$ is a connection. Let us give some examples of such partial connections:

- Let (E, d) be a metric space; given a partial connection $\mathcal{C}$ and some $s > 0$, take the set of elements of $\mathcal{C}$ whose diameter is at least s, plus $\emptyset$.
- In $\mathbf{Z}^n$, given a partial connection $\mathcal{C}$ and an integer $n > 0$, take the set of elements of $\mathcal{C}$ whose size at least is n, plus $\emptyset$.
- In $\mathbf{R}^n$, the set of connected open sets. (NB: topological and arc connectivity are equivalent for open sets.)

Heijmans [9] considered the connection $(\mathcal{C} \cap \mathrm{Ext}(\psi)) \cup \mathcal{S}(E)$, where ψ is an increasing operator on $\mathcal{P}(E)$ and $\mathrm{Ext}(\psi) = \{X \in \mathcal{P}(E) \mid X \subseteq \psi(X)\}$. But $\mathrm{Ext}(\psi) = \mathrm{Inv}(\mathbf{id} \wedge \psi) = \mathrm{Inv}(\alpha)$, where $\mathbf{id}$ is the identity operator and α is the greatest opening $\leq \mathbf{id} \wedge \psi$. Thus this example is a particular case of the above construction.

We showed in [12] that given two openings γ and α, the three equalities $\alpha\gamma\alpha = \gamma\alpha$, $\gamma\alpha\gamma = \gamma\alpha$ and $(\gamma\alpha)^2 = \gamma\alpha$ are equivalent. This applies in particular to the connection openings of a partial partition $\mathcal{C}$:

$$\forall p \in E, \quad \alpha\gamma_p\alpha = \gamma_p\alpha \iff \gamma_p\alpha\gamma_p = \gamma_p\alpha \iff (\gamma_p\alpha)^2 = \gamma_p\alpha. \tag{24}$$

Taking the leftmost expression of this equation, $\alpha\gamma_p\alpha = \gamma_p\alpha$, the fact that it holds for every point $p \in E$ means the following statement: *Every connected component of an invariant of α is itself an invariant of α.* We have then a very simple expression for the system of partial connection openings of $\mathcal{C} \cap \mathrm{Inv}(\alpha)$:

Proposition 34 *Let α be an opening on $\mathcal{P}(E)$ such that $\alpha\gamma_p\alpha = \gamma_p\alpha$ for every $p \in E$. Then the system of partial connection openings of $\mathcal{C} \cap \mathrm{Inv}(\alpha)$ is $(\gamma_p\alpha, p \in E)$.*

Proof Since γ_p and α are increasing and anti-extensive, $\gamma_p\alpha$ is increasing and anti-extensive. By (24), we have $(\gamma_p\alpha)^2 = \gamma_p\alpha$, so $\gamma_p\alpha$ is idempotent. Thus $\gamma_p\alpha$ is an opening for every $p \in E$. The system of partial connection openings of $\mathrm{Inv}(\alpha)$ is $(\alpha_p, p \in E)$, given by (22). For $p \in \alpha(X)$, $\alpha_p(X) = \alpha(X)$ by (22), so $\gamma_p(\alpha_p(X)) = \gamma_p(\alpha(X))$; for $p \notin \alpha(X)$, $\gamma_p(\alpha(X)) = \emptyset$, and $\alpha_p(X) = \emptyset$ by (22), so $\gamma_p(\alpha_p(X)) = \gamma_p(\emptyset) = \emptyset$. Thus $\gamma_p(\alpha_p(X)) = \gamma_p(\alpha(X))$ for any $X \in \mathcal{P}(E)$, hence $\gamma_p\alpha_p = \gamma_p\alpha$. Since γ_p and α_p are anti-extensive, $\gamma_p\alpha_p \leq \gamma_p$ and $\gamma_p\alpha_p \leq \alpha_p$. Now any opening σ such that $\sigma \leq \gamma_p$ and $\sigma \leq \alpha_p$ satisfies $\sigma = \sigma\sigma \leq \gamma_p\alpha_p$. Hence for every $p \in E$, $\gamma_p\alpha_p = \gamma_p\alpha$ is the greatest opening $\leq \gamma_p$ and $\leq \alpha_p$. By Proposition 29, $(\gamma_p\alpha, p \in E)$ is the system of partial connection openings of $\mathcal{C} \cap \mathrm{Inv}(\alpha)$. □

For $E = \mathbf{R}^n$ or $\mathbf{Z}^n$, a well-known example of opening α satisfying $\alpha\gamma_p\alpha = \gamma_p\alpha$, is the opening $X \mapsto X \circ B = \bigcup\{B_p \mid p \in E,\ B_p \subseteq E\}$ by a connected structuring element B (assuming a translation-invariant connection $\mathcal{C}$: for any $p \in E$, $B \in \mathcal{C} \Rightarrow B_p \in \mathcal{C}$). This example has been used in Figs. 1, 3 and 4. In [12] we generalized this to any opening based on a subset $\mathcal{B}$ of a connection $\mathcal{C}$, associating to every set the union of all elements of $\mathcal{B}$ included in that set. We have the following:

Proposition 35 *Let $\mathcal{C}$ be a partial connection. An operator α on $\mathcal{P}(E)$ satisfies*

$$\exists \mathcal{B} \subseteq \mathcal{C},\ \forall X \subseteq E, \quad \alpha(X) = \bigcup\{B \in \mathcal{B} \mid B \subseteq X\}, \tag{25}$$

iff α *is an opening such that* $\alpha\gamma_p\alpha = \gamma_p\alpha$ *for every* $p \in E$, *and*

$$\forall X \in \mathcal{P}(E), \quad \alpha(X) = \bigcup_{p \in E} \gamma_p\alpha(X). \tag{26}$$

The latter identity is always true if $\mathcal{C}$ *is a connection.*

Proof Let $\mathcal{B} \subseteq \mathcal{C}$, and for every $X \in \mathcal{P}(E)$, let $\mathcal{B}_X = \{B \in \mathcal{B} \mid B \subseteq X\}$. If α is given by (25), we have $\alpha(X) = \bigcup \mathcal{B}_X$, then α is typically an opening. For $X \in \mathcal{P}(E)$ and $p \in E$, $\gamma_p\alpha(X) \in \mathcal{C}$. If $\gamma_p\alpha(X) = \emptyset$, then $\alpha\gamma_p\alpha(X) = \emptyset$. Suppose thus that $\gamma_p\alpha(X) \neq \emptyset$, so $p \in \gamma_p\alpha(X)$. Let $B \in \mathcal{B}_X$ such that $B \cap \gamma_p\alpha(X) \neq \emptyset$; as $B, \gamma_p\alpha(X) \in \mathcal{C}$, we deduce that $B \cup \gamma_p\alpha(X) \in \mathcal{C}$; now $B, \gamma_p\alpha(X) \subseteq \alpha(X)$, so $p \in B \cup \gamma_p\alpha(X) \subseteq \alpha(X)$; by the maximality of $\gamma_p\alpha(X)$, cf. (17), we deduce that $B \subseteq \gamma_p\alpha(X)$. Hence

$$\begin{aligned}
\gamma_p\alpha(X) &= \gamma_p\alpha(X) \cap \alpha(X) = \gamma_p\alpha(X) \cap \bigcup \mathcal{B}_X \\
&= \bigcup \{B \cap \gamma_p\alpha(X) \mid B \in \mathcal{B}_X\} \\
&= \bigcup \{B \cap \gamma_p\alpha(X) \mid B \in \mathcal{B}_X,\ B \cap \gamma_p\alpha(X) \neq \emptyset\} \\
&= \bigcup \{B \cap \gamma_p\alpha(X) \mid B \in \mathcal{B}_X,\ B \subseteq \gamma_p\alpha(X)\} \\
&= \bigcup \{B \mid B \in \mathcal{B}_X,\ B \subseteq \gamma_p\alpha(X)\} \\
&= \bigcup \{B \mid B \in \mathcal{B},\ B \subseteq \gamma_p\alpha(X)\} = \alpha\gamma_p\alpha(X).
\end{aligned}$$

Therefore $\alpha\gamma_p\alpha = \gamma_p\alpha$ for all $p \in E$. For $B \in \mathcal{B}_X$, $B \subseteq \alpha(X)$; as $B \in \mathcal{C}$, $B = \bigcup_{p \in E} \gamma_p(B) \subseteq \bigcup_{p \in E} \gamma_p(\alpha(X))$. Thus $\alpha(X) = \bigcup \mathcal{B}_X \subseteq \bigcup_{p \in E} \gamma_p(\alpha(X)$; now the reciprocal inequality follows from the anti-extensivity of the γ_p, so we have (26).

Let α be an opening such that $\alpha\gamma_p\alpha = \gamma_p\alpha$ for every $p \in E$ and (26) holds. Let $\mathcal{B} = \mathcal{C} \cap \mathrm{Inv}(\alpha)$. For any $X \in \mathcal{P}(E)$, $\alpha(X) = \bigcup_{p \in E} \gamma_p\alpha(X)$. For any $p \in E$, $\gamma_p\alpha(X) \in \mathcal{C}$, and $\alpha(\gamma_p\alpha(X)) = \gamma_p\alpha(X)$, that is $\gamma_p\alpha(X) \in \mathrm{Inv}(\alpha)$. Thus $\gamma_p\alpha(X) \in \mathcal{B}$, and $\alpha(X)$ is a union of elements of $\mathcal{B}$. We deduce then that $\alpha(X) = \bigcup \{B \in \mathcal{B} \mid B \subseteq X\}$.

If $\mathcal{C}$ is a connection, every set is the union of its $\mathcal{C}$-components, so (26) holds. □

For α given by (25), every $B \in \mathcal{B}$ satisfies $B \in \mathcal{C} \cap \mathrm{Inv}(\alpha)$, hence we have $\mathrm{Con}^*(\mathcal{B}) \subseteq \mathcal{C} \cap \mathrm{Inv}(\alpha)$. The inclusion is often sharp: an element of $\mathrm{Con}^*(\mathcal{B})$ is obtained by chaining elements of $\mathcal{B}$, while an element of $\mathcal{C} \cap \mathrm{Inv}(\alpha)$ can be a union of disjoint adjacent elements of $\mathcal{B}$, see Fig. 9.

Now we can invert α and γ_p in (24):

$$\begin{aligned}
\forall p \in E, \quad \gamma_p\alpha\gamma_p = \alpha\gamma_p &\iff \alpha\gamma_p\alpha = \alpha\gamma_p \\
&\iff (\alpha\gamma_p)^2 = \alpha\gamma_p.
\end{aligned} \tag{27}$$

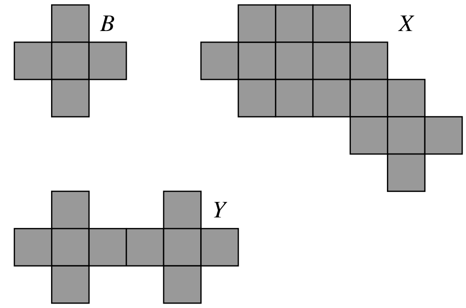

Fig. 9 Here $E = \mathbf{Z}^2$ and $\mathcal{C}$ is the family of all 4-connected sets. *Top left*: a 4-connected structuring element B; let $\mathcal{B}$ be the set of translates of B, thus $\mathcal{B} \subseteq \mathcal{C}$. Let α be the opening by B, in other words α is given by (25). *Top right*: a set X obtained by chaining translates of B, so $X \in \mathrm{Con}^*(\mathcal{B})$. *Bottom*: a set Y that is the union of two disjoint but 4-adjacent translates of B; thus $Y \in \mathcal{C} \cap \mathrm{Inv}(\alpha)$ but $Y \notin \mathrm{Con}^*(\mathcal{B})$

The fact that the leftmost expression $\gamma_p\alpha\gamma_p = \alpha\gamma_p$ holds for every point $p \in E$ means: for every $C \in \mathcal{C}$, $\alpha(C) = C$ or $\alpha(C) = \emptyset$. Indeed, as $\gamma_p(X) \in \mathcal{C}$, if $\alpha\gamma_p(X) = \gamma_p(X)$, then $\gamma_p\alpha\gamma_p(X) = \gamma_p\gamma_p(X) = \gamma_p(X) = \alpha\gamma_p(X)$, while if $\alpha\gamma_p(X) = \emptyset$, then $\gamma_p\alpha\gamma_p(X) = \gamma_p(\emptyset) = \emptyset$, so in any case $\gamma_p\alpha\gamma_p(X) = \alpha\gamma_p(X)$; conversely, given $C \in \mathcal{C}$ such that $\alpha(C) \subset C$, for $p \in C \setminus \alpha(C)$, $\gamma_p\alpha\gamma_p(C) = \gamma_p\alpha(C) = \emptyset$, so $\gamma_p\alpha\gamma_p = \alpha\gamma_p$ gives $\alpha(C) = \alpha\gamma_p(C) = \emptyset$. We obtain then the following analogue of Proposition 34:

Proposition 36 *Let* α *be an opening on* $\mathcal{P}(E)$ *such that* $\gamma_p\alpha\gamma_p = \alpha\gamma_p$ *for every* $p \in E$. *Then the system of partial connection openings of* $\mathcal{C} \cap \mathrm{Inv}(\alpha)$ *is* $(\alpha\gamma_p, p \in E)$.

Proof By (27), $\alpha\gamma_p$ is idempotent, hence it is an opening. The system of partial connection openings of $\mathrm{Inv}(\alpha)$ is $(\alpha_p, p \in E)$, given by (22). If $\alpha\gamma_p(X) = \emptyset$, then $p \notin \alpha\gamma_p(X)$, hence $\alpha_p\gamma_p(X) = \emptyset$ by (22); if $\alpha\gamma_p(X) \neq \emptyset$, as $\gamma_p\alpha\gamma_p = \alpha\gamma_p$, then $\gamma_p\alpha\gamma_p(X) \neq \emptyset$, so $p \in \alpha\gamma_p(X)$, hence $\alpha_p\gamma_p(X) = \alpha\gamma_p(X)$ by (22). Thus for any $p \in E$ and $X \in \mathcal{P}(E)$ we have $\alpha_p\gamma_p(X) = \alpha\gamma_p(X)$, therefore $\alpha_p\gamma_p = \alpha\gamma_p$. The same proof as the one of Proposition 34 shows that $\alpha_p\gamma_p = \alpha\gamma_p$ is the greatest opening $\leq \gamma_p$ and $\leq \alpha_p$, so that $(\alpha\gamma_p, p \in E)$ is the system of partial connection openings of $\mathcal{C} \cap \mathrm{Inv}(\alpha)$. □

An instance of opening satisfying $\gamma_p\alpha\gamma_p = \alpha\gamma_p$ for every $p \in E$ is a *grain opening* in the sense of [9]. Let ψ be an increasing map $\mathcal{C} \to \{0, 1\}$, then let α select from a set the union of all $\mathcal{C}$-components C with $\psi(C) = 1$:

$$\alpha(X) = \bigcup \{C \in \mathrm{PC}^{\mathcal{C}}(X) \mid \psi(C) = 1\}.$$

These $C \in \mathrm{PC}^{\mathcal{C}}(X)$ such that $\psi(C) = 1$ will be the $\mathcal{C} \cap \mathrm{Inv}(\alpha)$-components of X. Two well-known examples in $\mathbf{R}^n$ or $\mathbf{Z}^n$ are: (1) we choose a non-void structuring element B and set $\psi(C) = 1 \Leftrightarrow C \ominus B \neq \emptyset$, then α selects all connected components wide enough to contain a translate of B; (2) we set $\psi(C)=1$ iff the measure of C exceeds a threshold, and then α is the area opening.

Although grain openings have up to now been considered for $\mathcal{C}$ being a connection, they can be extended to the case where $\mathcal{C}$ is a partial connection. It is possible to combine successively Propositions 34 and 36: let α be an opening satisfying (25), and let β be a grain opening w.r.t. the partial connection $\mathcal{C}\cap\mathrm{Inv}(\alpha)$. Then $\mathcal{C}\cap\mathrm{Inv}(\alpha)\cap\mathrm{Inv}(\beta)$ will have the system of partial connection openings $(\beta\gamma_p\alpha,\ p\in E)$.

3.2 Partial Connections by Dilation or Closing

Serra [17] considered a connection $\mathcal{C}$ on $\mathcal{P}(E)$ and an extensive dilation δ on $\mathcal{P}(E)$. He required the condition that $\delta(\mathcal{C})\subseteq\mathcal{C}$ (this means: $\forall C\in\mathcal{C}$, $\delta(C)\in\mathcal{C}$); he noted first that it is equivalent to: for every $p\in E$, $\delta(\{p\})\in\mathcal{C}$. He then showed that under this condition, $\delta^{-1}(\mathcal{C})=\{X\in\mathcal{P}(E)\mid \delta(X)\in\mathcal{C}\}$ is a connection containing $\mathcal{C}$.

Similarly, given a closing φ on $\mathcal{P}(E)$ such that $\varphi(\mathcal{C})\subseteq\mathcal{C}$, $\varphi^{-1}(\mathcal{C})$ is a connection containing $\mathcal{C}$, see for instance [19]. Note that [19] also required the condition that $\varphi(\emptyset)=\emptyset$, but this condition is not necessary. Closings φ such that $\varphi(\mathcal{C})\subseteq\mathcal{C}$ and $\varphi(\emptyset)=\emptyset$ were already considered in [10] under the denomination of *connectivity-preserving closings*.

Now Heijmans [9] considered a connection $\mathcal{C}$ on $\mathcal{P}(E)$ and an increasing operator ψ on $\mathcal{P}(E)$, and defined the set

$$\mathcal{C}^{\psi}=\{X\in\mathcal{P}(E)\mid\exists C\in\mathcal{C},\ X\subseteq C\subseteq\psi(X)\}. \tag{28}$$

In order to have $\mathcal{S}(E)\subseteq\mathcal{C}^{\psi}$, he postulated that "$\psi$ is extensive on singletons", that is, $\forall p\in E$, $p\in\psi(\{p\})$; since ψ is increasing, this simply means that ψ is extensive. However this extensivity hypothesis is not necessary to show that $\mathcal{C}^{\psi}$ is a partial connection. When ψ is an extensive dilation mapping singletons into $\mathcal{C}$ (that is, $\forall p\in E$, $\psi(\{p\})\in\mathcal{C}$), or a closing preserving $\mathcal{C}$ (that is, $\psi(\mathcal{C})\subseteq\mathcal{C}$), we obtain $\mathcal{C}^{\psi}=\psi^{-1}(\mathcal{C})$ as above:

Proposition 37 *Let $\mathcal{C}$ be a partial connection of $\mathcal{P}(E)$ and ψ an increasing operator on $\mathcal{P}(E)$. Then $\mathcal{C}^{\psi}$ given by (28) is a partial connection. Furthermore*:

1. *If ψ is extensive, then $\mathcal{C}\cup\psi^{-1}(\mathcal{C})\subseteq\mathcal{C}^{\psi}$.*
2. *If ψ is a closing and $\psi(\mathcal{C})\subseteq\mathcal{C}$, then $\mathcal{C}^{\psi}=\psi^{-1}(\mathcal{C})$ and $\mathcal{C}\subseteq\mathcal{C}^{\psi}$.*
3. *If ψ is an extensive dilation and $\forall p\in E$, $\psi(\{p\})\in\mathcal{C}$, then $\mathcal{C}^{\psi}$ is a connection, $\psi(\mathcal{C})\subseteq\mathcal{C}$, $\mathcal{C}^{\psi}=\psi^{-1}(\mathcal{C})$ and $\mathcal{C}\subseteq\mathcal{C}^{\psi}$.*

Proof Note that in (28) $X\in\mathcal{C}^{\psi}$ contains as particular cases first $X=C\subseteq\psi(X)$, that is $X\in\mathcal{C}$ and $X\subseteq\psi(X)$, second $X\subseteq C=\psi(X)$, that is $\psi(X)\in\mathcal{C}$ and $X\subseteq\psi(X)$.

Clearly $\emptyset\in\mathcal{C}$ and $\emptyset\subseteq\psi(\emptyset)$; thus (28) gives $\emptyset\in\mathcal{C}^{\psi}$. Let $\mathcal{B}\subseteq\mathcal{C}^{\psi}$ with $\bigcap\mathcal{B}\neq\emptyset$; we can assume that $\mathcal{B}\neq\emptyset$. Let $p\in\bigcap\mathcal{B}$. For any $B\in\mathcal{B}$, as $B\in\mathcal{C}^{\psi}$, there is some $C\in\mathcal{C}$ such that $B\subseteq C\subseteq\psi(B)$; as $\mathcal{B}\neq\emptyset$, $B\subseteq\bigcup\mathcal{B}$, and as ψ is increasing, $\psi(B)\subseteq\psi(\bigcup\mathcal{B})$. Thus $p\in C\subseteq\psi(\bigcup\mathcal{B})$, hence $C\subseteq\gamma_p(\psi(\bigcup\mathcal{B}))\subseteq\psi(\bigcup\mathcal{B})$. Then

$$B\subseteq C\subseteq\gamma_p\Big(\psi\Big(\bigcup\mathcal{B}\Big)\Big)\subseteq\psi\Big(\bigcup\mathcal{B}\Big),$$

so that by taking the union of all such $B\in\mathcal{B}$ we get $\bigcup\mathcal{B}\subseteq\gamma_p(\psi(\bigcup\mathcal{B}))\subseteq\psi(\bigcup\mathcal{B})$, where $\gamma_p(\psi(\bigcup\mathcal{B}))\in\mathcal{C}$. Therefore $\bigcup\mathcal{B}\in\mathcal{C}^{\psi}$, so $\mathcal{C}^{\psi}$ is a partial connection.

1. If ψ is extensive, then for every $C\in\mathcal{C}$ we have $C\subseteq\psi(C)$, so that $C\in\mathcal{C}^{\psi}$; hence $\mathcal{C}\subseteq\mathcal{C}^{\psi}$. Now for $X\in\psi^{-1}(\mathcal{C})$, we have $\psi(X)\in\mathcal{C}$ and $X\subseteq\psi(X)$, so $X\in\mathcal{C}^{\psi}$; hence $\psi^{-1}(\mathcal{C})\subseteq\mathcal{C}^{\psi}$.

2. Let ψ be a closing such that $\psi(\mathcal{C})\subseteq\mathcal{C}$. For $X\in\mathcal{C}^{\psi}$, there is $C\in\mathcal{C}$ with $X\subseteq C\subseteq\psi(X)$. Since ψ is increasing, we have $\psi(X)\subseteq\psi(C)\subseteq\psi(\psi(X))$, but as ψ is idempotent, $\psi(\psi(X))=\psi(X)$, so $\psi(C)=\psi(X)$; as $\psi(\mathcal{C})\subseteq\mathcal{C}$, we have $\psi(C)\in\mathcal{C}$, thus $\psi(X)\in\mathcal{C}$, that is $X\in\psi^{-1}(\mathcal{C})$. Hence $\mathcal{C}^{\psi}\subseteq\psi^{-1}(\mathcal{C})$. Conversely, as ψ is extensive, $\psi^{-1}(\mathcal{C})\subseteq\mathcal{C}^{\psi}$, and we deduce the equality. As ψ is extensive, $\mathcal{C}\subseteq\mathcal{C}^{\psi}$.

3. Let ψ be an extensive dilation such that $\forall p\in E$, $\psi(\{p\})\in\mathcal{C}$. As ψ is a dilation, $\psi(\emptyset)=\emptyset\in\mathcal{C}$, so $\emptyset\in\psi^{-1}(\mathcal{C})$. Let $X\in\mathcal{C}^{\psi}$ such that $X\neq\emptyset$; there is $C\in\mathcal{C}$ such that $X\subseteq C\subseteq\psi(X)$. For any $p\in X$, $\psi(\{p\})\in\mathcal{C}$, and as ψ is extensive, $p\in\psi(\{p\})$; thus $\psi(\{p\}),C\in\mathcal{C}$ and $p\in\psi(\{p\})\cap C$, so that $\psi(\{p\})\cup C\in\mathcal{C}$; but $\bigcap_{p\in X}(\psi(\{p\})\cup C)\supseteq X\neq\emptyset$, so that $\bigcup_{p\in X}(\psi(\{p\})\cup C)\in\mathcal{C}$. As ψ is a dilation,

$$\bigcup_{p\in X}(\psi(\{p\})\cup C)=\Big(\bigcup_{p\in X}\psi(\{p\})\Big)\cup C$$
$$=\psi(X)\cup C=\psi(X).$$

Thus $\psi(X)\in\mathcal{C}$, so $X\in\psi^{-1}(\mathcal{C})$. Hence $\mathcal{C}^{\psi}\subseteq\psi^{-1}(\mathcal{C})$. Conversely, as ψ is extensive, $\psi^{-1}(\mathcal{C})\subseteq\mathcal{C}^{\psi}$, and we deduce the equality. As ψ is extensive, $\mathcal{C}\subseteq\mathcal{C}^{\psi}$. Thus $\mathcal{C}\subseteq\psi^{-1}(\mathcal{C})$, that is, $\psi(\mathcal{C})\subseteq\mathcal{C}$. For any $p\in E$, $\psi(\{p\})\in\mathcal{C}$, that is, $\{p\}\in\psi^{-1}(\mathcal{C})=\mathcal{C}^{\psi}$, so the partial connection $\mathcal{C}^{\psi}$ contains all singletons and is thus a connection. □

Concerning item 2, the main question is to guarantee that the closing ψ satisfies $\psi(\mathcal{C})\subseteq\mathcal{C}$. Characterizing such closings is a very difficult problem, see for instance [22].

A typical example of item 3 is a dilation by a structuring element B containing the origin, such that $B\in\mathcal{C}$ (assuming that the partial connection $\mathcal{C}$ is translation-invariant: $B\in\mathcal{C}\Rightarrow B_p\in\mathcal{C}$). For example if B is the closed ball of radius $r>0$ centered about the origin, the $\mathcal{C}^{\psi}$-components of a set X will be made of clusters of $\mathcal{C}$-components, where two $\mathcal{C}$-components are clustered together whenever the minimum distance between points of the two is $\leq 2r$, see [9, 17].

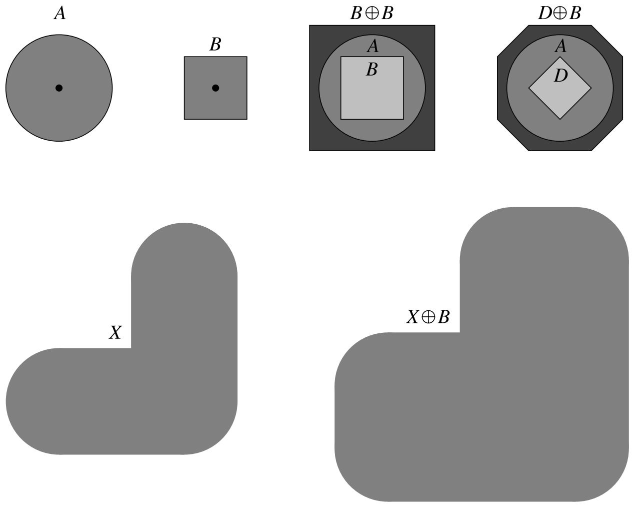

Fig. 10 Here $E=\mathbf{Z}^2$. Top, from *left* to *right*: the disk A of radius r and the square B of side s, where $r=0.85\,s$, both centered about the origin (shown as a *dot*); then $B\subseteq A\subseteq B\oplus B$; the diamond D inscribed in B satisfies also $D\subseteq A\subseteq D\oplus B$. Let $\mathcal{C}$ be partial connection made of all 4-connected sets that are open by A. *Bottom*: given $X\in\mathcal{C}$, its dilate $X\oplus B\in\mathcal{C}$, thus $\delta(\mathcal{C})\subseteq\mathcal{C}$. Now since $A\in\mathcal{C}$ and $B\subseteq A\subseteq B\oplus B$, $B\in\mathcal{C}^\delta$ for the dilation δ by B; but $B\oplus B\notin\mathcal{C}$, so $B\notin\delta^{-1}(\mathcal{C})$. Similarly, $D\in\mathcal{C}^\delta$, but $D\notin\delta^{-1}(\mathcal{C})$

Remark 38 Note that when $\mathcal{C}$ is a connection, every singleton belongs to $\mathcal{C}$, so in item 3 the condition $\forall p\in E$, $\psi(\{p\})\in\mathcal{C}$, follows from $\psi(\mathcal{C})\subseteq\mathcal{C}$; thus in this case we can take either condition. However if $\mathcal{C}$ is not a connection, we generally cannot replace the condition $\forall p\in E$, $\psi(\{p\})\in\mathcal{C}$ by $\psi(\mathcal{C})\subseteq\mathcal{C}$, as shows the following example, illustrated in Fig. 10.

Let $E=\mathbf{Z}^2$ and let $\mathcal{C}_0$ be the connection consisting of all 4-connected sets. Take two structuring elements A, B as follows: A is the disk of radius $r>0$ centered about the origin, while B is the square of side s centered about the origin, where $r\le s\le r\sqrt{2}$; thus $B\oplus B$ is the square of side $2s$ centered about the origin, and $B\subseteq A\subseteq B\oplus B$. Now $A,B\in\mathcal{C}_0$. Let $\alpha: X\mapsto X\circ A$ be the opening by A and let $\delta: X\mapsto X\oplus B$ be the dilation by B; as B contains the origin, δ is extensive. Then $\delta(\mathcal{C}_0)\subseteq\mathcal{C}_0$ (by item 3, in fact by [17]). Let $\mathcal{C}=\mathcal{C}_0\cap\mathrm{Inv}(\alpha)$; it is a partial connection, and $A\in\mathcal{C}$. For $X\in\mathcal{C}$, as $X\in\mathcal{C}_0$, we have $\delta(X)\in\mathcal{C}_0$; as $X\in\mathrm{Inv}(\alpha)$, and $\delta(X)$ is the union of translates of X by points of B, we have $\delta(X)\in\mathrm{Inv}(\alpha)$; thus $\delta(X)\in\mathcal{C}$. Hence $\delta(\mathcal{C})\subseteq\mathcal{C}$. For $p\in E$, $\delta(\{p\})=B_p$, where $B_p\circ A=\emptyset$, so there is no $C\in\mathcal{C}$ with $\{p\}\subseteq C\subseteq\delta(\{p\})$, hence $\{p\}\notin\mathcal{C}^\delta$, and $\mathcal{C}^\delta$ is not a connection. We have $B\subseteq A\subseteq B\oplus B=\delta(B)$, where $A\in\mathcal{C}$, so $B\in\mathcal{C}^\delta$; however $B\oplus B\notin\mathrm{Inv}(\alpha)$, so $B\notin\delta^{-1}(\mathcal{C})$. Therefore $\delta^{-1}(\mathcal{C})$ is a proper subset of $\mathcal{C}^\delta$.

Let us now investigate partial connections of the form $\delta^{-1}(\mathcal{C})$ for a dilation δ and a partial connection $\mathcal{C}$, but this time without the requirements that δ is extensive and that $\delta(\{p\})\in\mathcal{C}$ for every point p. In fact δ can be a dilation between two distinct spaces. We require the following property:

Lemma 39 *Given two spaces E_1 and E_2 (distinct or equal) and an adjunction $(\varepsilon,\delta):\mathcal{P}(E_2)\rightleftharpoons\mathcal{P}(E_1)$, the following three properties are equivalent*:

1. $\varepsilon(\emptyset)=\emptyset$.
2. $\varepsilon\delta(\emptyset)=\emptyset$.
3. $\forall X\in\mathcal{P}(E_1)$, $X\neq\emptyset\Rightarrow\delta(X)\neq\emptyset$.

If these properties are not satisfied, setting

$$E_1'=E_1\setminus\varepsilon(\emptyset),$$
$$\delta':\mathcal{P}(E_1')\to\mathcal{P}(E_2):X\mapsto\delta(X)\quad\textit{and}$$
$$\varepsilon':\mathcal{P}(E_2)\to\mathcal{P}(E_1'):Y\mapsto\varepsilon(Y)\setminus\varepsilon(\emptyset),$$

then (ε',δ') is an adjunction satisfying these properties.

Proof Since $\delta(\emptyset)=\emptyset$, $1\Leftrightarrow 2$. By adjunction, $\varepsilon(\emptyset)$ is the greatest $X\in\mathcal{P}(E_1)$ such that $\delta(X)=\emptyset$, hence $1\Leftrightarrow 3$.

Suppose that $\varepsilon(\emptyset)\neq\emptyset$, and let E_1', δ' and ε' be as above. For $X\in\mathcal{P}(E_1')$ and $Y\in\mathcal{P}(E_2)$, $\delta'(X)\subseteq Y$ means $\delta(X)\subseteq Y$ (since $\delta'(X)=\delta(X)$); by the adjunction (ε,δ), this is equivalent to $X\subseteq\varepsilon(Y)$; but since $X\subseteq E_1'=E\setminus\varepsilon(\emptyset)$, this is equivalent to $X\subseteq\varepsilon(Y)\cap(E_1\setminus\varepsilon(\emptyset))=\varepsilon(Y)\setminus\varepsilon(\emptyset)=\varepsilon'(Y)$; therefore (ε',δ') is an adjunction. Obviously $\varepsilon'(\emptyset)=\emptyset$. □

Definition 40 Consider an adjunction (ε,δ) satisfying the three equivalent properties in Lemma 39. Then the adjunction (ε,δ), the dilation δ and the erosion ε are called *regular*. In other words, an erosion ε is regular iff $\varepsilon(\emptyset)=\emptyset$, and a dilation δ is regular iff $X\neq\emptyset\Rightarrow\delta(X)\neq\emptyset$.

Proposition 41 *Consider two spaces E_1 and E_2 (distinct or equal), a regular dilation $\delta:\mathcal{P}(E_1)\to\mathcal{P}(E_2)$, and a partial connection $\mathcal{C}$ on $\mathcal{P}(E_2)$. Then $\delta^{-1}(\mathcal{C})=\{X\in\mathcal{P}(E_1)\mid\delta(X)\in\mathcal{C}\}$ is a partial connection on $\mathcal{P}(E_1)$.*

Proof As δ is a dilation, $\delta(\emptyset)=\emptyset\in\mathcal{C}$, so $\emptyset\in\delta^{-1}(\mathcal{C})$. Let $\mathcal{B}\subseteq\delta^{-1}(\mathcal{C})$ such that $\bigcap\mathcal{B}\neq\emptyset$. Let $\mathcal{D}=\{\delta(B)\mid B\in\mathcal{B}\}$; then $\mathcal{D}\subseteq\mathcal{C}$. As δ is increasing, $\delta(\bigcap\mathcal{B})\subseteq\bigcap_{B\in\mathcal{B}}\delta(B)=\bigcap\mathcal{D}$. As $\bigcap\mathcal{B}\neq\emptyset$ and δ is regular, we have $\delta(\bigcap\mathcal{B})\neq\emptyset$, hence $\bigcap\mathcal{D}\neq\emptyset$. As $\mathcal{D}\subseteq\mathcal{C}$ and $\mathcal{C}$ is a partial connection, we deduce that $\bigcup\mathcal{D}\in\mathcal{C}$. As δ is a dilation, $\delta(\bigcup\mathcal{B})=\bigcup\{\delta(B)\mid B\in\mathcal{B}\}=\bigcup\mathcal{D}\in\mathcal{C}$, so $\bigcup\mathcal{B}\in\delta^{-1}(\mathcal{C})$. □

For example, let $E_1=\mathbf{Z}^2$, $E_2=\mathbf{R}^2$, and let $\mathcal{C}$ be the family of topologically connected subsets of $\mathbf{R}^2$.

– If δ is the dilation by a closed ball of radius 1/2, then $\delta^{-1}(\mathcal{C})$ is the family of 4-connected subsets of $\mathbf{Z}^2$.
– If δ is the dilation by a closed square of side 1, then $\delta^{-1}(\mathcal{C})$ is the family of 8-connected subsets of $\mathbf{Z}^2$.

In relation to Proposition 26, note that for a dual Moore family $\mathcal{M}$ of $\mathcal{P}(E_2)$ and a dilation $\delta : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$, $\delta^{-1}(\mathcal{M})$ will be a dual Moore family of $\mathcal{P}(E_1)$, even if δ is not regular. Concerning Proposition 27, for any upper set $\mathcal{U} \subseteq \mathcal{P}(E_2)$ and any increasing operator $\psi : \mathcal{P}(E_1) \to \mathcal{P}(E_2)$, $\psi^{-1}(\mathcal{U})$ will be an upper set.

The problem with the partial connection $\delta^{-1}(\mathcal{C})$ is that, unless δ satisfies all conditions of item 3 of Proposition 37, it can be very poor, as show the following examples:

– Let $E_1 = E_2 = \mathbf{Z}$, and $\mathcal{C}$ be the usual connection consisting of all connected (equivalently, convex) subsets of $\mathbf{Z}$ (in other words, $\mathcal{C}$ arises from the adjacency relation $z \sim z+1$ on $\mathbf{Z}$). Take the dilation δ by the structuring element $\{n^2 \mid n \in \mathbf{N}\}$; δ is extensive. Then all non-empty elements of $\delta^{-1}(\mathcal{C})$ will be unbounded; indeed, for $X \subset \mathbf{Z}$ of width n, there will be a gap between $X + n^2$ and $X + (n+1)^2$, so $\delta(X) \notin \mathcal{C}$.
– Let $E_1 = E_2 = \mathbf{Z}^2$ and consider the horizontal connection $\mathcal{C}^h$ of Fig. 7. Let δ be the dilation by a ball of radius $r > 0$; δ is extensive. For every non-void $X \in \mathcal{P}(\mathbf{Z}^2)$, $\delta(X)$ will not be included in a line, so $\delta(X) \notin \mathcal{C}^h$; hence $\delta^{-1}(\mathcal{C}^h) = \{\emptyset\}$.

One can always extend $\delta^{-1}(\mathcal{C})$ into a connection by adding to it all singletons of E_1. We will give below two richer constructions. Recall the notion of chaining from Definition 12. Instead of requiring $\delta(X) \in \mathcal{C}$, we will take $\delta(X)$ to be chained by its $\mathcal{C}$-components and by sets $\delta(\{p\})$ ($p \in E$); in the first variant we restrict these $\delta(\{p\})$ to $p \in X$, while in the second variant the weaker restriction is $\delta(\{p\}) \subseteq \delta(X)$.

Given a set $X \in \mathcal{P}(E_1)$, we can consider $\mathcal{S}(X) = \{\{p\} \mid p \in X\}$ and $\delta(\mathcal{S}(X)) = \{\delta(\{p\}) \mid p \in X\}$, the family of dilates of singletons in X; note that $\delta(\mathcal{S}(E)) \cap \mathcal{P}(\delta(X)) = \{\delta(\{p\}) \mid p \in E,\ \delta(\{p\}) \subseteq \delta(X)\}$.

Theorem 42 *Consider two spaces E_1 and E_2 (distinct or equal), a regular adjunction $(\varepsilon, \delta) : \mathcal{P}(E_2) \rightleftharpoons \mathcal{P}(E_1)$, and a partial connection $\mathcal{C}$ on $\mathcal{P}(E_2)$. Define*

– *$\mathcal{C}^1_\delta$ to be the set of all $X \in \mathcal{P}(E_1)$ such that $\delta(X)$ is chained by $[\mathcal{C} \cap \mathcal{P}(\delta(X))] \cup \delta(\mathcal{S}(X))$, in other words, by $\mathcal{C}$-components of $\delta(X)$ and by sets $\delta(\{p\})$ for $p \in X$.*
– *$\mathcal{C}^2_\delta$ to be the set of all $X \in \mathcal{P}(E_1)$ such that $\delta(X)$ is chained by $[\mathcal{C} \cup \delta(\mathcal{S}(E))] \cap \mathcal{P}(\delta(X))$, in other words, by $\mathcal{C}$-components of $\delta(X)$ and by sets $\delta(\{p\})$ for $p \in E$ with $\delta(\{p\}) \subseteq \delta(X)$.*

Then:

1. *$\mathcal{C}^1_\delta$ and $\mathcal{C}^2_\delta$ are connections.*
2. *$\delta^{-1}(\mathcal{C}) \subseteq \mathcal{C}^1_\delta \subseteq \mathcal{C}^2_\delta$.*
3. *$(\varepsilon\delta)$ is a closing such that $(\varepsilon\delta)(\mathcal{C}^1_\delta) \subseteq \mathcal{C}^1_\delta$, and $\mathcal{C}^2_\delta = (\varepsilon\delta)^{-1}(\mathcal{C}^1_\delta)$ (cf. item 2 of Proposition 37).*
4. *$\mathcal{C}^2_\delta = \delta^{-1}(\mathrm{Con}^*[\mathcal{C} \cup \delta(\mathcal{S}(E))])$ (cf. Proposition 41).*

Proof Trivially, $\emptyset$ is chained, so $\emptyset \in \mathcal{C}^1_\delta$. For $p \in E_1$, $\delta(\{p\})$ is chained by $\{\delta(\{p\})\}$, hence it is chained by the larger set $[\mathcal{C} \cap \mathcal{P}(\delta(\{p\}))] \cup \{\delta(\{p\})\}$, so $\{p\} \in \mathcal{C}^1_\delta$. Let $\mathcal{B} \subseteq \mathcal{C}^1_\delta$ such that $\bigcap \mathcal{B} \neq \emptyset$. Since δ is increasing, $\delta(\bigcap \mathcal{B}) \subseteq \bigcap_{B \in \mathcal{B}} \delta(B)$. As $\bigcap \mathcal{B} \neq \emptyset$ and δ is regular, we get $\delta(\bigcap \mathcal{B}) \neq \emptyset$, hence $\bigcap_{B \in \mathcal{B}} \delta(B) \neq \emptyset$. As δ is a dilation, $\delta(\bigcup \mathcal{B}) = \bigcup\{\delta(B) \mid B \in \mathcal{B}\}$. Given $p, q \in \delta(\bigcup \mathcal{B})$, we have $p \in \delta(B)$ and $q \in \delta(B')$ for $B, B' \in \mathcal{B}$; let $r \in \bigcap_{B \in \mathcal{B}} \delta(B)$. By definition, p and r are chained by elements of $[\mathcal{C} \cap \mathcal{P}(\delta(B))] \cup \delta(\mathcal{S}(B))$, while r and q are chained by elements of $[\mathcal{C} \cap \mathcal{P}(\delta(B'))] \cup \delta(\mathcal{S}(B'))$; hence we can chain p and r, then r and q, by elements of the larger set $[\mathcal{C} \cap \mathcal{P}(\delta(\bigcup \mathcal{B}))] \cup \delta(\mathcal{S}(\bigcup \mathcal{B}))$; by transitivity of chaining, p and q will be chained, where p and q are arbitrary members of $\delta(\bigcup \mathcal{B})$; hence $\bigcup \mathcal{B} \in \mathcal{C}^1_\delta$. Therefore $\mathcal{C}^1_\delta$ is a connection. For $X \in \delta^{-1}(\mathcal{C})$, $\delta(X) \in \mathcal{C}$, so $\delta(X)$ is chained by $[\mathcal{C} \cap \mathcal{P}(\delta(X))]$, that is $X \in \mathcal{C}^1_\delta$; thus $\delta^{-1}(\mathcal{C}) \subseteq \mathcal{C}^1_\delta$.

For $p \in E$ and $X \in \mathcal{P}(E_1)$, by the adjunction (ε, δ) we have $\delta(\{p\}) \subseteq \delta(X) \Leftrightarrow p \in \varepsilon\delta(X)$, and $\delta(\varepsilon\delta(X)) = \delta(X)$. Thus

$$
\begin{aligned}
&\big[\mathcal{C} \cup \delta(\mathcal{S}(E))\big] \cap \mathcal{P}(\delta(X)) \\
&\quad = \big[\mathcal{C} \cap \mathcal{P}(\delta(X))\big] \cup \big[\delta(\mathcal{S}(E)) \cap \mathcal{P}(\delta(X))\big] \\
&\quad = \big[\mathcal{C} \cap \mathcal{P}(\delta(X))\big] \cup \delta(\mathcal{S}(\varepsilon\delta(X))) \\
&\quad = \big[\mathcal{C} \cap \mathcal{P}(\delta(\varepsilon\delta(X)))\big] \cup \delta(\mathcal{S}(\varepsilon\delta(X))).
\end{aligned}
$$

Thus $X \in \mathcal{C}^2_\delta$ iff $\delta(X)$ is chained by

$$\big[\mathcal{C} \cup \delta(\mathcal{S}(E))\big] \cap \mathcal{P}(\delta(X)),$$

iff $\delta(\varepsilon\delta(X))$ is chained by

$$\big[\mathcal{C} \cap \mathcal{P}(\delta(\varepsilon\delta(X)))\big] \cup \delta(\mathcal{S}(\varepsilon\delta(X))),$$

that is, $\varepsilon\delta(X) \in \mathcal{C}^1_\delta$; hence $\mathcal{C}^2_\delta = (\varepsilon\delta)^{-1}(\mathcal{C}^1_\delta)$. Obviously, $\varepsilon\delta$ is a closing. As $\delta(\mathcal{S}(X)) \subseteq \delta(\mathcal{S}(\varepsilon\delta(X)))$, we have

$$
\begin{aligned}
&\big[\mathcal{C} \cap \mathcal{P}(\delta(X))\big] \cup \delta(\mathcal{S}(X)) \\
&\quad \subseteq \big[\mathcal{C} \cap \mathcal{P}(\delta(X))\big] \cup \delta(\mathcal{S}(\varepsilon\delta(X)));
\end{aligned}
$$

thus for $X \in \mathcal{C}^1_\delta$, $\delta(X)$ is chained by

$$\big[\mathcal{C} \cap \mathcal{P}(\delta(X))\big] \cup \delta(\mathcal{S}(X)),$$

hence by the larger

$$\big[\mathcal{C} \cap \mathcal{P}(\delta(X))\big] \cup \delta(\mathcal{S}(\varepsilon\delta(X))),$$

and $X \in \mathcal{C}^2_\delta$; thus $\mathcal{C}^1_\delta \subseteq \mathcal{C}^2_\delta = (\varepsilon\delta)^{-1}(\mathcal{C}^1_\delta)$, so $(\varepsilon\delta)(\mathcal{C}^1_\delta) \subseteq \mathcal{C}^1_\delta$. By item 2 of Proposition 37, $\mathcal{C}^2_\delta$ is a partial connection containing $\mathcal{C}^1_\delta$, but as $\mathcal{C}^1_\delta$ is a connection, $\mathcal{C}^2_\delta$ is a connection. By Corollary 33 $\delta(X) \in \mathrm{Con}^*[\mathcal{C} \cup \delta(\mathcal{S}(E))]$ iff $\delta(X)$ is chained by $[\mathcal{C} \cup \delta(\mathcal{S}(E))] \cap \mathcal{P}(\delta(X))$, that is, $X \in \mathcal{C}^2_\delta$; hence $\mathcal{C}^2_\delta = \delta^{-1}(\mathrm{Con}^*[\mathcal{C} \cup \delta(\mathcal{S}(E))])$. □

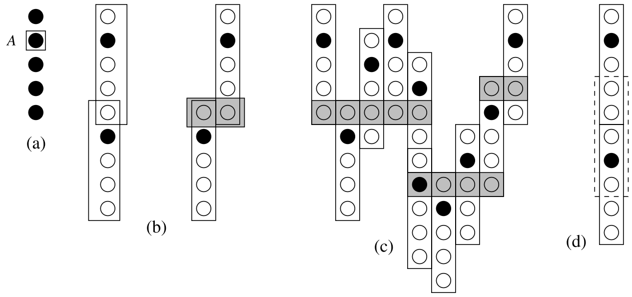

Fig. 11 Here $E_1 = E_2 = \mathbf{Z}^2$; pixels are shown as disks, *black* for foreground, *hollow* for background. (**a**) The structuring element A is a vertical segment of size n (here $n = 5$ and the origin is surrounded by a square); let δ be the dilation by A. (**b**) Two pixels p and q on the same column or on two adjacent columns, and whose heights differ by at most $n - 1$, form a pair in $[\mathcal{C}^h]^1_\delta$, because their dilates $\delta(\{p\}) = A_p$ and $\delta(\{q\}) = A_q$ overlap or can be joined by a horizontal segment (in grey). (**c**) A set in $[\mathcal{C}^h]^1_\delta$: its dilate is chained by the pixel dilates and by horizontal segments (in *grey*). (d) A pair $\{p, q\}$ of pixels on the same column whose heights differ by n, belongs to $[\mathcal{C}^h]^2_\delta$, because there is a pixel r whose dilate $\delta(\{r\})$ (shown *dashed*) chains $\delta(\{p\})$ with $\delta(\{q\})$ in $\delta(\{p, q\})$

These two connections are generally much richer than $\delta^{-1}(\mathcal{C}) \cup \mathcal{S}(E)$. Let us give here some examples. Consider a regular dilation $\delta : \mathcal{P}(E) \to \mathcal{P}(E)$. Here $\delta^{-1}(\{\emptyset\}) = \{\emptyset\}$, so $\delta^{-1}(\{\emptyset\}) \cup \mathcal{S}(E) = \{\emptyset\} \cup \mathcal{S}(E)$ (the least connection), but $\{\emptyset\}^1_\delta$ is the connection on $\mathcal{P}(E_1)$ arising from the adjacency relation $\overset{\delta}{\sim}$ on E_1, where for two points $p, q \in E_1$ we have $p \overset{\delta}{\sim} q$ iff $\delta(\{p\}) \cap \delta(\{q\}) \neq \emptyset$. For example let $E = \mathbf{R}^2$ and let δ be the dilation by the closed ball of radius $r > 0$; then $p \overset{\delta}{\sim} q$ iff $d(p, q) \leq 2r$.

Figure 11 illustrates the connections $\mathcal{C}^1_\delta$ and $\mathcal{C}^2_\delta$ in the case where $E_1 = E_2 = \mathbf{Z}^2$, $\mathcal{C}$ is the horizontal connection $\mathcal{C}^h$ of Fig. 7, and δ is the dilation by a vertical structuring element of size n. Here $\delta^{-1}(\mathcal{C}^h) = \{\emptyset\}$, so $\delta^{-1}(\mathcal{C}^h) \cup \mathcal{S}(E) = \{\emptyset\} \cup \mathcal{S}(E)$ (the least connection), while $[\mathcal{C}^h]^1_\delta$ and $[\mathcal{C}^h]^2_\delta$ are the connections arising from the adjacency relations $\overset{1}{\sim}$ and $\overset{2}{\sim}$, where

$$(i, j) \overset{1}{\sim} (i', j') \iff |i - i'| \leq n - 1 \text{ and } |j - j'| \leq 1,$$

$$(i, j) \overset{2}{\sim} (i', j') \iff (i, j) \overset{1}{\sim} (i', j') \text{ or } \big[|i - i'| = n \text{ and } j = j'\big],$$

i, i' being the row numbers, and j, j' the columns numbers, of the two pixels.

We saw that $\delta^{-1}(\mathcal{C}) \subseteq \mathcal{C}^1_\delta \subseteq \mathcal{C}^2_\delta$, and that $\mathcal{C}^1_\delta$ and $\mathcal{C}^2_\delta$ are connections, while $\delta^{-1}(\mathcal{C})$ is a partial connection. We can complement this comparison as follows:

Proposition 43 *Let E_1, E_2, δ and $\mathcal{C}$ be as above. Then the following three statements are equivalent*:

1. $\delta^{-1}(\mathcal{C})$ *is a connection*
2. *For every $p \in E_1$, $\delta(\{p\}) \in \mathcal{C}$.*
3. $\delta^{-1}(\mathcal{C}) = \mathcal{C}^1_\delta = \mathcal{C}^2_\delta$.

Proof $1 \Rightarrow 2$ If $\delta^{-1}(\mathcal{C})$ is a connection, then it contains the singletons, in other words $\{p\} \in \delta^{-1}(\mathcal{C})$ for each $p \in E_1$, that is, $\delta(\{p\}) \in \mathcal{C}$.

$2 \Rightarrow 3$ If $\delta(\{p\}) \in \mathcal{C}$ for all $p \in E_1$, then $\delta(\mathcal{S}(E)) \subseteq \mathcal{C}$, and $\mathcal{C}^2_\delta$ is the set of all $X \in \mathcal{P}(E_1)$ such that $\delta(X)$ is chained by $[\mathcal{C} \cup \delta(\mathcal{S}(E))] \cap \mathcal{P}(\delta(X)) = \mathcal{C} \cap \mathcal{P}(\delta(X))$, in other words $\delta(X) \in \mathcal{C}$. So $\mathcal{C}^2_\delta = \delta^{-1}(\mathcal{C})$, and as $\delta^{-1}(\mathcal{C}) \subseteq \mathcal{C}^1_\delta \subseteq \mathcal{C}^2_\delta$, $\mathcal{C}^1_\delta = \mathcal{C}^2_\delta$.

$3 \Rightarrow 1$ If $\delta^{-1}(\mathcal{C}) = \mathcal{C}^1_\delta = \mathcal{C}^2_\delta$, as $\mathcal{C}^2_\delta$ is a connection, so is $\delta^{-1}(\mathcal{C})$. □

Note that if $E_1 = E_2$, δ is extensive and for every $p \in E_1$, $\delta(\{p\}) \in \mathcal{C}$ (condition 2 above), then we are in the situation of item 3 of Proposition 37.

When the adjunction (ε, δ) is not regular, we do as in Lemma 39: we take $E'_1 = E_1 \setminus \varepsilon(\emptyset)$ and the restriction δ' of δ to E'_1. Then $\delta'^{-1}(\mathcal{C})$, $\mathcal{C}^1_{\delta'}$ and $\mathcal{C}^2_{\delta'}$ will be partial connections on E'_1, hence partial connections on E_1.

4 Serra's Segmentation Theorem and Partial Partitions

When establishing the theory of connective segmentation, Serra [23] showed that a family $\mathcal{C}$ of sets comprising $\emptyset$ is a connection iff the family of all partitions whose blocks belong to $\mathcal{C}$ is a dual Moore family. We generalize this result to partial partitions and partial connections, and extend this characterization with two new necessary and sufficient conditions. As a consequence, we show that for a partial connection $\mathcal{C}$, the dual Moore family made of all partial partitions with blocks in $\mathcal{C}$ is the invariance domain of the opening on partial partitions that splits each block into its $\mathcal{C}$-components.

Then we discuss the relevance of this result to segmentation, and explain how the theory of partial connections can be used to enhance segmentation algorithms.

4.1 Characterization of (Partial) Connections

For a family $\mathcal{C} \subseteq \mathcal{P}(E)$, let

$$\Pi(E, \mathcal{C}) = \Pi(E) \cap \mathcal{P}\big(\mathcal{C} \setminus \{\emptyset\}\big)$$

$$\text{and} \tag{29}$$

$$\Pi^*(E, \mathcal{C}) = \Pi^*(E) \cap \mathcal{P}\big(\mathcal{C} \setminus \{\emptyset\}\big),$$

be the families respectively of partitions and of partial partitions, whose blocks belong to $\mathcal{C}$ (in fact, blocks are non-void, so they belong to $\mathcal{C} \setminus \{\emptyset\}$).

Serra [23] showed that given a family $\mathcal{C} \subseteq \mathcal{P}(E)$ such that $\emptyset \in \mathcal{C}$, $\Pi(E, \mathcal{C})$ is a dual Moore family of $\Pi(E)$ iff $\mathcal{C}$ is a connection. We generalize this result as follows:

Theorem 44 *Let $\mathcal{C} \subseteq \mathcal{P}(E)$ such that $\emptyset \in \mathcal{C}$. Then the following four statements are equivalent*:

1. *$\mathcal{C}$ is a partial connection on $\mathcal{P}(E)$.*
2. *$\Pi^*(E,\mathcal{C})$ is a dual Moore family of $\Pi^*(E)$.*
3. *For every $A \in \mathcal{P}(E)$, $\Pi^*(A,\mathcal{C})$ is a dual Moore family of $\Pi^*(A)$.*
4. *For every $A \in \mathcal{P}(E)$, $\Pi^*(A,\mathcal{C})$ is non-void and has a greatest element.*

Furthermore, the following four statements are equivalent:

5. *$\mathcal{C}$ is a connection on $\mathcal{P}(E)$.*
6. *$\Pi(E,\mathcal{C})$ is a dual Moore family of $\Pi(E)$.*
7. *For every $A \in \mathcal{P}(E)$, $\Pi(A,\mathcal{C})$ is a dual Moore family of $\Pi(A)$.*
8. *For every $A \in \mathcal{P}(E)$, $\Pi(A,\mathcal{C})$ is non-void and has a greatest element.*

Proof Concerning items 3, 4, 7 and 8, we do not have to consider the specific case where $A = \emptyset$. Indeed, for any family $\mathcal{C} \subseteq \mathcal{P}(E)$, $\Pi^*(\emptyset,\mathcal{C}) = \Pi(\emptyset,\mathcal{C}) = \Pi^*(\emptyset) = \Pi(\emptyset) = \{\boldsymbol{\emptyset}\}$.

$1 \Rightarrow 2$ In $\Pi^*(E)$, the empty supremum is the least partial partition $\boldsymbol{\emptyset}$, having no block, thus $\boldsymbol{\emptyset} \in \Pi^*(E,\mathcal{C})$. Given a non-empty family $\{\pi_i \mid i \in I\}$ in $\Pi^*(E,\mathcal{C})$, the blocks of $\bigvee_{i\in I} \pi_i$ are obtained by chaining blocks of the π_i ($i \in I$), that are all in $\mathcal{C}$; hence (cf. Corollary 33) the blocks of $\bigvee_{i\in I} \pi_i$ belong to $\mathcal{C}$, thus $\bigvee_{i\in I} \pi_i \in \Pi^*(E,\mathcal{C})$. Therefore $\Pi^*(E,\mathcal{C})$ is a dual Moore family of $\Pi^*(E)$.

$2 \Rightarrow 3$ By Proposition 14, the supremum operation in $\Pi^*(A)$ is the one of $\Pi^*(E)$. Thus $\Pi^*(E,\mathcal{C})$ and $\Pi^*(A)$ are both dual Moore families in $\Pi^*(E)$, so their intersection $\Pi^*(A,\mathcal{C})$ is a dual Moore family in $\Pi^*(E)$, hence in $\Pi^*(A)$.

$3 \Rightarrow 4$ A dual Moore family is non-void (it contains the least element) and it has a greatest element (its supremum).

$4 \Rightarrow 1$ By hypothesis, $\emptyset \in \mathcal{C}$. Let $\mathcal{B} \subseteq \mathcal{C}$ such that $\bigcap \mathcal{B} \neq \emptyset$; we can assume that $\mathcal{B} \neq \emptyset$. Let $A = \bigcup \mathcal{B}$, and let π be the greatest element of $\Pi^*(A,\mathcal{C})$. For every $B \in \mathcal{B}$, $B \subseteq A$, so $\mathbf{1}_B \in \Pi^*(A,\mathcal{C})$, hence $\mathbf{1}_B \leq \pi$; by (15), $\mathbf{1}_A = \bigvee_{B\in\mathcal{B}} \mathbf{1}_B$, thus $\mathbf{1}_A \leq \pi$, and as $\pi \in \Pi^*(A)$, we deduce that $\pi = \mathbf{1}_A$. As $\mathbf{1}_A \in \Pi^*(A,\mathcal{C})$, $A \in \mathcal{C}$. Therefore $\mathcal{C}$ is a partial connection.

$5 \Rightarrow 6$ In $\Pi(E)$, the empty supremum is the least partition $\mathbf{0}_E$, made of singletons; as the connection $\mathcal{C}$ contains all singletons, $\mathbf{0}_E \in \Pi(E,\mathcal{C})$. Consider a non-empty family $\{\pi_i \mid i \in I\}$ in $\Pi(E,\mathcal{C})$; as the connection $\mathcal{C}$ is a partial connection, the equivalence $1 \Leftrightarrow 2$ implies that $\bigvee_{i\in I} \pi_i \in \Pi^*(E,\mathcal{C})$; by Proposition 15, this supremum is the same in $\Pi^*(E)$ and $\Pi(E)$, so $\bigvee_{i\in I} \pi_i \in \Pi(E,\mathcal{C})$. Therefore $\Pi(E,\mathcal{C})$ is a dual Moore family of $\Pi(E)$.

$6 \Rightarrow 7$ As $\Pi(E,\mathcal{C})$ is a dual Moore family of $\Pi(E)$, the least partition $\mathbf{0}_E$ belongs to $\Pi(E,\mathcal{C})$; as the blocks of $\mathbf{0}_E$ are the singletons, $\mathcal{C}$ comprises all singletons. Thus $\mathbf{0}_A \in \Pi(A,\mathcal{C})$, i.e., $\Pi(A,\mathcal{C})$ contains the least element of $\Pi(A)$. Consider a non-empty family $\{\pi_i \mid i \in I\}$ in $\Pi(A,\mathcal{C})$; by (12), for each $i \in I$, $\pi_i \vee \mathbf{0}_E = \pi \cup \mathbf{0}_{E\setminus A} \in \Pi(E,\mathcal{C})$. Now as $\Pi(E,\mathcal{C})$ is a dual Moore family, $\bigvee_{i\in I}(\pi_i \vee \mathbf{0}_E) \in \Pi(E,\mathcal{C})$. By Propositions 14 and 15, non-empty suprema in $\Pi(A)$ and $\Pi(E)$ are the same as in $\Pi^*(E)$, where we have $(\bigvee_{i\in I} \pi_i) \vee \mathbf{0}_E = \bigvee_{i\in I}(\pi_i \vee \mathbf{0}_E) \in \Pi^*(E,\mathcal{C})$; hence $\bigvee_{i\in I} \pi_i \in \Pi^*(E,\mathcal{C})$, so $\bigvee_{i\in I} \pi_i \in \Pi(A,\mathcal{C})$. Therefore $\Pi(A,\mathcal{C})$ is a dual Moore family of $\Pi(A)$.

$7 \Rightarrow 8$ Cf. $3 \Rightarrow 4$.

$8 \Rightarrow 5$ For any $p \in E$, $\Pi(\{p\},\mathcal{C})$ is non-void, so it contains the unique partition of $\{p\}$, namely $\{\{p\}\}$, thus $\{p\} \in \mathcal{C}$, and $\mathcal{C}$ comprises all singletons. For $A \in \mathcal{P}(E)$, let ξ_A be the greatest element of $\Pi(A,\mathcal{C})$. For $\pi \in \Pi^*(A,\mathcal{C})$, $\pi \cup \mathbf{0}_{A\setminus\mathrm{supp}(\pi)} \in \Pi(A,\mathcal{C})$, and we have $\pi \leq \pi \cup \mathbf{0}_{A\setminus\mathrm{supp}(\pi)} \leq \xi_A$. Thus ξ_A is the greatest element of $\Pi^*(A,\mathcal{C})$. Thus item 4 holds, hence we have item 1: $\mathcal{C}$ is a partial connection. As $\mathcal{C}$ comprises all singletons, it is a connection. □

Let us illustrate with a few counterexamples what this theorem does not mean:

– A dual Moore family of $\Pi(E)$ is not necessarily of the form $\Pi(E,\mathcal{C})$ for a connection $\mathcal{C}$. Take $E = \mathbf{Z}^2$ or $\mathbf{R}^2$, and let π_h be the partition of E into horizontal lines, and π_v the one into vertical lines. Then $\mathcal{M} = \{\mathbf{0}_E, \pi_h, \pi_v, \mathbf{1}_E\}$ is a dual Moore family of $\Pi(E)$, but the family $\mathcal{C}$ made of $\emptyset$ and the blocks of these partitions, namely the singletons, the horizontal and vertical lines, and E, is not a connection, and $\Pi(E,\mathcal{C})$ is not a dual Moore family. The Moore family generated by $\Pi(E,\mathcal{C})$ is $\Pi(E,\mathrm{Con}(\mathcal{C}))$. The same can be said for $\Pi^*(E)$, with the dual Moore family $\mathcal{M} \cup \{\boldsymbol{\emptyset}\}$.
– Comparing items 5 and 8 (or 1 and 4), in order for $\mathcal{C}$ to be a (partial) connection, it is not sufficient to require that $\Pi(E,\mathcal{C})$ (or $\Pi^*(E,\mathcal{C})$) has a greatest element. Consider again the previous example with $E = \mathbf{Z}^2$ or $\mathbf{R}^2$ and $\mathcal{C}$ the family comprising $\emptyset$, the singletons, the horizontal and vertical lines, and E. Then $\Pi(E,\mathcal{C})$ has a greatest element, namely $\mathbf{1}_E$, but $\mathcal{C}$ is not a connection.

Given a partial connection $\mathcal{C}$, let $\mathrm{CS}^{\mathcal{C}}$ be the operator on $\Pi^*(E)$ that splits each block of a partial partition into its $\mathcal{C}$-components:

$$\forall \pi \in \Pi^*(E),$$
$$\mathrm{CS}^{\mathcal{C}}(\pi) = \bigcup_{C\in\pi} \mathrm{PC}^{\mathcal{C}}(C) = \{\gamma_p(C) \mid C \in \pi,\, p \in C,\, \gamma_p(C) \neq \emptyset\}. \qquad (30)$$

It is known that when $\mathcal{C}$ is a connection, $\mathrm{CS}^{\mathcal{C}}$ is an opening on partitions. This remains true for a partial connection and partial partitions:

Proposition 45 *For any partial connection $\mathcal{C}$ on $\mathcal{P}(E)$, $\mathrm{CS}^{\mathcal{C}}$ is an opening on $\Pi^*(E)$, whose invariance domain is*

$\Pi^*(E,\mathcal{C})$. *When* $\mathcal{C}$ *is a connection, the restriction of* $\mathrm{CS}^{\mathcal{C}}$ *to* $\Pi(E)$ *is an opening whose invariance domain is* $\Pi(E,\mathcal{C})$.

Proof Let $\pi \in \Pi^*(E)$. Consider $\xi \in \Pi^*(E,\mathcal{C})$ such that $\xi \leq \pi$; for every class $C \in \xi$, we have $C \neq \emptyset$, $C \in \mathcal{C}$ and there is a class $D \in \pi$ such that $C \subseteq D$, thus C is included in a $\mathcal{C}$-component of D; hence $\xi \leq \mathrm{CS}^{\mathcal{C}}(\pi)$. Now clearly $\mathrm{CS}^{\mathcal{C}}(\pi) \in \Pi^*(E,\mathcal{C})$ and $\mathrm{CS}^{\mathcal{C}}(\pi) \leq \pi$. Therefore $\mathrm{CS}^{\mathcal{C}}(\pi)$ is the greatest $\xi \in \Pi^*(E,\mathcal{C})$ such that $\xi \leq \pi$. As $\Pi^*(E,\mathcal{C})$ is a dual Moore family of $\Pi^*(E)$, this implies [2, 8, 15] that $\mathrm{CS}^{\mathcal{C}}$ is an opening on $\Pi^*(E)$, with $\mathrm{Inv}(\mathrm{CS}^{\mathcal{C}}) = \Pi^*(E,\mathcal{C})$. When $\mathcal{C}$ is a connection, $\mathrm{CS}^{\mathcal{C}}(\pi) \in \Pi(E)$ for any $\pi \in \Pi(E)$, so $\mathrm{CS}^{\mathcal{C}}$ preserves $\Pi(E)$; then the restriction of $\mathrm{CS}^{\mathcal{C}}$ to $\Pi(E)$ is an opening whose invariance domain is $\Pi^*(E,\mathcal{C}) \cap \Pi(E) = \Pi(E,\mathcal{C})$. □

A typical example of the opening $\mathrm{CS}^{\mathcal{C}}$ arises when we choose for $\mathcal{C}$ a usual connection in the Euclidean or digital space: we split each block into its connected components. However, we have seen in Sect. 2.3 that there are many more types of partial connections. Let us describe what some of them can give:

– Take the partial connection $\mathcal{C} = \mathrm{Inv}(\alpha)$ for an opening α, cf. Proposition 26. By (22), $\mathrm{PC}^{\mathcal{C}}(X) = \{\alpha(X)\}$ if $\alpha(X) \neq \emptyset$, and $\mathbf{\emptyset}$ otherwise, so we get

$$\mathrm{CS}^{\mathcal{C}}(\pi) = \{\alpha(C) \mid C \in \pi,\ \alpha(C) \neq \emptyset\}.$$

  In other words, we apply the opening α to each block of the partial partition, and keep only non-void opened blocks.
– A particular case is when α is a trivial opening, in other words $\mathcal{C} = \mathcal{U} \cup \{\emptyset\}$ for an upper set $\mathcal{U}$, cf. Proposition 27. Here for $X \neq \emptyset$, $\mathrm{PC}^{\mathcal{C}}(X) = \{X\}$ if $X \in \mathcal{U}$, and $\mathbf{\emptyset}$ otherwise, so we get $\mathrm{CS}^{\mathcal{C}}(\pi) = \pi \cap \mathcal{P}(\mathcal{U})$. In other words, in a partial partition we keep only the blocks belonging to $\mathcal{U}$. For instance, if $\mathcal{U} = \mathcal{P}_{\geq n}(E)$, the set of all $X \in \mathcal{P}(E)$ such that $|X| \geq n$, this means that we remove all blocks of size $< n$.
– For a partial connection $\mathcal{C}$ and an upper set $\mathcal{U}$, $\mathcal{C} \cup \mathcal{U}$ is a partial connection where

$$\mathrm{PC}^{\mathcal{C}\cup\mathcal{U}}(X) = \begin{cases} X & \text{if } X \in \mathcal{U}, \\ \mathrm{PC}^{\mathcal{C}}(X) & \text{if } X \notin \mathcal{U}. \end{cases}$$

  Thus in a partial partition π, $\mathrm{CS}^{\mathcal{C}\cup\mathcal{U}}$ will preserve all blocks of π belonging to $\mathcal{U}$, and split all other blocks into their $\mathcal{C}$-components.
– Let $\mathcal{C} = \{\emptyset\} \cup \mathcal{S}(E)$, the least connection. The $\mathcal{C}$-components of a set are its singletons, so $\mathrm{CS}^{\mathcal{C}}$ is the block grinding opening **grind**: $\mathrm{CS}^{\mathcal{C}}(\pi) = \mathbf{0}_{\mathrm{supp}(\pi)}$.

4.2 Discussion: Relevance to Segmentation

Theorem 44 is at the basis of the theory of connective segmentation [15, 23]. Recall from Sect. 1 that:

– a *criterion* σ is a Boolean predicate associating to every function $F : E \to V$ and every subset $A \subseteq E$ a value $\sigma[F, A]$ that can be 1 or 0;
– we write $\mathcal{C}_\sigma^F$ for the set of all $A \in \mathcal{P}(E)$ such that $\sigma[F, A] = 1$;
– σ is *connective* if for any $F : E \to V$, the set $\mathcal{C}_\sigma^F$ is a connection.

Let us say that:

– given a function $F : E \to V$, a *pre-segmentation of* F *according to* σ is a partition of E such that each block A of it satisfies $\sigma[F, A] = 1$, in other words an element of $\Pi(E, \mathcal{C}_\sigma^F)$;
– if $\Pi(E, \mathcal{C}_\sigma^F)$ has a greatest element, it is called the *segmentation of* F *according to* σ;
– the criterion *segments all functions* if for any $F : E \to V$, the set $\Pi(E, \mathcal{C}_\sigma^F)$ of pre-segmentations of F according to σ is a dual Moore family of $\Pi(E)$ [23].

The equivalence 6 ⇔ 5 proved by Serra in [23] means then that the criterion σ segments all functions iff it is connective.

In [15], the framework of connective segmentation was extended to partial partitions and partial connections. No proofs were given then, they appear only here. The above definitions admit partial counterparts:

– a criterion σ is *partially connective* if $\mathcal{C}_\sigma^F$ is a partial connection;
– a *partial pre-segmentation of* F *according to* σ is any element of $\Pi^*(E, \mathcal{C}_\sigma^F)$;
– if $\Pi^*(E, \mathcal{C}_\sigma^F)$ has a greatest element, it is called the *partial segmentation of* F *according to* σ;
– a criterion σ *partially segments all functions* if for any $F : E \to V$, the set $\Pi^*(E, \mathcal{C}_\sigma^F)$ of partial pre-segmentations of F according to σ is a dual Moore family of $\Pi^*(E)$.

Then the equivalence 2 ⇔ 1 means the following assertion made in [15]: the criterion σ partially segments all functions iff it is partially connective.

On the other hand, properties 7 and 8 for a connection (respectively, 3 and 4 for a partial connection), were not considered by Serra [23]. At first sight, one could think that property 7 means "for every $A \in \mathcal{P}(E)$, σ segments all functions $A \to V$", but this is misleading. Indeed, given $A \in \mathcal{P}(E)$, $F : E \to V$ and $F_A : A \to V$ the restriction of F to A, for any $B \in \mathcal{P}(A)$, whether F satisfies criterion σ on B can depend on the knowledge of the whole of F, and not only on its restriction F_A to A. Thus for $B \in \mathcal{P}(A)$, $\sigma[F, B]$ is not necessarily equal to $\sigma[F_A, B]$, in other words $\mathcal{C}_\sigma^F \cap \mathcal{P}(A)$ does not necessarily coincide with $\mathcal{C}_\sigma^{F_A}$. Later on, we will discuss further this question of regional and global knowledge.

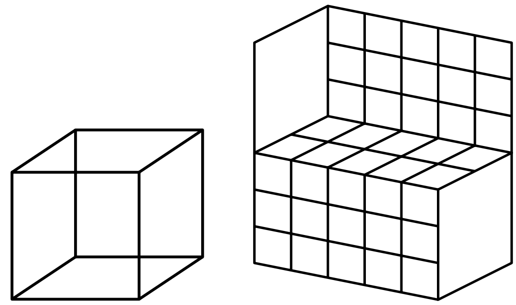

Fig. 12 Two ambiguous images: *left*, the *Necker cube*; *right*, a bench/inverted bench. They both have two incompatible 3D interpretations

For $A \in \mathcal{P}(E)$ and $F : E \to V$, let us call *segmentation of F on A according to* σ the greatest element of $\Pi(A, \mathcal{C}_\sigma^F)$ (if it exists), that is, the greatest partition of A such that each block B of it satisfies $\sigma[F, B] = 1$. Then property 8 means: for every $A \in \mathcal{P}(E)$, every function has a segmentation on A according to σ. Now property 7 (that $\Pi(A, \mathcal{C}_\sigma^F)$ is a dual Moore family) can be expressed as: for every $A \in \mathcal{P}(E)$, σ *segments all functions on* A. Both are equivalent to property 5 (that σ is connective). We have then similar interpretations of properties 3 and 4 in the partial case.

Why is Theorem 44 important? In a preliminary version of [23], Serra gave some examples of so-called *ambiguous images*; we show two such images in Fig. 12. They have two incompatible visual interpretations, so their perception switches back and forth between these two interpretations. In other words, the two interpretations *cannot be unified*.

Now turning to image segmentation, if several segmentation algorithms extract from a function $F : E \to V$ distinct segmentation partitions $\pi_1, \dots, \pi_n$, how do we unify them? The answer is that a unification of $\pi_1, \dots, \pi_n$ is a segmentation partition π coarser than each of them, in other words $\pi \geq \bigvee_{i=1}^n \pi_i$. The fact that all segmentations of F according to criterion σ can be unified, means that $\Pi(E, \mathcal{C}_\sigma^F)$ has a greatest element. This is not sufficient for σ to be connective: indeed, take the second counterexample after Theorem 44, that is, $E = \mathbf{Z}^2$ and $\mathcal{C}_\sigma^F$ comprising $\emptyset$, the singletons, the horizontal and vertical lines, and E, then $\Pi(E, \mathcal{C}_\sigma^F)$ has a greatest element, namely $\mathbf{1}_E$, but it is not a connection. On the other hand if for all $A \in \mathcal{P}(E)$ the segmentations on A can be unified, in other words $\Pi(A, \mathcal{C}_\sigma^F)$ has a greatest element, then by $8 \Leftrightarrow 5$ the criterion σ is connective. We can say the same for partial segmentations (partial partitions) with the equivalence $4 \Leftrightarrow 1$.

Therefore *connective segmentation corresponds to segmentation by partitioning the space into homogeneous regions according to a criterion, in such a way that inside any region of the space, any set of partitions can be unified by a coarser one*.

Given a (partially) connective criterion σ, to every function $F : E \to V$ corresponds the (partial) connection $\mathcal{C}_\sigma^F$ made of all $A \in \mathcal{P}(E)$ such that $\sigma[F, A] = 1$. Every (partial) pre-segmentation of F according to σ, i.e., every (partial) partition in with blocks in $\mathcal{C}_\sigma^F$ can be considered as a valid but sub-optimal (partial) segmentation of the function F induced by the criterion σ; thus in practice the optimal one should be the greatest one, that we called the (partial) segmentation of F according to σ, in other words the (partial) partition $\mathrm{PC}^{\mathcal{C}_\sigma^F}(E)$ of E into its $\mathcal{C}_\sigma^F$-components. Let us write $\mathbf{seg}_\sigma(F, E)$ for this greatest (partial) segmentation. Why do we need to consider the whole (partial) connection $\mathcal{C}_\sigma^F$ instead of just taking the greatest (partial) segmentation $\mathbf{seg}_\sigma(F, E)$?

One important reason, outlined in Sect. 1, is that we can combine (partially) connective criteria. In particular a Boolean conjunction $\sigma_1 \wedge \cdots \wedge \sigma_n$ of (partially) connective criteria $\sigma_1, \dots, \sigma_n$, will be (partially) connective; indeed

$$\mathcal{C}_{\sigma_1 \wedge \cdots \wedge \sigma_n}^F = \mathcal{C}_{\sigma_1}^F \cap \cdots \cap \mathcal{C}_{\sigma_n}^F,$$

which is an intersection of (partial) connections, hence a (partial) connection. However $\mathbf{seg}_{\sigma_1 \wedge \cdots \wedge \sigma_n}(F, E)$ is in general not obtained as the infimum of the partitions $\mathbf{seg}_{\sigma_i}(F, E)$ $(i = 1, \dots, n)$, as we saw in Fig. 3; to be built it requires the whole (partial) connection $\mathcal{C}_{\sigma_1}^F \cap \cdots \cap \mathcal{C}_{\sigma_n}^F$.

On the other hand the Boolean disjunction $\sigma_1 \vee \cdots \vee \sigma_n$ is in general not (partially) connective. Writing $\mathrm{con}(\sigma_1 \vee \cdots \vee \sigma_n)$ for the least connective criterion $\geq \sigma_1 \vee \cdots \vee \sigma_n$, we have

$$\mathcal{C}_{\mathrm{con}(\sigma_1 \vee \cdots \vee \sigma_n)}^F = \mathrm{Con}\big(\mathcal{C}_{\sigma_1}^F \cup \cdots \cup \mathcal{C}_{\sigma_n}^F\big),$$

the least connection containing $\mathcal{C}_{\sigma_1}^F \cup \cdots \cup \mathcal{C}_{\sigma_n}^F$. By Proposition 32, we have then

$$\mathbf{seg}_{\mathrm{con}(\sigma_1 \vee \cdots \vee \sigma_n)}(F, E) = \bigvee_{i=1}^n \mathbf{seg}_{\sigma_i}(F, E). \tag{31}$$

In the partially connective case, we have a similar result with $\mathrm{con}^*(\sigma_1 \vee \cdots \vee \sigma_n)$, the least partially connective criterion $\geq \sigma_1 \vee \cdots \vee \sigma_n$, which corresponds to $\mathrm{Con}^*(\mathcal{C}_{\sigma_1}^F \cup \cdots \cup \mathcal{C}_{\sigma_n}^F)$, the least partial connection containing $\mathcal{C}_{\sigma_1}^F \cup \cdots \cup \mathcal{C}_{\sigma_n}^F$.

The consideration of the whole connection $\mathcal{C}_\sigma^F$ (instead of only the greatest partition $\mathbf{seg}_\sigma(F, E)$) is also relevant for the segmentation of F on a subset A of E, that we mentioned above in relation to properties 3, 4, 7 and 8. Here the fact that $\mathcal{C}_\sigma^F$ is a connection has interesting consequences. If A is a $\mathcal{C}_\sigma^F$-component of a set Y, then A is a $\mathcal{C}_\sigma^F$-component of Z for any set Z such that $A \subseteq Z \subseteq Y$. More generally, given two sets Y and Z and a point $p \in Y \cap Z$, if $\gamma_p(Y) \subseteq Z$ and $\gamma_p(Z) \subseteq Y$, then $\gamma_p(Y) = \gamma_p(Z)$. For segmentation, this means that an object extracted in the segmentation of a function on E, will remain identically segmented if we segment the function on a bounded mask, provided that the mask is large enough. This is called *class permanence* in [23].

It is now time to discuss for a connective criterion σ, what is the information from F needed to determine the value of $\sigma[F, A]$ for $A \in \mathcal{P}(E)$ and $F : E \to V$. Is it the whole of F, the restriction of F to A, or to an intermediate set? For example, in segmentation methods based on the minimization of a global energy function, the segmentation of an object changes according to the mask on which the energy is computed (J. Serra, personal communication); in other words, the segmentation of F on A requires the knowledge of F on the whole space E.

To make the matter more provocative, let us note that *to every segmentation algorithm one can associate a connective criterion* (M. Tajine, personal communication): *let the criterion be satisfied by any connected subset of a segmentation class*. More precisely, suppose that an algorithm associates to a function F its (partial) segmentation on E, that is a (partial) partition $\mathrm{Seg}(F, E)$. We define the criterion σ by $\sigma[F, A] = 1$ iff $A \in \mathbf{subbl}(\mathrm{Seg}(F, E))$, in other words, $\mathcal{C}_\sigma^F = \mathbf{subbl}(\mathrm{Seg}(F, E))$, the family of all subsets of blocks of the partition $\mathrm{Seg}(F, E)$. If the algorithm is designed to produce connected segmentation classes (according to a standard connection $\mathcal{C}_0$), then one can take $\mathcal{C}_\sigma^F = \mathcal{C}_0 \cap \mathbf{subbl}(\mathrm{Seg}(F, E))$. Since $\mathbf{subbl}(\mathrm{Seg}(F, E))$ is a connection on the support of $\mathrm{Seg}(F, E)$, the criterion σ is (partially) connective.

Of courses, such a connective criterion is defined *a posteriori*, after the segmentation is effectively realized, and one is rather interested in the definition of an *a priori* criterion that allows to build the segmentation of a function.

Let us examine in this respect the connective segmentations described in Sect. 1. Let $\mathcal{C}_0$ be the standard connection on $\mathcal{P}(E)$, we assume that only segmentations with classes in $\mathcal{C}_0$ are taken into account, in other words $\mathcal{C}_\sigma^F \subseteq \mathcal{C}_0$; if we do not make such an assumption, we can put $\mathcal{C}_0 = \mathcal{P}(E)$. In the *segmentation by flat zones*, $\sigma[F, A] = 1$ iff $A \in \mathcal{C}_0$ and F is constant on A. In *thresholding* of interval U, $\sigma[F, A] = 1$ iff $A \in \mathcal{C}_0$ and $F(p) \in U$ for all $p \in A$. Thus in these two examples, only the set A and the restriction F_A of F to A are needed to determine $\sigma[F, A]$, so the criterion uses only local information. In the *regional Lipschitz segmentation*, $\sigma[F, A] = 1$ iff $A \in \mathcal{C}_0$ and for all $p \in A$, F is Lipschitz on the neighbourhood $B(p)$; this requires the knowledge of the values of F only on the neighbourhood of A given by $\bigcup_{p \in A} B(p)$. Again, the criterion uses only local information.

In the *jump segmentation* of parameter k, the connection is generated by seeds. Here a seed is a set $A \in \mathcal{C}_0$ such that there is a regional minimum M of F intersecting A, and such that if m is the level of M, then for every $p \in A$, we have $m \leq F(p) < m + k$. Thus the only knowledge necessary to determine $\sigma[F, A]$ is: (a) whether A is connected, (b) the restriction F_A of F to A, (c) the set of regional minima of F. The latter (c) is the only global information. Let us note that if A is a seed, the union of all flat zones of the points of A is also a seed. One can thus take the graph of flat zones of F [5], where each vertex corresponds to a flat zone, and an edge links two vertices if the corresponding flat zones are adjacent; then the function F becomes a function G on the set V of vertices. Hence the jump segmentation can be applied to G, but here each regional minimum of F becomes a local minimum of G (a vertex v such that $G(v) < G(w)$ for any adjacent vertex w); thus a seed is determined by the restriction of G to its neighbourhood. Hence at the level of flat zones, the criterion uses only local information.

These examples constitute indeed instances of *a priori* connective criteria based on more or less local information. Note that the working on the graph of flat zones that we suggested for jump segmentation, can be generalized. Assume a (partially) connective criterion σ such that for every set A and function F, the union A' of flat zones of F containing points of A satisfies $\sigma[F, A'] \geq \sigma[F, A]$; then one can apply the segmentation by σ to the image defined on the graph of flat zones.

In [23], Serra defined an *a posteriori* connective segmentation for watershed segmentation: $\sigma[F, A] = 1$ iff A is connected and is included in the catchment basin of a single regional minimum of F. To determine $\sigma[F, A]$, one needs in practice to determine all paths of steepest descent from A to minima of F. Said in another way, a catchment basin C is determined not only by the values of F on C, but also by those of F on the portion of watershed surrounding it, and on some parts of the neighbouring catchment basins. This represents a rather global information, so the construction of watersheds by an *a priori* connective criterion is probably impossible.

Let us end this section by suggesting ways to improve existing segmentation algorithms with the use of our theory of partial partitions and partial connections. In [15, 23] several examples are given of improved segmentations obtained by an infimum of two or more connective criteria. Since every segmentation algorithm determines an *a posteriori* connective criterion σ, we can combine this criterion with an *a priori* connective criterion σ' to obtain the segmentation according to $\sigma \wedge \sigma'$ or $\mathrm{con}(\sigma \vee \sigma')$ (both connective criteria). One can also modify the underlying standard connection $\mathcal{C}_0$ by replacing it with a second-generation partial connection, cf. Sect. 3, see also Figs. 1 and 3.

If the resulting segmentation is only partial, it can be completed by a segmentation of the residual, as done in Figs. 4 and 5. If one wants to limit oversegmentation, one can do as in Fig. 4: take as markers the classes of the first segmentation, and regroup the classes of the further segmentations into influence zones of the markers. Note that Fig. 4 is also an example of a reduction of undersegmentation: a connected region made of two wide parts separated

Table 1 Notation (in the order of first appearance)

$\mathcal{C}_0$	A standard connection on $\mathcal{P}(E)$ (e.g., arc, topological, 4- or 8-connectivity)
$\mathrm{Inv}(\psi)$	Invariance domain of the operator ψ
$(\alpha, \beta) : A \rightleftharpoons B$	$\alpha : A \to B$ and $\beta : B \to A$
$\mathcal{B}$	A family of subsets of E
$\mathrm{supp}(R)$	Support of the binary relation R on E
$\mathrm{supp}(\mathcal{B})$	Support of the family $\mathcal{B}$
π	A partial partition
$\mathrm{PE}(\pi)$	Partial equivalence corresponding to π
Cl_π	Partial partition class map associated to π
cl	A partial partition class map
PP(cl)	Partial partition associated to cl
$\mathcal{E}^*(E)$	Set of all partial equivalences on E
$\Pi(E)$	Set of all partitions of E
$\Pi^*(E)$	Set of all partial partitions of E
$\mathbf{0}_A$	Identity partition of A into its singletons
$\mathbf{1}_A$	Universal partition of A into a single block
$\mathbf{0}_{\mathrm{supp}(\pi)}$	$= \mathbf{grind}(\pi)$, block grinding of π
$\mathbf{1}_{\mathrm{supp}(\pi)}$	$= \mathbf{blend}(\pi)$, block blending of π
$\mathcal{C}$	A partial connection on $\mathcal{P}(E)$
$\mathcal{S}(X)$	Family of all singletons in X
γ_p	Partial connection opening on $\mathcal{P}(E)$
$(\gamma_p, p \in E)$	System of partial connection openings on $\mathcal{P}(E)$
$\mathrm{PC}^{\mathcal{C}}(X)$	Partial partition of all $\mathcal{C}$-components of X
$\mathbf{subbl}(\pi)$	partial connection of all subsets of blocks of π
$\Gamma(E)$	Set of all connections on $\mathcal{P}(E)$
$\Gamma^*(E)$	Set of all partial connections on $\mathcal{P}(E)$
$\mathrm{Con}(\mathcal{B})$	Connection generated by the family $\mathcal{B}$
$\mathrm{Con}^*(\mathcal{B})$	Partial connection generated by the family $\mathcal{B}$
$\mathcal{C}^\psi$	$\{X \in \mathcal{P}(E) \mid \exists C \in \mathcal{C},\ X \subseteq C \subseteq \psi(X)\}$
$\psi^{-1}(\mathcal{C})$	$\{X \in \mathcal{P}(E) \mid \psi(X) \in \mathcal{C}\}$
$\psi(\mathcal{C})$	$\{\psi(X) \mid X \in \mathcal{C}\}$
$\delta(\mathcal{S}(X))$	$\{\delta(\{p\}) \mid p \in X\}$
$\mathcal{C}^1_\delta$	Set of all X with $\delta(X)$ chained by $[\mathcal{C} \cap \mathcal{P}(\delta(X))] \cup \delta(\mathcal{S}(X))$
$\mathcal{C}^2_\delta$	Set of all X with $\delta(X)$ chained by $[\mathcal{C} \cup \delta(\mathcal{S}(E))] \cap \mathcal{P}(\delta(X))$
$\Pi(E, \mathcal{C})$	Set of all partitions of E with blocks in $\mathcal{C} \setminus \{\emptyset\}$
$\Pi^*(E, \mathcal{C})$	Set of all partial partitions of E with blocks in $\mathcal{C} \setminus \{\emptyset\}$
$\mathrm{CS}^{\mathcal{C}}$	Opening on $\Pi^*(E)$ splitting blocks into $\mathcal{C}$-components
$\mathcal{C}^F_\sigma$	$\{A \in \mathcal{P}(E) \mid \sigma[F, A] = 1\}$
$\mathbf{seg}_\sigma(F, E)$	$\mathrm{PC}^{\mathcal{C}^F_\sigma}(E)$

by a narrow isthmus is split into two regions, the isthmus becomes the border between them. This approach could be used (in combination with other methods, such as the one in Sect. 8 of [22]) in the design of methods for closing broken contours.

5 Concluding Remarks

We have recalled the framework of connective segmentation, and we have argued for the necessity to broaden it to partial partitions and partial connections. This work is devoted to the study of these two concepts.

One of the merits of [23] was to highlight some links between connections and partitions. We have investigated further such links, and generalized this analysis to partial connections and partial partitions. We have studied in depth the properties of partial connections and partial partitions, the lattices that they constitute, and the relations between these two lattices.

Given a supremum of partial connections, it associates to a set a partial partition of connected components, which will be the supremum of the partial partitions of the set associated to each individual partial connection. On the other hand, for a non-void infimum of partial connections, its partial connection opening at a point will be the infimum, in the lattice of openings, of the partial connection openings of each individual partial connection.

We have generalized to partial connections known methods for generating a new connection from a given connection and an operator. We have also introduced the two new connections $\mathcal{C}^1_\delta$ and $\mathcal{C}^2_\delta$ built from a partial connection $\mathcal{C}$ and a regular dilation δ.

Serra's theorem states that a family of sets is a connection iff the set of all partitions with blocks belonging to that family, is a dual Moore family. We have generalized it to partial connections and partial partitions. The opening corresponding to that dual Moore family is the operator on partial partitions that splits each block into its connected components.

We have argued the relevance of our theory of partial connections to segmentation. The set of connective segmentation criteria is stable under the infimum operation, while a supremum of connective criteria generates a connective criterion for which the standard segmentation is the supremum of the ones of the individual criteria, cf. (31). Partially connective segmentations can be combined sequentially, cf. Figs. 4 and 5. Since every segmentation algorithm leads to an *a posteriori* connective criterion, this gives various possibilities for modifying existing algorithms.

In the same way as connections led to a progress in the processing of binary and grey-level images, thanks to the introduction of connected operators [24], partial connections can lead to new morphological operations on partial partitions. For example we saw that for any partial connection $\mathcal{C}$, the operator on partial partitions that splits each block into its $\mathcal{C}$-components, is an opening. It can be shown that every opening on partial partitions that acts by splitting each block separately, is of this form. Partial connections can also be involved in closings that cluster blocks of a partial partition.

The author has initiated further work on morphological and geodesic operations on partial partitions. Some preliminary ideas have been presented (without proofs) in [13]. A collaborative research project is planned on connective segmentation criteria and algorithms, as well as on image filtering adapted to this framework (see for example [25]).

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