J Math Imaging Vis (2008) 32: 1–22
DOI 10.1007/s10851-008-0084-3

Equivalence between Closed Connected *n*-*G*-Maps without Multi-Incidence and *n*-Surfaces

Sylvie Alayrangues · Xavier Daragon · Jacques-Olivier Lachaud · Pascal Lienhardt

Published online: 15 April 2008

Abstract Many combinatorial structures have been designed to represent the topology of space subdivisions and images. We focus here on two particular models, namely the n-G-maps used in geometric modeling and computational geometry and the n-surfaces used in discrete imagery. We show that a subclass of n-G-maps is equivalent to n-surfaces. To achieve this, we provide several characterizations of n-surfaces. Finally, the proofs being constructive, we show how to switch from one representation to another effectively.

Keywords Comparison of combinatorial structures · Subdivisions · Generalized maps · n-surfaces · Geometric modeling · Computational geometry · Discrete imagery

S. Alayrangues (✉) · P. Lienhardt
Laboratoire SIC-XLIM, UMR CNRS 6172, Université de Poitiers, bât SP2MI, Bd M. et P. Curie, BP 30179, 86962 Futuroscope Chasseneuil Cedex, France
e-mail: sylvie.alayrangues@sic.univ-poitiers.fr

P. Lienhardt
e-mail: pascal.lienhardt@sic.univ-poitiers.fr

X. Daragon
ESIEE—Laboratoire A²SI, 2 bd Blaise Pascal, cité DESCARTES, BP 99, 93162 Noisy le Grand Cedex, France
e-mail: daragonx@esiee.fr

J.-O. Lachaud
Laboratoire de Mathématiques, UMR CNRS 5127, Université de Savoie, 73776 Le-Bourget-du-Lac Cedex, France
e-mail: jacques-olivier.lachaud@univ-savoie.fr

1 Introduction

The representation of space subdivisions and the study of their topological properties are significant topics in various fields of research such as geometric modeling, computational geometry and discrete imagery. A lot of combinatorial structures have already been defined to represent such topologies and specific tools have been developed on each of them to perform operations on the subdivisions they represent (see for instance [8, 11, 14, 17, 25]). Although most of them aim at representing the same underlying space (generally manifold-like), they have very variable definitions. This variety mainly comes from the way they represent the topology (e.g., point-set topology, cellular decomposition, mappings defining relations). Moreover each model is equipped with a set of operations which is essentially determined by the field of applications where the structure is used (e.g. modeling, image analysis).

Comparing these structures, and highlighting their similarities or specificities are important for several reasons. It can first create bridges between them and offer the possibility to switch from one framework to another according to the needs of a given application. It may also lead to a more general framework which unifies most of these structures. Theoretical results and practical algorithms can also be transferred from one to another. However, these structures are most likely not interchangeable. Indeed, there is no complete combinatorial characterization of manifolds. The structures found in the literature generally propose local combinatorial properties that can approach the properties of space subdivisions but without capturing their complete nature. In designing algorithms involving topological and geometric models, it is therefore extremely important to know precisely what class of objects is associated with each structure. Several studies have already been carried out in this direction. Quad-edge, facet-edge and cell-tuples were compared by Brisson in [6]. Lienhardt [21] studied their relations with several structures used in geometric modeling like the n-dimensional (generalized or not) map. The relation between a subclass of orders [3] and cell complexes was also studied in [2]. A similar work was done on dual graphs and maps by Brun and Kropatsch in [7].

We focus here mainly on two structures: the n-surface and the n-dimensional generalized map. The n-surface is a specific subclass of orders defined by Bertrand and Couprie in [4] which is equivalent to the notion previously defined by Evako *et al.* on graphs in [16]. It is essentially an order relation over a set together with a finite recursive property. It is designed to represent the topology of images and objects within. The generalized map introduced by Lienhardt in [21] is an effective tool in geometric modeling and is also used in computational geometry. It is defined by a set of $n+1$ involutions joining elements dimension by dimension. Although the definitions of these two structures are very different, we manage to show that a subclass of generalized maps, namely *closed connected n-G-maps without multi-incidence*, is equivalent to n-surfaces. This may have various nice consequences. From a theoretical point of view, some proofs may be simplified by expressing them rather on a model than on the other, some notions can also be extended. Moreover the operators defined on each model may be translated onto the other. A possible application would consist in using the tools defined on orders: homotopic thinning, marching chains using frontier orders [11, 12] to obtain n-surfaces. They can then be transformed into n-G-maps which can easily be handled with their associated construction operators, e.g. identification, extrusion, split, merge.

In a previous work [1], we used an intermediary model, a subclass of incidence graphs, to achieve this comparison. In order to clarify the proof and highlight intrinsic properties of n-surfaces, we only deal here with subclasses of orders. We first give a static characterization of n-surfaces, which we use then to distinguish such orders through properties of their maximal chains. The latter characterization is the key point towards the expression of the equivalence between n-surfaces and closed connected n-G-maps without multi-incidence.

The paper is organized as follows. In Sect. 2, we define some essential notions[1] related to the models we intend to compare (orders, n-surfaces and closed connected n-G-maps). We present and prove then two characterizations of n-surfaces (Sect. 3, Theorem 9 and Theorem 19). The first characterization is required to prove the second characterization, which in turns is used to prove the equivalence of models. We go on with defining and characterizing the class of closed connected n-G-maps without multi-incidence (Sect. 4, Theorem 21) that will be proved to be equivalent to n-surfaces. We detail then the whole demonstration (Sect. 5). We first show how to construct an n-G-map from an n-surface (Sect. 5.1, Theorem 22). We then exhibit the construction of an n-surface from an n-G-map (Sect. 5.2, Theorem 23). We finally prove that both operations are inverse to each other up to an isomorphism (Sect. 5.3, Theorem 24) and hence prove the isomorphism between the set of n-surfaces and the set of closed connected n-G-maps without multi-incidence.

2 Model Description

This section begins with a slight introduction about the representation of subdivided objects with combinatorial models in an image context. It focuses then on the structures we are more precisely interested in, orders and n-G-maps, and recalls the main notions related to them. It also describes the simplicial interpretations associated with each model which helps visualizing their relationship.

2.1 Combinatorial Models for Image Representation[2]

Many image applications in geometric modeling or image analysis require to represent subdivided objects, i.e. objects partitioned in cells of different dimensions (e.g. vertices, edges, faces, volumes). Such a representation is necessary, either to represent an object naturally made of elements of different dimensions (e.g. a building is made of rooms ($3D$-volume) separated by walls ($2D$-faces)) or to discretize an object while keeping its structure (e.g. triangulation of an object). In both cases, relations between cells have to be kept to grant the integrity of the subdivided object. Neighborhood relations between cells of same dimension (*adjacency relations*) as well as hierarchical relations between cells of different dimensions (*incidence relations*) have to be preserved.

Many combinatorial models have been designed to encode such subdivisions. Each model is adapted to a given context of application (geometric modeling, image analysis, computational geometry…) and optimized for the associated set of operations. These models mainly differ on two aspects. They can first be made of different kinds of cells: regular cells (e.g. simplices, cubical simplices, simploids), or more general cells. Their second difference lies on the way these cells are glued together. Some models are not able for instance to encode non "manifold"[3] subdivisions.

[1] An index gathering most notations and notions used in this paper and referring the pages of the corresponding definitions has been inserted at the end of the article, after the bibliographical references.

[2] A more detailed description can be found in [25].

[3] "Manifold" has to be understood here, according to the terminology often used in geometric modeling, as a generalization of the notion of

Other models forbid some cell configurations, such as an edge looping on a single vertex.

Brisson [6] suggests to classify these models into two categories: incidence-graph-like models and models containing ordering information. n-surfaces (see Sect. 2.2) and n-G-maps (see Sect. 2.3), which are the main subject of this paper, respectively belong to the first and second class of models.

Incidence-graph-like models explicitly encode the set of cells of the subdivision and the incidence relations between them. Adjacency relations are implicit and can be deduced from the incidence relations.[4] These models are able to represent cellular subdivision that does not contain multiply incident cells. But this limitation is not the main drawback of such structures. Indeed, their genericity makes it hard to define accurate consistency constraints that grant the preservation of the topological properties characterizing the subdivision. For instance, the definition of such models do not prevent a 1-cell to be incident to more than two 0-cells. Not many solutions have been proposed yet to overcome this problem. An option is to deal with subclasses of such models verifying additional properties and representing hence a more limited set of subdivisions. The set of n-surfaces is an example of such a subclass. But the difficulty is then to grant that the subclass is stable under the operations applied on it.

Models containing ordering information overcome most problems of incidence-graph-like models. They rely on an implicit representation of cells, adjacency relations and incidence relations. The basic elements encoded by these models are not the cells themselves but more elementary objects. Combinatorial maps and generalized maps belong to this class of models. Their basic elements are called darts. Informally a dart can be seen as a vertex of the subdivision "viewed" from an edge incident to this vertex, "viewed" from a face incident to this edge... The structure of the subdivision is encoded with applications (e.g. permutations or involutions) that glue the darts together. In $2D$ for instance, gluing two darts that share a vertex and an edge implies gluing two faces of the subdivision along a common edge and a common vertex. Subdivisions encoded by such models can contain multiply incident cells. Moreover consistency constraints can be easily added on the applications linking the darts to avoid incorrect configurations of cells (e.g. a 1-cell being incident to more than two 0-cells).

This article shows how to build a link between both classes of models by explicitly constructing conversion operators and showing the equivalence of two subclasses of incidence-graph-like models and models containing ordering information.

surface to any dimension. It is different from the topological notion of manifold which cannot be combinatorially characterized.

[4] Cells of a same dimension are adjacent if they share at least a common incident face.

2.2 Orders and n-Surfaces

Orders are used by Bertrand *et al.* [3] to study topological properties of images. The main advantages of this model are its genericity and its simplicity. Orders can be used to represent images of any dimension, whether they are regularly sampled or not.

Definition 1 (CF-order) An *order* is a pair $|X| = (X, \alpha)$, where X is a set and α a reflexive, antisymmetric, and transitive binary relation. We denote β the inverse of α and θ the union of α and β. CF *orders* are orders which are *countable*, i.e. X is countable, and *locally finite*, i.e. $\forall x \in X, \theta(x) = \{x' \in X,\ x' \in \alpha(x) \text{ or } x' \in \beta(x)\}$ is finite.

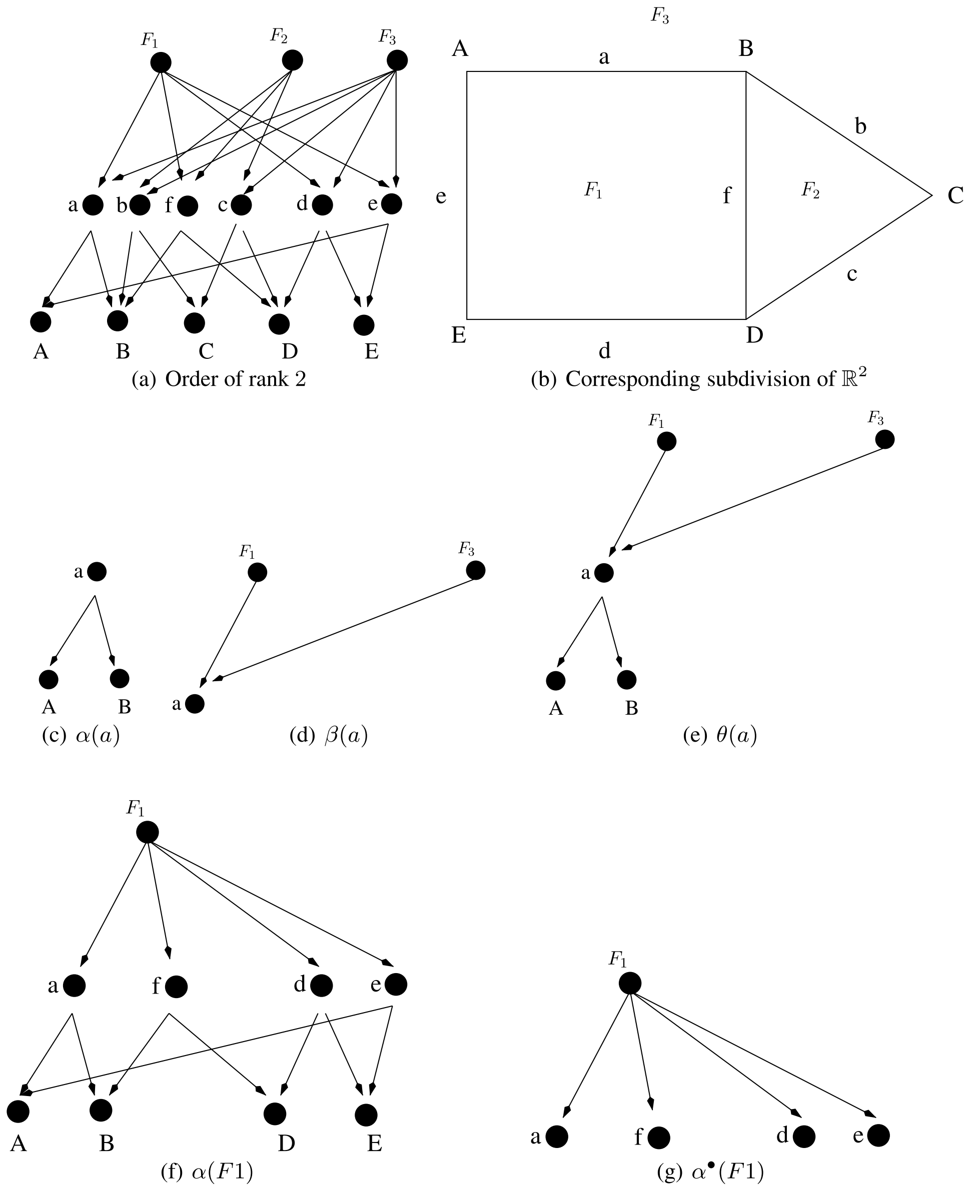
There are many ways to represent such orders. We choose here to represent them as simple directed acyclic graphs (DAG), where each node is associated with an element of the order and the transitive closure of the incidence relation between nodes is α (see Fig. 1(a)). This notion is formalized below.

The following notions are illustrated on Fig. 1. The set $\alpha(x)$ is called the α-adherence of x (see Fig. 1(a) and 1(f)). We denote respectively by $\alpha^{\square}(x)$, $\beta^{\square}(x)$ and $\theta^{\square}(x)$ the sets $\alpha(x)\backslash\{x\}$, $\beta(x)\backslash\{x\}$ and $\theta(x)\backslash\{x\}$ (see Fig. 1(e)). Moreover we call α-*closeness* of an element x the set $\alpha^{\bullet}(x) = \{y \in \alpha^{\square}(x),\ \alpha^{\square}(x) \cap \beta^{\square}(y) = \emptyset\}$. Intuitively this set contains the elements that are the "closest" to x with respect to the relation α. It is represented by the set of children of its associated node (see Fig. 1(g)). The DAG representing the order $|X|$ is simply the graph $(X, \alpha^{\bullet})$. And α can be seen as the transitive closure of $\alpha^{\bullet}$ for the set of nodes associated with X. Finally we call *path* or θ-*chain* of length n (often abbreviated as n-θ-chain), on an order, any sequence $x_0, x_1, \dots, x_n$ such that $x_{k+1} \in \theta^{\square}(x_k)$. An order is connected if it is path-connected. α-chains, β-chains, $\alpha^{\bullet}$-chains and $\beta^{\bullet}$-chains are defined in a similar way. Such a chain is said to be *maximal* if no other element of the order may be included in or added to it. There exists of course an isomorphism between the set of α-chains and the set of β-chains, and an isomorphism between the set of $\alpha^{\bullet}$-chains and the set of $\beta^{\bullet}$-chains.

An implicit dimension may be associated with each element of an order [2, 16]. It is generally called the rank of the element. The rank of an element x in an order $|X| = (X, \alpha)$, denoted by $\rho(x, |X|)$ is the length of the longest $\alpha^{\bullet}$-chain beginning at it. In the following, an element of rank k is called a k-element, and its name is often followed by the superscript k. By extension, the rank of an order is the maximal rank of its elements, $\mathtt{max}_{x \in X}(\rho(x, |X|))$, i.e. the length of the longest $\alpha^{\bullet}$-chain of the order.

If S is a subset of X, we denote by $|S| = (S, \alpha_{|S})$ the suborder of $|X|$ relative to S where $\alpha_{|S}$ is the restriction of α

Fig. 1 Representation of an order with a DAG and illustration of the notions of α-, β-, θ-adherences and α-closeness



(a) Order of rank 2 (b) Corresponding subdivision of $\mathbb{R}^2$

(c) $\alpha(a)$ (d) $\beta(a)$ (e) $\theta(a)$

(f) $\alpha(F1)$ (g) $\alpha^{\bullet}(F1)$

on the elements of S, i.e. $\alpha_{|S} = \alpha \cap (S \times S)$. In the following, we will often study suborders built on the $\theta^{\square}$-adherence of some element x. We note that the rank of any element in such a suborder may be easily deduced from the rank of the same element in the original order. Actually if the element belongs to the $\alpha^{\square}$-adherence of x then its rank remains the same. But if it belongs to the $\beta^{\square}$-adherence of x then its rank is equal to its rank in the original order minus one.

We introduce below two particular kinds of orders defined in [10], namely pure orders and closed orders. An order which is both pure and closed is depicted on Fig. 1(a). Three other orders which lacks at least one of these properties are displayed on Fig. 2.

Informally, a pure order is such that any of its elements belongs to the α-adherence of at least one element of rank n.

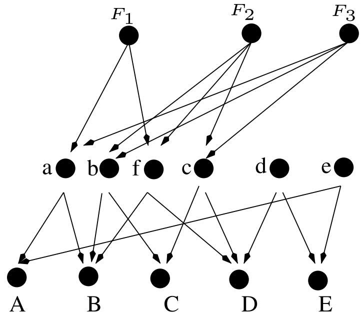
Definition 2 (Pure order) Let $|X|$ be an order of rank n, $|X|$ is said to be *pure* if each of its elements belongs to an n-$\beta^{\bullet}$-chain.

A closed order is such that between an element of rank k and an element of rank l which are related by α there are at least $l-k-1$ elements of rank $k+1 \dots l-1$. Informally, there is no "dimensional gap" between two elements related by α.

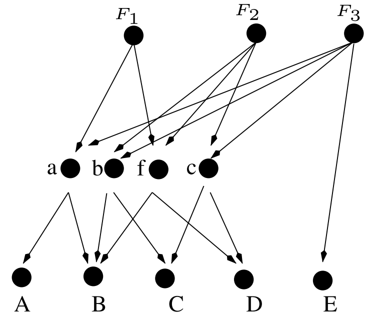
Definition 3 (Closed order) Let $|X| = (X, \alpha)$ be an order, $|X|$ is said to be *closed* if for any $x \in X$ and $y \in \alpha^{\square}(x)$:

$$\forall i \in]\rho(y, |X|), \rho(x, |X|)[, \ \exists z \in \alpha^{\square}(x) \cap \beta^{\square}(y),$$
$$\rho(z, |X|) = i.$$

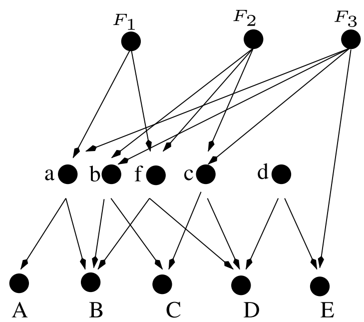
Fig. 2 Counter-examples of purity and closeness properties



(a) Order of rank 2 which is closed but not pure: E, e and d do not belong to an n-$\alpha^\bullet$-chain.



(b) Order of rank 2 which is pure but not closed: there is no element of rank 1 between E and F_3.



(c) Order of rank 2 which is neither closed nor pure: there is no element of rank 1 between E and F_3 and d does not belong to an n-$\alpha^\bullet$-chain.

The closure property may equivalently be expressed as a property on maximal chains of the order.

Proposition 4 *Let $|X| = (X, \alpha)$ be an order, $|X|$ is* closed *if and only if for any $x \in X$ whose rank is equal to $k \geq 0$, x has position $k + 1$ in any maximal $\beta^\bullet$-chain of $|X|$ containing it.*

From here we focus on a subclass of orders first defined by Evako [16] and brought to image analysis by Bertrand *et al.* [4] which is close to the notion of manifold proposed by Kovalevsky [19]. Such objects are called n-surfaces and are recursively defined. They form a subclass of the well-known closed pseudo-manifolds (Theorem 17 of [13]).

Definition 5 (n-surface) Let $|X| = (X, \alpha)$ be a non-empty CF-order.

- The order $|X|$ is a 0-*surface* if X is composed exactly of two elements x and y such that $y \notin \alpha(x)$ and $x \notin \alpha(y)$.
- The order $|X|$ is an n-*surface*, $n > 0$, if $|X|$ is connected and if, for each $x \in X$, the order $|\theta^\square(x)|$ is an $(n-1)$-surface.

The order depicted on Fig. 1(a) is a 2-surface. The recursion relation is illustrated on Fig. 3.

For convenience we consider from here that any order $|X| = (X, \alpha)$ of rank n has two additional virtual elements x^{-1} of rank -1 and x^{n+1} of rank $(n+1)$, such that $\beta^\square_X(x^{-1}) = X \cup \{x^{n+1}\}$ and $\alpha^\square_X(x^{n+1}) = X \cup \{x^{-1}\}$ (see Fig. 4). They are useful to avoid side effects problems in the statement of some definitions and properties (e.g. see Def 15). But they have no other function. For instance, none of them are taken into account when studying the connectedness of an order. This trick is similar to the one used by Brisson in [6] to deal with n-dimensional augmented incidence graphs.

2.3 n-G-Maps

The n-dimensional generalized maps or n-G-maps defined by Lienhardt [21, 22] are used to represent the topology of subdivisions of topological spaces. Similarly to orders they are general enough to deal with spaces of any dimension. However they can only represent quasi-manifolds (see Sect. 2.4 and [22]), orientable or not, with or without boundary. We may note here that as n-surfaces, quasi-manifolds are a subset of pseudo-manifolds (see §2.1.5 of [22]). The n-G-map combinatorial structure is particularly adapted to geometric modeling since many basic and advanced modeling operators have been designed for it: merge, split, extru-

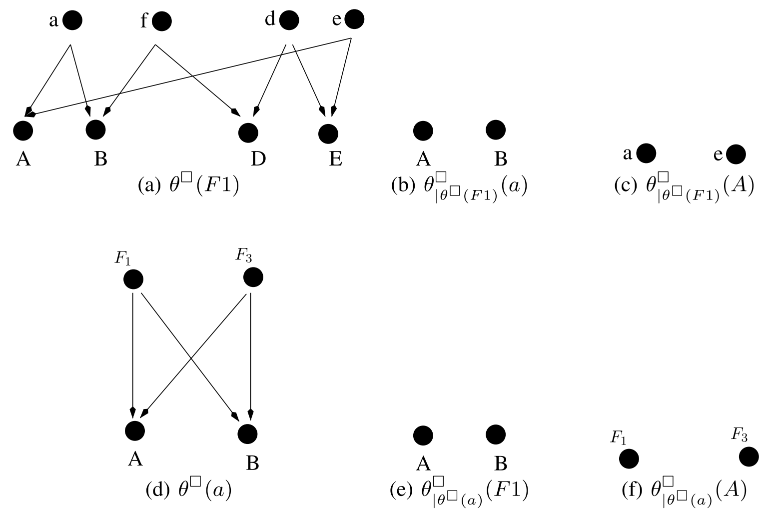


Fig. 3 The $\theta^{\square}$-adherence of any element of a 2-surface must be a 1-surface. For instance, $|\theta^{\square}(F1)|$ and $|\theta^{\square}(a)|$ are 1-surfaces, because each element of them has a $\theta^{\square}$-adherence made of two disconnected elements

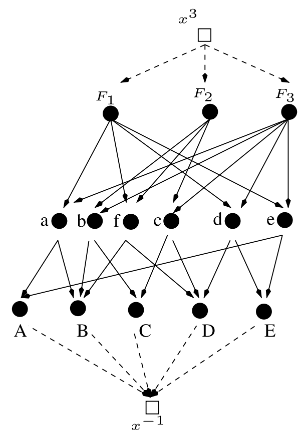


Fig. 4 An order of rank 2 represented with its two fictive elements x^{-1} and x^3

sion, chamfering, Cartesian product... (see for instance [8, 9, 20, 23]).

We begin with the general definition of n-G-maps.

Definition 6 (n-G-map) Let $n \geq 0$, an *n-G-map* is an $(n+2)$-tuple $G = (D, \alpha_0, \dots, \alpha_n)$ such that:

- D is a finite set of darts
- α_i, $i \in \{0, \dots, n\}$ are permutations on D such that:
  - $\forall i \in \{0, \dots, n\}$, α_i is an involution.[5]
    If $D = \emptyset$, α_i is not defined.
  - $\forall i,\ j$ such that $0 \leq i < i + 2 \leq j \leq n$, $\alpha_i \alpha_j$ is an involution.

An example of a 2-G-map is given in Fig. 5(a). This map is made of 24 darts and α_0, α_1, α_2 are defined as follows:

$$\begin{aligned}\alpha_0 = \{&(1,\ 2),\ (3,\ 4),\ (5,\ 6),\ (7,\ 8),\ (9,\ 10),\ (11,\ 12),\\ &(13,\ 14),\ (15,\ 16),\ (17,\ 18),\ (19,\ 20),\ (21,\ 22),\\ &(23,\ 24)\},\\ \alpha_1 = \{&(1,\ 8),\ (2,\ 3),\ (4,\ 5),\ (6,\ 7),\ (9,\ 14),\ (10,\ 11),\\ &(12,\ 13),\ (15,\ 24),\ (16,\ 17),\ (18,\ 19),\ (20,\ 21),\\ &(22,\ 23)\},\\ \alpha_2 = \{&(1,\ 15),\ (2,\ 16),\ (3,\ 9),\ (4,\ 10),\ (5,\ 21),\ (6,\ 22),\\ &(7,\ 23),\ (8,\ 24),\ (11,\ 20),\ (12,\ 19),\ (13,\ 18),\\ &(14,\ 17)\}.\end{aligned}$$

An n-G-map represents a subdivision of a topological space, otherwise said a set of i-cells, $i \in \{0, \dots, n\}$. Each i-cell of this subdivision is represented by a particular subset of D. To express this formally, we recall now the notion of orbit.

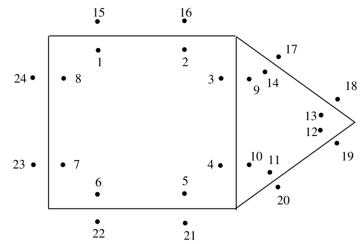
Definition 7 (Orbit) Let $\Phi = \{\pi_0, \dots, \pi_n\}$ be a set of permutations over a set of elements D. Let $\langle \Phi \rangle = \langle \pi_0, \dots, \pi_n \rangle$ be the group of permutations of D generated by Φ. The *orbit* of an element d of D related to the group $\langle \Phi \rangle$, denoted by $\langle \Phi \rangle(d)$ is the set $\{\phi(d), \phi \in \langle \Phi \rangle\}$.

Informally the orbit of an element d related to a group of permutations is the set of elements which are images of d by a composition of permutations that belong to Φ. When no confusion may arise $\langle \pi_0, \dots, \pi_n \rangle(d)$ is denoted by $\langle \pi \rangle_N(d)$ with $N = \{0, \dots, n\}$. More generally, given a list of permutations, the orbit related to a subgroup of these permutations is denoted by $\langle \pi \rangle$ indexed by the set of indices of the involved permutations.

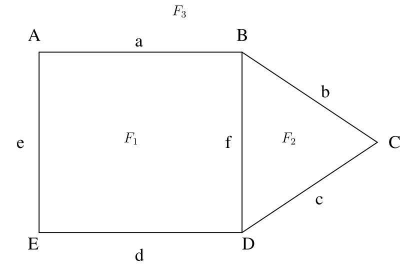
The i-cells of the subdivision represented by an n-G-map are hence determined by the orbits of the darts of D. Exam-

[5] A permutation π on the domain set D is an involution if and only if $\pi \circ \pi$ is the identity map on D. In the following, we use the notation $d\alpha_i\alpha_j$ for $\alpha_j \circ \alpha_i(d)$.

Fig. 5 A 2-G-map together with the subdivision it represents. An explicit description of the cells of the subdivision based on the orbits of the 2-G-map is depicted on Fig. 6

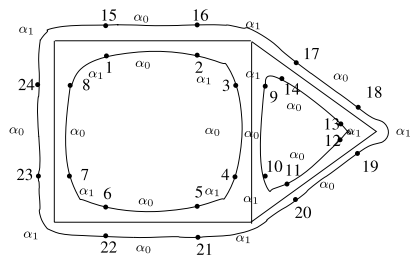


(a) 2-G-map, $G = (D, \alpha_0, \alpha_1, \alpha_2)$, with $D = \{1, \cdots, 24\}$

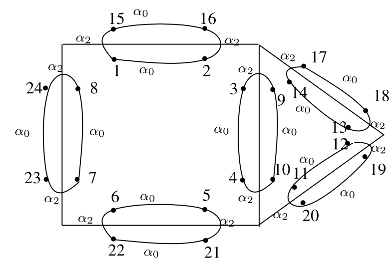


(b) Corresponding subdivision of $\mathbb{R}^2$

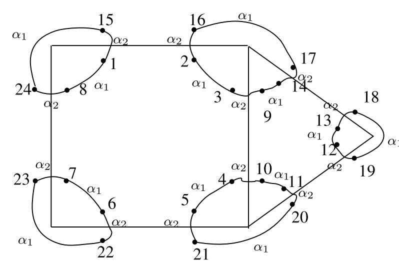
Fig. 6 Illustration of the link between orbits of a 2-G-map and the cells of the corresponding subdivision (depicted on Fig. 5(b))



(a) $<\alpha_0, \alpha_1>$ orbits = 2-cells on a 2-G-map
$<\alpha_0, \alpha_1>$ $(15) \Leftrightarrow F_3$
$<\alpha_0, \alpha_1>$ $(9) \Leftrightarrow F_2$
$<\alpha_0, \alpha_1>$ $(1) \Leftrightarrow F_1$



(b) $<\alpha_0, \alpha_2>$ orbits = 1-cells on a 2-G-map
$<\alpha_0, \alpha_2>$ $(1) \Leftrightarrow a$, $<\alpha_0, \alpha_2>$ $(13) \Leftrightarrow b$
$<\alpha_0, \alpha_2>$ $(11) \Leftrightarrow c$, $<\alpha_0, \alpha_2>$ $(5) \Leftrightarrow d$
$<\alpha_0, \alpha_2>$ $(7) \Leftrightarrow e$, $<\alpha_0, \alpha_2>$ $(3) \Leftrightarrow f$



(c) $<\alpha_1, \alpha_2>$ orbits = 0-cells on a 2-G-map
$<\alpha_1, \alpha_2>$ $(1) \Leftrightarrow A$, $<\alpha_1, \alpha_2>$ $(2) \Leftrightarrow B$
$<\alpha_1, \alpha_2>$ $(12) \Leftrightarrow C$, $<\alpha_1, \alpha_2>$ $(4) \Leftrightarrow D$
$<\alpha_1, \alpha_2>$ $(6) \Leftrightarrow E$

ples of orbits and of the cells they represent are displayed on Figs. 5(b), 6(a) and 6(b).

Definition 8 (i-cell) Let $G = (D, \alpha_0, \ldots, \alpha_n)$ be an n-G-map. Each i-cell of the associated subdivision corresponds to a connected component of the $(n-1)$-generalized map: $(D, \alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$.[6]

The set of i-cells is hence a partition of the darts of the n-G-map, for each i between 0 and n. Each i-cell is associated to an orbit $\langle\alpha\rangle_{N-\{i\}}(d)$, where d is a dart of G. All darts belonging to such an orbit are said to be incident to the corresponding i-cell. The incidence relations between the cells of the subdivision can be easily computed. A cell $c^j = \langle\alpha\rangle_{N-\{j\}}(d)$ is a face of a cell $c^i = \langle\alpha\rangle_{N-\{i\}}(d)$ if $j \leq i$ and $\langle\alpha\rangle_{N-\{j\}}(d) \cap \langle\alpha\rangle_{N-\{i\}}(d) \neq \emptyset$ [21]. Two such cells are said to be consecutive if $i = j + 1$. Therefore an incidence graph or incidence multigraph can easily be associated with any n-G-map.

Most classical properties that may be attached to subdivisions have an interpretation in terms of involutions and orbits. We give below the expression of two of them.

i. *connectedness*: $\forall d \in D,\ \langle\alpha\rangle_N(d) = D$
ii. *closeness or without boundaries*: $\forall i \in N = \{0, \ldots, n\}, \alpha_i$ is without fixed point[7]

[6] See [22] for more details.

[7] α_i is without fixed point if $\forall d \in D, \alpha_i(d) \neq d$.

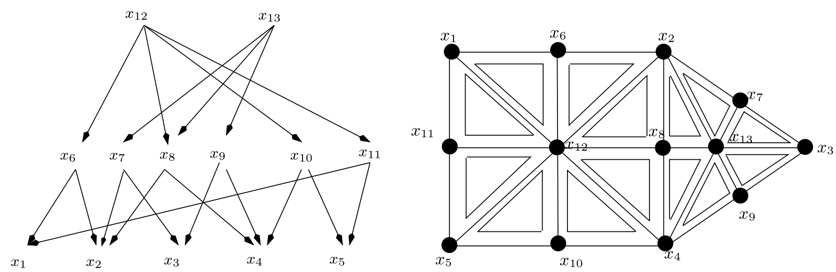


Fig. 7 Example of an order (which is a suborder of Fig. 1(a)) and its associated abstract simplicial complex

A n-G-map fulfilling both properties is called a closed connected n-G-map.

Contrary to orders, n-G-maps may represent unambiguously subdivisions with multi-incidence, that is subdivisions containing cells multiply incident to some other cell. Orders fail to characterize such subdivisions because they cannot provide information on how the different cells are glued together unless there exists explicit cells between them. A simple example is the torus made from a single square with opposite borders glued together. We propose and justify in Sect. 4 a characterization of n-G-maps that represent only subdivisions without multi-incidence.

2.4 Simplicial Interpretation

Traditionally both orders and generalized maps have been associated with a particular kind of abstract simplicial set.

The simplicial set usually built on orders is an *abstract simplicial complex*: it is a set V of vertices and a family Δ of finite subsets of V, called simplices and such that $\emptyset \neq \sigma \subseteq \tau \in \Delta \Rightarrow \sigma \in \Delta$. The dimension of a simplex is equal to the number of its elements less one.

More precisely the *order complex* (see [5]) $\Delta(|X|)$ of an order $|X|$ is the abstract simplicial complex induced by the chain order associated with $|X|$. The vertices of this complex are exactly the elements of X, and the k-simplices are the k-α-chains on $|X|$. The incidence relations between the simplices correspond to the inclusion relations between chains, as shown in Fig. 7.

As orders, each n-G-map may be associated to a semi-simplicial set [24], where each dart is represented by an n-dimensional simplex. More precisely, the numbered semi-simplicial set associated to an n-G-map is defined as follows. Each 0-simplex numbered by i corresponds to an orbit $\langle\alpha\rangle_{N-\{i\}}(d)$. Each k-simplex represents an orbit $\langle\alpha\rangle_{N-I}(d)$ where I is a subset of N made of $k+1$ elements. And an i-simplex, s^i, is a face of a j-simplex, s^j, $j>i$, if there exists a dart d such that s^i and s^j are respectively associated to $\langle\alpha\rangle_{N-I}(d)$ and $\langle\alpha\rangle_{N-J}(d)$ with $I \subset J$ (see Fig. 8). The set of n-G-maps is precisely equivalent to the set of numbered simplicial quasi-manifolds (see [21]). Quasi-manifolds are

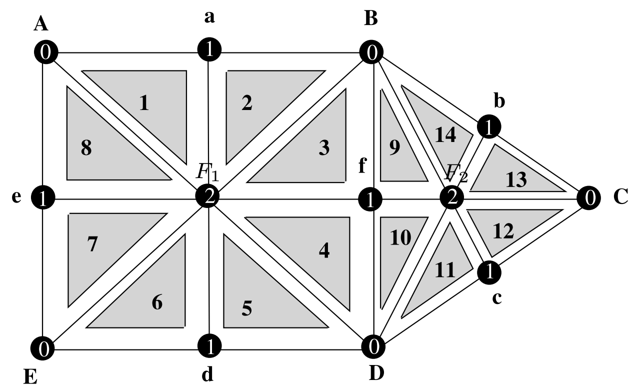


Fig. 8 Numbered simplicial set associated to the submap $G'=(D', \alpha_{0_{|D'}}, \alpha_{1_{|D'}}, \alpha_{2_{|D'}})$ with $D'=\{1,\dots,14\}$ of the 2-G-map displayed on Fig. 5.
Vertices: $\langle\alpha\rangle_{N-\{2\}}(9) \Leftrightarrow F_2$, $\langle\alpha\rangle_{N-\{2\}}(1) \Leftrightarrow F_1$, $\langle\alpha\rangle_{N-\{1\}}(1) \Leftrightarrow a$, $\langle\alpha\rangle_{N-\{1\}}(13) \Leftrightarrow b$, $\langle\alpha\rangle_{N-\{1\}}(11) \Leftrightarrow c$, $\langle\alpha\rangle_{N-\{1\}}(5) \Leftrightarrow d$, $\langle\alpha\rangle_{N-\{1\}}(7) \Leftrightarrow e$, $\langle\alpha\rangle_{N-\{1\}}(3) \Leftrightarrow f$, $\langle\alpha\rangle_{N-\{0\}}(1) \Leftrightarrow A$, $\langle\alpha\rangle_{N-\{0\}}(2) \Leftrightarrow B$, $\langle\alpha\rangle_{N-\{0\}}(12) \Leftrightarrow C$, $\langle\alpha\rangle_{N-\{0\}}(4) \Leftrightarrow D$, $\langle\alpha\rangle_{N-\{0\}}(6) \Leftrightarrow E$.
Edges: $\langle\alpha\rangle_{N-\{0,1\}}(1) \Leftrightarrow Aa$, $\langle\alpha\rangle_{N-\{0,1\}}(2) \Leftrightarrow Ba$, $\langle\alpha\rangle_{N-\{0,1\}}(3) \Leftrightarrow Bf$, $\langle\alpha\rangle_{N-\{0,1\}}(4) \Leftrightarrow Df$, $\langle\alpha\rangle_{N-\{0,1\}}(5) \Leftrightarrow Dd$, $\langle\alpha\rangle_{N-\{0,1\}}(6) \Leftrightarrow Ed$, $\langle\alpha\rangle_{N-\{0,1\}}(7) \Leftrightarrow Ee$, $\langle\alpha\rangle_{N-\{0,1\}}(8) \Leftrightarrow Ae$, $\langle\alpha\rangle_{N-\{0,1\}}(11) \Leftrightarrow Dc$, $\langle\alpha\rangle_{N-\{0,1\}}(12) \Leftrightarrow Cc$, $\langle\alpha\rangle_{N-\{0,1\}}(13) \Leftrightarrow Cb$, $\langle\alpha\rangle_{N-\{0,1\}}(14) \Leftrightarrow Bb$, $\langle\alpha\rangle_{N-\{0,2\}}(2) \Leftrightarrow F_1B$, $\langle\alpha\rangle_{N-\{0,2\}}(4) \Leftrightarrow F_1D$, $\langle\alpha\rangle_{N-\{0,2\}}(6) \Leftrightarrow F_1E$, $\langle\alpha\rangle_{N-\{0,2\}}(8) \Leftrightarrow F_1A$, $\langle\alpha\rangle_{N-\{0,2\}}(10) \Leftrightarrow F_2D$, $\langle\alpha\rangle_{N-\{0,2\}}(12) \Leftrightarrow F_2C$, $\langle\alpha\rangle_{N-\{0,2\}}(14) \Leftrightarrow F_2B$, $\langle\alpha\rangle_{N-\{1,2\}}(2) \Leftrightarrow F_1a$, $\langle\alpha\rangle_{N-\{1,2\}}(4) \Leftrightarrow F_1f$, $\langle\alpha\rangle_{N-\{1,2\}}(6) \Leftrightarrow F_1d$, $\langle\alpha\rangle_{N-\{1,2\}}(8) \Leftrightarrow F_1e$, $\langle\alpha\rangle_{N-\{1,2\}}(10) \Leftrightarrow F_2f$, $\langle\alpha\rangle_{N-\{1,2\}}(12) \Leftrightarrow F_2c$, $\langle\alpha\rangle_{N-\{1,2\}}(14) \Leftrightarrow F_2b$.
Faces: $\langle\alpha\rangle_{N-\{0,1,2\}}(1) \Leftrightarrow F_1aA$, $\langle\alpha\rangle_{N-\{0,1,2\}}(2) \Leftrightarrow F_1aB$, $\langle\alpha\rangle_{N-\{0,1,2\}}(3) \Leftrightarrow F_1fB$, $\langle\alpha\rangle_{N-\{0,1,2\}}(4) \Leftrightarrow F_1fD$, $\langle\alpha\rangle_{N-\{0,1,2\}}(5) \Leftrightarrow F_1dD$, $\langle\alpha\rangle_{N-\{0,1,2\}}(6) \Leftrightarrow F_1dE$, $\langle\alpha\rangle_{N-\{0,1,2\}}(7) \Leftrightarrow F_1eE$, $\langle\alpha\rangle_{N-\{0,1,2\}}(8) \Leftrightarrow F_1eA$, $\langle\alpha\rangle_{N-\{0,1,2\}}(9) \Leftrightarrow F_2fB$, $\langle\alpha\rangle_{N-\{0,1,2\}}(10) \Leftrightarrow F_2fD$, $\langle\alpha\rangle_{N-\{0,1,2\}}(11) \Leftrightarrow F_2cD$, $\langle\alpha\rangle_{N-\{0,1,2\}}(12) \Leftrightarrow F_2cC$, $\langle\alpha\rangle_{N-\{0,1,2\}}(13) \Leftrightarrow F_2bC$, $\langle\alpha\rangle_{N-\{0,1,2\}}(14) \Leftrightarrow F_2bB$

a special kind of numbered simplicial sets that can be constructively defined as a set of n-dimensional cells, glued together along their $(n-1)$-faces such that each $(n-1)$-cell is incident to at most two n-cells.[8]

Semi-simplicial sets are a more general structure than ab-

[8]Note that this notion of quasi-manifold is different from the definition proposed by Kovalevsky in [18].

stract simplical complexes because several simplices may share exactly the same faces or be multiply incident. When such a configuration occurs, the associated cellular subdivision contains multi-incident cells. Examples of subdivisions containing multi-incidence are depicted on Figs. 14, 15, and Fig. 18(a). For instance, the 2-G-map depicted on Fig. 14(a) encodes the minimal subdivision of a sphere (see Fig. 14(b)). This subdivision is made of one vertex, one edge and one face, where the vertex is multiply incident to the edge and the edge multiply-incident to the face. The corresponding 2-G-map has 2 darts. The associated semi-simplicial set has hence two 2-simplices (see Fig. 14(c)). Moreover both darts belong to the same orbits: $\langle\alpha\rangle_{N-\{0,1\}}$, $\langle\alpha\rangle_{N-\{0,2\}}$, $\langle\alpha\rangle_{N-\{1,2\}}$,$\langle\alpha\rangle_{N-\{0\}}$, $\langle\alpha\rangle_{N-\{1\}}$ and $\langle\alpha\rangle_{N-\{2\}}$. The two associated 2-simplices share hence three edges and three vertices, which means, as we deal with simplices, that both simplices share exactly the same faces.

A subdivision without multi-incidence is displayed on Fig. 5(b). An n-G-map associated to such a subdivision is said to be *without multi-incidence*. This notion will be formally defined and characterized in Sect. 4.

3 Characterizations of n-Surfaces

3.1 Static Characterization of n-Surfaces

This section contains the proof of the following theorem which may be understood as a static characterization of n-surfaces. The point 9 of Theorem 9 is illustrated on Fig. 9.

Theorem 9 *An order $|X| = (X, \alpha)$ of rank n is an n-surface if and only if*:

i. *$|X|$ is pure and closed*,
ii. *The intersection of the $\theta^{\square}$-adherences of the elements of every β-chain $\mathcal{C} \cup \{x^{-1}\} \cup \{x^{n+1}\}$ of $|X|$ (with $\mathcal{C}$ eventually empty)*[9] *is*
   (a) *empty if* $\mathtt{card}(\mathcal{C}) = n + 1$,[10]
   (b) *made of two elements having the same rank if* $\mathtt{card}(\mathcal{C}) = n$,
   (c) *connected if* $\mathtt{card}(\mathcal{C}) \leq n - 1$.
   We note that case iic may occur only if $n \geq 1$. From here, Properties i, iia, iib *and* iic *are called* static surface properties.

We just note here that the point 9 guarantees that whenever the rank of an order fulfilling static surface properties is strictly greater than 0 then it is connected. Similarly point 9 implies that an order of rank 0 having static surface properties is made of two distinct elements of rank 0 and is hence a 0-surface.

The proof essentially relies on two lemmas: Lemma 13 and Lemma 14. Lemmas 11 and 12 are used to prove Lemma 13 whereas Lemma 10 is used to prove Lemma 11.

We first give a useful characterization of pure and closed orders.

Lemma 10 *$|X| = (X, \alpha)$ is a pure and closed order of rank n if and only if any maximal $\beta^{\bullet}$-chain of $|X|$ has length n.*

Proof Let $|X|$ be an order of rank n.

$\Rightarrow$ if $|X|$ is pure and closed, then any element of X belongs to at least one n-$\beta^{\bullet}$-chain (purity), otherwise said to at least one maximal $\beta^{\bullet}$-chain. Moreover let $\mathcal{C}$ be a maximal $\beta^{\bullet}$-chain of $|X|$. Let x^k be the greatest element of $\mathcal{C}$ according to $\beta^{\bullet}$. As the order is closed, x^k has position $k + 1$ in the chain, otherwise said $\mathcal{C}$ has length k. If k is strictly less than n, then the purity of the order would imply that $\mathcal{C}$ is not maximal. $\mathcal{C}$ has hence length n.

$\Leftarrow$ By definition of maximal chains, any element of X belongs to at least one maximal $\beta^{\bullet}$-chain. If $|X|$ is such that

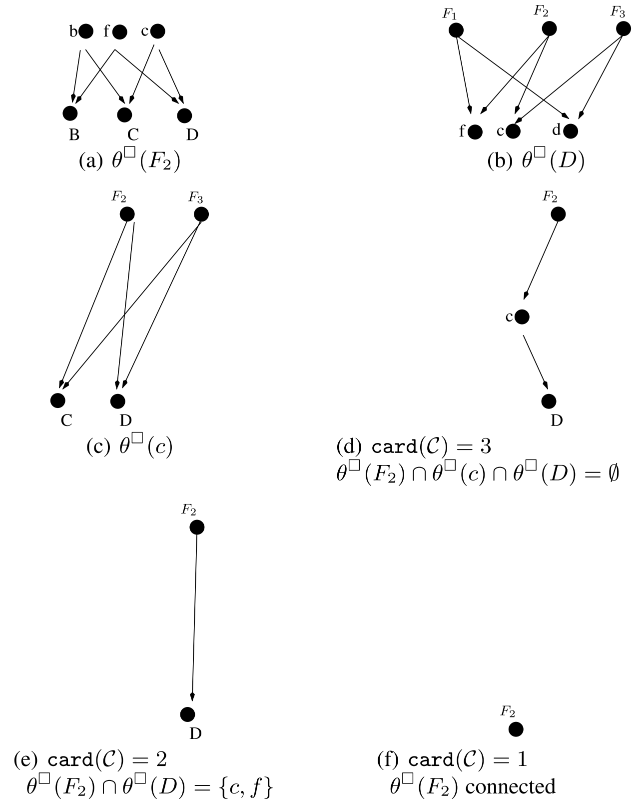
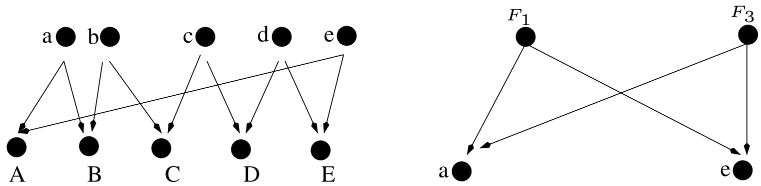


Fig. 9 The static surface properties of the 2-surface shown on Fig. 1(a). The $\theta^{\square}$-adherences of three elements, F_2,c and D are depicted on (**a**), (**b**), (**c**). Three β-chains containing some of these elements are displayed on (**d**), (**e**), and (**f**)

[9] As $\theta^{\square}(x^{-1}) \cap \theta^{\square}(x^{n+1})$ is, by definition, the set X, adding x^{-1} and x^{n+1} to any β-chain $\mathcal{C}$ does not modify the intersection if $\mathcal{C}$ is not empty and makes the intersection equal to X if $\mathtt{card}(\mathcal{C})$ is equal to 0, i.e. if the chain is reduced to $\{x^{-1}, x^{n+1}\}$.

[10] iia is actually true for every order.



(a) $\theta^{\square}_{|X}(F_3)$ (b) $\theta^{\square}_{|X}(A)$

Fig. 10 $|X|$ is the order depicted on Fig. 1(a). $\theta^{\square}_{|X}(F_3) \cap \theta^{\square}_{|X}(A) = \theta^{\square}_{|\theta^{\square}_{|X}(F_3)}(A) = \theta^{\square}_{|\theta^{\square}_{|X}(A)}(F_3) = \{a, e\}$

any maximal $\beta^{\bullet}$-chain has length n, then $|X|$ is pure. Let us show that it is also closed. Let x be an element of X of rank k. Assume that x belongs to at least two maximal $\beta^{\bullet}$-chains $\mathcal{C}$ and $\mathcal{C}'$ such that x has position l and m respectively in $\mathcal{C}$ and $\mathcal{C}'$. Let us assume without loss of generality that $l \leq m$. Then there exists a $\beta^{\bullet}$-chain containing the m elements of $\mathcal{C}'$ strictly lower than x (according to $\beta^{\bullet}$), x and the $n - l$ elements of $\mathcal{C}$ strictly greater than x (according to $\beta^{\bullet}$). Such a chain has length $n + m - l$. As no $\beta^{\bullet}$-chain has length strictly greater than n then $m = l$. Moreover by definition of the rank, there exists at least one $\beta^{\bullet}$-chain of $|X|$ of length k ending at x. From it, it is possible to build a maximal $\beta^{\bullet}$-chain where x has position $k + 1$. x has hence position $k + 1$ in any maximal $\beta^{\bullet}$-chain of $|X|$ containing it. □

We state below two properties related to such orders that are straightforward consequences of the characterization of pure and closed orders. The second property is simply a generalization of the first property.

The first lemma is illustrated on Fig. 10. Informally it simply means that an element of an order which is in the neighborhood of two others, is also a neighbor of each of them in the subgraph built on the neighborhood of the other. It also says that closure and purity properties are passed from an order to any suborder built on some neighborhood.

These properties are used to prove Lemma 13 and Lemma 16.

Lemma 11 *Let $|X| = (X, \alpha)$ be a pure and closed order of rank n and x be an element of X then $(\theta^{\square}(x), \alpha_{|\theta^{\square}(x)})$ is a pure and closed order of rank $n - 1$ and*:

$$\forall x' \in \theta^{\square}(x), \quad \theta^{\square}_{|\theta^{\square}(x)}(x') = \theta^{\square}(x') \cap \theta^{\square}(x).$$

Proof By definition of suborders, $\theta^{\square}_{|\theta^{\square}(x)} = \theta^{\square}_{|X} \cap (\theta^{\square}(x) \times \theta^{\square}(x))$, where $\theta^{\square}(x)$ is a simplified notation for $\theta^{\square}_{|X}(x)$. Let x' be any element of $\theta^{\square}(x)$. Then:

$$\begin{aligned}
&\theta^{\square}_{|\theta^{\square}(x)}(x') \\
&\quad = \{y \in \theta^{\square}(x), (x', y) \in \theta^{\square}_{|X} \cap \theta^{\square}(x) \times \theta^{\square}(x)\} \\
&\quad \overset{x' \in \theta^{\square}(x)}{=} \{y \in \theta^{\square}(x), (x', y) \in \theta^{\square}_{|X}\} \\
&\quad \overset{\theta^{\square}(x) \subseteq X}{=} \{y \in X, (x', y) \in \theta^{\square}_{|X}\} \cap \theta^{\square}(x) \\
&\quad = \theta^{\square}(x') \cap \theta^{\square}(x).
\end{aligned}$$

Let us now prove that $|\theta^{\square}(x)| = (\theta^{\square}(x), \alpha_{|\theta^{\square}(x)})$ is pure and closed. By definition of θ, the suborder $|\theta^{\square}(x)|$ has rank $n - 1$. By Lemma 10, it is necessary and sufficient to prove that any maximal $\beta^{\bullet}_{|\theta^{\square}(x)|}$-chain of $|\theta^{\square}(x)|$ has length $(n - 1)$.

Let $\mathcal{C}$ be a maximal $\beta^{\bullet}_{|\theta^{\square}(x)|}$-chain of $|\theta^{\square}(x)|$. The length of $\mathcal{C}$ is at most $n - 1$. By definition of $\theta^{\square}(x)$, $\mathcal{C} \cup \{x\}$ is a $\beta^{\bullet}$-chain of $|X|$. Let us suppose that the length of $\mathcal{C}$ is strictly less than $n - 1$. Then $\mathcal{C} \cup \{x\}$ has a length strictly less than n and hence is not a maximal $\beta^{\bullet}$-chain of $|X|$. Let us complete this chain with elements $y_1, \ldots, y_k$ of X to obtain a maximal $\beta^{\bullet}$-chain. By definition of $\theta^{\square}(x)$, $y_1 \ldots y_k$ also belong to $\theta^{\square}(x)$. Moreover $\mathcal{C} \cup \{y_1\} \cup \ldots \cup \{y_k\}$ is a $\beta^{\bullet}_{|\theta^{\square}(x)|}$-chain of $|\theta^{\square}(x)|$. It contradicts the assumption that $\mathcal{C}$ is a maximal $\beta^{\bullet}_{|\theta^{\square}(x)|}$-chain. Hence, $\mathcal{C}$ has length $n - 1$. □

Lemma 12 *Let $|X| = (X, \alpha)$ be a pure and closed order of rank n. Let $\{x^{i_j}\}_{j \in \{0, \ldots, p\}}$ be an α-chain of $|X|$, then α induces a relation $\alpha_{|X'}$ on $X' = \bigcap_{j \in \{0, \ldots, p\}} \theta^{\square}(x^{i_j})$, such that $(X', \alpha_{|X'})$ is a pure and closed order of rank $(n - (p + 1))$ and*:

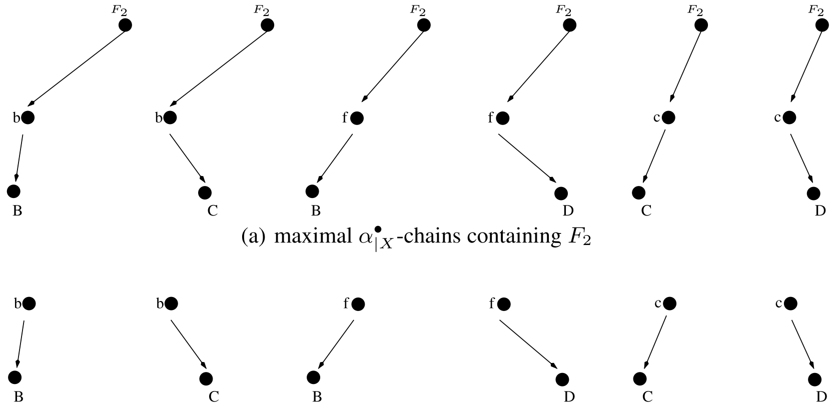
$$\forall x \in X', \quad \theta^{\square}_{|X'}(x) = X' \cap \theta^{\square}(x).$$

Proof This proof can be achieved by induction on p. The case $p = 0$ comes from Lemma 11. The proof of the validity of the induction hypothesis is very similar to the proof for the case $p = 0$. It is left to the reader. □

We need two more lemmas to effectively show that an order verifying all static surface properties may be recursively defined in the same way as n-surfaces.

The first lemma expresses that the "neighbourhood" of any cell of an order with surface properties also has the same properties. Informally this means that the neighborhood of a cell of a surface is also a surface.

Lemma 13 *Let $|X|$ be an order of rank $n \geq 1$ fulfilling the static surface properties then $\forall x \in X, \theta^{\square}(x)$ is an order of rank $n - 1$ that fulfills the static surface properties too.*



(a) maximal $\alpha^\bullet_{|X}$-chains containing F_2

(b) maximal $\alpha^\bullet_{|\theta^\Box(F_2)}$-chains

Fig. 11 Illustration of the isomorphism between the maximal $\alpha^\bullet_{|X}$-chains of $|X|$ (depicted on Fig. 1(a)) containing some element x (here F_2) and the maximal $\alpha^\bullet_{|\theta^\Box(x)}$-chains of $|\theta^\Box(x)|$ (here $|\theta^\Box(F_2)|$ depicted on Fig. 9(a))

Proof Let x^i be an element of X of rank i, $i \in \{0,\dots,n\}$. We consider $(\theta^\Box(x^i), \alpha_{|\theta^\Box(x^i)})$ which is an order of rank $(n-1)$ (see Lemma 11). We must prove that each surface property holds for $|\theta^\Box(x^i)|$. Property 9 is guaranteed by Lemma 11. Now, let $\mathcal{C}_i$ be some $\beta_{|\theta^\Box(x^i)}$-chain of $|\theta^\Box(x^i)|$. To prove 9, we need to study the properties of the set $\bigcap_{x\in\mathcal{C}_i\cup\{x^{-1}\}\cup\{x^{n+1}\}} \theta^\Box_{|\theta^\Box(x^i)}(x)$ according to the cardinal of $\mathcal{C}_i$.

By definition of $\theta^\Box$, $\mathcal{C}_i \cup \{x^i\}$ is a $\beta_{|X}$-chain of $|X|$. Moreover, by Lemma 11 and classical properties of intersection, we have:

$$\bigcap_{x\in\mathcal{C}_i\cup\{x^i\}\cup\{x^{-1}\}\cup\{x^{n+1}\}} \theta^\Box_{|X}(x)$$

$$= \bigcap_{x\in\mathcal{C}_i\cup\{x^{-1}\cup\{x^{n+1}\}\}} (\theta^\Box_{|X}(x)) \cap \theta^\Box_{|X}(x^i)$$

$$\overset{\text{Lemma 11}}{=} \bigcap_{x\in\mathcal{C}_i\cup\{x^{-1}\cup\{x^{n+1}\}\}} \theta^\Box_{|\theta^\Box_{|X}(x^i)}(x).$$

And as $|X|$ fulfills the static surface properties, we know the properties of $\bigcap_{x\in\mathcal{C}_i\cup\{x^i\}\cup\{x^{-1}\}\cup\{x^{n+1}\}}\theta^\Box_{|X}(x)$ according to the cardinal of $\mathcal{C}_i \cup \{x^i\}$, i.e. $card(\mathcal{C}_i)+1$. We can therefore deduce that $\bigcap_{x\in\mathcal{C}_i\cup\{x^{-1}\}\cup\{x^{n+1}\}}\theta^\Box_{|\theta^\Box_{|X}(x^i)}(x)$ is:

- empty if $\mathtt{card}(\mathcal{C}_i)+1 = n+1$, i.e. $\mathtt{card}(\mathcal{C}_i) = n$
- made of two elements having the same rank if $\mathtt{card}(\mathcal{C}_i)+1 = n$, i.e. $\mathtt{card}(\mathcal{C}_i) = n-1$
- connected if $\mathtt{card}(\mathcal{C}_i)+1 \le n-1$, i.e. $\mathtt{card}(\mathcal{C}_i) \le n-2$

The static surface property ii hence holds for the order $(\theta^\Box(x^i), \alpha_{|\theta^\Box(x^i)})$ of rank $(n-1)$.

□

The next lemma shows that a connected order with dimension at least 1, which is locally everywhere an order fulfilling the static surface properties, is also itself an order fulfilling the static surface properties.

Lemma 14 *Let $|X|$ be an order of rank $n \ge 1$, such that $\forall x \in X$, $\theta^\Box(x)$ is an order of rank $(n-1)$ fulfilling the static surface properties then $|X|$ also fulfills the static surface properties.*

Proof We successively prove property i and ii:

Property (i). Let x be an element of $|X|$ with rank k. As $|\theta^\Box(x)|$ is a pure and closed order of rank $n-1$, all of its maximal $\alpha^\bullet_{|\theta^\Box(x)}$-chains have length $(n-1)$. Let $\mathcal{C}$ be such a maximal chain of $\theta^\Box(x)$. Then there exists an $\alpha^\bullet$-chain in $|X|$ containing all the elements of $\mathcal{C}$ plus x. Hence x belongs to a chain with length n. Moreover by definition of $\theta^\Box$, it is clear that there exists a bijection between the set of maximal $\alpha^\bullet$-chains of $|X|$ containing some element x and the set of maximal $\alpha^\bullet_{|\theta^\Box(x)}$-chains of $|\theta^\Box(x)|$ (an illustration is given on Fig. 11). Any maximal $\alpha^\bullet$-chain of X containing x may be obtained from some maximal $\alpha^\bullet_{|\theta^\Box(x)}$-chain of $|\theta^\Box(x)|$ by inserting the element x. All maximal $\alpha^\bullet$-chains of X have hence length n.

Property (ii). Let $\mathcal{C}_I$ be a β-chain of $|X|$, $\mathcal{C}_I = \{x^i\}_{i\in I}$ where $I \subset \{0,\dots,n\}$, $card(I) \ge 1$. Let us choose some $j \in I$. We have naturally: $\bigcap_{i\in I\setminus\{j\}}(\theta^\Box(x^i) \cap \theta^\Box(x^j)) = \bigcap_{i\in I}\theta^\Box(x^i)$. From Lemma 11, $\forall i \in I\setminus\{j\}$, $\theta^\Box(x^i) \cap \theta^\Box(x^j) = \theta^\Box_{|\theta^\Box(x^j)}(x^i)$. Otherwise said $\bigcap_{i\in I}\theta^\Box(x^i) = \bigcap_{i\in I\setminus\{j\}}\theta^\Box_{|\theta^\Box(x^j)}(x^i)$. And as $|\theta^\Box(x^j)|$ verifies all static surface properties, this intersection has the expected property. □

Last two lemmas imply that an order equipped with static surface properties may be recursively defined with the same recursive property as n-surfaces. Moreover an order of rank equal to 0 fulfilling static surface properties is clearly a 0-surface. Hence Theorem 9 holds.

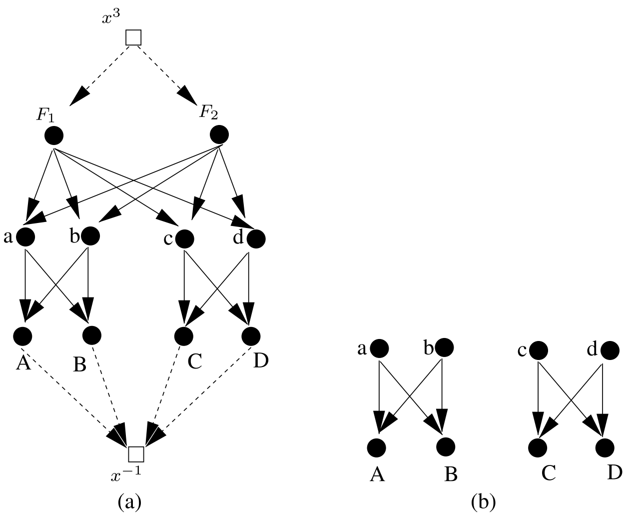


Fig. 12 The order of (**a**) is an example of a `switch`-order which is not an n-surface. It fulfills properties iia, iib but not property iic. $\theta^{\square}(F_1)$ displayed on (**b**) is not connected

3.2 Characterization of an n-Surface through Properties of Its Maximal $\beta^{\bullet}$-Chains

The purpose of this section is to prove that n-surfaces can be characterized by transformations on their maximal $\beta^{\bullet}$-chains (Theorem 19). We need further definitions to achieve this goal.

We define first the notion of `switch`-orders. It can be easily proved that the set of `switch`-orders is isomorphic to the set of augmented incidence graphs defined by Brisson in [6] and used in the first attempts to prove the isomorphism between the set of n-surfaces and a subset of generalized maps [1].

Definition 15 (Switch-order) A pure and closed order $|X|$ is a *switch-order* if

$\forall (x^{i-1}, x^i, x^{i+1}) \in X \times X \times X,$

such that $\{x^{i-1}, x^i, x^{i+1}\}$ is a $\beta^{\bullet}$-chain of $|X|$,

$\exists ! x'^i \in X, \quad \alpha_X^{\square}(x^{i+1}) \cap \beta_X^{\square}(x^{i-1}) = \{x^i, x'^i\}.$

The property characterizing switch-orders is called `switch`-property. We note here that the static surface properties i and iib guarantee that n-surfaces are `switch`-orders. The converse is generally not true (see Fig. 12). Nevertheless such orders have a common characteristic with n-surfaces, they are locally everywhere `switch`-orders.

Lemma 16 *Let $|X|$ be a* `switch`*-order of rank $n \geq 1$ then $\forall x \in X, \theta^{\square}(x)$ is a* `switch`*-order of rank $n-1$.*

Moreover let $\{x^{i_j}\}_{j \in \{0,\ldots,p\}}$ be an α-chain of $|X|$, then the suborder induced on $\bigcap_{j \in \{0,\ldots,p\}} \theta^{\square}(x^{i_j})$ is a `switch`*-order of rank $(n-(p+1))$.*

This lemma is a straightforward consequence of Lemmas 11 and 12, and of the intrinsic properties of $\theta^{\square}$.

Moreover $(n+1)$ involutions may be straightforwardly defined on the n-$\beta^{\bullet}$-chains of a switch-order of rank n.

Proposition 17 (`switch`$_i$-operators) *Let $|X|$ be a switch-order of rank n. The* `switch` *property induces $(n+1)$ involutions without fixed point* `switch`$_i$*, $i \in \{0,\ldots,n\}$ on the set of the n-$\beta^{\bullet}$-chains of $|X|$, $(x^0,\ldots,x^i,\ldots,x^n)$, defined by*:

$$\begin{aligned}&\mathtt{switch}_i((x^0,\ldots,x^{i-1},x^i,x^{i+1},\ldots,x^n))\\&\quad=(x^0,\ldots,x^{i-1},x'^i,x^{i+1},\ldots,x^n)\\&\quad\text{where } \alpha_X^{\square}(x^{i+1}) \cap \beta_X^{\square}(x^{i-1}) = \{x^i, x'^i\}.\end{aligned}$$

We also note that any pure and closed order that may be equipped with such a set of operators is a `switch`-order. We remark here that any order built on a non empty subset $\bigcap_{i \in I} \theta^{\square}(x^i)$ is a `switch`-order with rank $n - \mathrm{card}(I)$ (Lemma 16). It may hence be equipped with $n - \mathrm{card}(I) + 1$ `switch`$_i$-operators naturally deduced from the operators defined on the whole order. In the following, in order to avoid any ambiguity, we subscript the operators of such suborders with their index in the original order: $\{\mathtt{switch}_k\}_{k \in N \setminus I}$.

Definition 18 (Chain connected order) A switch-order $|X|$ of rank n is a *chain connected order* if for any β-chain (eventually empty) of X, $\{x^{i_j}\}_{j \in \{0,\ldots,k\}}$, $0 \leq k \leq n$, any n-$\beta^{\bullet}$-chain of X containing these k elements is the image of any other n-$\beta^{\bullet}$-chain containing the k-elements by a composition of `switch`$_i$, $i \in N - \{i_0,\ldots,i_k\}$.

The order depicted in Fig. 12 is not chain connected as there is for instance no composition of `switch`-operators transforming the chain AaF_1 into the chain CcF_1.

We prove below that any n-surface is a chain connected order which is also pure and closed and that the converse is also true. In order to obtain this result we use both the recursive and static characterization of n-surfaces.

Theorem 19 *Let $|X|$ be an order then the following statements are equivalent*:

i. *$|X|$ is an n-surface,*
ii. *$|X|$ is a chain connected order of rank n.*

Proof The proof is achieved in two steps. The first step mainly uses the recursive definition of n-surfaces whereas the second step is based on their static characterization. As the proof is quite tricky, we first give the main underlying ideas.

Proving that n-surfaces are chain connected ((i) $\Rightarrow$ (ii)) is essentially achieved through a recursive proof. 1-surfaces are first proved to be chain-connected. Then a sequence of switch-operators connecting any two chains of an n-surface is deduced from the switch-operators of well-chosen suborders of lower rank.

Proving that chain-connected orders of rank n are n-surfaces ((ii) ⇒ (i)) mainly requires to check point iic of the static characterization of n-surfaces. In other words, each suborder built on the intersection of $\theta^{\square}$-adherences of a finite set of elements must be connected. The proof exhibits how to construct a path between any two elements of such a suborder.

(i) ⇒ (ii). $|X|$ is an n-surface first implies that $|X|$ is pure and closed (see static surface property i in Theorem 9). We have also already remarked that any n-surface is a switch-order. In order to prove that it is chain connected we use the static surface property ii.

Let $\{x^{i_j}\}_{j\in\{1,\dots,p\}}$ be p elements of $|X|$. And let U_p be the set of n-$\beta^\bullet$-chains of $|X|$ containing these p elements.[11] We note that U_0 is simply the set made of all n-$\beta^\bullet$-chains of $|X|$. We have to prove that two n-$\beta^\bullet$-chains of U_p can be related by a composition of involutions $\mathtt{switch}_i$, $i \in N - \{i_0, \dots, i_p\}$.

- If $p = n+1$, U_p contains only one element.
- For $p \le n$, we proceed by complete induction on the rank n of the order.

  If the rank of $|X|$ is equal to 0, $p \le n$ implies $p = 0$. U_0 is the set of all 0-$\beta^\bullet$-chains of the order, and contains hence two elements related by $\mathtt{switch}_0$.

  Let us suppose the property true for all n-surfaces, $n \le k$. What happens for a $(k+1)$-surface ? In this case, p is lower than or equal to $k+1$ and elements of U_p are $(k+1)$-$\beta^\bullet$-chains. Let us consider two elements of U_p. By definition, each of them contains the elements $\{x^{i_j}\}_{j\in\{1,\dots,p\}}$. Let us denote respectively $\{e^{i_j}\}_{j\in\{p+1,\dots,k+2\}}$ and $\{e'^{i_j}\}_{j\in\{p+1,\dots,k+2\}}$ the $k-p+2$ cells completing these $(k+1)$-$\beta^\bullet$-chains.

  The recursive definition of n-surfaces and the definition of $\theta^{\square}$ imply that the suborder built on $\bigcap_{j\in\{1,\dots,p\}} \theta^{\square}(x^{i_j})$ has rank $k+1-p$ and is a $(k-p+1)$-surface. $(e^{i_{p+1}}, \dots, e^{i_{k+2}})$ and $(e'^{i_{p+1}}, \dots, e'^{i_{k+2}})$ are $(k-p+1)$-$\beta^\bullet$-chains of this order. As $p \le k+1$, $k+1-p \ge 0$ and the suborder has a dimension greater than or equal to 0. Moreover as it is a suborder of $|X|$, its rank is at most k. The recursion hypothesis hence holds. Hence, there exists a composition of operators $\mathtt{switch}_{i_j}$, $i_j \in \{i_{p+1}, \dots, i_{k+2}\}$ relating both $(k-p+1)$-$\beta^\bullet$-chains. This same composition of involutions, whose indices belong to $N - \{i_1, \dots, i_p\}$, relates the corresponding k-$\beta^\bullet$-chains in the whole order.

[11] To be more formal, we should denote U_p by $U_p(\{x^{i_j}\}_{j\in\{1,\dots,p\}})$

This process is illustrated in Fig. 13 where the rank of the order is equal to 2 and p to 1. Let us study the link between 2-$\beta^\bullet$-chains (B, a, F_1) and (B, b, F_3). The rank of the fixed element B is 0. We aim hence at proving that there is a composition of $\mathtt{switch}_1$ and $\mathtt{switch}_2$ that links (B, a, F_1) and (B, b, F_3). Let us consider the suborder built on $\theta^{\square}(B)$. (a, F_1) and (b, F_3) are both 1-$\beta^\bullet$-chains of this order. The recursion hypothesis holds for them. There exists an alternative sequence of $\mathtt{switch}_1$ and $\mathtt{switch}_2$ relating both 1-$\beta^\bullet$-chains: (a, F_1), (f, F_1), (f, F_2), (b, F_2), (b, F_3). This same sequence applied on (B, a, F_1) leads to (B, b, F_3).

$$\begin{aligned}(B, a, F_1) &\xrightarrow{\mathtt{switch}_1} (B, f, F_1) \xrightarrow{\mathtt{switch}_2} (B, f, F_2)\\ &\xrightarrow{\mathtt{switch}_1} (B, b, F_2) \xrightarrow{\mathtt{switch}_2} (B, b, F_3).\end{aligned}$$

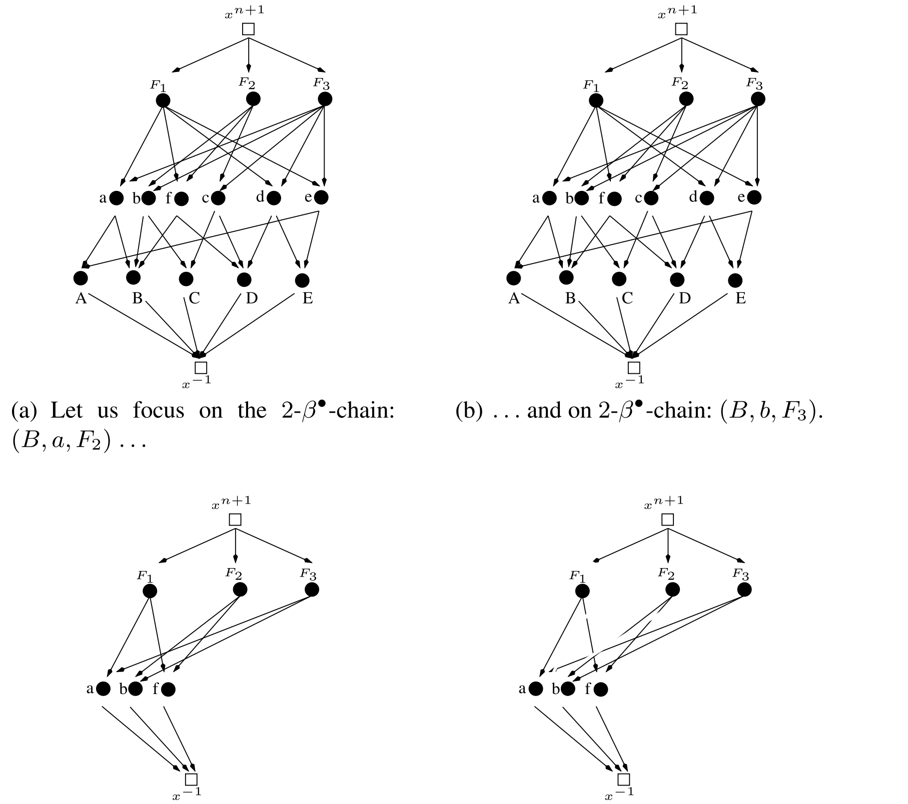
(ii) ⇒ (i). $|X|$ is a chain connected order. As $|X|$ is pure and closed, we only need to prove that $|X|$ fulfills static surface property i of Theorem 9.

iia holds for any order.

iib is a direct consequence of the switch-property of chain connected orders.

In order to prove iic, we consider a k-β-chain $\{x^{i_j}\}_{j\in\{0,\dots,k\}}$ of $|X|$ with $k+1 \le n-1$. Let y and y' be two distinct elements of the suborder $|X'|$ built on $\bigcap_{j\in\{0,\dots,k\}} \theta^{\square}(x^{i_j})$. This suborder is a switch-order with rank $n-k-1$ which is strictly greater than 0. We have to show that there exists a $\theta_{|X'}$-chain between y and y'. We explain below how to construct such a path.

Let $\mathcal{C}_y$ and $\mathcal{C}_{y'}$ be two n-$\beta^\bullet$-chains of $|X|$ respectively containing y and y'. There exists a finite sequence of $\mathtt{switch}_j$-operators, $j \in N - \{i_0, \dots, i_k\}$, relating $\mathcal{C}_y$ to $\mathcal{C}_{y'}$. If no operator is indexed by the rank of y then it is obvious that y belongs to $\mathcal{C}_{y'}$, which means that y belongs to $\theta^{\square}(y')$. Otherwise let us traverse the sequence of operators until the first one indexed by the rank of y. Let $\mathcal{C}_p$ and $\mathcal{C}_{p+1}$ be the two n-$\beta^\bullet$-chains related by this operator. The element y belongs to $\mathcal{C}_p$ but not to $\mathcal{C}_{p+1}$ and all other elements of $\mathcal{C}_p$ and $\mathcal{C}_{p+1}$ are identical. Let us choose one of these elements such that its rank does not belong to $\{i_0, \dots, i_k\}$, and call it y_0. The element y_0 belongs to $|X'|$ and $\{y, y_0\}$ is hence a $\theta_{|X'}$-chain. If y_0 belongs to $\theta^{\square}(y')$, we have just exhibited a path between y and y'. Otherwise we continue moving on along the sequence until finding an operator indexed by the rank of y_0 and we add another element to the path in the same way as previously. We iterate the process until finding an element which is in the $\theta^{\square}$-adherence of y'. The termination of the process is guaranteed by the definition of the involved sequence of operators. □



(a) Let us focus on the 2-$\beta^\bullet$-chain: (B, a, F_2) ...

(b) ... and on 2-$\beta^\bullet$-chain: (B, b, F_3).

(c) Let us consider the 1-surface built on $\theta^\Box(B)$.

(d) The recursion hypothesis implies that there exists a path made of 1-$\beta^\bullet$-chain between (a, F_1) and (b, F_3), for example: (a, F_1), (f, F_1), (f, F_2), (b, F_2), (b, F_3).

Fig. 13 Illustration of part one of the proof of Theorem 19 for a 2-surface. The 2-$\beta^\bullet$-chain (B, a, F_1) may be related to the 2-$\beta^\bullet$-chain (B, b, F_3) $(BaF_1, BfF_1, BfF_2, BbF_2, BbF_3)$ by using a composition of operators $\texttt{switch}_1$ and $\texttt{switch}_2$

4 Definition and Characterization of *n*-*G*-Maps without Multi-Incidence

As previously said, orders can only represent subdivisions without multi-incidence. In this section, we characterize the class of generalized maps that also represent such subdivisions.

The most intuitive characterization of generalized maps without multi-incidence is simply a translation in terms of orbits that its associated simplicial object is an abstract simplicial complex.

This characterization is a consequence of the link between the cellular subdivision represented by a generalized map and the associated semi-simplicial set. Let us remind that an i-cell of the cellular subdivision is associated to an orbit $\langle\alpha\rangle_{N-\{i\}}(d)$ which also corresponds to a 0-simplex (i.e. a vertex) of the associated semi-simplicial set. More generally, each i-simplex is related to an orbit $\langle\alpha\rangle_{N-I}(d)$ for some dart d where I has cardinality $i+1$. This connection leads to a duality between multi-incidence in the cellular subdivision and identifications in the semi-simplicial set. An identification occurs if two k-simplices are incident to precisely the same set of $(k+1)$ vertices. As each vertex of the semi-simplicial set corresponds to a cell of the associated cellular subdivision, the $(k+1)$ cells of the cellular subdivision associated to these $(k+1)$ vertices are incident in at least two ways (each one corresponding to a k-simplex).

The classical characterization of n-G-maps without multi-incidence simply says that a given set of $k+1$ vertices of the associated numbered semi-simplicial set defines a unique k-simplex.

Theorem 20 (Classical characterization of n-G-maps without multi-incidence) *An n-G-map is without multi-incidence if and only if*

$$\forall d \in D,\ \forall I \subseteq N, \quad \langle\alpha\rangle_{N-I}(d) = \bigcap_{i\in I} \langle\alpha\rangle_{N-\{i\}}(d). \tag{1}$$

The previous theorem is not very useful in practice as deciding whether an n-G-map contains multi-incident cells requires a lot of tests. Actually, for each dart of an n-G-map, nearly 2^{n+1} orbits have to be computed to ensure that the generalized map is without multi-incidence. We give below a simpler characterization which requires around n^2 computations of orbits per dart, and prove that both characterizations are equivalent. This new characterization only relies on the observation of the 1-simplices and of the n-simplices. It says that whenever there are no identification in dimensions 1 and n, there are no identification at all in the semi-simplicial set. We first prove this theorem. Then we show through counter-examples that both conditions are necessary to ensure the absence of multi-incidence. We call hence this characterization "optimal".

Theorem 21 (Optimal characterization of n-G-maps without multi-incidence) *An n-G-map $G = (D, \alpha_0, \dots, \alpha_n)$ is without multi-incidence if*:

$$\forall d \in D, \quad \bigcap_{i \in N} \langle\alpha\rangle_{N-\{i\}}(d) = \{d\}, \tag{2}$$

$$\forall i, j \in N, \forall d \in D,$$
$$\langle\alpha\rangle_{N-\{i\}}(d) \cap \langle\alpha\rangle_{N-\{j\}}(d) = \langle\alpha\rangle_{N-\{i,j\}}(d). \tag{3}$$

Conditions (2) *and* (3) *are respectively called* maximal simplicity *and* minimal simplicity.

Proof We prove below the equivalence between (2) and (3) of Theorem 21 and (1) of Theorem 20.

i. (1) $\Rightarrow$ (2 and 3): is straightforward ((2) and (3) are particular cases of (1)),
ii. (2 and 3) $\Rightarrow$ (1): is recursively proved.

To prove ii, we first note that the following property is always true:

$$\forall d \in D, \forall I \subseteq N, \quad \langle\alpha\rangle_{N-I}(d) \subset \bigcap_{i \in I} \langle\alpha\rangle_{N-\{i\}}(d). \tag{4}$$

Hence the condition (1) is equivalent to:

$$\forall d \in D, \forall I \subseteq N, \quad \langle\alpha\rangle_{N-I}(d) \supset \bigcap_{i \in I} \langle\alpha\rangle_{N-\{i\}}(d). \tag{5}$$

We shall therefore prove that: (2 and 3) $\Rightarrow$ (5).

The property holds for $\mathrm{card}(I) = 2$ (see property (3)). Let us suppose it true for any I such that $\mathrm{card}(I) \le k$ and observe what happens if $\mathrm{card}(I) = k + 1$.

Let $I = \{i_0, \dots, i_k\}$ be a subset of N,

Let d and d' be elements of D such that

$$d' \in \bigcap_{i \in I} \langle\alpha\rangle_{N-\{i\}}(d),$$

does d' belong to $\langle\alpha\rangle_{N-I}(d)$?

$d' \in \bigcap_{i \in I} \langle\alpha\rangle_{N-\{i\}}(d)$, then

$$d' \in \bigcap_{i \in I-\{i_k\}} \langle\alpha\rangle_{N-\{i\}}(d) = \bigcap_{i \in \{i_0,\dots,i_{k-1}\}} \langle\alpha\rangle_{N-\{i\}}(d) \tag{6}$$

and

$$d' \in \bigcap_{i \in I-\{i_0\}} \langle\alpha\rangle_{N-\{i\}}(d) = \bigcap_{i \in \{i_1,\dots,i_k\}} \langle\alpha\rangle_{N-\{i\}}(d). \tag{7}$$

As $\mathrm{card}(\{i_0, \dots, i_{k-1}\}) = \mathrm{card}(\{i_1, \dots, i_k\}) = k$, recursion hypothesis holds and:

$$d' \in \langle\alpha\rangle_{N-\{i_0,\dots,i_{k-1}\}}(d) \tag{8}$$

and

$$d' \in \langle\alpha\rangle_{N-\{i_1,\dots,i_{k-1},i_k\}}(d), \tag{9}$$

$$(8) \Rightarrow d' = d \underbrace{\alpha_{j_1} \cdots \alpha_{j_m}}_{\substack{j_s < i_{k-1} \\ \text{and} \\ j_s \notin \{i_0,\dots,i_{k-1}\}}} \underbrace{\alpha_{j_{m+1}} \cdots \alpha_{j_p}}_{j_s > i_{k-1}}, \tag{10}$$

$$(9) \Rightarrow d' = d \underbrace{\alpha_{l_1} \dots \alpha_{l_q}}_{\substack{l_s < i_{k-1} \\ \text{and} \\ l_s \notin \{i_1,\dots,i_{k-1}\}}} \underbrace{\alpha_{l_{q+1}} \dots \alpha_{l_r}}_{\substack{l_s > i_{k-1} \\ \text{and} \\ l_s \neq i_k}}. \tag{11}$$

Equations (10) and (11) imply that:

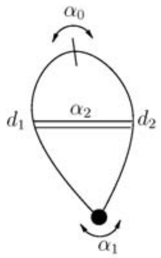
$$\begin{aligned}(d' =)d &\underbrace{\alpha_{j_1} \cdots \alpha_{j_m}}_{\substack{j_s < i_{k-1} \\ \text{and} \\ j_s \notin \{i_0,\dots,i_{k-1}\}}} \underbrace{\alpha_{j_{m+1}} \cdots \alpha_{j_p}}_{j_s > i_{k-1}} \\ = d &\underbrace{\alpha_{l_1} \cdots \alpha_{l_q}}_{\substack{l_s < i_{k-1} \\ \text{and} \\ l_s \notin \{i_1,\dots,i_{k-1}\}}} \underbrace{\alpha_{l_{q+1}} \dots \alpha_{l_r}}_{\substack{l_s > i_{k-1} \\ \text{and} \\ l_s \neq i_k}}.\end{aligned} \tag{12}$$

Let $d'' = d\alpha_{j_1} \cdots \alpha_{j_m}$. As no $j_1 \cdots j_m$ belongs to I, $d'' \in \langle\alpha\rangle_{N-I}(d)$. By letting d'' appear on both sides of (12), we have:

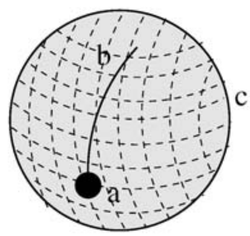
$$\begin{aligned}(d' =)d'' &\underbrace{\alpha_{j_{m+1}} \cdots \alpha_{j_p}}_{j_s > i_{k-1}} \\ = \quad &\underbrace{d''\alpha_{j_m} \cdots \alpha_{j_1}}_{\substack{=d \\ \text{and} \\ j_s < i_{k-1} \text{ and } j_s \notin \{i_0,\dots,i_{k-1}\}}} \underbrace{\alpha_{l_1} \cdots \alpha_{l_q}}_{\substack{l_s < i_{k-1} \\ \text{and} \\ l_s \notin \{i_1,\dots,i_{k-1}\}}} \underbrace{\alpha_{l_{q+1}} \cdots \alpha_{l_s}}_{\substack{l_s > i_{k-1} \\ \text{and} \\ l_s \neq i_k}}.\end{aligned} \tag{13}$$

Equation (13) is equivalent to:

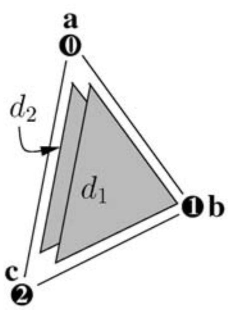
$$d'' \underbrace{\alpha_{j_{m+1}} \cdots \alpha_{j_p}}_{j_s > i_{k-1}} \underbrace{\alpha_{l_r} \cdots \alpha_{l_{q+1}}}_{\substack{l_s > i_{k-1} \\ \text{and} \\ l_s \neq i_k}}$$



(a) 2-G-map. . .



(b) . . .encoding the minimal subdivision of a sphrere.



(c) Associated semi-simplicial set: two 2-simplices share the same faces.

Fig. 14 A closed 2-G-map containing multiply incident cells which encodes the topology of a minimal subdivision of a sphere. $\langle\alpha\rangle_{N-\{0\}}(d_1)=\{d_1,d_2\}$, $\langle\alpha\rangle_{N-\{1\}}(d_1)=\{d_1,d_2\}$, $\langle\alpha\rangle_{N-\{2\}}(d_1)=\{d_1,d_2\}$, $\bigcap_{i\in\{0,1,2\}}\langle\alpha\rangle_{N-\{i\}}(d_1)=\{d_1,d_2\}\neq\langle\alpha\rangle_{N-\{0,1,2\}}(d_1)=\{d_1\}$

$$= d''\underbrace{\alpha_{j_m}\cdots\alpha_{j_1}}_{\substack{j_s<i_{k-1}\\ \text{and}\\ j_s\notin\{i_0,\dots,i_{k-1}\}}}\ \underbrace{\alpha_{l_1}\cdots\alpha_{l_q}}_{\substack{l_s<i_{k-1}\\ \text{and}\\ l_s\notin\{i_1,\dots,i_{k-1}\}}}. \tag{14}$$

Let

$$d^3 = d''\underbrace{\alpha_{j_{m+1}}\cdots\alpha_{j_p}}_{j_s>i_{k-1}}\underbrace{\alpha_{l_r}\cdots\alpha_{l_{q+1}}}_{\substack{l_s>i_{k-1}\\ \text{and}\\ l_s\neq i_k}}.$$

By construction, $d^3\in\langle\alpha\rangle_{N-\{j\}}(d'')$ for all $j\leq i_{k-1}$. Equation (14) also implies that $d^3\in\langle\alpha\rangle_{N-\{j\}}(d'')$ for all $j\geq i_{k-1}$. Then $d^3\in\bigcap_{i\in N}\langle\alpha\rangle_{N-\{i\}}(d'')$. By property 2, we have hence: $d^3=d''$.

Construction of d^3 and (14) lead to:

$$d'' = d''\underbrace{\alpha_{j_m}\cdots\alpha_{j_1}}_{\substack{j_s<i_{k-1}\\ \text{and}\\ j_s\notin\{i_0,\dots,i_{k-1}\}}}\ \underbrace{\alpha_{l_1}\cdots\alpha_{l_q}}_{\substack{l_s<i_{k-1}\\ \text{and}\\ l_s\notin\{i_1,\dots,i_{k-1}\}}}. \tag{15}$$

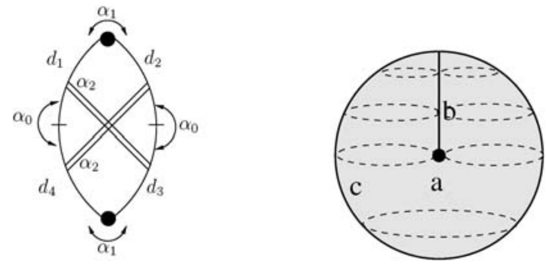
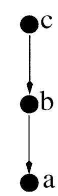


Fig. 15 A closed 2-G-map containing multiply incident cells, which represents the topology of a minimal subdivision of a projective plane (see Fig. 17 for an illustration of the construction process of such a subdivision). $\langle\alpha\rangle_{N-\{0\}}(d_1)=\{d_1,d_2,d_3,d_4\}$, $\langle\alpha\rangle_{N-\{2\}}(d_1)=\{d_1,d_2,d_3,d_4\}$, $\bigcap_{i\in\{0,2\}}\langle\alpha\rangle_{N-\{i\}}(d_1)=\{d_1,d_2,d_3,d_4\}\neq\langle\alpha\rangle_{N-\{0,2\}}(d_1)=\{d_1,d_2\}$

Fig. 16 Order representing both subdivisions of Figs. 14 and 15. Two different subdivisions are represented by a single order: in both cases the pair (a,b) is twice incident to face c and in both cases the condition of maximal simplicity is not fulfilled



Let us now replace $d''\alpha_{j_m}\cdots\alpha_{j_1}\alpha_{l_1}\cdots\alpha_{l_q}$ by d'' (according to (15)) in the right side of (13):

$$d' = d''\underbrace{\alpha_{l_{q+1}}\cdots\alpha_{l_r}}_{\substack{l_s>i_{k-1}\\ \text{and}\\ l_s\neq i_k}}. \tag{16}$$

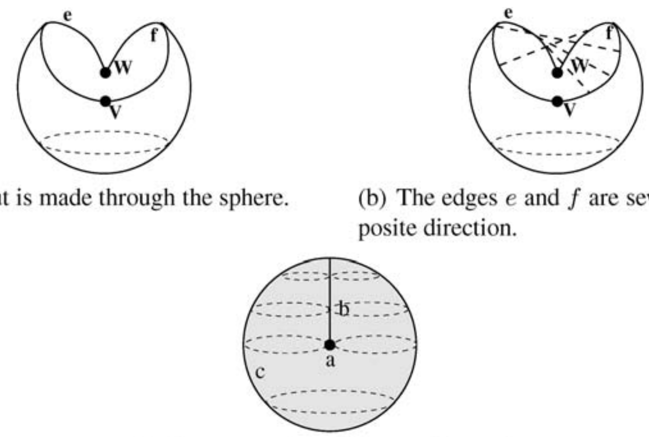
Finally we have $d'\in\langle\alpha\rangle_{N-I}(d'')=\langle\alpha\rangle_{N-I}(d)$. □

The need for maximal simplicity to avoid multi-incidence is illustrated on Figs. 14, 15 and 16. And despite what have been conjectured in [21], this sole condition is not sufficient as depicted on Figs. 18 and 19.

5 Statement and Proof of the Isomorphism

We shall prove in this section the following theorem: the set of n-surfaces and the set of closed connected n-G-maps without multi-incidence are equivalent. The proof is conducted as follows. With the characterization of n-surfaces through properties of their maximal $\beta^{\bullet}$-chains, we are able to prove that a closed n-G-map without multi-incidence may be built from any n-surface (Theorem 22). Reciprocally we also use it to show that an n-surface may be obtained from any n-G-map (Theorem 23). We conclude by proving that both transformations are inverse to each other up to isomorphism (Theorem 24).

Fig. 17 Construction process of a subdivision of the projective plane made of 1 vertex, 1 edge and 1 face

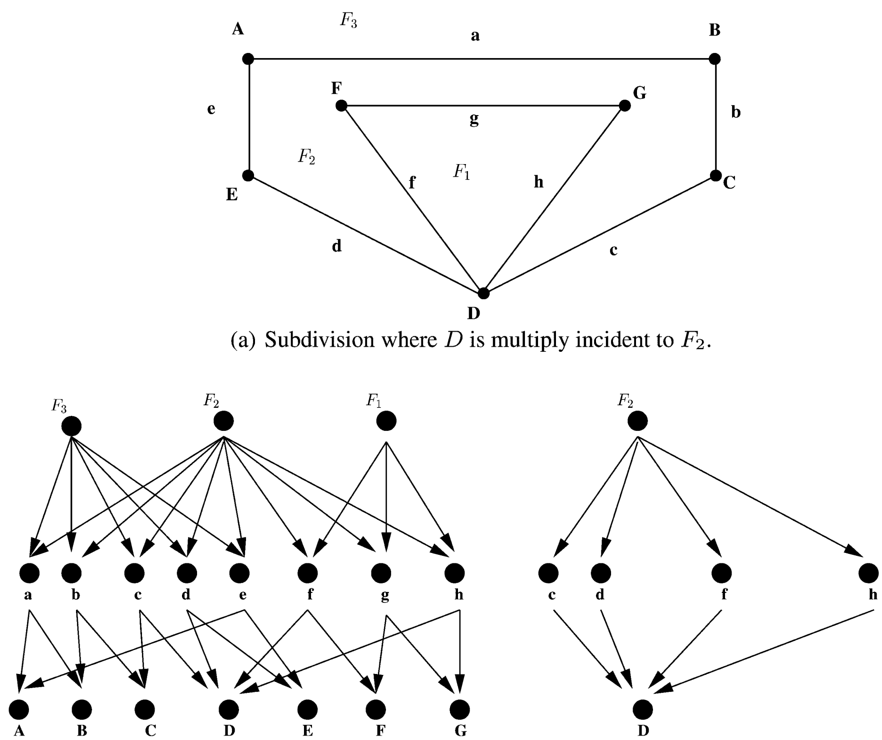


(a) A cut is made through the sphere.

(b) The edges e and f are sewed in opposite direction.

(c) Representation of the resulting subdivision of the projective plane.

Fig. 18 Example of a subdivision with multi-incidence whose associated order does not fulfill the `switch`-property. There are indeed *four* 1-elements (instead of the required *two*) between the 0-element D and the 2-element F_2 (see (**c**)). As the construction process of an n-G-map essentially relies on the `switch`-operator (see Theorem 22), the information carried by the order alone is not sufficient to construct the 2-G-map corresponding to the subdivision. Figure 19 shows indeed how two different 2-G-maps may be constructed from this order by arbitrarily defining two `switch`-operators on it



(a) Subdivision where D is multiply incident to F_2.

(b) Associated order...

(c) ... that is not a `switch`-order.

Figure 20 exhibits a 2-G-map and an equivalent 2-surface.

5.1 Construction and Characterization of the n-G-Map Associated to an n-Surface

We show now that it is possible to build a closed connected n-G-map without multi-incidence from a chain connected order, otherwise said an n-surface.

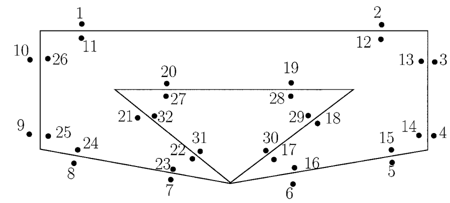
We both exhibit a construction process of an n-G-map from any n-surface and prove that the resulting n-G-map is closed, connected and without multi-incidence.

Theorem 22 (Construction of a closed connected n-G-map without multi-incidence from an n-surface) *Let $|X|$ be an n-surface. Let us define*

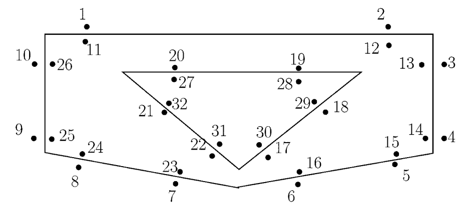
- *D as the set of n-$\beta^\bullet$-chains of $|X|$*
- $\forall i \in \{0, \dots, n\}$, $\alpha_i = \mathtt{switch}_i$

Then $(D, \alpha_0, \dots, \alpha_n)$ is a closed connected n-G-map without multi-incidence.

Proof The proof is decomposed in four parts. We first prove that α_i are involutions without fixed points. We then show that $\forall 0 \le i < i+2 \le j \le n, \alpha_i\alpha_j$ is an involution. These two

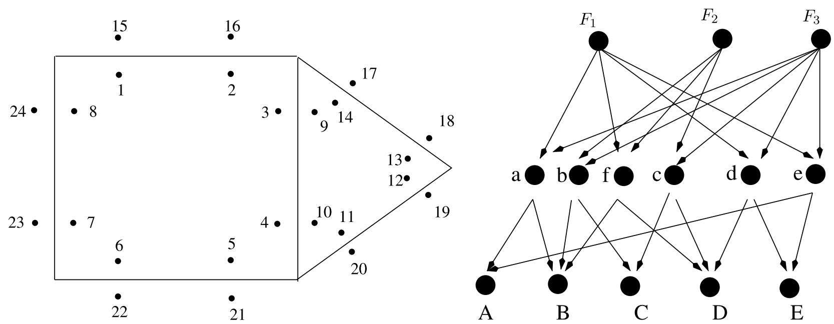


(a) Let us choose:
$\texttt{switch}_1(D,d,F_2) = (D,f,F_2)$ and $\texttt{switch}_1(D,h,F_2) = (D,c,F_2)$
Then the obtained 2-G-map also represents the subdivision depicted on Fig. 18(a).



(b) Let us choose:
$\texttt{switch}_1(D,d,F_2) = (D,c,F_2)$ and $\texttt{switch}_1(D,h,F_2) = (D,f,F_2)$
Then the obtained 2-G-map represents a different subdivision.
0-cell D of the initial subdivision is split into two 0-cells which are respectively associated to orbits : $\{7,23,16,6\}$ and $\{22,31,30,17\}$.
Moreover this $2G$-map has two connected components instead of one and four 2-cells instead of three.

Fig. 19 Construction of two 2-G-maps from the order depicted on Fig. 18(b). On this order, two different `switch`-operators may be *arbitrarily* defined. One `switch`-operator leads to the 2-G-map representing the initial subdivision (see (**a**)). The 2-G-map constructed with the second operator (see (**b**)) encodes a completely different subdivision



(a) 2-G-map. (b) 2-surface.

Fig. 20 a 2-G-map and a 2-surface, that encode isomorphic subdivisions

steps prove that $(D,\alpha_0,\ldots,\alpha_n)$ is a closed n-G-map. Moreover as any n-surface is connected we can also deduce that this n-G-map is connected. The third step shows that it fulfills the minimal simplicity requirement. We finish by proving that it also has the maximal simplicity property. We note first that any element of D may be written as $(x^0,\ldots,x^n)$.

i. **closeness:** by construction, $\forall i\in\{0,\ldots,n\}, \alpha_i = \texttt{switch}_i$ and is hence an involution without fixed point.
ii. **commutativity:** we shall prove now that $\alpha_i\alpha_j$ is also an involution for $i<j-1$ or $i>j+1$. As α_i and α_j are involutions, this is equivalent to prove that $\alpha_i\alpha_j = \alpha_j\alpha_i$. The condition on i and j guarantees that α_i will not modify any of the x^{j-1}, x^j, x^{j+1} and that α_j will also not change any of the x^{i-1}, x^i, x^{i+1}.

   This implies that $\alpha_i\alpha_j$ is equal to $\alpha_j\alpha_i$.
iii. **maximal simplicity:** we prove that:

$$\forall d\in D, \bigcap_{i=0}^{i=n}\langle\alpha\rangle_{N-\{i\}}(d) = \{d\}.$$

   Let $d=(x^0,\ldots,x^n)$ and $d'=(x'^0,\ldots,x'^n)$ be two elements of D such that:

$$d'\in\bigcap_{i=0}^{i=n}\langle\alpha\rangle_{N-\{i\}}(d). \tag{17}$$

   From (17), for all $j\in N$, d and d' belong both to $\langle\alpha\rangle_{N-\{j\}}(d)$, which means that d and d' agree on their jth element: $\forall j\in N, x^j = x'^j$ and hence $d=d'$.
iv. **minimal simplicity:** Let x^i and x^j be two cells of the subdivision and d a dart associated to a maximal

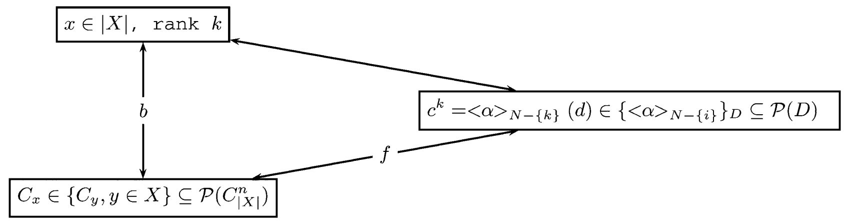


Fig. 21 Illustration of bijections b and f

β-chain, i.e. an n-β-chain, containing both x^i and x^j. The property of chain connectedness of an n-surface states that any two n-β-chains containing both x^i and x^j are connected by a composition of $\mathtt{switch}_k$, $k \in N - \{i, j\}$. In terms of orbits this simply means that: $\langle\alpha\rangle_{N-\{i\}}(d) \cap \langle\alpha\rangle_{N-\{j\}}(d) = \langle\alpha\rangle_{N-\{i,j\}}(d)$. □

5.2 Characterization of the Order Associated to a Closed Connected n-G-Map without Multi-Incidence

We show now that a natural order may be associated to any n-G-map and prove that it is an n-surface. Let us first denote $\{\langle\alpha\rangle_{N-\{i\}}\}_D$ the set of i-cells of the subdivision associated to any n-G-map $(D, \alpha_0, \ldots, \alpha_n)$. And let us denote $<_G$ the incidence relations between the cells of this subdivision.

Theorem 23 *Let G be a closed connected n-G-map without multi-incidence. Let us define*:

- *X as the set* $\bigcup_{i\in N}\{\langle\alpha\rangle_{N-\{i\}}\}_D$,
- $\alpha^{\Box} = <_G$

Then $|X| = (X, \alpha)$ is an n-surface.

Proof $|X|$ is obviously an order. We prove below that it is a chain connected one and hence an n-surface. By construction, for any element x of X, there exists at least one dart d and one integer $i \in N$, such that $x = \langle\alpha\rangle_{N-\{i\}}(d)$. x belongs hence to at least one n-$\beta^\bullet$-chain: $(\langle\alpha\rangle_{N-\{0\}}(d), \langle\alpha\rangle_{N-\{1\}}(d), \ldots, \langle\alpha\rangle_{N-\{n\}}(d))$. Moreover all maximal chains of $|X|$ have obviously length n. $|X|$ is hence a pure and closed order.

We also remark that there exists a link between the darts of the n-G-map and the n-$\beta^\bullet$-chains of the associated order. Any dart is by construction associated to a unique n-$\beta^\bullet$-chain of the order.

Moreover the definition of $<_G$ implies that any n-$\beta^\bullet$-chain $\mathcal{C}$ corresponds to at least one dart of G. Let d be such a dart. The set of darts that may be associated to $\mathcal{C}$ is $\bigcap_{i\in N}\langle\alpha\rangle_{N-\{i\}}(d)$. As G is without multi-incidence this set only contains d itself.

There is hence a bijection between the set of darts of G and the set of n-$\beta^\bullet$-chains of $|X|$.

This bijection implies that each α_i induces an operator $\mathtt{switch}_i$ on the set of n-$\beta^\bullet$-chains of $|X|$. $|X|$ is hence a $\mathtt{switch}$-order.

Finally let $\{x^{i_j}\}_{j\in\{0,\ldots,k\}}$, $0 \le k \le n$, be a β-chain of $|X|$. And let $\mathcal{C}$ be an n-$\beta^\bullet$-chain containing it. There exists a unique dart d corresponding to $\mathcal{C}$. And the set of n-$\beta^\bullet$-chains containing these $(k+1)$ elements corresponds to the set of darts $\bigcap_{j\in\{0,\ldots,k\}}\langle\alpha\rangle_{N-\{i_j\}}(d)$. Non multi-incidence implies that it is equal to $\langle\alpha\rangle_{N-\{i_0,i_1,\ldots,i_k\}}(d)$. This simply means that there exists a sequence of $\mathtt{switch}_i$-operators, $i \in N - \{i_0, i_1, \ldots, i_k\}$ relating any two n-$\beta^\bullet$-chains containing $\{x^{i_j}\}_{j\in\{0,\ldots,k\}}$.

Theorem 19 concludes. □

5.3 Stability

We call respectively $nGMnS$-conversion and $nSnGM$-conversion the construction of an n-surface from a closed connected n-G-map without multi-incidence and the construction of an n-G-map from an n-surface, which have been previously described (Theorem 23 and Theorem 22 respectively).

Let us first recall that two generalized maps are said to be isomorphic if there exists a bijection between their sets of darts that preserves the α_i involutions for i in $\{0, \ldots, n\}$. Two orders are said to be isomorphic if there exists a bijection between their sets of elements that preserves the order relation α.

Theorem 24 *$nGMnS$-conversion and $nSnGM$-conversion are inverse to each other up to an isomorphism.*[12]

Proof The proof is done in two steps. The first step is illustrated by Fig. 21,[13] the second step by Fig. 22.

i. Let $|X| = (X, \alpha)$ be an n-surface and $|X'| = (X', \alpha')$ the image of $|X|$ by an $nSnGM$-conversion followed by a $nGMnS$-conversion. By construction, Theorem 22 and Theorem 23, $|X'|$ also is an n-surface. Let us denote by $G = (D, \alpha_0, \ldots, \alpha_n)$ the intermediary closed connected n-G-map without multi-incidence.

   We only need to prove that there exists an isomorphism between X and X' because there is a natural morphism between $(X, \alpha^{\Box})$ and $(X', \alpha'^{\Box})$.

[12] By this we mean that an $nGMnS$-conversion followed by $nSnGM$-conversion gives an n-G-map isomorphic to the initial n-G-map. And reciprocally an $nSnGM$-conversion followed by $nGMnS$-conversion produces an n-surface isomorphic to the initial n-surface.

[13] $\mathcal{P}(D)$ is the power set of D

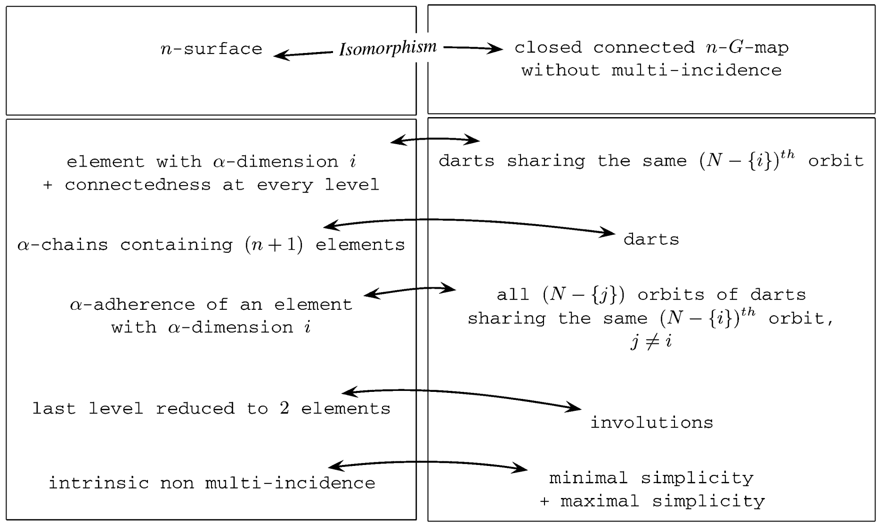


Fig. 23 Overview of the correspondence between notions defined on n-surfaces and notions defined on closed connected n-G-maps without multi-incidence

$$d \in D \longleftarrow g \longrightarrow C_d \in C^n_{|X|}$$

Fig. 22 Illustration of bijection g

Let x be an element of X. We denote by $\mathcal{C}_x$ the set of the n-$\beta^\bullet$-chains of $|X|$ that contain x. There exists a bijection, b, between X and the set $\{\mathcal{C}_x, x \in X\}$. The characterization of n-surfaces through properties of their maximal $\beta^\bullet$-chains (see Theorem 19 and Definition 18) implies that any element of $\mathcal{C}_x$ may be obtained from any other element of $\mathcal{C}_x$ by a composition of $\texttt{switch}_i$, where each index i belongs to $N\backslash\{\dim_\alpha(x)\}$. Let x_1 be an element of $\mathcal{C}_x$, there exists d_1 image of x_1 in D such that the image of $\mathcal{C}_x$ in G is the orbit $\langle\alpha\rangle_{N-\{\dim_\alpha(x)\}}(d_1)$.

Moreover if $\langle\alpha\rangle_{N-\{\dim_\alpha(x)\}}(d_1) \neq \langle\alpha\rangle_{N-\{\dim_\alpha(x)\}}(d_2)$, then the antecedents of d_1 and d_2, respectively x_1 and x_2 are not connected by a composition of $\texttt{switch}_\texttt{i}$, $i \neq \dim_\alpha(x)$. Hence x_2 does not belong to $\mathcal{C}_x$. The set $\{\mathcal{C}_x, x \in X\}$ is hence in bijection with $\{\{\langle\alpha\rangle_{N-\{i\}}\}_D, i \in \{0, \ldots, n\}\}$. We note f this bijection. The construction of X' implies that there is a bijection between $\{\mathcal{C}_x, x \in X\}$ and $\{\mathcal{C}_x, x \in X'\}$, and hence a one-to-one correspondence between X and X'.

ii. Let $G = (D, \alpha_0, \ldots, \alpha_n)$ be a closed connected n-G-map without multi-incidence and $G' = (D', \alpha'_0, \ldots, \alpha'_n)$ the image of G by an $nGMnS$-conversion followed by an $nSnGM$-conversion. By construction G' also is a closed connected n-G-map without multi-incidence. Let us denote $|X|$ the intermediary n-surface.

Theorem 22 implies that there is a bijection between the set of darts of G and the set of n-$\beta^\bullet$-chains of $|X|$. Moreover we have shown when proving Theorem 23 that there is also a bijection between the set of n-$\beta^\bullet$-chains of $|X|$ and the set of darts of G'. There is hence a bijection, g, between D and D'. Moreover for any $i \in \{0, \ldots, n\}$, the involution α_i on the darts of D corresponds to the involution $\texttt{switch}_i$ on the n-$\beta^\bullet$-chains of $|X|$ (Theorem 22). And we saw during the proof of Theorem 23 that each $\texttt{switch}_i$-operator corresponds to α'_i.

□

This last theorem concludes the proof of the equivalence between n-surfaces and closed connected n-G-maps without multi-incidence. Links between notions defined on these two models are summarized on Fig. 23.

To conclude, this work has to be related to a previous study achieved by Brisson in [6] which has inspired part of our approach. Both studies deal indeed with data structures used to represent n-dimensional objects. Brisson investigates a class of d-dimensional "geometrical" objects (the so-called subdivided d-manifolds), which cannot be fully characterized through their combinatorial properties. Nevertheless he provides a new structure, called the *cell-tuple structure* to encode the combinatorial properties of such objects. This structure is conceptually very close to the generalized maps as it is made of elements (tuples of $d + 1$ consecutive cells) related by involutions ($\texttt{switch}$ operators). Moreover the link between this structure and two classical representations of cellular subdivisions (the associated incidence graph and the simplicial complex obtained by barycentric subdivision) is exhibited.

In the present work we deal with purely combinatorial objects, which are hence more general than those studied by Brisson, even if their combinatorial properties are very similar (e.g. $\texttt{switch}$-property). Besides, by focusing on combinatorial properties, we gain a deeper understanding of them and show how they can be differently expressed. We also achieve to formalize some properties that were most likely implicit on the structure proposed by Brisson such as the static surface properties, or the notion of chain-connectedness. Finally, although it is impossible to characterize manifolds purely combinatorially, this works highlights that some combinatorial properties are particularly well-suited to characterize n-dimensional "surface-like" objects. The structures studied in this paper have indeed been used in very different contexts: image analysis (n-surfaces [4, 10, 11]) and topological modeling (n-G-maps [8, 21]).

6 Conclusion and Future Works

After having proposed two original characterizations of n-surfaces, we proved the equivalence between n-surfaces and n-G-maps without multi-incidence. We also provide a simpler characterization of n-G-maps without multi-incidence. Besides we exhibited conversion operators between both models and proved their stability.

From this equivalence, interesting properties of both generalized maps and n-surfaces may be guessed and studied. The recursive definition of n-surfaces implies for instance that there is also a recursive decomposition of generalized maps without multi-incidence. The neighborhood of any cell of a subdivision represented by a generalized map can indeed be encoded by a generalized map. And join operators used to construct n-surfaces [10] may be adapted to n-G-maps. Moreover generalized maps are able to encode open surfaces and they are well adapted to detect whether the surfaces they represent are orientable or not. Such capabilities may be transferred on orders through the equivalence.

Finally this equivalence could be a good starting point to study more general structures such as chains of maps [15], which represent less constrained subdivisions.

Acknowledgements The authors wish to thank the anonymous referees for their useful comments and suggestions that greatly help improving the quality of this paper.

References

1. Alayrangues, S., Daragon, X., Lachaud, J.-O., Lienhardt, P.: Equivalence between regular n-g-maps and n-surfaces. In: Proceedings of the 10th International Workshop on Combinatorial Image Analysis, IWCIA 2004, Auckland, New Zealand, 1–3 December 2004, pp. 122–136 (2004)
2. Alayrangues, S., Lachaud, J.-O.: Equivalence between order and cell complex representations. In: Wildenauer, H., Kropatsch, W. (eds.) Proceedings of the Computer Vision Winter Workshop (CVWW), Bad Aussee, Austria, February 2002, pp. 222–233 (2002)
3. Bertrand, G.: New notions for discrete topology. In: Bertrand, G., Couprie, M., Perroton, L. (eds.) Proceedings of 8th Discrete Geometry for Computer Imagery (DGCI'99), Marne-la-Vallée, France. Lecture Notes in Computer Science, vol. 1568, pp. 218–228. Springer, Berlin (1999)
4. Bertrand, G., Couprie, M.: A model for digital topology. In: Bertrand, G., Couprie, M. (eds.) Proceedings of 8th Discrete Geometry for Computer Imagery (DGCI'99), Marne-la-Vallée, France. Lecture Notes in Computer Science, vol. 1568, pp. 229–241. Springer, Berlin (1999)
5. Björner, A.: Topological methods. Handbook of Combinatorics, vol. 2, pp. 1819–1872 (1995)
6. Brisson, E.: Representing geometric structures in d dimensions: topology and order. Discrete Comput. Geom. **9**, 387–426 (1993)
7. Brun, L., Kropatsch, W.: Contraction kernels and combinatorial maps. In: Jolion, J.M., Kropatsch, W., Vento, M. (eds. 3rd IAPR-TC15 Workshop on Graph-Bsed Representations in Pattern Recognition, Ischia Italy, May 2001, pp. 12–21. IAPR-TC15, CUEN (2001)
8. Damiand, G., Lienhardt, P.: Removal and contraction for n-dimensional generalized maps. In: Nyström, I., Sanniti di Baja, G., Svensson, S. (eds.) DGCI. Lecture Notes in Computer Science, vol. 2886, pp. 408–419. Springer, Berlin (2003)
9. Damiand, G., Resch, P.: Split and merge algorithms defined on topological maps for 3d image segmentation. Graph. Models **65**(1–3), 149–167 (2003)
10. Daragon, X.: Surfaces discrètes et frontières d'objets dans les ordres. PhD thesis, Université de Marne-la-Vallée, October 2005
11. Daragon, X., Couprie, M., Bertrand, G.: New "marching-cubes-like" algorithm for Alexandroff-Khalimsky spaces. In Proceedings of SPIE: Vision Geometry XI (2002)
12. Daragon, X., Couprie, M., Bertrand, G.: Discrete frontiers. In: Discrete Geometry for Computer Imagery. Lecture Notes in Computer Science, vol. 2886, pp. 236–245. Springer, Berlin (2003)
13. Daragon, X., Couprie, M., Bertrand, G.: Discrete surfaces and frontier orders. J. Math. Imaging Vis. **23**, 379–399 (2005)
14. De Floriani, L., Hui, A.: Update operations on 3d simplicial decompositions of non-manifold objects. In: SM '04: Proceedings of the Ninth ACM Symposium on Solid Modeling and Applications, Aire-la-Ville, Switzerland, Switzerland, pp. 169–180. Eurographics Association (2004)
15. Elter, H., Lienhardt, P.: Cellular complexes as structured semi-simplicial sets. Int. J. Shape Model. **1**(2), 191–217 (1995)
16. Evako, A.V., Kopperman, R., Mukhin, Y.V.: Dimensional properties of graphs and digital spaces. J. Math. Imaging Vis. **6**, 109–119 (1996)
17. Grasset-Simon, C., Damiand, G., Lienhardt, P.: nd-generalized map pyramids: definition, representations and basic operations. Pattern Recognit. **39**(4), 527–538 (2006)
18. Kovalevsky, V.: A topological method of surface representation. In: Discrete Geometry for Computer Imagery, 8th International Conference, DCGI'99, pp. 118–135. Springer, Berlin (1999)
19. Kovalevsky, V.A.: Topological foundations of shape analysis. In: Shape in Picture: Mathematical Description of Shape in Grey-level Images, p. 21 (1994)
20. Ledoux, F., Mota, J.-M., Arnould, A., Dubois, C., Le Gall, P., Bertrand, Y.: Formal specification for a mathematics-based application domain: geometric modeling. Technical Report, Université d'Evry-Val d'Essonne, Rapport LaMI n51, Université d'Evry-Val d'Essonne, Evry (2000)
21. Lienhardt, P.: Topological models for boundary representation: a comparison with n-dimensional generalized maps. Comput.-Aided Des. **23**(1), 59–82 (1991)
22. Lienhardt, P.: N-dimensional generalized combinatorial maps and cellular quasi-manifolds. Int. J. Comput. Geom. Appl. **4**(3), 275–324 (1994)
23. Lienhardt, P., Skapin, X., Bergey, A.: Cartesian product of simplicial and cellular structures. Int. J. Comput. Geom. Appl. **14**(3), 115–159 (2004)
24. May, P.: Simplicial Objects in Algebraic Topology. von Nostrand, Princeton (1967)
25. Péroche, B., Bechmann, D.: Informatique graphique, modélisation géométrique et animation. Traitement du Signal et de l'Image, vol. 1. Hermès, Paris (2007)

Sylvie Alayrangues graduated from ENSEIRB engineering school in Computer Science in 1997 and received a Ph.D. degree in computer science from Bordeaux I University in 2005. She is currently an Associated Professor at the university of Poitiers and works at the department SIC of XLIM research institute. Her research interests include the study and comparison of topological models used to structure digital images and the computation and use of topological invariants (e.g. homology groups) in image applications.

Xavier Daragon received his Ph.D. degree in computer science from Marne-La- Vallee University in 2005, under the guidance of Michel Couprie and Gilles Bertrand from ESIEE, A2SI laboratory. After some introductory studies in economics and Japanese civilization at INALCO, he is currently experiencing the ways of growing private companies at ULLINK. His research focused on order theory and its applications to image analysis and computer graphics, mainly in the field of 3D medical imaging (segmentation of the cerebral neo-cortex).

Jacques-Olivier Lachaud graduated from ENSIMAG engineering school in Computer Science in 1994 and received a Ph.D. degree in computer science from Joseph Fourier Grenoble 1 University in 1998. He is currently a Professor in Computer Science at the university of Chambery (France) and works in the LAMA laboratory. His research interests are in image segmentation and analysis, more specifically deformable models, energy-minimizing techniques, digital geometry, topological models and invariants. He has written about thirty papers in international journals or conferences on these topics.

Pascal Lienhardt is Professor in Computer Science at the University of Poitiers, France. His research interests include the study of topological models (simplicial, simploidal and cellular ones) and operations for constructing such models (e.g. cartesian product) or for computing topological properties (e.g. homology groups) and the application of such models and operations in different fields as C.A.D., geosciences, etc.