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Flat Morphology on Power Lattices*

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Abstract. *Flat morphological operators*, also called *stack filters*, are the natural extension of increasing set operators to grey-level images. The latter are usually modeled as functions $E \to T$, where T is a closed subset of $\bar{\mathbf{R}}$ (for instance, $\overline{\mathbf{Z}}$ or $[a, b]$).

We give here a general theory of flat morphological operators for functions defined on a space E of points and taking their values in an arbitrary complete lattice V of values. Several examples of such lattices have been considered in the litterature, and we illustrate our therory with them. Our approach relies on the usual techniques of thresholding and stacking. Some of the usual properties of flat operators for numerical functions extend unconditionally to this general framework. Others do not, unless the lattice V is completely distributive.

Keywords: flat operators, mathematical morphology, lattice theory

1. Introduction

Most morphological operations used for processing and filtering grey-level images are *flat operators*. This means [16] that they are grey-level extensions of operators for binary images, and they can be obtained by: (a) thresholding the grey-level image for all threshold values, (b) applying the binary operator to each thresholded image set, and (c) superposing the resulting sets. For example the flat operators corresponding to dilation and erosions are the max and min filters.

The genesis of flat operators lies within the first studies towards extending set morphology to numerical functions by the use of umbras. These were developed in the late seventies by Matheron, Serra and Meyer in France, and by Sternberg in the USA. A summary of these early works can be found in Chapter 12 of [37].

Under the name of *stack filters*, flat operators were investigated in [44] through an operation called threshold decomposition, which is a variant of the standard stacking procedure used for flat operators [16] (thresholded images being arithmetically summed instead of combined by lattice-theoretical operations). This work inspired several others [25, 26 42], relying deeply on the thresholding and stacking paradigm. An alternative view of flat operators, under the name of *order-configuration filters*, was given in [31]: such operators "choose" one grey-level value in the pixel window by using only the order relations between these grey-levels. In fact, one has for flat operators an analogue of the Matheron decomposition theorem [20, 37, 38], so that a flat operator computes a combination of infima and suprema of grey-level values in pixel neighbourhoods (see Section 3.3).

Hardware implementations of flat operators have been described in the litterature, see for example [4]. Flat operators have been applied to the analysis of fingerprints, leading to the creation of start-ups (like

*This paper is dedicated to Henk Heijmans, who made major contributions to the theory of Mathematical Morphology, until a health accident in March 2004 ended his scientific career.

Morpho-systems in Fontainebleau, France, now a subsidiary of SAGEM).

An extensive theory of flat operators on grey-level images was made by Heijmans [16, 17, 18]. He considered grey-level images as numerical functions $E \to T$ where $T = \overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty, -\infty\}$ or $\overline{\mathbf{Z}} = \mathbf{Z} \cup \{+\infty, -\infty\}$, or more generally T is a closed subset of $\overline{\mathbf{R}}$ (for example, $T = [a, b]$ or $T = [a, b] \cap \mathbf{Z}$ for $a, b \in \mathbf{R}$ with $a < b$).

Flat operators share some fundamental properties [18], however it is known that there are variations between the two cases where the set of grey-levels is discrete, and where it is analog. For example, with bounded discrete grey-levels (T being a finite subset of $\overline{\mathbf{R}}$, say $T = [a, b] \cap \mathbf{Z}$ for $a < b$), flat operators commute with increasing grey-level transformations and with thresholding. This is very interesting conceptually, because it allows a binary interpretation of the behaviour of flat operators in terms of "bright" and "dark" zones of an image, but also in practical situations, because flat operators are compatible with the compression of the range of image values. On the other hand, in the case of analog grey-levels (T is not included in $\overline{\mathbf{Z}}$), flat operators commute in general only with *continuous* increasing grey-level transformations; also, when the underlying set operator is *upper semi-continuous*, the corresponding flat operator commutes with all increasing grey-level transformations and with thresholding (as in the case of discrete grey-levels). A general theory of flat morphology for images with real grey-levels is made in [14], where it is shown in particular that the commutation with thresholding holds "almost everywhere".

There is a growing need for a general theory of flat operators for images $E \to V$, where E is a space of points, and V is a lattice of values. One reason is the need to reach a common understanding of flat morphology for grey-level images in both cases of discrete and analog grey-levels; in particular there should be an explanation of why some properties like commutation with thresholding hold unconditionally in the case of discrete grey-levels, and only for upper semi-continuous operators in the case of analog grey-levels.

Another reason is the use of other types of images than sets (binary images) or grey-level images with the numerical ordering on grey-levels. Let us describe some of them.

The first example is given by multivalued images, where each pixel has a vector of values. Examples include multimodal medical images, multispectral images in remote sensing and astronomy, or colour images coded in RGB. Here the space of vector values is considered as a subset of $\overline{\mathbf{R}}^n$, more precisely it is the set $C = T^n$, where T is the same numerical scale used for each scalar value (e.g., the R, G and B components of colour). This set $C = T^n$ has a componentwise ordering inherited from the numerical ordering on T (that is, $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ iff $x_1 \leq y_1, \dots, x_n \leq y_n$), and is thus a complete lattice. Flat operators can then be defined on multivalued images in the same way as is usually done [16] for grey-level images. In fact, it is easily shown that this amounts to applying the usual (grey-level) flat operator on each of the scalar components of the image (see Section 3.2). However, as pointed out by Serra [39] in the case of RGB images (but this is true for any type of multivalued images), we lose some properties, even for discrete RGB values, in particular the commutation with thresholding. Indeed let the image I be made of red and green points (i.e., with coulours $(255, 0, 0)$ and $(0, 255, 0)$), take the thresholding θ selecting all points with colour above yellow (i.e., $\geq (255, 255, 0)$), and let δ be a dilation (for sets); as red and green are not above yellow, the thresholded image $\theta(I)$ will be empty, so dilating it by δ remains empty: $\delta(\theta(I)) = \emptyset$; on the other hand, if we dilate the image first with the flat dilation δ^C (for colour images), the dilated image $\delta^C(I)$ will mix some red and green points into yellow ones, which will be selected by thresholding, so $\theta(\delta^C(I)) \neq \emptyset$. Why does dilation not commute with thresholding, that is, why do we have $\theta(\delta^C(I)) \neq \delta(\theta(I))$? (Note that some authors have tried to palliate this problem by an artificial linearization of the RGB space, or by using various HLS-type colour spaces, which contradict each other and do not conform to standard colorimetry [29]).

Second, in order to process video sequences, Keshet (Kresch) introduced [23, 24] the *reference order* on grey-level images. Take a closed grey-level set $T \subseteq \mathbf{R}$. Choosing a *reference* grey-level $r \in T$, we define the *reference order* $\leq_r$ *w.r.t.* r as follows: for two grey-levels a, b we have $a \leq_r b$ if a is between r and b, i.e., if either $r \leq a \leq b$ or $r \geq a \geq b$ (numerically). This ordering on the set T of grey-levels turns it into a *complete inf-semilattice*, in other words every nonvoid family of grey-levels has an infimum, but not necessarily a supremum. By adding a greatest element ∞ to T, $\overline{T} = T \cup \{\infty\}$ becomes a complete lattice. In Fig. 1 we illustrate the construction of this lattice for $T = \mathbf{Z}$. Given a space E of points, the sets T^E and $\overline{T}^E$ of functions $E \to T$ or $E \to \overline{T}$ can be ordered with reference to a fixed function $Ref \in T^E$: for every point $p \in E$, the values $F(p)$, $G(p)$ of functions

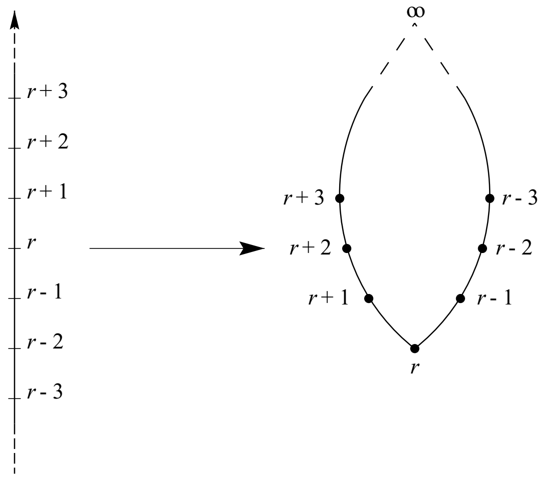

Figure 1. Transformation of the numerically ordered lattice **Z** into the complete lattice $\mathbf{Z} \cup \{\infty\}$ for the reference order w.r.t. r.

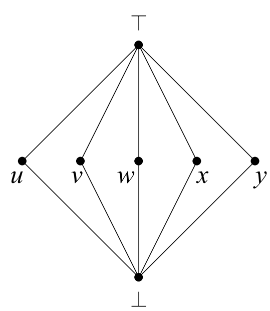

Figure 2. Hasse diagram of the lattice U of labels with 5 proper labels u, v, w, x, y, and the 2 dummy labels $\perp$ and $\top$.

$F, G \in T^E$ are ordered with reference to $Ref(p)$; then T^E is a complete inf-semilattice, called the *reference semi-lattice*, while $\overline{T}^E$ is a complete lattice. The reference semilattice has been studied in detail in [19]. Can we define flat operators on T^E or on $\overline{T}^E$, and in this case what are their properties?

Third, in an algorithm for segmenting video sequences, Agnus [1, 2, 3] defined "object-oriented" variants of flat erosion and geodesical reconstruction, where grey-levels are considered as labels of objects, and all image portions having distinct grey-levels are processed independently. It turns out [34, 35] that these "object-oriented" variants of flat morphological operators are simply the standard flat operators on *label images*, that is functions $E \to U$, where U is the lattice of labels made of n mutually incomparable proper labels ($n \geq 3$), to which one adds, as least and greatest elements, two dummy labels $\perp$ and $\top$ (meaning respectively "no label" and "conflicting labels"); we illustrate in Fig. 2 the *Hasse diagram* of this lattice for $n = 5$ (the Hasse diagram [6, 18] is a graph whose nodes are the elements of the lattice and where an edge between nodes a above and b below indicates that a *covers* b, that is $a > b$ but there is no m such that $a > m > b$). We will briefly discuss this lattice U later, because it will appear in several counterexamples. As explained in [34], that approach can be used for the processing of the hue component of colour images, and indeed something relatively similar has been done in [15], under the name of *labeled openings*.

These three examples show the need of a general theory of flat morphological operators on the power lattice V^E of functions $E \to V$, where E is a space of points and V is an arbitrary complete lattice of image values. However, with the notable exceptions of [39], Chapter 10 of [18], and to a lesser extent [45, 46] for finite window operators, the theory of flat morphological operators on non-binary images has been restricted to the case of grey-level images having grey-level values in a subset of $\overline{\mathbf{Z}}$ or $\overline{\mathbf{R}}$.

The purpose of this paper is to give such a general theory. Some preliminary results have been published in [33], and we applied this theory for label images in [34]. Let us briefly review the approaches of Heijmans (Chap. 10 of [18]), Serra [39] and Wild [45, 46], and then explain how we proceed.

In Chap. 10 and 11 of [18], Heijmans defines flat operators through the usual operations of thresholding and superposition. Let E be the space of points and let V be a complete lattice of image values. Consider a function $F : E \to V$. For every $v \in V$, we define the *threshold set*

$$X_v(F) = \{p \in E \mid F(p) \geq v\}. \tag{1}$$

Then, given an operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ on sets, the *flat operator corresponding to* ψ is the operator $\psi^V : V^E \to V^E$ on functions defined by setting for any function F and point p:

$$\psi^V(F)(p) = \bigvee\{v \in V \mid p \in \psi\,(X_v(F))\}. \tag{2}$$

However Heijmans makes some drastic assumptions on the lattice V of values (see in [18] Definition 10.4 of *an admissible* and of a *strongly admissible complete lattice*). They are introduced because of the intricate constructions that he uses to analyse properties of flat operators. We will provide a detailed analysis of these assumptions in the Appendix, where we show that a complete lattice is admissible iff it is completely distributive. Moreover, a product of two or more complete chains (e.g., the lattice of RGB colours) is in general not strongly admissible. Thus all results of [18] are restricted to a very particular case for the lattice V of values; in practice, V will have to be a complete chain of numerical values ($\overline{\mathbf{R}}$, $[a, b]$, $\overline{\mathbf{Z}}$, or $[a, b] \cap \mathbf{Z}$).

On the other hand, in an attempt to obtain in a general framework the fundamental property that flat operators commute with continuous increasing transformations on the lattice of values, Serra [39] takes an arbitrary lattice V of values, but in the definition (1) of thresholding he replaces the order $\geq$ by the negation $\not\leq$ of the dual order $\leq$:

$$Y_v(F) = \{p \in E \mid F(p) \not\leq v\}.$$

When V is a chain (i.e., totally ordered), which is the case for subsets of $\overline{\mathbf{R}}$ or $\overline{\mathbf{Z}}$, the relation $\not\leq$ means $>$. With such a definition of thresholding, the above definition (2) of the flat operator ψ^V must also be modified. Serra's approach did not lead to a complete theory as the one of Heijmans.

Wild [45, 46] considers images $E \to V$, where V is a lattice (not necessarily complete), and operators on such images which compute on each point a composition of finite suprema and infima of values of points. When the lattice V is distributive, some decomposition formulas are given. This corresponds in fact to the particular case of a flat operator ψ^V arising from a *finite window operator* ψ on sets: this means that for each point $p \in E$ there is a finite window $W(p) \subseteq E$ such that for every set $X \subseteq E$, whether $p \in \psi(X)$ or not depends only on $X \cap W(p)$; then for an image $F : E \to V$, $\psi^V(F)(p)$ depends only on the values of F on $W(p)$.

In this paper, we will provide a general theory of flat operators on functions $E \to V$, where V is an arbitrary lattice of values, using the classical construction of (1, 2). From this point of view, we follow Heijmans. However, following Serra, we do not put any assumption on the lattice V. We will see that some known results for grey-level functions in $\overline{\mathbf{R}}$ or $\overline{\mathbf{Z}}$ extend to the general case, while some others require the complete lattice V to be completely distributive (which is in fact the admissibility condition of Heijmans). We will obtain an analogue of Matheron's theorem, which shows how to compute the value $\psi^V(F)(p)$ by a supremum of infima of values $F(q)$ for some points q; this is similar to what Wild does, but we do not restrict ourselves to finite window operators, so the suprema and infima of our formula can be infinite (that is why we require the lattice to be complete).

The paper is organized as follows. Section 2 recalls some lattice-theoretical concepts and terminology for mathematical morphology; in particular we discuss in detail distributivity, and its specializations called infinite and complete distributivity. It introduces also our notation, and recalls some known facts about adjunctions and Galois connections. Section 3 defines flat operators and gives their main properties. The Conclusion summarizes our results and links them to further perspectives.

This paper being already very long, we have left out some questions, which will be dealt with in further papers. We mentioned above that the commutation with thresholding, and with increasing mappings $V \to V$ (also called anamorphoses or contrast functions), takes two different forms for discrete grey-levels and for continuous ones; this will be analysed in a general framework. There are also some generalizations of flat operators, namely flat operators in several variables, and *semi-flat* operators (according to Heijmans [18]).

2. Lattices, Distributivity and Images

We assume that the reader is familiar with the notions of partial order, poset (partially ordered set), lattice, complete lattice, and power lattice, used in mathematical morphology. See Section 1.3 of [20] for a brief overview. Chapter 2 of [18] gives a broader exposition. A standard reference on posets and lattices is [6]. We will recall here some classical concepts not always dealt with in [20] (but which are described in [6, 18]).

Let us first introduce our notation. We consider mainly three types of lattices. First lattices of image values; they are denoted with capital letters V, W (for a general lattice), or T, U, R (for some specific lattices, see below), and we write their elements by lower-case letters $a, b, \ldots, y, z$, except for the least and greatest elements, written $\perp$ and $\top$ respectively. The order relation on the lattice, and the corresponding supremum and infimum operations are written $\leq$, $\bigvee$ and $\bigwedge$. Next, we consider lattices whose members are images, which can be of two types:

- The lattice $\mathcal{P}(E)$ of subsets of a space E, ordered by inclusion $\subseteq$, with supremum and infimum operations given by the union $\bigcup$ and the intersection $\bigcap$, and $\emptyset$ and E as least and greatest elements. Otherwise, parts of E are written by capital letters $X, Y, Z, \ldots$
- The *power lattice* V^E of functions $E \to V$, where V is a complete lattice of values, with the pointwise ordering: $F \leq G$ iff $F(p) \leq G(p)$ for all $p \in E$, and

the pointwise supremum and infimum operations:

$$\bigvee_{i \in I} F_i : E \to V : p \mapsto \bigvee_{i \in I} F_i(p) \quad \text{and}$$
$$\bigwedge_{i \in I} F_i : E \to V : p \mapsto \bigwedge_{i \in I} F_i(p).$$

The least and greatest functions are the ones with constant values $\perp$ and $\top$ respectively, we write them $C_\perp$ and $C_\top$, see (15) below. Otherwise functions $E \to V$ are written by capital letters $F, G, H, \ldots$

Let $2 = \{0, 1\}$, the binary lattice; then we have an isomorphism between $\mathcal{P}(E)$ and the power lattice 2^E of binary functions $E \to 2$.

Finally, we consider operators, that is maps transforming an image into an image, written with Greek letters (generally lowercase), except for some special operators, like the identity $\mathbf{id} : F \mapsto F$. For a lattice L of images, the operators constitute the power lattice L^L, with componentwise order, supremum and infimum operations: $\psi \leq \xi$ iff $\psi(F) \leq \xi(F)$ for all $F \in L$, $\left[\bigvee_{i \in I} \psi_i\right](F) = \left[\bigvee_{i \in I} \psi_i(F)\right]$, and similarly for $\bigwedge$. There is another operation on operators, the *composition*: the composition of ξ followed by ψ is $\psi\xi : F \mapsto \psi(\xi(F))$.

When $L = \mathcal{P}(E)$, the notation for the order relation and the supremum and infimum operations, will be the same for set operators $\mathcal{P}(E) \to \mathcal{P}(E)$ as for sets: we will thus write $\psi \subseteq \xi$, $\bigcup_{i \in I} \psi_i$ and $\bigcap_{i \in I} \psi_i$. In the power lattice $\mathcal{P}(E)^W$ of maps $W \to \mathcal{P}(E)$ (for any set W), we will adopt the same set-theoretical notation for the order, supremum and infimum, which are given by the componentwise inclusion, union and intersection; the same convention holds as well as for operators on $\mathcal{P}(E)^W$.

Let us now describe the specific lattices of values denoted by T, U and R.

1. T designates a lattice of numerically ordered grey-levels; it can be any subset of $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$ which is closed under nonvoid supremum and infimum operations (or equivalently: topologically closed); in practice, T will usually be $\overline{\mathbf{R}}$, $[a, b] = \{x \in \overline{\mathbf{R}} \mid a \leq x \leq b\}$ (with $a, b \in \overline{\mathbf{R}}$ and $a < b$), $\overline{\mathbf{Z}} = \mathbf{Z} \cup \{-\infty, +\infty\}$, or $[a \ldots b] = [a, b] \cap \overline{\mathbf{Z}}$ (with $a, b \in \overline{\mathbf{Z}}$ and $a < b$). In classical mathematical morphology [18], grey-level images are usually considered as numerical functions $E \to T$.
2. U designates the lattice of labels illustrated in Fig. 2. It is made of a finite family U_* of *proper labels*, to which two *dummy labels* $\perp$ and $\top$ are added; we assume that $|U_*| \geq 3$, so that $|U| \geq 5$. The order relation $\leq$ on U reduces to the following:

$$\forall u \in U, \quad u \leq u, \quad \perp \leq u, \quad u \leq \top.$$

   In particular, two distinct elements of U_* (proper labels) are not comparable for the order: we never have $u < u'$ for $u, u' \in U_*$. For more details see [34], where it is shown that the application of flat morphological operators to label images amounts to processing independently all zones having a given proper label.
3. R designates the set $\mathbf{R} \cup \{\infty\}$ or $\mathbf{Z} \cup \{\infty\}$ with the reference order w.r.t. a reference $r \in R \setminus \{\infty\}$, illustrated in Fig. 1, in other words for $a, b \in R \setminus \{\infty\}$, we have $a \leq_r b$ if a is between r and b, that is $r \leq a \leq b$ or $r \geq a \geq b$ (numerically); for every $a \in R$, we set $a \leq_r \infty$. Then R is a complete lattice with r and ∞ as least and greatest elements. When we want to specify the chosen reference element, we will write R_r; otherwise we can assume that $r = 0$.

Sometimes we will give definitions and properties for an abstract complete lattice (which can be a lattice of values, of images, of operators, or anything else). In this case we will write this lattice L or M, and its elements with lower-case letters.

We recall some standard notions concerning complete lattices [6, 13, 18, 20]. Let L be a complete lattice. Notice first the following identity, which will be used freely later on:

$$\forall X_i \in \mathcal{P}(L)\ (i \in I), \quad \bigvee\left(\bigcup_{i \in I} X_i\right) = \bigvee_{i \in I}\left(\bigvee X_i\right). \tag{3}$$

A subset S of L is called a *sup-generating family* of L if every element x of L is a supremum of elements of S; in fact we have then $x = \bigvee\{s \in S \mid s \leq x\}$. A *complete sublattice* of L is a part M of L which is a complete lattice for the order $\leq$ on L, *with the same supremum and infimum operations as in* L, and *with the same least and greatest elements as those of* L. In other words, for any part S of M, $\bigvee S$ and $\bigwedge S$ are elements of M (including for $S = \emptyset$, for which $\bigvee \emptyset$ and $\bigwedge \emptyset$ give the least and greatest element of L).

Let L, M be two complete lattices (which may be either distinct or equal) with least and greatest elements $\perp_L, \top_L$ in L and $\perp_M, \top_M$ in M. An increasing map

$\psi : L \to M$ verifies:

$$\forall x_i \in L\ (i \in I), \qquad \begin{cases} \psi\left(\bigvee_{i\in I} x_i\right) \geq \bigvee_{i\in I} \psi(x_i), \\ \psi\left(\bigwedge_{i\in I} x_i\right) \leq \bigwedge_{i\in I} \psi(x_i). \end{cases} \tag{4}$$

A *dilation* is a map $\delta : L \to M$ which commutes with the supremum operation, and an *erosion* is a map $\varepsilon : L \to M$ which commutes with the infimum operation:

$$\forall x_i \in L\ (i \in I), \quad \delta\left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I} \delta(x_i) \quad \text{and} \quad \varepsilon\left(\bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I} \varepsilon(x_i).$$

In particular $\delta(\perp_L) = \perp_M$ and $\varepsilon(\top_L) = \top_M$ (since the least element is equal to the empty supremum, and the greatest element is equal to the empty infimum). Dilations and erosions are increasing.

Given two maps $\delta : L \to M$ and $\varepsilon : M \to L$, the ordered pair (ε, δ) is an *adjunction* if

$$\forall x \in L, \quad \forall y \in M, \qquad \delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y).$$

It is well-known [13, 18, 20] that given $\delta : L \to M$ and $\varepsilon : M \to L$ such that (ε, δ) is an adjunction, ε is an erosion and δ is a dilation, $\delta\varepsilon\delta = \delta$, $\varepsilon\delta\varepsilon = \varepsilon$, $\delta\varepsilon$ is an opening on M and $\varepsilon\delta$ is a closing on L; conversely, given a dilation $\delta : L \to M$, there is a unique erosion $\varepsilon : M \to L$ such that (ε, δ) is an adjunction, and given an erosion $\varepsilon : M \to L$, there is a unique dilation $\delta : L \to M$ such that (ε, δ) is an adjunction.

There is a classical variant of the adjunction, which arises when we replace in M the order $\leq$ by its dual $\geq$. A map $\alpha : L \to M$ is an *anti-dilation* if it transforms the supremum into an infimum:

$$\forall x_i \in L\ (i \in I), \qquad \alpha\left(\bigvee_{i\in I} x_i\right) = \bigwedge_{i\in I} \alpha(x_i).$$

In particular $\delta(\perp_L) = \top_M$ (the empty supremum in L becomes the empty infimum in M). A pair of maps $\alpha : L \to M$ and $\beta : M \to L$ forms a *Galois connection* if

$$\forall x \in L, \quad \forall y \in M, \qquad y \leq \alpha(x) \Leftrightarrow x \leq \beta(y)\ .$$

Note that α and β play here symmetrical roles (contrarily to δ and ε in an adjunction), so we do not order the pair $\{\alpha, \beta\}$. Now the above adjunction property becomes the following:

Property 1. *Let L and M be two complete lattices (not necessarily distinct).*

1. *Given $\alpha : L \to M$ and $\beta : M \to L$ which form a Galois connection, α and β are anti-dilations. Furthermore, $\alpha\beta\alpha = \alpha$ and $\beta\alpha\beta = \beta$, $\alpha\beta$ and $\beta\alpha$ are closings (respectively on M and on L).*
2. *Given an anti-dilation $\alpha : L \to M$, there is a unique anti-dilation $\beta : M \to L$ such that α and β form a Galois connection; β is defined by setting for any $y \in M$: $\beta(y)$ is the greatest $x \in L$ such that $y \leq \alpha(x)$.*

2.1. Distributivity, Infinite and Complete

We consider here various properties of lattices, that will be relevant for the lattice V of values taken by a function. Most of these properties will extend to the lattice V^E of functions $E \to V$.

In a poset L, the order $\leq$ is *total* if for every $a, b \in L$, we have always $a \leq b$ or $b \leq a$; we say then that L is a *chain*, and we call a *complete chain* a totally ordered complete lattice.

Let L be any lattice; for any $a_0, a_1, x \in L$ we have

$$a_0 \leq a_1 \quad \Rightarrow \quad a_0 \vee (x \wedge a_1) \leq (a_0 \vee x) \wedge a_1.$$

One says that L is *modular* [6, 18] if this inequality is in fact an equality:

$$a_0 \leq a_1 \quad \Rightarrow \quad a_0 \vee (x \wedge a_1) = (a_0 \vee x) \wedge a_1. \tag{5}$$

For any $a, b, c \in L$ we have always

$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c) \ \text{ and}$$
$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c).$$

L is *distributive* [6, 18] if

$$\forall a, b, c \in L, \qquad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

It is known [6] that this is equivalent to requiring the dual condition:

$$\forall a, b, c \in L, \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Every distributive lattice is modular, but the converse is not true. An example of modular lattice which is not distributive is the one of vector subspaces (ordered by inclusion) of a vector space of dimension >1 [6]. The lattice U of labels (see Fig. 2) is also modular but non-distributive [34]. On the other hand the reference lattice R is not modular, hence not distributive.

There are several generalizations of distributivity for complete lattices [6, 18]. Two of them are often used in morphology [18], namely *infinite supremum distributivity* (in brief, *ISD*), and its dual, *infinite infimum distributivity* (in brief, *IID*). Let L be a complete lattice; we say that L is *infinitely supremum distributive*, or *ISD*, if

$$\forall a \in L,\ \forall b_i \in L\ (i \in I),\quad a \wedge \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \wedge b_i), \tag{6}$$

where I is a nonempty index set; we say that L is *infinitely infimum distributive*, or *IID*, if

$$\forall a \in L,\quad \forall b_i \in L\ (i \in I),\quad a \vee \left(\bigwedge_{i \in I} b_i \right) = \bigwedge_{i \in I} (a \vee b_i). \tag{7}$$

The two are not equivalent. For example the family of open sets of $\mathbf{R}^n$ constitutes a complete lattice which is ISD but not IID, while the family of closed sets is an IID complete lattice which is not ISD.

Note that ISD implies that for any integer $n > 1$ we have

$$\bigwedge_{i=1}^{n} \left(\bigvee_{j \in J_i} a_{i,j} \right) = \bigvee_{\substack{(j_1,\ldots,j_n) \\ \in J_1 \times \cdots \times J_n}} \left(a_{1,j_1} \wedge \cdots \wedge a_{n,j_n} \right) = \bigvee_{\substack{(j_1,\ldots,j_n) \\ \in J_1 \times \cdots \times J_n}} \bigwedge_{i=1}^{n} a_{i,j_i}. \tag{8}$$

The dual formula (with $\vee$ and $\wedge$ interverted) holds for IID. This suggests a stronger distributivity law, where (8) or its dual would be extended to n being infinite, in other words a distributivity between infinite suprema and infinite infima. This is called *extended distributivity* or *complete distributivity* [6, 18]. Consider a set I indexing a family of index sets J_i for $i \in I$; a *choice map* associates to each $i \in I$ an element of J_i, in other words it is a function $\varphi : I \to \bigcup_{i \in I} J_i$ such that for every $i \in I$ we have $\varphi(i) \in J_i$; write $\Phi(I)$ for the set of choice maps. Clearly, a complete lattice L verifies for any $a_{i,j} \in L$ $(i \in I,\ j \in J_i)$:

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} a_{i,j} \geq \bigvee_{\varphi \in \Phi(I)} \bigwedge_{i \in I} a_{i,\varphi(i)} \quad \text{and} \quad \bigvee_{i \in I} \bigwedge_{j \in J_i} a_{i,j} \leq \bigwedge_{\varphi \in \Phi(I)} \bigvee_{i \in I} a_{i,\varphi(i)}. \tag{9}$$

Complete distributivity means that the above inequalities are in fact equalities. More precisely: the *extended supremum distributivity* law is

$$\forall a_{i,j} \in L\ (i \in I,\ j \in J_i),\quad \bigwedge_{i \in I} \bigvee_{j \in J_i} a_{i,j} = \bigvee_{\varphi \in \Phi(I)} \bigwedge_{i \in I} a_{i,\varphi(i)}, \tag{10}$$

and the dual *extended infimum distributivity* law is

$$\forall a_{i,j} \in L\ (i \in I,\ j \in J_i),\quad \bigvee_{i \in I} \bigwedge_{j \in J_i} a_{i,j} = \bigwedge_{\varphi \in \Phi(I)} \bigvee_{i \in I} a_{i,\varphi(i)}. \tag{11}$$

Here we can always suppose that $I \neq \emptyset$ and $J_i \neq \emptyset$ for all $i \in I$, otherwise (10) and (11) are trivially verified (under the form $\top = \top$ or $\bot = \bot$). We illustrate extended supremum distributivity in Fig. 3. Note that for I finite, (10) reduces to (8).

In fact, the two laws are equivalent. This can be shown directly by applying (10) to the right-hand side of (11) (and vice-versa), see for example [27], pp. 78, 79. Earlier, Raney [30] showed that each of (10) and (11) is equivalent to the following autodual condition: "L is a complete homomorphic image of a complete ring of sets"; this means that there is a

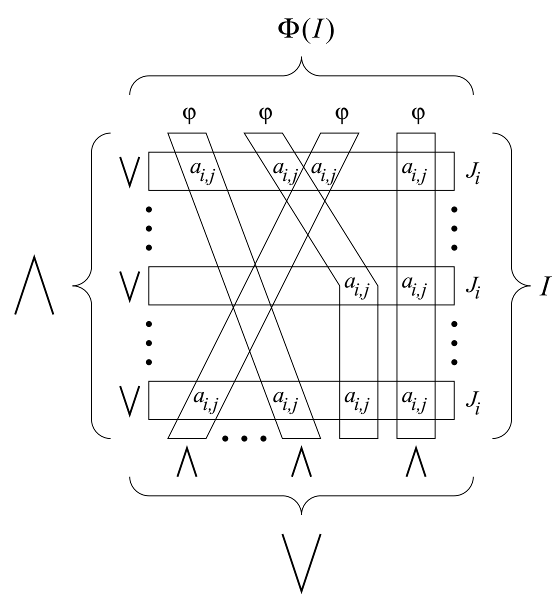

Figure 3. Extended supremum distributivity: on each line, we take the supremum, then we take the infimum of the results; alternatively, we can take the infimum on each transversal crossing each line once, then take the supremum of results.

set A, a family $\mathcal{B}$ of parts of A which is closed under arbitrary unions and intersections (for $B_i \in \mathcal{B}$, $i \in I$, $\bigcup_{i\in I} B_i, \bigcap_{i\in I} B_i \in \mathcal{B}$), and a surjective map $\psi : \mathcal{B} \to L$ such that for $B_i \in \mathcal{B}, i \in I$, $\psi(\bigcup_{i\in I} B_i) = \bigvee_{i\in I} \psi(B_i)$ and $\psi(\bigcap_{i\in I} B_i) = \bigwedge_{i\in I} \psi(B_i)$. We say thus that the complete lattice L is *completely distributive* if (10) or equivalently (11) holds. Note that Matheron [27] says "complete" and "total" distributivity to mean infinite and complete distributivity respectively.

We can express (10, 11) without index sets. Given a family $\mathcal{X}$ of parts of L, a *choice map* on $\mathcal{X}$ is a map $\varphi : \mathcal{X} \to \bigcup \mathcal{X}$ such that for any $X \in \mathcal{X}$ we must have $\varphi(X) \in X$; let $\Phi(\mathcal{X})$ be the set of choice maps on $\mathcal{X}$; then (10) can be written as

$$\forall \mathcal{X} \subseteq \mathcal{P}(L), \qquad \bigwedge_{X\in\mathcal{X}} \bigvee X = \bigvee_{\varphi\in\Phi(\mathcal{X})} \bigwedge_{X\in\mathcal{X}} \varphi(X), \qquad (12)$$

while (11) becomes

$$\forall \mathcal{X} \subseteq \mathcal{P}(L), \qquad \bigvee_{X\in\mathcal{X}} \bigwedge X = \bigwedge_{\varphi\in\Phi(\mathcal{X})} \bigvee_{X\in\mathcal{X}} \varphi(X). \qquad (13)$$

It is known [6] that every complete chain is completely distributive. The same holds then for a product of chains. The lattice $\mathcal{P}(E)$ is also completely distributive.

The above expressions (10, 11), or (12, 13), for complete distributivity are not always easy to use. We will give an equivalent definition, which does not rely on choice functions. Our approach derives from the work of Bruns [8], and to a lesser extent from that of Papert [28].

Define the relation $\lhd$ on L as follows [8]: for $w, x \in L$,

$$w \lhd x \Leftrightarrow \Big[\forall Y \subseteq L, \quad x \leq \bigvee Y \Rightarrow \exists y \in Y, w \leq y\Big]. \qquad (14)$$

Heijmans [18] wrote $w \ll x$ and said that w *is below* x.

For $w \lhd x$, we have $w \leq x$ [8, 18]: this follows by taking $Y = \{x\}$ in (14). Also, $v \leq w \lhd x \leq y$ implies $v \lhd y$ [8]. Finally, the least element $\perp$ of L verifies $\perp \lhd x \Leftrightarrow \perp < x$ [18]. We say that w is a *sup-factor* of x if $w \lhd x$ and $w > \perp$.

Let us give a few examples:

- For $L = \mathcal{P}(E)$, the sup-factors of a subset X of E are precisely the singletons included in X.
- For L being the set of numerical functions $E \to \bar{\mathbf{Z}}$, define the impulse $i_{h,v}$ ($h \in E, v \in \mathbf{Z}$) by $i_{h,v}(h) = v$ and $i_{h,v}(p) = -\infty$ for $p \neq h$; then the sup-factors of a function F are the impulses $i_{h,v}$ such that $i_{h,v} \leq F$, that is $v \leq F(h)$.
- In a complete lattice L, an *atom* is some $a \in L$ which covers $\perp$, that is such that $\perp < a$ but there is no $b \in L$ with $\perp < b < a$. The lattice L is *atomic* if its atoms constitute a sup-generating family. As noticed in [18], p. 339, in an atomic lattice L, a sup-factor of $x \in L$ must necessarily be an atom $\leq x$; Heijmans states also the converse (for an atom a such that $a \leq x$, we have $a \lhd x$), but this is wrong in general, this property holds iff $(L, \leq)$ is isomorphic to $(\mathcal{P}(A), \subseteq)$, where A is the set of atoms.
- An element x of a complete lattice L is a *strong coprime* [27] if for any $Y \subseteq L$, $x \leq \bigvee Y$ implies that there is some $y \in Y$ with $x \leq y$. By (14), this is equivalent to $x \lhd x$; in particular, $x \neq \perp$, so x is its own sup-factor.

Other examples can be found in [18], p. 340. We give now our characterization of complete distributivity:

Lemma 2. *The complete lattice L is completely distributive iff every element of L is the supremum of its sup-factors:* $\forall x \in L$, $x = \bigvee\{w \in L \mid \perp < w \lhd x\}$.

Proof: Suppose that L is completely distributive, and let $x \in L$; as $\perp$ is the supremum of the empty set of its sup-factors, we can assume that $x > \perp$. Let $\mathcal{Z} = \{Z \subseteq L \mid x \leq \bigvee Z\}$. Note that $\{x\} \in \mathcal{Z}$, with $\bigvee\{x\} = x$, while $x \leq \bigvee Z$ for any other $Z \in \mathcal{Z}$ (in particular, $Z \neq \emptyset$). Hence $\bigwedge_{Z\in\mathcal{Z}} \bigvee Z = x$. For every choice map $\varphi \in \Phi(\mathcal{Z})$, let $w_\varphi = \bigwedge_{Z\in\mathcal{Z}} \varphi(Z)$. Applying (12), we get

$$x = \bigwedge_{Z\in\mathcal{Z}} \bigvee Z = \bigvee_{\varphi\in\Phi(\mathcal{Z})} \bigwedge_{Z\in\mathcal{Z}} \varphi(Z) = \bigvee_{\varphi\in\Phi(\mathcal{Z})} w_\varphi.$$

Take any $\varphi \in \Phi(\mathcal{Z})$, and let $Y \subseteq L$ with $x \leq \bigvee Y$; then $Y \in \mathcal{Z}$, and as $w_\varphi = \bigwedge_{Z\in\mathcal{Z}} \varphi(Z)$, we get $w_\varphi \leq \varphi(Y)$, but as $\varphi \in \Phi(\mathcal{Z})$, we have $\varphi(Y) \in Y$; we have thus shown that $w_\varphi \lhd x$ for all $\varphi \in \Phi(\mathcal{Z})$. Thus

$$\begin{aligned} x = \bigvee_{\varphi\in\Phi(\mathcal{Z})} w_\varphi &\leq \{w \in L \mid w \lhd x\} \\ &= \{w \in L \mid \perp < w \lhd x\}. \end{aligned}$$

But we explained above that for $w \lhd x$ we have $w \leq x$, so the reverse inequality $\{w \in L \mid \perp < w \lhd x\} \leq x$ holds, and we derive the equality.

Suppose now that $x = \bigvee\{w \in L \mid \perp < w \lhd x\}$ for all $x \in L$. Take any $a_{i,j} \in L$ ($i \in I$, $j \in J_i$), and let

$x = \bigwedge_{i \in I} \bigvee_{j \in J_i} a_{i,j}$. We can assume that $I \neq \emptyset$ and $J_i \neq \emptyset$ for all $i \in I$, otherwise (10) is trivially verified. Then for a fixed $i \in I$ we have $x \leq \bigvee_{j \in J_i} a_{i,j}$, so for any $w \lhd x$, there is some $j \in J_i$ with $w \leq a_{i,j}$. We associate thus to each sup-factor w of x a choice map φ_w where for every $i \in I$, $\varphi_w(i)$ is a $j \in J_i$ chosen such that $w \leq a_{i,j}$; in other words $w \leq a_{i,\varphi_w(i)}$ for all $i \in I$, so we get $w \leq \bigwedge_{i \in I} a_{i,\varphi_w(i)}$. As $x = \bigvee\{w \in L \mid \bot < w \lhd x\}$, we get

$$x = \bigvee_{\bot < w \lhd x} w \leq \bigvee_{\bot < w \lhd x} \bigwedge_{i \in I} a_{i,\varphi_w(i)}.$$

But the choice functions φ_w constitute only a part of $\Phi(I)$, so we get $x \leq \bigvee_{\varphi \in \Phi(I)} \bigwedge_{i \in I} a_{i,\varphi(i)}$, that is:

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} a_{i,j} \leq \bigvee_{\varphi \in \Phi(I)} \bigwedge_{i \in I} a_{i,\varphi(i)}.$$

Given the reverse inequality in (9), we obtain the extended supremum distributivity law (10). □

A consequence of this lemma is the well-known fact [27] that a complete lattice having a sup-generating family of strong coprimes (elements which are their own sup-factors) is completely distributive.

In this paper, we will be considering the power lattice V^E of functions $E \to V$ for a space E and a complete lattice V of values. It is clear that whenever V is modular, distributive, infinitely supremum or infimum distributive, or completely distributive, then V^E will share that property. In particular, if V is a chain, V^E will be completely distributive. This is for example the case if we take for V the lattice T of numerically ordered grey-levels; similarly, for multivalued images we take $V = T^n$, so V^E is isomorphic to $T^{n \times E}$, which is completely distributive. Also the binary lattice $2 = \{0, 1\}$ is a chain, so 2^E, which is isomorphic to $\mathcal{P}(E)$, is completely distributive.

3. Flat Operators

In this section, we will show how the construction of increasing flat operators on grey-level images, given in [16] and Chapters 10 and 11 of [18] (see (1, 2) above), can be extended to functions having values in an arbitrary complete lattice V; we will see that many features of flat operators known in the grey-level case remain valid in this general framework. Some properties will be obtained without any requirement on the complete lattice V of values, so they are valid for multivalued images, label images, images with values in the completed reference lattice, etc. On the other hand, a few properties do not hold unless we assume the lattice V to be infinitely or completely distributive (a property satisfied by binary, grey-level and multivalued images).

We stress the following point: throughout this section, we restrict ourselves to *increasing* set operators (and the flat operators derived from them will also be increasing). Indeed, the standard construction for flat operators does not work correctly for non-increasing operators (see [16]).

Several of our results stated here were proved in [33], and we indicate this source in their statement; however for some of them, we nevertheless include a proof, which means that this is a new proof, different from the one given in [33].

Let us now introduce some general notation. We consider a space E of points (which can in fact be any set), and an arbitrary complete lattice V of image values, whose least and greatest elements are written $\bot$ and $\top$ respectively. Images will then be functions $E \to V$, we call such functions V*-images*. The *power lattice* V^E is the complete lattice of V-images $E \to V$, with the componentwise ordering. Image processing transformations are called *operators*; in the case of binary images, they correspond to maps $\mathcal{P}(E) \to \mathcal{P}(E)$, while in the case of V-images, they are considered as maps $V^E \to V^E$.

For every $v \in V$, write C_v for the function $E \to V$ with constant value v:

$$\forall p \in E, \quad C_v(p) = v. \tag{15}$$

We see in particular that the least and greatest elements of the lattice V^E are the constant functions $C_\bot$ and $C_\top$ respectively. For any $B \subseteq E$ and $v \in V$, the *cylinder of base B and level v* is the function $C_{B,v}$ defined by

$$\forall p \in E, \qquad C_{B,v}(p) = \begin{cases} v & \text{if } p \in B, \\ \bot & \text{if } p \notin B. \end{cases} \tag{16}$$

Note in particular that $C_v = C_{E,v}$. Also, for $h \in E$ and $v \in V$, the *impulse* $i_{h,v}$ is the cylinder $C_{\{h\},v}$, thus

$$\forall p \in E, \qquad i_{h,v}(p) = \begin{cases} v & \text{if } p = h, \\ \bot & \text{if } p \neq h. \end{cases} \tag{17}$$

3.1. Thresholding, Stacking and Flat Operators

We introduce now our main concepts: thresholding and stacking, whose properties are investigated. This leads then to the definition of flat operators.

For a V-image F (a function $F : E \to V$) and a value $v \in V$, we define the *threshold set* $X_v(F)$ as in the definition of Heijmans, see (1) above:

$$\forall v \in V, \quad X_v(F) = \{p \in E \mid F(p) \geq v\}. \tag{18}$$

It is easily seen that $X_v(F)$ increases with F (that is, $F \leq G$ implies $X_v(F) \subseteq X_v(G)$), but decreases with v:

$$v \leq w \Rightarrow X_w(F) \subseteq X_v(F). \tag{19}$$

The following is related to an argument on p. 8 of [27]:

Lemma 3. *Let $F \in V^E$ be fixed. Define the maps $\theta_F : V \to \mathcal{P}(E)$ and $\eta_F : \mathcal{P}(E) \to V$ by*

$$\forall v \in V, \quad \theta_F(v) = X_v(F) \text{ and}$$
$$\forall Z \in \mathcal{P}(E), \quad \eta_F(Z) = \bigwedge_{p \in Z} F(p).$$

Then θ_F and η_F form a Galois connection. Furthermore:

1. *For any $v \in V$, we set*

$$s(v, F) = \bigwedge \{F(p) \mid p \in X_v(F)\}; \tag{20}$$

then $s(v, F) = \eta_F \theta_F(v)$, $s(v, F) \geq v$ and $X_{s(v,F)}(F) = X_v(F)$.

2. *For any $Z \in \mathcal{P}(E)$, we set*

$$S(Z, F) = X_v(F) \quad \text{for} \quad v = \bigwedge_{p \in Z} F(p); \tag{21}$$

then $S(Z, F) = \theta_F \eta_F(Z)$, $S(Z, F) \supseteq Z$ and $\bigwedge_{p \in S(Z,F)} F(p) = \bigwedge_{p \in Z} F(p)$.

Proof: For $v \in V$ and $Z \in \mathcal{P}(E)$, $Z \subseteq \theta_F(v)$ means that $Z \subseteq X_v(F)$, in other words $\forall p \in Z$, $F(p) \geq v$, which is equivalent to $\bigwedge_{p \in Z} F(p) \geq v$, that is $v \leq \eta_F(Z)$. Hence θ_F and η_F form a Galois connection. This implies in particular (see Property 1) that $\eta_F \theta_F$ and $\theta_F \eta_F$ are closings on V, and that $\theta_F \eta_F \theta_F = \theta_F$ and $\eta_F \theta_F \eta_F = \eta_F$. For any $v \in V$,

$$\eta_F \theta_F(v) = \eta_F(\theta_F(v)) = \eta_F(X_v(F)) = \bigwedge_{p \in X_v(F)} F(p) = s(v, F).$$

As $\eta_F \theta_F$ is a closing, $s(v, F) = \eta_F \theta_F(v) \geq v$, and the equality $\theta_F \eta_F \theta_F = \theta_F$ gives

$$X_{s(v,F)}(F) = \theta_F(s(v, F)) = \theta_F(\eta_F \theta_F(v)) = \theta_F(v) = X_v(F).$$

For any $Z \in \mathcal{P}(E)$, $S(Z, F) = X_v(F) = \theta_F(v)$ for $v = \eta_F(Z)$, so $S(Z, F) = \theta_F(\eta_F(Z)) = \theta_F \eta_F(Z)$. As $\theta_F \eta_F$ is a closing, $S(Z, F) \supseteq Z$, and the equality $\eta_F \theta_F \eta_F = \eta_F$ gives

$$\bigwedge_{p \in S(Z,F)} F(p) = \eta_F(S(Z, F)) = \eta_F(\theta_F \eta_F(Z)) = \eta_F(Z) = \bigwedge_{p \in Z} F(p).$$

□

This Lemma will be used for proving several results in this paper and in further ones. By Property 1, θ_F is an anti-dilation, which means that for $W \subseteq V$,

$$w = \bigvee W \Rightarrow \quad X_w(F) = \bigcap_{v \in W} X_v(F). \tag{22}$$

This generalizes (19).

Definition 4.

1. A *stack* (on V) is a decreasing map $\Xi : V \to \mathcal{P}(E)$, i.e., it associates to every $v \in V$ a subset $\Xi(v)$ of E, and for $v, w \in V$ with $v \leq w$ we have $\Xi(w) \subseteq \Xi(v)$. Write $\mathcal{S}(V, E)$ for the family of stacks on V.
2. Given a V-image F, the *threshold stack of F* is the map $\Theta F : V \to \mathcal{P}(E)$ given by

$$\forall v \in V, \qquad \Theta F(v) = X_v(F);$$

by (19), it is a stack on V (i.e., for $v \leq w$ we have $\Theta F(w) \subseteq \Theta F(v)$).
3. Given a stack Ξ on V and an increasing operator ψ on $\mathcal{P}(E)$, $\psi_V \Xi$ is the map $V \to \mathcal{P}(E)$ defined by

$$\psi_V \Xi(v) = \psi\left(\Xi(v)\right);$$

it is a stack on V. The map $\psi_V : \mathcal{S}(V, E) \to \mathcal{S}(V, E) : \Xi \mapsto \psi_V \Xi$ is called the *action of ψ on $\mathcal{S}(V, E)$*.

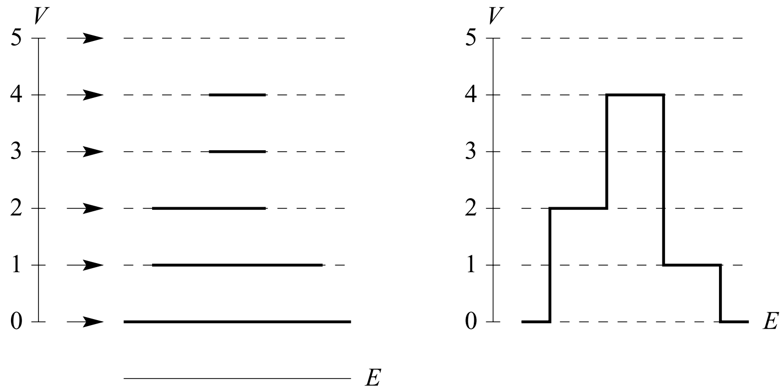

Figure 4. Left: $V = \{0, 1, 2, 3, 4, 5\}$, E is a segment, and we show a stack on V. Right: its superposition.

4. Given a stack Ξ on V, the *superposition* of Ξ is the V-image $\Sigma\,\Xi$ defined by

$$\Sigma\,\Xi = \bigvee_{v \in V} C_{\Xi(v),v}; \tag{23}$$

in other words, for every point $p \in E$ we have

$$\Sigma\,\Xi(p) = \bigvee\{v \in V \mid p \in \Xi(v)\}. \tag{24}$$

This definition calls several comments:

1. The stack ΘF is the map θ_F of Lemma 3.
2. $\mathcal{P}(E)^V$ is a power lattice for the order given by componentwise inclusion, with supremum and infimum given by componentwise union and intersection. $\mathcal{S}(V, E)$ being a part of $\mathcal{P}(E)^V$, it inherits thus this ordering: $\Xi_0 \subseteq \Xi_1$ iff $\Xi_0(v) \subseteq \Xi_1(v)$ for all $v \in V$. In fact $\mathcal{S}(V, E)$ is a *complete sublattice* of $\mathcal{P}(E)^V$, in the sense that a (componentwise) union or intersection of stacks remains a stack.
3. The operator ψ must necessarily be increasing in order to guarantee that for $v \leq w$, $\Xi(w) \subseteq \Xi(v)$ will lead to $\psi(\Xi(w)) \subseteq \psi(\Xi(v))$. Then the action ψ_V of ψ on $\mathcal{S}(V, E)$ is increasing, in the sense that for two stacks Ξ and Ξ', $\Xi \subseteq \Xi'$ implies $\psi_V\,\Xi \subseteq \psi_V\,\Xi'$. This action ψ_V is a homomorphism for both the composition operation and complete lattice structure of operators, in the sense that $[\xi\psi]_V = \xi_V\psi_V$, the identity operator **id** has its action $\mathbf{id}_V$ equal to the identity on stacks, and for a family ψ_i $(i \in I)$ of operators, $[\bigcup_{i \in I} \psi_i]_V = \bigcup_{i \in I} \psi_{i\,V}$ and $[\bigcap_{i \in I} \psi_i]_V = \bigcap_{i \in I} \psi_{i\,V}$.

We illustrate a stack and its superposition in Fig. 4.

Lemma 5. *The maps* $\Theta : V^E \to \mathcal{S}(V, E) : F \mapsto \Theta F$ *and* $\Sigma : \mathcal{S}(V, E) \to V^E : \Xi \mapsto \Sigma\,\Xi$ *form an adjunction* (Θ, Σ).

Proof: For $\Xi \in \mathcal{S}(V, E)$ and $F \in V^E$, $\Sigma\,\Xi \leq F$ means $\bigvee_{v \in V} C_{\Xi(v),v} \leq F$, that is $\forall v \in V$, $C_{\Xi(v),v} \leq F$, in other words $\forall v \in V$, $\forall p \in \Xi(v)$, $v \leq F(p)$, equivalently $\forall v \in V$, $\Xi(v) \subseteq X_v(F) = \Theta F(v)$, which means $\Xi \subseteq \Theta F$. □

Thus Θ is an erosion, which means that for a family F_i $(i \in I)$ of V-images, we have for every $v \in V$:

$$X_v\left(\bigwedge_{i \in I} F_i\right) = \bigcap_{i \in I} X_v(F_i). \tag{25}$$

In particular, X_v is increasing: $F \leq G$ implies $X_v(F) \subseteq X_v(G)$ (as said above).

Lemma 6. *The map* $\Theta : V^E \to \mathcal{S}(V, E) : F \mapsto \Theta F$ *is injective, the map* $\Sigma : \mathcal{S}(V, E) \to V^E : \Xi \mapsto \Sigma\,\Xi$ *is surjective, and* $\Sigma\Theta$ *is the identity on* V^E.

Proof: Let $F \in V^E$ and $p \in E$; set $V_{p,F} = \{v \in V \mid p \in X_v(F)\}$. By (24) we have $\Sigma\Theta F(p) = \bigvee\{v \in V \mid p \in X_v(F)\} = \bigvee V_{p,F}$. For $v \in V_{p,F}$, we have $F(p) \geq v$; however $p \in X_{F(p)}(F)$, that is $F(p) \in V_{p,F}$; we get thus $\bigvee V_{p,F} = F(p)$. Hence $\Sigma\Theta F = F$ for all $F \in V^E$. This implies that Σ is surjective and Θ is injective. □

The following is implicitly contained in Chapter 10 of [18] and in [27], p. 8:

Lemma 7. *Let* Ξ *be a stack on* V. *Then the following are equivalent:*

1. *There is some* $F \in V^E$ *such that* $\Xi = \Theta F$.
2. $\Xi = \Theta\Sigma\Xi$.
3. Ξ *is an anti-dilation.*

Proof: Item 2 implies item 1 by taking $F = \Sigma\Xi$. Item 1 implies item 3, because ΘF is the map θ_F of Lemma 3, forming a Galois connection with η_F, so that it is an anti-dilation by Property 1.

Let us show that item 3 implies item 2. Let the stack Ξ be an anti-dilation. By Lemma 6 and the adjunction property, $\Theta\Sigma$ is a closing, so $\Xi \subseteq \Theta\Sigma\Xi$. Let $v \in V$ and $p \in \Theta\Sigma\Xi(v)$; then $p \in X_v(\Sigma\Xi)$, that is $v \leq \Sigma\Xi(p) = \bigvee\{w \in V \mid p \in \Xi(w)\}$. As Ξ is an anti-dilation, we get

$$\begin{aligned}\Xi(v) &\supseteq \Xi\Big(\bigvee\{w \in V \mid p \in \Xi(w)\}\Big) \\ &= \bigcap\{\Xi(w) \mid w \in V,\ p \in \Xi(w)\}\end{aligned}$$

which implies that $p \in \Xi(v)$. Therefore $\Theta\Sigma\Xi(v) \subseteq \Xi(v)$ for every $v \in V$, and from the double inclusion $\Xi \subseteq \Theta\Sigma\Xi$ and $\Theta\Sigma\Xi \subseteq \Xi$, the equality follows. □

Combining the last two lemmas, we get:

Corollary 8. *Let* $\mathcal{S}_{ad}(V, E)$ *be the set of stacks on V which are anti-dilations. Then* $\mathcal{S}_{ad}(V, E)$ *is a complete lattice isomorphic to* V^E*, the isomorphism being given by the map* $\Theta : V^E \to \mathcal{S}_{ad}(V, E) : F \mapsto \Theta F$ *and the restriction of the map* Σ *to* $\mathcal{S}_{av}(V, E)$*:* $\mathcal{S}_{av}(V, E) \to V^E : \Xi \mapsto \Sigma\Xi$.

We can now construct flat operators on V^E as in [16] (see (2) above):

Definition 9. Let ψ be an increasing operator on $\mathcal{P}(E)$. The *flat operator corresponding to* ψ, or the *flat extension of* ψ, is the operator $\psi^V : V^E \to V^E$ on V-images, defined by setting for any V-image F:

$$\psi^V(F) = \Sigma\psi_V\Theta F; \tag{26}$$

in other words (see (23)),

$$\psi^V(F) = \bigvee_{v \in V} C_{\psi(X_v(F)),v}, \tag{27}$$

so that for every point $p \in E$ we have by (24):

$$\psi^V(F)(p) = \bigvee\{v \in V \mid p \in \psi(X_v(F))\}. \tag{28}$$

This latter Eq. (28) is (up to a change of notation) the same formula as in Section 5 of [16]. When V is the lattice T of numerically ordered grey-levels, one associates to a function F its *umbra* (or *hypograph*)

$$U(F) = \{(h, v) \mid h \in E, v \in T, v \leq F(h)\};$$

then $X_v(F)$ is the horizontal cross-section of $U(F)$ at height v. Note that (25) translates in terms of umbras as:

$$U\left(\bigwedge_{i \in I} F_i\right) = \bigcap_{i \in I} U(F_i).$$

Intuitively, we apply ψ to each horizontal cross-section of the umbra $U(F)$ of the function F, and take the upper envelope of the modified umbra.

We illustrate this in Fig. 5 for a one-valued (constant) function, and for a two-valued function. We will discuss such functions further in Section 3.2.

The following result is an immediate consequence of the facts that Θ and Σ are increasing, while an increasing operator on sets has an increasing action on stacks:

Proposition 10 ([33]). *For an increasing operator* ψ *on* $\mathcal{P}(E)$*,* ψ^V *is an increasing operator on* V^E.

3.2. Images with Specific Sets of Values

In Definition 4 we have taken threshold sets $X_v(F)$ for all values $v \in V$. In fact, some values of v are not necessary for this purpose:

- The value $v = \bot$ is redundant in all the above formulas.
- For $T = \overline{\mathbf{R}}$ or $\overline{\mathbf{Z}}$, let $T' = T \setminus \{\pm\infty\}$. It has been stressed in [32] that we should not take into account infinite grey-levels $t = \pm\infty$ for the umbra $U(F) = \{(h, t) \mid t \leq F(h)\}$, the impulses $i_{h,t}$ (see (17)), and the threshold sets $X_t(F)$ (see (18)). Indeed, not only $\bot = -\infty$, but also $\top = +\infty$ is redundant, because $+\infty = \sup T'$, so that $X_{+\infty}(F) = \bigcap_{t \in T'} X_t(F)$ by (22). Thus we can restrict such formulas to $t \in T'$.
- For RGB colour images, the flat extension of a set operator can be constructed by applying the grey-level flat operator to each of the red, green, and blue components of the image, and joining the results. This amounts to considering only red $(r, \bot, \bot)$, green $(\bot, g, \bot)$, and blue $(\bot, \bot, b)$ threshold values v for the threshold sets $X_v(F)$. A similar remark applies to multivalued images.

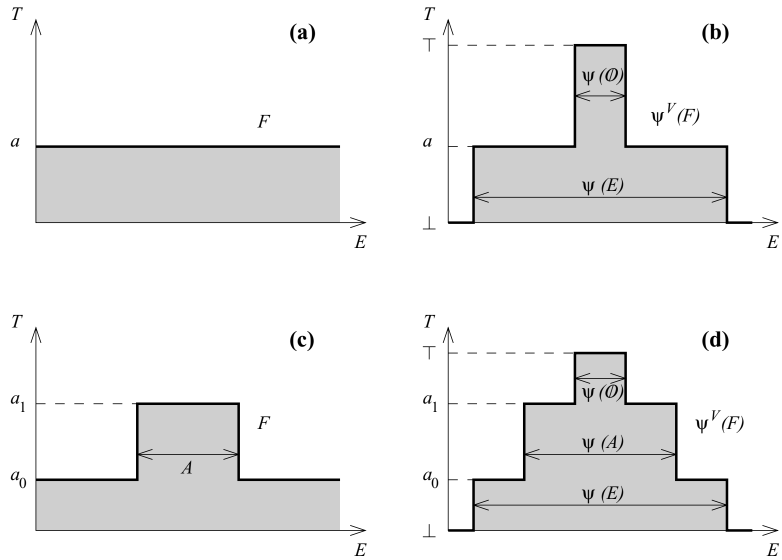

Figure 5. (a) The function $F = C_a$. (b) The function $\psi^T(F)$ is obtained by applying ψ to the horizontal cross-sections of the umbra of F (shown in light grey), and superposing the results. We get $\psi^T(F) = C_{\psi(E),a} \vee C_{\psi(\emptyset),\top}$. (c) The function $F = C_{a_0} \vee C_{A,a_1}$, where $a_0 < a_1$. (d) We get $\psi^T(F) = C_{\psi(E),a_0} \vee C_{\psi(A),a_1} \vee C_{\psi(\emptyset),\top}$.

More generally, it is easily derived from (22) that in Definition 9 one can replace V by a sup-generating family:

Proposition 11 ([33]). *Let V_s be a sup-generating subset of V. Then for any increasing operator ψ on $\mathcal{P}(E)$ and any V-image F, the construction of $\psi^V(F)$ in Definition 9 can be based on thresholds $X_v(F)$ only for $v \in V_s$, in other words (see (27))*

$$\psi^V(F) = \bigvee_{v \in V_s} C_{\psi(X_v(F)),v}, \tag{29}$$

so that for every point $p \in E$ we have (see (28))

$$\psi^V(F)(p) = \bigvee\{v \in V_s \mid p \in \psi(X_v(F))\}. \tag{30}$$

For example, with $V = T = \overline{\mathbf{R}}$, we take $V_s = T' = \mathbf{R}$ or $V_s = \mathbf{Q}$, while with $V = T = \overline{\mathbf{Z}}$, we take $V_s = T' = \mathbf{Z}$.

A consequence of this result deals with the case where V is a product lattice: $V = W_1 \times \cdots \times W_n$, with componentwise ordering $(w_1, \ldots, w_n) \leq (w'_1, \ldots, w'_n)$ iff $w_i \leq w'_i$ for all $i = 1, \ldots, n$. We define for $i = 1, \ldots, n$ the *i-th projection* $\pi_i : V \to W_i : (w_1, \ldots, w_n) \mapsto w_i$; it can naturally be extended to V-images as $\Pi_i : V^E \to W_i^E : F \mapsto \Pi_i(F)$ by applying it pointwise: $\Pi_i(F)(p) = \pi_i(F(p))$. Now a flat operator on V-images amounts to applying to each i-th projection the flat operator on W_i-images:

Proposition 12. *Let $V = W_1 \times \cdots \times W_n$ and let ψ be an increasing operator on $\mathcal{P}(E)$. Then for every V-image F we have $\Pi_i(\psi^V(F)) = \psi^{W_i}(\Pi_i(F))$ for all $i = 1, \ldots, n$.*

Proof: Note first that the i-th projection distributes both supremum and infimum operations: $\pi_i(\bigvee X) = \bigvee_{x \in X} \pi_i(x)$ and $\pi_i(\bigwedge X) = \bigwedge_{x \in X} \pi_i(x)$. For $i = 1, \ldots, n$, let $\rho_i : W_i \to V$ be defined by

$$\forall w \in W_i, \qquad \pi_j(\rho_i(w)) = \begin{cases} w & \text{if } j = i, \\ \bot & \text{if } j \neq i. \end{cases}$$

In other words, $\rho_i(w)$ is the vector having w at i-th position, and $\bot$ everywhere else. Then (π_i, ρ_i) is an adjunction: for $w \in W_i$ and $v \in V$, $\rho_i(w) \leq v$ iff $w \leq \pi_i(v)$.

For every $v \in V$, we see easily that $v = \bigvee_{i=1}^n \rho_i(\pi_i(v))$, because the left and right terms have the same j-th projection for $j = 1, \ldots, n$:

$$\pi_j\left(\bigvee_{i=1}^n \rho_i(\pi_i(v))\right) = \bigvee_{i=1}^n \pi_j(\rho_i(\pi_i(v))) = \pi_j(v).$$

Thus $W = \bigcup_{i=1}^n \{\rho_i(w) \mid w \in W_i\}$ is a sup-generating family of V. Applying Proposition 11, we get for each

$F \in V^E$:

$$\psi^V(F) = \bigvee_{w \in W} C_{\psi(X_w(F)),w} = \bigvee \bigcup_{i=1}^{n} \{C_{\psi(X_{\rho_i(w)}(F)),\rho_i(w)} \mid w \in W_i\}.$$

For $p \in E$, we have $p \in X_{\rho_i(w)}(F)$ iff $\rho_i(w) \leq F(p)$, which is, by the adjunction (π_i, ρ_i), equivalent to $w \leq \pi_i(F(p))$, that is $p \in X_w(\Pi_i(F))$. Thus $X_{\rho_i(w)}(F) = X_w(\Pi_i(F))$, and combining this with (3), we get:

$$\psi^V(F) = \bigvee_{i=1}^{n} \bigvee \{C_{\psi(X_w(\Pi_i(F))),\rho_i(w)} \mid w \in W_i\}.$$

Then for $j = 1, \ldots, n$, as Π_j distributes the supremum, we get

$$\Pi_j(\psi^V(F)) = \bigvee_{i=1}^{n} \bigvee \{\Pi_j(C_{\psi(X_w(\Pi_i(F))),\rho_i(w)}) \mid w \in W_i\}.$$

We check easily from (16) that for $B \subseteq E$ and $v \in V$, $\Pi_j(C_{B,v}) = C_{B,\pi_j(v)}$, thus for $v = \rho_i(w)$ we have $\Pi_j(C_{B,\rho_i(w)}) = C_{B,\pi_j(\rho_i(w))}$, which gives $C_{B,w}$ for $i = j$ and $C_\perp$ for $i \neq j$; so for $B = \psi(X_w(\Pi_i(F)))$ we get:

$$\Pi_j\big(C_{\psi(X_w(\Pi_i(F))),\rho_i(w)}\big) = \begin{cases} C_{\psi(X_w(\Pi_j(F))),w} & \text{if } i = j, \\ C_\perp & \text{if } i \neq j. \end{cases}$$

Therefore

$$\Pi_j(\psi^V(F)) = \bigvee \{C_{\psi(X_w(\Pi_j(F))),w} \mid w \in W_j\} = \psi^{W_j}(\Pi_j(F)).$$

□

For example, in multivalued images, vector values form the lattice T^n, where T is the complete chain of grey-levels, thus applying a flat operator $\psi^{(T^n)}$ to a multivalued image amounts to applying the corresponding grey-level flat operator ψ^T to each one of the scalar components of the image.

Flat operators are reputed not to create new values in a grey-level image. Note first that even when $\perp$ and $\top$ do not belong the set of values of F, these values can appear in $\psi^V(F)$ if some conditions are not met:

Proposition 13. *Let ψ be an increasing operator on $\mathcal{P}(E)$.*

1. *If $\psi(\emptyset) \neq \emptyset$, then for all $F \in V^E$ and $p \in \psi(\emptyset)$ we have $\psi^V(F)(p) = \top$.*
2. *If $\psi(E) \neq E$, then for all $F \in V^E$ and $p \notin \psi(E)$ we have $\psi^V(F)(p) = \perp$.*
3. *If $\psi(\emptyset) = \emptyset$ and $\psi(E) = E$, then:*
   - *For any $a \in V$, $\psi^V(C_a) = C_a$.*
   - *For any $a, b \in V$ with $a < b$ and for every $F \in V^E$ such that $a \leq F(p) \leq b$ for all $p \in E$, then $a \leq \psi^V(F)(p) \leq b$ for all $p \in E$.*

Proof:

1. Let $p \in \psi(\emptyset)$; as ψ is increasing, for every $v \in V$ we have $p \in \psi(X_v(F))$, so (28) gives $\psi^V(F)(p) = \bigvee V = \top$.
2. Let $p \notin \psi(E)$; as ψ is increasing, for every $v \in V$ we have $p \notin \psi(X_v(F))$, so (28) gives $\psi^V(F)(p) = \bigvee \emptyset = \perp$.
3. Suppose that $\psi(\emptyset) = \emptyset$ and $\psi(E) = E$. For every $v \in V$, we have $X_v(C_a) = E$ for $v \leq a$, and $\emptyset$ otherwise; thus $\psi(X_v(C_a)) = E$ for $v \leq a$, and $\emptyset$ otherwise. Hence (28) gives for every $p \in E$:
   $$\psi^V(C_a)(p) = \bigvee \{v \in V \mid p \in \psi(X_v(C_a))\} = \bigvee \{v \in V \mid v \leq a\} = a,$$
   that is $\psi^V(C_a) = C_a$. Now for $a, b \in V$ with $a < b$, the fact that $a \leq F(p) \leq b$ for all $p \in E$ means $C_a \leq F \leq C_b$; as ψ^V is increasing (Proposition 10), we get $C_a = \psi^V(C_a) \leq \psi^V(F) \leq \psi^V(C_b) = C_b$, that is $a \leq \psi^V(F)(p) \leq b$ for all $p \in E$.

□

Note that for $E = R^n$ or $\mathbf{Z}^n$, the usual operators, like the dilation, erosion, opening and closing by a *nonvoid* structuring element A, verify that condition $\psi(\emptyset) = \emptyset$ and $\psi(E) = E$; in fact this is the case for any *translation-invariant* operator, except the constant E and constant $\emptyset$ operators, in other words the erosion/closing and the dilation/opening by the empty structuring element $\emptyset$. This is why this condition is not considered in practice.

It is well-known that in the case of images with finitely many grey-levels (say, $V = T = [a, b] \cap \mathbf{Z}$), for an increasing set operator ψ verifying $\psi(\emptyset) = \emptyset$ and $\psi(E) = E$, the flat operator ψ^V will not add new grey-levels in an image. On the other hand, for $T = \overline{\mathbf{Z}}$, it can introduce the grey-level $\pm\infty$, while for continuous grey-levels, it can introduce new finite grey-levels, but in both cases it is only as limits of existing grey-levels in the original image.

Here in the general case of V-images, new values in $\psi^V(F)$ can be obtained by combinations of suprema and infima of existing values $F(q)$ (including the empty supremum $\bot$ and the empty infimum $\top$). In other words if the image F has its values in a complete sublattice W of V, those of $\psi^V(F)$ will also be in W. More precisely, the following is a straightforward consequence of item 1 of Lemma 3 (namely that $s(v, F) \geq v$ and $X_{s(v,F)}(F) = X_v(F)$):

Proposition 14 ([33]). *Let W be a complete sublattice of V and let F be a W-image. Let ψ be an increasing operator on $\mathcal{P}(E)$. Then $\psi^V(F) = \psi^W(F)$.*

It means in particular that for any V-image F, the values of $\psi^V(F)$ will belong to the complete sublattice W of V generated by the values of F.

Let us give a few applications of Propositions 11 and 14.

As $\{\bot, \top\}$ is a complete sublattice of V, isomorphic to $\{0, 1\}$, the lattice $\{\bot, \top\}^E$ of binary images $E \to \{\bot, \top\}$ (ordered by $\leq$) is isomorphic to the lattice 2^E, hence to the lattice $\mathcal{P}(E)$ of parts of E (ordered by inclusion); this isomorphism is given by the map $\mathcal{P}(E) \to \{\bot, \top\}^E : B \mapsto C_{B,\top}$. Now this isomorphism extends to operators:

Proposition 15. *For any increasing operator ψ on $\mathcal{P}(E)$, ψ^V behaves like ψ on binary images $E \to \{\bot, \top\}$: for any $B \subseteq E$,*

$$\psi^V(C_{B,\top}) = C_{\psi(B),\top}. \tag{31}$$

In particular, two distinct increasing operators ψ, ξ on $\mathcal{P}(E)$ have distinct flat extensions: $\psi \neq \xi \Rightarrow \psi^V \neq \xi^V$.

Proof: Take $W = \{\bot, \top\}$ and let $W_s = \{\top\}$; now W_s is a sup-generating family of W, and we have $X_\top(C_{B,\top}) = B$, with $C_{B,\top} \in W^E$. Apply Proposition 14, then Proposition 11 for the sup-generating family W_s:

$$\psi^V(C_{B,\top}) = \psi^W(C_{B,\top}) = C_{\psi(X_\top(C_{B,\top})),\top} = C_{\psi(B),\top}.$$

This shows (31). Now if $\psi \neq \xi$, there is some $B \subseteq E$ such that $\psi(B) \neq \xi(B)$, so $\psi^V(C_{B,\top}) \neq \xi^V(C_{B,\top})$. $\square$

For cylinders, the behaviour of a flat operator is as follows:

Proposition 16. *Take $B \subseteq E$ and $a \in V$. Let ψ be an increasing operator on $\mathcal{P}(E)$. Then $\psi^V(C_{B,a}) = C_{\psi(B),a} \vee C_{\psi(\emptyset),\top}$. In particular, if $\psi(\emptyset) = \emptyset$, then $\psi^V(C_{B,a}) = C_{\psi(B),a}$.*

Proof: If $a = \bot$, then $C_{B,a} = C_{\emptyset,\top}$, and Proposition 15 gives $\psi^V(C_{\emptyset,\top}) = C_{\psi(\emptyset),\top}$, which is indeed the same as $C_{\psi(B),\bot} \vee C_{\psi(\emptyset),\top}$. If $a = \top$, then $C_{B,a} = C_{B,\top}$, and Proposition 15 again gives $\psi^V(C_{B,\top}) = C_{\psi(B),\top}$, which is indeed the same as $C_{\psi(B),\top} \vee C_{\psi(\emptyset),\top}$ (because $\psi(\emptyset) \subseteq \psi(B)$). We can thus suppose that $a \neq \bot, \top$. The values of $C_{B,a}$ are in the complete sublattice $W = \{\bot, a, \top\}$ of V; the set $W_s = \{a, \top\}$ is a sup-generating family of W. Apply Proposition 14, then Proposition 11 for the sup-generating family W_s of W:

$$\begin{aligned}\psi^V(C_{B,a}) &= C_{\psi(X_a(C_{B,a})),a} \vee C_{\psi(X_\top(C_{B,a})),\top} \\ &= C_{\psi(B),a} \vee C_{\psi(\emptyset),\top},\end{aligned}$$

because $X_a(C_{B,a}) = B$ and $X_\top(C_{B,a}) = \emptyset$. We have thus shown the equality $\psi^V(C_{B,a}) = C_{\psi(B),a} \vee C_{\psi(\emptyset),\top}$, whatever the value of a. If $\psi(\emptyset) = \emptyset$, the term $C_{\psi(\emptyset),\top}$ becomes the least V-image $C_\bot$, which is redundant in the formula, and we get $\psi^V(C_{B,a}) = C_{\psi(B),a}$. $\square$

As a particular case, for $B = E$, we obtain $\psi^V(C_a) = C_{\psi(E),a} \vee C_{\psi(\emptyset),\top}$, and indeed if $\psi(\emptyset) = \emptyset$ and $\psi(E) = E$, we get $\psi^V(C_a) = C_a$, as in item 3 of Proposition 13. When the conditions $\psi(\emptyset) = \emptyset$ and $\psi(E) = E$ are not met, for $a \in V \setminus \{\bot, \top\}$, $\psi^V(C_a)$ will be like what we showed in Fig. 5(b) for $V = T$.

We can now describe the behaviour of a flat operator on an image whose values form a finite chain:

Proposition 17. *Let $n > 0$, let $a_0, \ldots, a_n \in V$ and $A_0, \ldots, A_n \in \mathcal{P}(E)$ such that $\bot < a_0 < \cdots < a_n = \top$ and $A_0 \supset \cdots \supset A_n$. Let*

$$F = \bigvee_{i=0}^{n} C_{A_i,a_i},$$

in other words the V-image F is given by

$$\forall p \in E, \quad F(p) = \begin{cases} \bot & \text{if } p \in E \setminus A_0, \\ a_i & \text{if } p \in A_i \setminus A_{i+1} \\ & \quad (i = 0, \ldots, n-1), \\ a_n = \top & \text{if } p \in A_n. \end{cases}$$

Then

$$\psi^V(F) = \bigvee_{i=0}^{n} C_{\psi(A_i),a_i},$$

in other words

$$\forall p \in E,\ \psi^V(F)(p) = \begin{cases} \bot & \text{if } p \in E \setminus \psi(A_0), \\ a_i & \text{if } p \in \psi(A_i) \setminus \psi(A_{i+1}) \\ & \quad (i{=}0, \dots, n-1), \\ a_n = \top & \text{if } p \in \psi(A_n). \end{cases}$$

Proof: We verify that for $i = 0, \dots, n$, $X_{a_i}(F) = A_i$. Now $W = \{\bot, a_0, \dots, a_n\}$ is a complete sub-lattice of V, and $W_s = \{a_0, \dots, a_n\}$ is a sup-generating family of W. Applying Propositions 14 and 11 we get

$$\psi^V(F) = \bigvee_{w \in W_s} C_{\psi(X_w(F)),w} = \bigvee_{i=0}^{n} C_{\psi(X_{a_i}(F)),a_i} = \bigvee_{i=0}^{n} C_{\psi(A_i),a_i},$$

and the result follows. □

For example in Fig. 5(c), we have $n = 2$, $A_0 = E$, $A_1 = A$ and $A_2 = \emptyset$, so $\psi^V(F)$ will indeed be as in Fig. 5(d).

3.3. Matheron's Theorem: Sup-Inf Decomposition

In mathematical morphology for sets, Matheron's theorem [37] states that a translation-invariant increasing operator on $\mathcal{P}(E)$ is a union of erosions; more precisely, we have $\psi = \bigcup_{B \in \mathcal{K}(\psi)} \varepsilon_B$, where $\mathcal{K}(\psi) = \{B \in \mathcal{P}(E) \mid o \in \psi(B)\}$ (o being the origin); the set $\mathcal{K}(\psi)$ is called the *kernel* of ψ. As noticed in [31], it is linked to the well-known fact that an increasing function $\{0, 1\}^n \to \{0, 1\}$ can be expressed as a maximum of minima. There have been several generalizations, for example for translation-invariant increasing operators in a complete lattice where Minkowski operations are defined [20], or increasing operators in an arbitrary complete lattice [38]. Matheron's kernel has the following property:

Definition 18. Given $\mathcal{B} \subseteq \mathcal{P}(E)$, we say that $\mathcal{B}$ is an *upper family* if for every $B \in \mathcal{B}$ and for every $C \in \mathcal{P}(E)$ such that $B \subset C$, we necessarily have $C \in \mathcal{B}$. For any $\mathcal{B} \subseteq \mathcal{P}(E)$, the least upper family containing $\mathcal{B}$ is

$$\mathcal{U}(\mathcal{B}) = \{C \in \mathcal{P}(E) \mid \exists B \in \mathcal{B},\ B \subseteq C\}; \qquad (32)$$

we call it the *upper closure* of $\mathcal{B}$.

The kernel $\mathcal{K}(\psi)$ is an upper family, which makes it redundant: for $B \in \mathcal{K}(\psi)$ and $C \supset B$, we have $C \in \mathcal{K}(\psi)$, but for every $X \in \mathcal{P}(E)$, we have $\varepsilon_C(X) \subseteq \varepsilon_B(X)$, so that ε_C is redundant in the union $\bigcup_{B \in \mathcal{K}(\psi)} \varepsilon_B$. Given $\mathcal{B} \subseteq \mathcal{K}(\psi)$ such that $\mathcal{U}(\mathcal{B}) = \mathcal{K}(\psi)$, we have then $\psi = \bigcup_{B \in \mathcal{B}} \varepsilon_B$.

We will prove here an analogue of Matheron's theorem. As we do not assume translation-invariance, we will have a kernel at each point, and we will see that the flat operator gives at each point the supremum, for all sets in the kernel, of the infima of values of the function on that set.

Given an increasing operator ψ on $\mathcal{P}(E)$, for every point $p \in E$ we define the *kernel of ψ at p* as the set

$$\mathcal{K}(\psi, p) = \{B \in \mathcal{P}(E) \mid p \in \psi(B)\}. \qquad (33)$$

This generalizes Matheron's kernel in the following sense: in the case where $E = \mathbf{R}^n$ or $\mathbf{Z}^n$ and ψ is translation-invariant, at the origin o we have $\mathcal{K}(\psi, o) = \mathcal{K}(\psi)$, while for any other point p, $\mathcal{K}(\psi, p)$ is the set of translates by p of elements of $\mathcal{K}(\psi)$: $\mathcal{K}(\psi, p) = \{B_p \mid B \in \mathcal{K}(\psi)\}$.

Theorem 19. *Let ψ be an increasing operator on $\mathcal{P}(E)$. Then for every $F \in V^E$ and $p \in E$, we have*

$$\psi^V(F)(p) = \bigvee_{B \in \mathcal{K}(\psi, p)} \bigwedge_{q \in B} F(q). \qquad (34)$$

Furthermore, the following hold for every point $p \in E$:

1. *$\mathcal{K}(\psi, p)$ is an upper family.*
2. *$\mathcal{K}(\psi, p) = \emptyset \Leftrightarrow p \notin \psi(E) \Leftrightarrow [\forall F \in V^E, \psi^V(F)(p) = \bot]$.*
3. *$\mathcal{K}(\psi, p) = \mathcal{P}(E) \Leftrightarrow p \in \psi(\emptyset) \Leftrightarrow [\forall F \in V^E,\ \psi^V(F)(p) = \top]$.*

Proof: Fix $F \in V^E$ and $p \in E$. Let

$$P = \{v \in V \mid p \in \psi\,(X_v(F))\} \qquad \text{and}$$

$$Q = \left\{ \bigwedge_{q \in B} F(q) \mid B \in \mathcal{K}(\psi, p) \right\}.$$

By (28), we have $\psi^V(F)(p) = \bigvee P$. The result (34) that we have to show states that $\psi^V(F)(p) = \bigvee Q$; we will thus prove that $\bigvee P = \bigvee Q$.

Let $w \in Q$; thus there is some $B \in \mathcal{K}(\psi, p)$ such that $w = \bigwedge_{q \in B} F(q)$; by (21), $S(B, F) = X_w(F)$, and by Lemma 3, we have $S(B, F) \supseteq B$; as $B \in \mathcal{K}(\psi, p)$, we have $p \in \psi(B)$, and as ψ is increasing and $B \subseteq S(B, F) = X_w(F)$, we get $p \in \psi(X_w(F))$, so $w \in P$. Therefore $Q \subseteq P$ and so $\bigvee Q \leq \bigvee P$. Now let $v \in P$; by (20), $s(v, F) = \bigwedge_{q \in X_v(F)} F(q)$, and by Lemma 3, we have $v \leq s(v, F)$. As $v \in P$, we have $p \in \psi(X_v(F))$, so $X_v(F) \in \mathcal{K}(\psi, p)$, hence $s(v, F) \in Q$. Therefore $\bigvee P \leq \bigvee_{v \in P} s(v, F) \leq \bigvee Q$. From the two inequalities $\bigvee Q \leq \bigvee P$ and $\bigvee P \leq \bigvee Q$, the equality follows, and (34) holds.

Let $B \in \mathcal{K}(\psi, p)$; thus $p \in \psi(B)$. As ψ is increasing, for any $C \in \mathcal{P}(E)$ such that $B \subset C$, we have $p \in \psi(C)$, that is $C \in \mathcal{K}(\psi, p)$. Hence $\mathcal{K}(\psi, p)$ is an upper family.

If $\mathcal{K}(\psi, p) = \emptyset$, then $E \notin \mathcal{K}(\psi, p)$, that is $p \notin \psi(E)$. Conversely, if $p \notin \psi(E)$, then as $\mathcal{K}(\psi, p)$ is an upper family, for any $B \in \mathcal{P}(E)$ we have $B \notin \mathcal{K}(\psi, p)$, thus $\mathcal{K}(\psi, p) = \emptyset$. Hence $\mathcal{K}(\psi, p) = \emptyset \Leftrightarrow p \notin \psi(E)$. We showed in Proposition 13 that if $p \notin \psi(E)$, then for every $F \in V^E$ we have $\psi^V(F)(p) = \bot$. On the other hand, by Proposition 15, $\psi^V(C_\top) = \psi^V(C_{E,\top}) = C_{\psi(E),\top}$, so that if $p \in \psi(E)$, then $\psi^V(C_\top)(p) = C_{\psi(E),\top}(p) = \top \neq \bot$. Hence $p \notin \psi(E) \Leftrightarrow \left[\forall F \in V^E,\ \psi^V(F)(p) = \bot\right]$. Therefore item 2 holds.

If $\mathcal{K}(\psi, p) = \mathcal{P}(E)$, then $\emptyset \in \mathcal{K}(\psi, p)$, that is $p \in \psi(\emptyset)$. Conversely, if $p \in \psi(\emptyset)$, then as $\mathcal{K}(\psi, p)$ is an upper family, for any $B \in \mathcal{P}(E)$ we have $B \in \mathcal{K}(\psi, p)$, thus $\mathcal{K}(\psi, p) = \mathcal{P}(E)$. Hence $\mathcal{K}(\psi, p) = \mathcal{P}(E) \Leftrightarrow p \in \psi(\emptyset)$. We showed in Proposition 13 that if $p \in \psi(\emptyset)$, then for every $F \in V^E$ we have $\psi^V(F)(p) = \top$. On the other hand, by Proposition 15, $\psi^V(C_\bot) = \psi^V(C_{\emptyset,\top}) = C_{\psi(\emptyset),\top}$, so that if $p \notin \psi(\emptyset)$, then $\psi^V(C_\bot)(p) = C_{\psi(\emptyset),\top}(p) = \bot \neq \top$. Hence $p \in \psi(\emptyset) \Leftrightarrow \left[\forall F \in V^E,\ \psi^V(F)(p) = \top\right]$. Therefore item 3 holds. □

As we will see in Section 3.6, there is generally no dual decomposition of $\psi^V(F)(p)$ as an infimum of suprema; this requires complete distributivity.

The fact that for $p \in E$ the kernel $\mathcal{K}(\psi, p)$ is an upper family, implies that the decomposition (34) is redundant: given $B \in \mathcal{K}(\psi, p)$ and $C \supset B$, we have $C \in \mathcal{K}(\psi, p)$, but $\bigwedge_{q \in C} F(q) \leq \bigwedge_{q \in B} F(q)$, so that C is redundant in the formula (34). We can thus replace $\mathcal{K}(\psi, p)$ by any $\mathcal{B} \subseteq \mathcal{P}(E)$ such that $\mathcal{U}(\mathcal{B}) = \mathcal{K}(\psi, p)$.

Lemma 20. *Let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{P}(E)$. Then*

1. *$\mathcal{B} \subseteq \mathcal{U}(\mathcal{C}) \Leftrightarrow \mathcal{U}(\mathcal{B}) \subseteq \mathcal{U}(\mathcal{C})$, and this holds iff for any $F \in V^E$, $\bigvee_{B \in \mathcal{B}} \bigwedge_{q \in B} F(q) \leq \bigvee_{C \in \mathcal{C}} \bigwedge_{q \in C} F(q)$.*
2. *$\mathcal{U}(\mathcal{B}) = \mathcal{U}(\mathcal{C})$ iff for any $F \in V^E$, $\bigvee_{B \in \mathcal{B}} \bigwedge_{q \in B} F(q) = \bigvee_{C \in \mathcal{C}} \bigwedge_{q \in C} F(q)$.*

Proof:

1. The equivalence $\mathcal{B} \subseteq \mathcal{U}(\mathcal{C}) \Leftrightarrow \mathcal{U}(\mathcal{B}) \subseteq \mathcal{U}(\mathcal{C})$ follows from the fact that $\mathcal{U}(\mathcal{C})$ is an upper family and $\mathcal{U}(\mathcal{B})$ is the least upper family containing $\mathcal{B}$. Suppose that $\mathcal{B} \subseteq \mathcal{U}(\mathcal{C})$, and let $F \in V^E$: for $B \in \mathcal{B}$, there is some $C \in \mathcal{C}$ such that $C \subseteq B$, so $\bigwedge_{q \in B} F(q) \leq \bigwedge_{q \in C} F(q)$; thus $\bigvee_{B \in \mathcal{B}} \bigwedge_{q \in B} F(q) \leq \bigvee_{C \in \mathcal{C}} \bigwedge_{q \in C} F(q)$. Suppose now that $\mathcal{B} \not\subseteq \mathcal{U}(\mathcal{C})$: there is some $B \in \mathcal{B}$ such that for every $C \in \mathcal{C}$, $C \not\subseteq B$; define $F \in V^E$ by $F(q) = \top$ for $q \in B$ and $F(q) = \bot$ for $q \notin B$; then $\bigwedge_{q \in B} F(q) = \top$ but for every $C \in \mathcal{C}$ we have $\bigwedge_{q \in C} F(q) = \bot$, and we get $\bigvee_{B \in \mathcal{B}} \bigwedge_{q \in B} F(q) = \top \not\leq \bot = \bigvee_{C \in \mathcal{C}} \bigwedge_{q \in C} F(q)$.
2. Interverting $\mathcal{B}$ and $\mathcal{C}$ in item 1, we get $\mathcal{U}(\mathcal{B}) \supseteq \mathcal{U}(\mathcal{C})$ iff for any $F \in V^E$, $\bigvee_{B \in \mathcal{B}} \bigwedge_{q \in B} F(q) \geq \bigvee_{C \in \mathcal{C}} \bigwedge_{q \in C} F(q)$. Combining this with item 1, the result follows. □

Combining item 2 with (34) and the fact that $\mathcal{K}(\psi, p)$ is an upper family, we obtain:

Corollary 21. *Given an increasing operator ψ on $\mathcal{P}(E)$, a point $p \in E$, and a family $\mathcal{B} \subseteq \mathcal{P}(E)$, we have:*

$$\left[\forall F \in V^E,\ \psi^V(F)(p) = \bigvee_{B \in \mathcal{B}} \bigwedge_{q \in B} F(q)\right] \Leftrightarrow \quad \mathcal{U}(\mathcal{B}) = \mathcal{K}(\psi, p).$$

We see also that every operator on V-images that applies at each point a supremum of infima of image values, is a flat operator:

Corollary 22. *Let $\Psi : V^E \to V^E$ be such that for every $p \in E$ there is some $\mathcal{B}_p \in \mathcal{P}(\mathcal{P}(E))$ such that for every $F \in V^E$ we have*

$$\Psi(F)(p) = \bigvee_{B \in \mathcal{B}_p} \bigwedge_{q \in B} F(q).$$

Define the operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ by setting for every $X \in \mathcal{P}(E)$:

$$\psi(X) = \{p \in E \mid X \in \mathcal{U}(\mathcal{B}_p)\}.$$

Then ψ is increasing, $\mathcal{U}(\mathcal{B}_p) = \mathcal{K}(\psi, p)$ for every $p \in E$, and $\Psi = \psi^V$.

Let us illustrate the above results with the example of the median operation used by median filters. It is the flat extension of the "majority vote" operation on binary variables (given n binary variables $x_1, \ldots, x_n$, n odd, select the binary value represented by the majority of the x_i's). Thus, for $x_1, \ldots, x_n \in V$, the median verifies

$$med(x_1, \ldots, x_n) = \bigvee_{B \in \mathcal{B}} \bigwedge_{i \in B} x_i,$$

where $\mathcal{B}$ is the set of parts of $\{1, \ldots, n\}$ having size $(n+1)/2$. Let us consider some specific examples for V (but not the standard lattice T of numerically ordered grey-levels):

1. For $V = T^n$ (colour or multivalued images), the median is obtained by taking the median in each coordinate, cf. Proposition 12.
2. For $V = U$ (the lattice of labels) and $x_1, \ldots, x_n \neq \top$, if there is a label u represented at least $(n+1)/2$ times in the sequence $x_1, \ldots, x_n$, then the median is u, otherwise it is $\perp$ [34].
3. For $V = R_r$ (the lattice $\mathbf{R} \cup \{\infty\}$ or $\mathbf{Z} \cup \{\infty\}$ ordered w.r.t. a reference $r \neq \infty$) and $x_1, \ldots, x_n \neq \infty$, the median for the reference ordering will coincide with the usual median for the numerical ordering. Indeed, we have 3 cases: (*a*) at least $(n+1)/2$ of the x_i's are numerically above r, and the median is the $(n+1)/2$-th in (numerically) descending order; (*b*) at least $(n+1)/2$ of the x_i's are numerically below r, and the median is the $(n+1)/2$-th in (numerically) ascending order; (*c*) neither holds, and the median is the reference r. This means that if we take images with finite values ordered w.r.t. reference r (as in the original works of Keshet [19, 23, 24], where the value ∞ was excluded), we can define the median filter in the same way as for numerically ordered values, and it will be a flat operator.

3.4. Flat Extension of Some Particular Operators

The following is a rewording of Lemma 6:

Proposition 23 ([33]). $\mathbf{id}^V = \Sigma\Theta$, *it is is the identity on* V^E.

We can interpret this result according to Section 3.3: for every $p \in E$ we have $B \in \mathcal{K}(\mathbf{id}, p) \Leftrightarrow p \in B$, so $\mathcal{K}(\mathbf{id}, p) = \mathcal{U}(\{\{p\}\})$, hence by Corollary 21, $\mathbf{id}^V(F)(p) = F(p)$.

Let us now consider dilations, erosions and adjunctions on $\mathcal{P}(E)$. A *variable structuring element* (or *neighbourhood function*, or *windowing function* [33, 34]) is a map $\mathbf{A} : E \to \mathcal{P}(E)$ associating to every point $p \in E$ a *pointwise structuring element* (or *neighbourhood*, or *window*) $\mathbf{A}(p) \subseteq E$. Given a variable structuring element $\mathbf{A}$, its *dual* is the variable structuring element $\widetilde{\mathbf{A}}$ defined by

$$\forall p, q \in E, \qquad q \in \widetilde{\mathbf{A}}(p) \Leftrightarrow p \in \mathbf{A}(q).$$

The *dilation by* $\mathbf{A}$ and the *erosion by* $\mathbf{A}$ are the operators $\delta_{\mathbf{A}}$ and $\varepsilon_{\mathbf{A}}$ on $\mathcal{P}(E)$ defined by setting for $Z \subseteq E$:

$$\begin{aligned} \delta_{\mathbf{A}}(Z) &= \textstyle\bigcup_{z \in Z} \mathbf{A}(z), \\ \varepsilon_{\mathbf{A}}(Z) &= \{p \in E \mid \mathbf{A}(p) \subseteq Z\}. \end{aligned} \tag{35}$$

Note that

$$\delta_{\mathbf{A}}(Z) = \{p \in E \mid \widetilde{\mathbf{A}}(p) \cap \mathbf{Z} \neq \emptyset\},$$

and that $(\delta_{\mathbf{A}}(Z^c))^c = \varepsilon_{\widetilde{\mathbf{A}}}(Z)$ and $(\varepsilon_{\mathbf{A}}(Z^c))^c = \delta_{\widetilde{\mathbf{A}}}(Z)$. A "folk theorem" states that

- $(\varepsilon_{\mathbf{A}}, \delta_{\mathbf{A}})$ is an adjunction on $\mathcal{P}(E)$, and
- conversely, given an adjunction (ε, δ) on $\mathcal{P}(E)$, there is a unique variable structuring element $\mathbf{A}$ such that $\varepsilon = \varepsilon_{\mathbf{A}}$ and $\delta = \delta_{\mathbf{A}}$; for every $p \in E$, we have $\mathbf{A}(p) = \delta(\{p\})$.

We can now consider the flat operators corresponding to dilations and erosions by a variable structuring element. We see that we obtain the same formulas as in the usual case of numerical functions [16, 17]:

Proposition 24 ([33]). *Let* $\delta_{\mathbf{A}}$ *and* $\varepsilon_{\mathbf{A}}$ *be the dilation and erosion (on* $\mathcal{P}(E)$*) by a neighboorhood function* $\mathbf{A}$. *Then:*

1. *For* $F \in V^E$ *and* $p \in E$, *we have:*

$$\begin{aligned} \delta_{\mathbf{A}}^V(F)(p) &= \bigvee_{q \in \widetilde{\mathbf{A}}(p)} F(q), \\ \varepsilon_{\mathbf{A}}^V(F)(p) &= \bigwedge_{q \in \mathbf{A}(p)} F(q). \end{aligned} \tag{36}$$

2. $(\varepsilon_{\mathbf{A}}^V, \delta_{\mathbf{A}}^V)$ *is an adjunction on* V^E.

Interpreting this result according to Section 3.3, for every $p \in E$ we have $\mathcal{K}(\delta_{\mathbf{A}}, p) = \mathcal{U}(\{\{q\} \mid q \in \widetilde{\mathbf{A}}(p)\})$ (i.e., $B \in \mathcal{K}(\delta_{\mathbf{A}}, p) \Leftrightarrow B \cap \widetilde{\mathbf{A}}(p) \neq \emptyset$), and $\mathcal{K}(\varepsilon_{\mathbf{A}}, p) = \mathcal{U}(\{\mathbf{A}(p)\})$ (i.e., $B \in \mathcal{K}(\varepsilon_{\mathbf{A}}, p) \Leftrightarrow \mathbf{A}(p) \subseteq B$), so that (36) follows.

Note that the dilation $\delta_{\mathbf{A}}$ verifies $\delta_{\mathbf{A}}(\emptyset) = \emptyset$; therefore Proposition 16 gives

$$\forall B \subseteq E,\ \forall v \in V, \qquad \delta_{\mathbf{A}}^V(C_{B,v}) = C_{\delta_{\mathbf{A}}(B),v}. \tag{37}$$

It is known that if π is a symmetry (translation, rotation, etc.) of the Euclidean space $E = \mathbf{R}^n$, we can apply π to grey-level or multivalued images by setting $\pi(F) : \pi(p) \mapsto F(p)$, that is $\pi(F)(p) = F(\pi^{-1}(p))$. In fact the action of π on V-images is the flat extension of its action on sets:

Proposition 25. *Let the map $\pi : E \to E$ act on subsets of E: for $X \in \mathcal{P}(E)$, $\pi(X) = \{\pi(x) \mid x \in X\}$. Then for $F \in V^E$ and $p \in E$ we have $\pi^V(F)(p) = \bigvee\{F(q) \mid q \in E, \pi(q) = p\}$. In particular, if π is bijective (a permutation of E), then $\pi^V(F)(p) = F(\pi^{-1}(p))$, where π^{-1} is the inverse of π, in other words π^V corresponds to the action of π on V-images.*

Proof: In fact, π acting on sets is $\delta_{\mathbf{A}}$ for $\mathbf{A}(p) = \{\pi(p)\}$, with $\widetilde{\mathbf{A}}(p) = \{q \in E \mid \pi(q) = p\}$. Applying (36), we obtain $\pi^V(F)(p) = \bigvee\{F(q) \mid q \in E, \pi(q) = p\}$. When π is bijective, $\{q \in E \mid \pi(q) = p\} = \{\pi^{-1}(p)\}$, where π^{-1} is the inverse of π, so that $\pi^V(F)(p) = F(\pi^{-1}(p))$. □

An important class of morphological operations are the *connected operators* [10, 11, 36, 40, 41] (see [43] for a brief description). Suppose that we have a *connection* (or *connectivity class*) $\mathcal{C}$ in $\mathcal{P}(E)$ (see Chapter 2 of [38]); the elements of $\mathcal{C}$ are the connected sets. Examples of connections are the family of 4-connected subsets of $E = \mathbf{Z}^2$, or the one of topologically connected subsets of $E = \mathbf{R}^n$. For $F \in V^E$, a *flat zone* of F is a connected component (according to $\mathcal{C}$) of $F^{-1}(t) = \{p \in E \mid F(p) = t\}$ for $t \in \{F(p) \mid p \in E\}$. For $X \in \mathcal{P}(E)$, a *flat zone* of X is a flat zone of its characteristic function $\chi_X : E \to \{0, 1\}$, in other words, it is a connected component (according to $\mathcal{C}$) of X or X^c. A *connected operator* is an operator which coarsens the partition of E into flat zones, in other words such that a flat zone in the initial image will be part of a flat zone of the resulting image. We will rather use the following equivalent definition:

Definition 26. Let $\mathcal{C}$ be a connection in $\mathcal{P}(E)$.

- An operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is *connected* if for every $C \in \mathcal{C}$ and $X \in \mathcal{P}(E)$ such that $C \subseteq X$ or $C \subseteq X^c$, we must have $C \subseteq \psi(X)$ or $C \subseteq \psi(X)^c$.
- An operator $\Psi : V^E \to V^E$ is *connected* if for every $C \in \mathcal{C}$ and $F \in V^E$ such that F has constant value on C, $\Psi(F)$ must have constant value on C.

The following is known in the case of grey-level images [43]:

Proposition 27. *For an increasing connected operator ψ on $\mathcal{P}(E)$, ψ^V is connected.*

Proof: Let $C \in \mathcal{C}$ and $F \in V^E$ such that there is some $w \in V$ with $F(p) = w$ for all $p \in C$. Thus for every $v \in V$, we have either $v \leq w$ and $C \subseteq X_v(F)$, or $v \not\leq w$ and $C \subseteq X_v(F)^c$. As ψ is connected, we will get either $C \subseteq \psi(X_v(F))$, or $C \subseteq \psi(X_v(F))^c$. This means that the cylinder $C_{\psi(X_v(F)),v}$ will have its values for $p \in C$, either all equal to v, or all equal to $\perp$. From (27) we get that $\psi^V(F) = \bigvee_{v \in V} C_{\psi(X_v(F)),v}$ has a constant value on C. Hence ψ^V is connected. □

3.5. *Combinations of Operators*

Now we give some results and counterexamples concerning the flat extension of a union, intersection, and composition of set operators, as well as on the flat extension of openings and closings. These properties were obtained by Heijmans [18] in the case where the lattice V is strongly admissible.

The flat extension of a union of set operators is the join of their individual flat extensions:

Proposition 28 ([33]). *Given a family of increasing operators ψ_i ($i \in I$) on $\mathcal{P}(E)$,*

$$\left(\bigcup_{i \in I} \psi_i\right)^V = \bigvee_{i \in I} \psi_i^V. \tag{38}$$

Proof: For any V-image F we have:

$$\begin{aligned}\left(\bigcup_{i \in I} \psi_i\right)^V(F) &= \Sigma\left(\bigcup_{i \in I} \psi_i\right)_V \Theta F \\ &= \Sigma\left(\bigcup_{i \in I} \psi_{i\,V} \Theta F\right) = \bigvee_{i \in I} \Sigma \psi_{i\,V} \Theta F \\ &= \bigvee_{i \in I} \psi_i^V(F).\end{aligned}$$

Here we used successively: (26) in Definition 9, the fact that the action of operators on stacks is a homomorphism, the fact that Σ is a dilation (thanks to Lemma 5), and again (26). □

Corollary 29. *Given two increasing operators η, ζ on $\mathcal{P}(E)$, $\eta \subseteq \zeta$ if and only if $\eta^V \leq \zeta^V$.*

Proof: The map $\psi \to \psi^V$ is injective by Proposition 15, and combining this with the above Proposition 28 , we get

$$\eta \subseteq \zeta \Leftrightarrow \zeta = \eta \cup \zeta \Leftrightarrow \zeta^V = \eta^V \vee \zeta^V \Leftrightarrow \eta^V \leq \zeta^V.$$

□

Now the dual form of Proposition 28 does not hold in general: the flat extension of an intersection of set operators does not always coincide with the meet of their individual flat extensions:

Proposition 30 ([33]). *Given a family ψ_i ($i \in I$) of increasing operators on $\mathcal{P}(E)$,*

$$\left(\bigcap_{i\in I} \psi_i\right)^V \leq \bigwedge_{i\in I} \psi_i^V. \tag{39}$$

Furthermore, the equality

$$\left(\bigcap_{i\in I} \psi_i\right)^V = \bigwedge_{i\in I} \psi_i^V \tag{40}$$

holds in the following two cases:

1. *V is ISD and I is finite;*
2. *V is completely distributive.*

Proof: Consider a family Ξ_i ($i \in I$) of stacks on V. Since the map $\Sigma : \mathcal{S}(V, E) \to V^E$ is increasing, from (9) we deduce:

$$\Sigma\left[\bigcap_{i\in I} \Xi_i\right] \leq \bigwedge_{i\in I} [\Sigma\,\Xi_i]. \tag{41}$$

Let us prove that in the cases 1 and 2 we have the equality:

$$\Sigma\left[\bigcap_{i\in I} \Xi_i\right] = \bigwedge_{i\in I} [\Sigma\,\Xi_i]. \tag{42}$$

If $I = \emptyset$, the empty intersection $\bigcap_{i\in I} \Xi_i$ is the greatest stack $\Xi_E : v \mapsto E$, with $\Sigma\,\Xi_E = C_\top$, while the empty infimum $\bigwedge_{i\in I} [\Sigma\,\Xi_i]$ is the greatest function $C_\top$, so that (42) holds. Assume now that $I \neq \emptyset$. Take any point $p \in E$. For each $i \in I$, let $W_i = \{v \in V \mid p \in \Xi_i(v)\}$. Then we have $(\forall i \in I,\ v \in W_i) \Leftrightarrow (\forall i \in I,\ p \in \Xi_i(v))$, that is:

$$\bigcap_{i\in I} W_i = \left\{v \in V \mid p \in \bigcap_{i\in I} \Xi_i(v)\right\}.$$

Hence (24) in Definition 4 gives

$$\forall i \in I, \quad \Sigma\,\Xi_i(p) = \bigvee W_i \qquad \text{and}$$

$$\Sigma\left[\bigcap_{i\in I} \Xi_i\right](p) = \bigvee\left[\bigcap_{i\in I} W_i\right]. \tag{43}$$

Let $\Phi(I)$ be the set of choice maps $\varphi : I \to \bigcup_{i\in I} W_i : i \mapsto \varphi(i) \in W_i$. Then

$$\bigwedge_{i\in I} \bigvee W_i = \bigvee_{\varphi\in\Phi(I)} \bigwedge_{i\in I} \varphi(i).$$

Indeed, this follows from the extended distributivity law (10, 12), which is verified in case 2 (V completely distributive), and which in case 1 (V ISD and I finite) reduces to (8), as for $I = \{i_1, \ldots, i_n\}$, a choice map φ amounts to an n-tuple $(\varphi(i_1), \ldots, \varphi(i_n)) \in W_{i_1} \times \cdots \times W_{i_n}$. For each $\varphi \in \Phi(I)$, let $w_\varphi = \bigwedge_{i\in I} \varphi(i)$. For any $i \in I$, we have $\varphi(i) \in W_i$, that is $p \in \Xi_i(\varphi(i))$; now $w_\varphi \leq \varphi(i)$, and as Ξ_i is a stack, we have $\Xi_i(\varphi(i)) \subseteq \Xi_i(w_\varphi)$, so that $p \in \Xi_i(w_\varphi)$, that is $w_\varphi \in W_i$. As $w_\varphi \in W_i$ for each $i \in I$, we get $w_\varphi \in \bigcap_{i\in I} W_i$. Therefore

$$\bigwedge_{i\in I} \bigvee W_i = \bigvee_{\varphi\in\Phi(I)} w_\varphi \leq \bigvee\left[\bigcap_{i\in I} W_i\right].$$

Applying (43), we get $\bigwedge_{i\in I}[\Sigma\,\Xi_i](p) \leq \Sigma[\bigcap_{i\in I} \Xi_i](p)$; as this holds for any $p \in E$, we obtain the inequality

$$\bigwedge_{i\in I}[\Sigma\,\Xi_i] \leq \Sigma\left[\bigcap_{i\in I} \Xi_i\right],$$

and combining it with the reverse equality (41), the equality (42) follows.

Consider now increasing operators ψ_i ($i \in I$). For any V-image F, we have by (26) and the fact that the action of operators on stacks is a homomorphism:

$$\left(\bigcap_{i\in I} \psi_i\right)^V (F) = \Sigma\left[\bigcap_{i\in I} \psi_i\right]_V \Theta F = \Sigma\left[\bigcap_{i\in I} \psi_{i\,V} \Theta F\right]$$

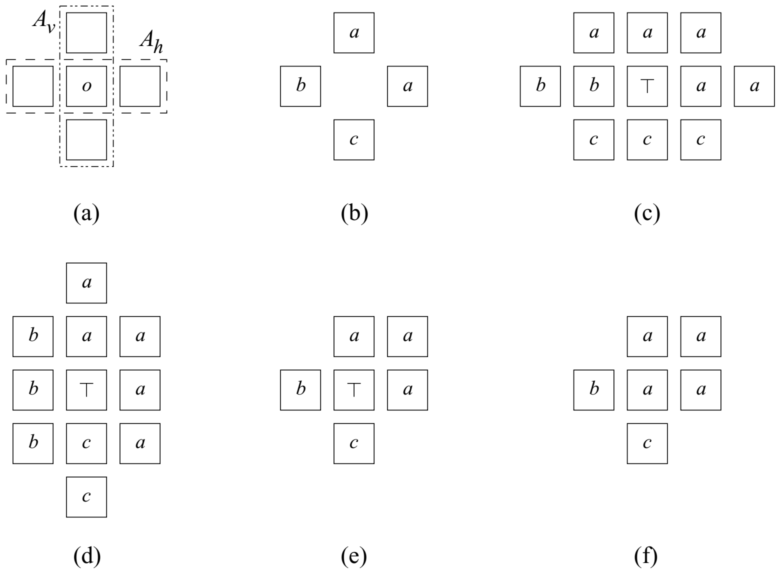

Figure 6. $E = \mathbf{Z}^2$ and $V = U$ with $|U_*| \geq 3$. (a) The two structuring elements A_h (horizontal) and A_v (vertical), centered about the origin o; let δ_h and δ_v be the dilations by A_h and A_v respectively (for sets). (b) The function $F \in U^E$, where a, b, c are pairwise distinct proper labels, and with value $\perp$ outside the support shown here. (c) $\delta_h^U(F)$. (d) $\delta_v^U(F)$. (e) $(\delta_h^U \wedge \delta_v^U)(F)$. (f) $(\delta_h \cap \delta_v)^U(F)$ is obtained by applying $\delta_h \cap \delta_v$ separately to the portions marked a, b and c, and joining the results; it has a smaller value on the central pixel.

and

$$\left(\bigwedge_{i\in I} \psi_i^V\right)(F) = \bigwedge_{i\in I} \left(\psi_i^V(F)\right) = \bigwedge_{i\in I} (\Sigma \psi_{i\,V} \Theta F).$$

The result follows then by applying (41, 42) to the stacks $\Xi_i = \psi_{i\,V}\Theta F$. □

In the proof we showed that when V is completely distributive, Σ is an erosion. It can be verified that the adjoint dilation Λ maps a V-image F to the stack ΛF defined by setting for each $v \in V$:

$$\Lambda F(v) = \{p \in E \mid v \lhd F(p)\}.$$

The usual lattices 2, T and T^n of image values (corresponding to sets, or to numerical or multivalued functions) are completely distributive, and so is any finite distributive lattice. The equality (40) is then verified. However it does not always hold when V is not distributive. Taking for V the non-distributive lattice U of labels (with $|U_*| \geq 3$), we show in Fig. 6 an example where $(\delta_1 \cap \delta_2)^V < \delta_1^V \wedge \delta_2^V$ for two dilations δ_1 and δ_2.

Let us consider now the flat extension of a composition of set operators; it should be the composition of their individual flat extensions: $(\psi_1\psi_2)^V = \psi_1^V\psi_2^V$; however we will see that this is verified only in three cases: ψ_1 is a dilation, ψ_2 is an erosion, or V is completely distributive.

Lemma 31. *Given an increasing operator ψ and an adjunction (ε, δ) on $\mathcal{P}(E)$, we have:*

1. $\delta^V \Sigma = \Sigma \delta_V$ *and* $\Theta \varepsilon^V = \varepsilon_V \Theta$.
2. $\Sigma \psi_V \leq \psi^V \Sigma$.
3. *If V is completely distributive, then* $\psi^V \Sigma = \Sigma \psi_V$.

Proof: Let Ξ be a stack on V. Applying (23), the fact that δ^V is a dilation (Proposition 24), and (37), we get

$$\delta^V(\Sigma\Xi) = \delta^V\left(\bigvee_{v\in V} C_{\Xi(v),v}\right) = \bigvee_{v\in V} \delta^V\left(C_{\Xi(v),v}\right) = \bigvee_{v\in V} C_{\delta(\Xi(v)),v} = \Sigma\delta_V\Xi.$$

Thus $\delta^V\Sigma = \Sigma\delta_V$. Now $(\varepsilon_V, \delta_V)$, $(\varepsilon^V, \delta^V)$ and (Θ, Σ) are adjunctions (see Lemma 5 and Proposition 24), so that for a stack Ξ and a V-image F we have:

$$\delta^V(\Sigma\Xi) \leq F \Leftrightarrow \Sigma\Xi \leq \varepsilon^V(F) \Leftrightarrow \Xi \subseteq \Theta\varepsilon^V(F)$$

and

$$\Sigma\delta_V\Xi \leq F \Leftrightarrow \delta_V\Xi \subseteq \Theta F \Leftrightarrow \Xi \subseteq \varepsilon_V\Theta F;$$

as $\delta^V(\Sigma\Xi) = \Sigma\delta_V\Xi$, we get $\Xi \subseteq \Theta\varepsilon^V(F) \Leftrightarrow \Xi \subseteq \varepsilon_V\Theta F$, hence $\Theta\varepsilon^V(F) = \varepsilon_V\Theta F$ for every $F \in V^E$; therefore $\Theta\varepsilon^V = \varepsilon_V\Theta$.

As (Θ, Σ) is an adjunction, $\Theta\Sigma$ is a closing on stacks, so $\Xi \subseteq \Theta\Sigma\Xi$. As ψ_V and Σ are increasing, we get $\Sigma\psi_V\Xi \leq \Sigma\psi_V\Theta\Sigma\Xi$. As $\psi^V = \Sigma\psi_V\Theta$ by (26), we obtain $\Sigma\psi_V\Xi \leq \psi^V\Sigma\Xi$ for every stack Ξ, that is $\Sigma\psi_V \leq \psi^V\Sigma$.

Suppose that V is completely distributive. Let $x \in V$ and w be a sup-factor of x. For $q \in [\Theta\Sigma\Xi](x)$, by Definition 4 we have $[\Sigma\Xi](q) \geq x$, that is $x \leq \bigvee\{v \in V \mid q \in \Xi(v)\}$, and as $w \lhd x$, we have $w \leq v$ for some $v \in V$ with $q \in \Xi(v)$; but as Ξ is a stack, $\Xi(v) \subseteq \Xi(w)$, so $q \in \Xi(w)$. Hence $[\Theta\Sigma\Xi](x) \subseteq \Xi(w)$, and as ψ is increasing, $\psi([\Theta\Sigma\Xi](x)) \subseteq \psi(\Xi(w))$, that is $[\psi_V\Theta\Sigma\Xi](x) \subseteq [\psi_V\Xi](w)$. Let $p \in E$. If $p \in [\psi_V\Theta\Sigma\Xi](x)$, we have $p \in [\psi_V\Xi](w)$ for every sup-factor w of x, but by Lemma 2, x is the supremum of its sup-factors, so (24) gives then

$$[\Sigma\psi_V\Xi](p) = \bigvee\{w \in V \mid p \in [\psi_V\Xi](w)\} \geq \bigvee\{w \in V \mid \perp < w \lhd x\} \geq x.$$

We deduce that

$$[\Sigma\psi_V\Xi](p) \geq \bigvee\{x \in V \mid p \in [\psi_V\Theta\Sigma\Xi](x)\} = [\Sigma\psi_V\Theta\Sigma\Xi](p) = [\psi^V\Sigma\Xi](p).$$

Therefore $\Sigma\psi_V\Xi \geq \psi^V\Sigma\Xi$ for every stack Ξ, that is $\Sigma\psi_V \geq \psi^V\Sigma$. Combining with the reverse inequality, the equality $\psi^V\Sigma = \Sigma\psi_V$ holds. □

Proposition 32. *Given two increasing operators ψ, ξ and an adjunction (ε, δ) on $\mathcal{P}(E)$, we have:*

1. $(\delta\psi)^V = \delta^V\psi^V$ *and* $(\psi\varepsilon)^V = \psi^V\varepsilon^V$.
2. $(\psi\xi)^V \leq \psi^V\xi^V$.
3. *If V is completely distributive, then* $(\psi\xi)^V = \psi^V\xi^V$.

Proof: Apply (26) and the preceding Lemma. Item 1 gives

$$\delta^V\psi^V = \delta^V\Sigma\psi_V\Theta = \Sigma\delta_V\psi_V\Theta = \Sigma[\delta\psi]_V\Theta = (\delta\psi)^V$$

and

$$\psi^V\varepsilon^V = \Sigma\psi_V\Theta\varepsilon^V = \Sigma\psi_V\varepsilon_V\Theta = \Sigma[\psi\varepsilon]_V\Theta = (\psi\varepsilon)^V.$$

Taking the stack $\Xi = \xi_V\Theta F$, item 2 gives

$$(\psi\xi)^V(F) = \Sigma[\psi\xi]_V\Theta F = \Sigma\psi_V\xi_V\Theta F \leq \psi^V\Sigma\xi_V\Theta F = \psi^V\xi^V(F),$$

and here item 3 provides the equality when V is completely distributive. □

Note that this result was given in [33], except for the identity $(\psi\varepsilon)^V = \psi^V\varepsilon^V$. When V is completely distributive, Proposition 32 reduces to item 3 (item 1 becomes then a particular case). This guarantees in particular that the flat extension of an idempotent operator will be idempotent. As it was the case with Proposition 30, item 3 is verified for the usual lattices 2, T and T^n of image values (which are completely distributive).

However it does not always hold when V is not distributive, in particular flat extension does not necessarily preserve the property of idempotence. We show in Fig. 7 an example where we take again for V the non-distributive lattice U (with $|U_*| \geq 3$); here ψ and ξ are respectively the erosion ε and dilation δ by a 2×2 square; thus $\varepsilon\delta$ is a closing. The second row shows that that $(\varepsilon\delta)^U$ is not idempotent, so it is not a closing. In fact, if we extended the image G upwards by repeating the sequence c, b, a of labels, then we would have $\varepsilon^U\delta^U(G) = [(\varepsilon\delta)^U]^\infty(G)$ (where the exponent ∞ means the limit for $n \to \infty$, that is $\Psi^\infty = \bigvee_{n\in\mathrm{N}}\Psi^n$). Since ε and δ use a 2×2 window, and a label can increase no more than 2 times (from $\perp$ to a proper label u, then to $\top$), it is clear that the repeated application of $(\varepsilon\delta)^U$ to an image will at every point reach a stable result after a finite number of iterations. Hence $[(\varepsilon\delta)^U]^\infty$ is idempotent, so it is the least closing $\geq (\varepsilon\delta)^U$. The first row gives $(\varepsilon\delta)^U(F) = F < \varepsilon^U\delta^U(F)$, so that even by repeating the application of $(\varepsilon\delta)^U$ to F, we would not reach $\varepsilon^U\delta^U(F)$, that is $[(\varepsilon\delta)^U]^\infty < \varepsilon^U\delta^U$; hence $\varepsilon^U\delta^U$ is strictly greater than the least closing $\geq (\varepsilon\delta)^U$.

We saw here that the flat extension of a closing is not always a closing. However we have a positive counterpart for openings:

Corollary 33. *Let γ be an opening on $\mathcal{P}(E)$. Then γ^V is an opening on V^E.*

Proof: By item 2 of Proposition 32, we have $\gamma^V = (\gamma\gamma)^V \leq \gamma^V\gamma^V$. As $\gamma \subseteq \mathbf{id}$, Corollary 29 gives $\gamma^V \leq \mathbf{id}^V$, where $\mathbf{id}^V$ is the identity on V^E by Proposition 23; hence γ^V is anti-extensive, and we get $\gamma^V\gamma^V \leq \gamma^V$. From the two inequalities $\gamma^V \leq \gamma^V\gamma^V$ and $\gamma^V\gamma^V \leq \gamma^V$, it follows that γ^V is idempotent. □

Another interesting consequence of Proposition 28 and item 1 of Proposition 32 is for the flat extension of:

- an increasing operator ψ on sets expressed as a union of erosions, or

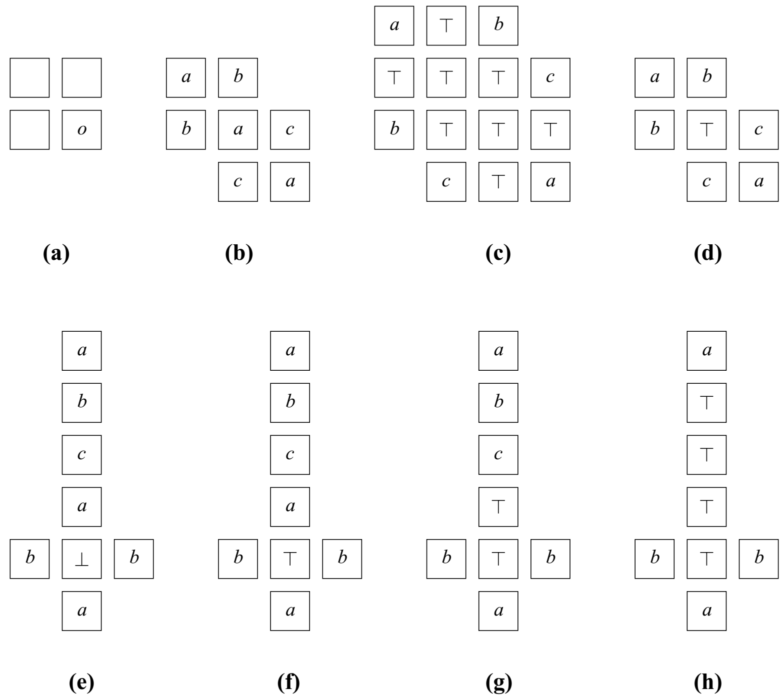

Figure 7. $E = \mathbf{Z}^2$ and $V = U$ with $|U_*| \geq 3$. (a) The structuring element A is a 2×2 square with the origin o in the bottom right corner; let δ and ε be respectively the dilation and erosion by A (for sets). (b) The function $F \in U^E$, where a, b, c are pairwise distinct proper labels, and with value $\perp$ outside the support shown here; we have $(\varepsilon\delta)^U(F) = F$. (c) $\delta^U(F)$. (d) $\varepsilon^U\delta^U(F)$, which has a greater value than $(\varepsilon\delta)^U(F)$ on the central pixel. (e) The function $G \in U^E$. (f) $(\varepsilon\delta)^U(G)$. (g) $[(\varepsilon\delta)^U]^2(G)$; iterating $(\varepsilon\delta)^U$ will spread the label $\top$ upwards. (h) $[(\varepsilon\delta)^U]^4(G) = \varepsilon^U\delta^U(G)$.

- an opening γ on sets expressed as a union of openings of the form $\delta\varepsilon$ arising from adjunctions (ε, δ),

(thanks to the theorems of Matheron and Serra [18, 38]). From the decompositions

$$\psi = \bigcup_{i \in I} \varepsilon_i \qquad \text{and} \qquad \gamma = \bigcup_{j \in J} \delta_j \varepsilon_j,$$

we deduce [33] the decompositions

$$\psi^V = \bigvee_{i \in I} \varepsilon_i^V \qquad \text{and} \qquad \gamma^V = \bigvee_{j \in J} \delta_j^V \varepsilon_j^V.$$

This gives in particular an alternate proof of Corollary 33. Indeed, as each $(\varepsilon_j^V, \delta_j^V)$ is an adjunction by Proposition 24, γ^V will be a supremum of openings, hence an opening [18].

At first sight, it is not evident why the comparison of

1. the flat extension of a combination (union, intersection, composition) of set operators, and
2. the corresponding combination (supremum, infimum, composition) of flat operators,

we have in some cases the equality between the two, and in some other cases an inequality, where the first one is always below the second one. This becomes clear in light of the results of Section 3.3. Let us consider first the 3 cases with unconditional equality (Proposition 28, and item 1 of Proposition 32). Given $F \in V^E$ and the adjunction $(\varepsilon_\mathbf{A}, \delta_\mathbf{A})$ for a variable structuring element $\mathbf{A}$, we have:

$$\begin{aligned} \left(\bigcup_{i \in I} \psi_i\right)^V (F) &= \bigvee_{i \in I} \psi_i^V(F), \\ (\delta_\mathbf{A}\psi)^V(F) &= \delta_\mathbf{A}^V(\psi^V(F)), \\ (\psi\varepsilon_\mathbf{A})^V(F) &= \psi^V\big(\varepsilon_\mathbf{A}^V(F)\big), \end{aligned} \tag{44}$$

which means by (34, 36) that for every $p \in E$ we have:

$$\begin{aligned} \bigvee_{B \in \mathcal{K}(\bigcup_{i \in I} \psi_i, p)} \bigwedge_{q \in B} F(q) &= \bigvee_{i \in I} \bigvee_{B \in \mathcal{K}(\psi_i, p)} \bigwedge_{q \in B} F(q), \\ \bigvee_{B \in \mathcal{K}(\delta_\mathbf{A}\psi, p)} \bigwedge_{r \in B} F(r) &= \bigvee_{q \in \widetilde{\mathbf{A}}(p)} \bigvee_{B \in \mathcal{K}(\psi, q)} \bigwedge_{r \in B} F(r), \\ \bigvee_{B \in \mathcal{K}(\psi\varepsilon_\mathbf{A}, p)} \bigwedge_{r \in B} F(r) &= \bigvee_{B \in \mathcal{K}(\psi, p)} \bigwedge_{q \in B} \bigwedge_{r \in \mathbf{A}(q)} F(r). \end{aligned} \tag{45}$$

Suppose (provisionally) that F is a binary image with values in the sublattice $\{\bot, \top\}$. As seen in Proposition 15, under the isomorphism $\mathcal{P}(E) \to \{\bot, \top\}^E$ between sets and binary images, set operators correspond to their flat counterparts. Hence (44) trivially holds for a binary F, it simply expresses such an isomorphism. It follows then that (45) holds for any binary F. We see here that each member of the 3 equalities takes the form of a supremum of infima. But we saw in Lemma 20 that the equality between two suprema of infima of image values, depends only on the sets on which these suprema and infima are taken, and *not* on the lattice in which they are taken. This implies that *since (45) holds for any $F : E \to \{\bot, \top\}$, it must then necessarily hold for any $F : E \to V$*. Therefore every $F \in V^E$ will verify (44).

Consider now Proposition 30. Combining (34) with (9),

$$\begin{aligned}\bigwedge_{i\in I} \psi_i^V(F)(p) &= \bigwedge_{i\in I} \bigvee_{B\in\mathcal{K}(\psi_i,p)} \bigwedge_{q\in B} F(q) \\ &\geq \bigvee_{\varphi\in\Phi(I)} \bigwedge_{i\in I} \bigwedge_{q\in\varphi(i)} F(q), \end{aligned} \tag{46}$$

where $\Phi(I)$ is the set of choice maps $\varphi : I \to \bigcup_{i\in I} \mathcal{K}(\psi_i, p) : i \mapsto \varphi(i) \in \mathcal{K}(\psi_i, p)$. Note that when one of the two condition given in Proposition 30 is verified (V is ISD and I is finite, or V is completely distributive), we can apply (10, 12) instead of (9), so the last inequality becomes an equality. Now (34) gives

$$\left(\bigcap_{i\in I} \psi_i\right)^V (F)(p) = \bigvee_{B\in\mathcal{K}(\bigcap_{i\in I}\psi_i,p)} \bigwedge_{q\in B} F(q). \tag{47}$$

Given a binary image $F : E \to \{\bot, \top\}$, by Proposition 15 we must have $\bigwedge_{i\in I} \psi_i^V(F) = (\bigcap_{i\in I} \psi_i)^V(F)$, and the lattice $\{\bot, \top\}$ is completely distributive. It follows thus that all expressions in (46) and (47) must be equal, in particular

$$\bigvee_{\varphi\in\Phi i(I)} \bigwedge_{i\in I} \bigwedge_{q\in\varphi(i)} F(q) = \bigvee_{B\in\mathcal{K}(\bigcap_{i\in I}\psi_i,p)} \bigwedge_{q\in B} F(q).$$

By Lemma 20, this identity being verified for any $F : E \to \{\bot, \top\}$, it must then hold for every $F : E \to V$. Combining it with (46, 47), we deduce that

$$\bigwedge_{i\in I} \psi_i^V(F)(p) \geq \left(\bigcap_{i\in I} \psi_i\right)^V (F)(p),$$

with equality when V is ISD and I is finite, or V is completely distributive.

A similar argument applies for items 2 and 3 of Proposition 32. Here

$$\begin{aligned}\psi^V\xi^V(F)(p) &= \bigvee_{B\in\mathcal{K}(\psi,p)} \bigwedge_{q\in B} \bigvee_{C\in\mathcal{K}(\xi,q)} \bigwedge_{r\in C} F(r) \\ &\geq \bigvee_{B\in\mathcal{K}(\psi,p)} \bigvee_{\varphi\in\Phi(B)} \bigwedge_{q\in B} \bigwedge_{r\in\varphi(q)} F(r), \end{aligned} \tag{48}$$

where $\Phi(B)$ is the set of choice maps $\varphi : B \to \bigcup_{q\in B} \mathcal{K}(\xi, q) : q \mapsto \varphi(q) \in \mathcal{K}(\xi, q)$, and with the equality for V completely distributive. We have also

$$(\psi\xi)^V(F)(p) = \bigvee_{B\in\mathcal{K}(\psi\xi,p)} \bigwedge_{r\in B} F(r). \tag{49}$$

For $F : E \to \{\bot, \top\}$, the equality $(\psi\xi)^V(F) = \psi^V\xi^V(F)$ and the complete distributivity of $\{\bot, \top\}$ imply that all expressions in (48, 49) are equal, in particular

$$\bigvee_{B\in\mathcal{K}(\psi,p)} \bigvee_{\varphi\in\Phi(B)} \bigwedge_{q\in B} \bigwedge_{r\in\varphi(q)} F(r) = \bigvee_{B\in\mathcal{K}(\psi\xi,p)} \bigwedge_{r\in B} F(r).$$

Then this equality must hold for every $F : E \to V$, hence

$$\psi^V\xi^V(F)(p \geq (\psi\xi)^V(F)(p),$$

with equality when V is completely distributive.

To summarize, in order to compare the flat extension of a combination of set operators with the corresponding combination of flat operators, we express the behaviour of the latter at a point p, obtaining thus a formula combining suprema and infima. If this combination is a supremum of infima, the two operators are equal; otherwise, applying the appropriate distributivity law transforms the formula into a supremum of infima, and here the two operators are equal only when V satisfies the required distributivity law.

Let us note that both operators $(\bigcap_{i\in I} \psi_i)^V$ and $\bigwedge_{i\in I} \psi_i^V$ have the same behaviour on binary images $F : E \to \{\bot, \top\}$ (the one corresponding to $\bigcap_{i\in I} \psi_i$ on sets). It follows from Proposition 15 that if $\bigwedge_{i\in I} \psi_i^V$ is flat, it must them be identical to the flat operator $(\bigcap_{i\in I} \psi_i)^V$. Therefore if V is not completely distributive and $(\bigcap_{i\in I} \psi_i)^V < \bigwedge_{i\in I} \psi_i^V$, then the operator $\bigwedge_{i\in I} \psi_i^V$ is *not flat*. Similarly for the operator $\psi^V\xi^V$, when $(\psi\xi)^V < \psi^V\xi^V$ (this is for instance the case with the operator $\varepsilon^U\delta^U$ of Fig. 7).

By Theorem 19 and Corollary 22, flat operators on V-images are precisely those that apply at each point a supremum of infima of image values. Suppose that V is not completely distributive. Let us call *flatoid* an operator on V-images which applies at each point a combination of suprema and infima of image values (in any order). When the operators $\bigwedge_{i\in I}\psi_i^V$ and $\psi^V\xi^V$ are not flat, they are flatoids, see (46, 48). The set of flatoids is closed under the composition, supremum and infimum operations, and it contains all flat operators.

3.6. Duality

Given a partial order $\leq$, its reciprocal $\geq$ is also a partial order, for which the supremum and infimum are exchanged w.r.t. the order $\leq$. From this elementary fact follows the general principle of *duality*, that every property or theorem about complete lattices has a dual, where we exchange $\leq\leftrightarrow\geq$, $\bigwedge\leftrightarrow\bigvee$, $\bot\leftrightarrow\top$, etc. We will apply this principle to the construction of flat operators, obtaining thus *dual flat* operators, which will then be compared to flat ones. Our results for V completely distributive were obtained by Heijmans in the restricted case where V is strongly admissible.

Definition 34. The *dual cylinder of base B and level v* is the function $C^*_{B,v}$ defined by

$$\forall p\in E,\qquad C^*_{B,v}(p)=\begin{cases} v & \text{if } p\in B,\\ \top & \text{if } p\notin B.\end{cases}\tag{50}$$

For a V-image F and a value $v\in V$, we define the *dual threshold set* $X^*_v(F)$ by

$$X^*_v(F)=\{p\in E\,|\,F(p)\leq v\}.\tag{51}$$

A *dual stack* on V is an increasing map $\Xi: V\to\mathcal{P}(E)$. We have in particular for every $F\in V^E$ the *dual threshold stack of F*, namely the map $\Theta^*F: V\to\mathcal{P}(E)$ given by

$$\forall v\in V,\qquad \Theta^*F(v)=X^*_v(F);$$

Operators act on dual stacks in the same way as they do on stacks. Given a dual stack Ξ on V, the *dual superposition* of Ξ is the V-image $\Sigma^*\Xi$ defined by

$$\Sigma^*\Xi=\bigwedge_{v\in V}C^*_{\Xi(v),v};\tag{52}$$

in other words, for every point $p\in E$ we have

$$\Sigma^*\Xi(p)=\bigwedge\{v\in V\,|\,p\in\Xi(v)\}.\tag{53}$$

Given an increasing operator ψ on $\mathcal{P}(E)$, the *dual flat operator corresponding to ψ*, or the *dual flat extension of ψ*, is the operator $\psi^{V*}: V^E\to V^E$ on V-images, defined by setting for any V-image F:

$$\psi^{V*}(F)=\Sigma^*\psi_V\Theta^*F;\tag{54}$$

in other words by (52):

$$\psi^{V*}(F)=\bigwedge_{v\in V}C^*_{\psi(X^*_v(F)),v},\tag{55}$$

so that for every point $p\in E$ we have by (53):

$$\psi^{V*}(F)(p)=\bigwedge\{v\in V\,|\,p\in\psi(X^*_v(F))\}.\tag{56}$$

Note that in [34], where we took for V the lattice U of labels, we wrote $\psi^{\widetilde{U}}$ instead of ψ^{U*}.

There is another view of duality, namely *duality by inversion*. In a lattice L, an *autmorphism* is an increasing bijection $\alpha: L\to L$ whose inverse α^{-1} is also increasing (i.e., $a\leq b\Leftrightarrow\alpha(a)\leq\alpha(b)$), while a *dual automorphism* is a decreasing bijection $\beta: L\to L$ whose inverse β^{-1} is also decreasing (i.e., $a\leq b\Leftrightarrow\beta(a)\geq\beta(b)$). An *inversion* (or *involution*) of L is a dual automorphism which is equal to its inverse, in other words a decreasing map η such that η^2 is the identity on L. Then, for a fixed inversion η, any operator ψ on L has its *dual by inversion*, namely $\eta\psi\eta$; the properties of $\eta\psi\eta$ are dual to those of ψ, and reciprocally ψ is the dual by inversion of $\eta\psi\eta$. If one does not take an involution, but more generally a dual automorphism β, then the dual of an operator ψ will be $\beta\psi\beta^{-1}$.

For the lattice $\mathcal{P}(E)$, the standard inversion is the set complementation $X\mapsto X^c$; then every operator ψ on $\mathcal{P}(E)$ has a *dual by complementation* $\psi^*: X\mapsto\psi(X^c)^c$. For the lattice V^E of images, an inversion is built from an inversion of V. Given a map $\lambda: V\to V$, write λ_E for the map $V^E\to V^E$ that applies λ to the value of each point: for $F\in V^E$, $\lambda_E(F): p\mapsto\lambda(F(p))$. Then for an inversion η of V, η_E will be an inversion of V^E, and every operator Ψ on V^E has a *dual by inversion* η_E, namely $\eta_E\Psi\eta_E$.

As expected, the dual flat extension of an operator is indeed the dual of the flat extension, flat dilation and erosion are dual, and identity is autodual:

Proposition 35. *Consider an increasing operator ψ on $\mathcal{P}(E)$ and a variable structuring element $\mathbf{A}: E\to\mathcal{P}(E)$. Then*

1. *For any automorphism α of V, $\alpha_E\psi^V\alpha_E^{-1}=\psi^V$.*
2. *For any dual automorphism β of V, $\beta_E\psi^V\beta_E^{-1}=\psi^{V*}$.*

3. $\delta_{\mathbf{A}}^{V*} = \varepsilon_{\tilde{\mathbf{A}}}^{V}$ *and* $\varepsilon_{\mathbf{A}}^{V*} = \delta_{\tilde{\mathbf{A}}}^{V}$.
4. $\mathbf{id}^{V*} = \mathbf{id}^{V}$, *the identity on* V^E

Proof: 1. Let $F \in V^E$ and $p \in E$. Then (28) gives

$$\alpha\big(\psi^V\big[\alpha_E^{-1}(F)\big](p)\big) = \alpha\Big(\bigvee\big\{v \in V \mid p \in \psi\big(X_v\big[\alpha_E^{-1}(F)\big]\big)\big\}\Big).$$

As α is an automorphism, for every $q \in E$ and $v \in V$, we have $\alpha^{-1}(F(q)) \geq v \Leftrightarrow F(q) \geq \alpha(v)$, and $\alpha_E^{-1}(F)(q) = \alpha^{-1}(F(q))$; hence $X_v(\alpha_E^{-1}(F)) = X_{\alpha(v)}(F)$. Thus, since $[\alpha_E\psi^V\alpha_E^{-1}](F)(p) = \alpha(\psi^V[\alpha_E^{-1}(F)](p))$, we get:

$$\big[\alpha_E\psi^V\alpha_E^{-1}\big](F)(p) = \alpha\Big(\bigvee\big\{v \in V \mid p \in \psi\big(X_{\alpha(v)}(F)\big)\big\}\Big) = \bigvee\big\{\alpha(v) \mid v \in V,\ p \in \psi\big(X_{\alpha(v)}(F)\big)\big\}.$$

Since α is a bijection, the set of $\alpha(v)$ ($v \in V$) is V, so

$$\big[\alpha_E\psi^V\alpha_E^{-1}\big](F)(p) = \bigvee\{w \in V \mid p \in \psi(X_w(F))\} = \psi^V(F)(p).$$

2. Here $X_v(\beta_E^{-1}(F)) = X^*_{\beta(v)}(F)$, and we get

$$\begin{aligned}\big[\beta_E\psi^V\beta_E^{-1}\big](F)(p) &= \beta\big(\psi^V\big[\beta_E^{-1}(F)\big](p)\big)\\ &= \beta\Big(\bigvee\big\{v \in V \mid p \in \psi\big(X_v\big[\beta_E^{-1}(F)\big]\big)\big\}\Big)\\ &= \beta\Big(\bigvee\big\{v \in V \mid p \in \psi\big(X^*_{\beta(v)}(F)\big)\big\}\Big)\\ &= \bigwedge\big\{\beta(v) \mid v \in V,\ p \in \psi\big(X^*_{\beta(v)}(F)\big)\big\}\\ &= \bigwedge\big\{w \in V \mid p \in \psi\big(X^*_w(F)\big)\big\} = \psi^{V*}(F)(p).\end{aligned}$$

3. Using (36) and the fact that duality exchanges $\bigvee$ and $\bigwedge$, we get:

$$\delta_{\mathbf{A}}^{V*}(F)(p) = \varepsilon_{\tilde{\mathbf{A}}}^{V}(F)(p) = \bigwedge_{q \in \tilde{\mathbf{A}}(p)} F(q),$$
$$\varepsilon_{\mathbf{A}}^{V*}(F)(p) = \delta_{\tilde{\mathbf{A}}}^{V}(F)(p) = \bigvee_{q \in \mathbf{A}(p)} F(q).$$

4. By Proposition 23, $\mathbf{id}^V$ is the identity on V^E, so dually $\mathbf{id}^{V*}$ is also the identity. □

In the case of grey-level images, it is known that the dual by inversion of a flat operator is the flat operator corresponding to the dual binary operator: $\psi^{T*} = (\psi^*)^T$. For an arbitrary lattice V of values, the equality requires complete distributivity (as for the intersection and composition of operators, see Propositions 30 and 32):

Proposition 36. *Let ψ be an increasing operator on $\mathcal{P}(E)$, and let ψ^* be its dual by complementation. Then $(\psi^*)^V \leq \psi^{V*}$; when V is completely distributive, we have the equality $(\psi^*)^V = \psi^{V*}$.*

Proof: Let $F \in V^E$ and $p \in E$. Define

$$A = \{v \in V \mid p \notin \psi(X_v(F)^c)\} \text{ and}$$
$$B = \{v \in V \mid p \in \psi(X^*_v(F))\}.$$

Note that $p \notin \psi\,(X_v(F)^c) \Leftrightarrow p \in \psi^*\,(X_v(F))$; thus (28) applied to ψ^* gives $(\psi^*)^V(F)(p) = \bigvee A$. Similarly (56) applied to ψ gives $\psi^{V*}(F)(p) = \bigwedge B$.

Let $v \in A$ and $w \in B$. We have $p \notin \psi\,(X_v(F)^c)$ and $p \in \psi(X^*_w(F))$; hence $\psi(X^*_w(F)) \not\subseteq \psi\,(X_v(F)^c)$; as ψ is increasing, we deduce that $X^*_w(F) \not\subseteq X_v(F)^c$; in other words there is some $q \in X^*_w(F) \cap X_v(F)$, which means (18, 51) that $F(q) \leq w$ and $F(q) \geq v$. Hence $v \leq w$ for all $v \in A$ and $w \in B$, and we get

$$(\psi^*)^V(F)(p) = \bigvee A \leq \bigwedge B = \psi^{V*}(F)(p).$$

Thus $(\psi^*)^V \leq \psi^{V*}$.

Write $b = \bigwedge B$, and let $g \in V \setminus A$. Thus $p \in \psi(X_g(F)^c)$. Let $h = \bigvee\{F(q) \mid q \in X_g(F)^c\}$; then for each $q \in X_g(F)^c$ we have $F(q) \leq h$, so that $X_g(F)^c \subseteq X^*_h(F)$. As ψ is increasing, $\psi(X_g(F)^c) \subseteq \psi(X^*_h(F))$, and $p \in \psi(X^*_h(F))$. Hence $h \in B$, and $b \leq h = \bigvee\{F(q) \mid q \in X_g(F)^c\}$. If $g \lhd b$, by (14) there is some $q \in X_g(F)^c$ with $g \leq F(q)$, that is $q \in X_g(F)$, a contradiction. Therefore a sup-factor of b may not be outside A. If V is completely distributive, then (by Lemma 11) b is the supremum of its sup-factors, all of them belonging to A, so $b \leq \bigvee A$. This means that $\psi^{V*}(F)(p) \leq (\psi^*)^V(F)(p)$, so we have the inequality $\psi^{V*} \leq (\psi^*)^V$, which, combined with the reverse inequality, leads to the equality $(\psi^*)^V = \psi^{V*}$. □

The counterexample to item 3 of Proposition 32, given in Fig. 7, applies also to the last result. Let $V = U$ with $|U_*| \geq 3$, and take δ and ε to be respectively the dilation and erosion by a 2×2 square A. Consider their duals by complementation $\delta' = \varepsilon^*$ and $\varepsilon' = \delta^*$; then δ' and ε' are the dilation and erosion by the symmetrical 2×2 square $\check{A} = \{-a \mid a \in A\}$. Note that

$(\delta'\varepsilon')^* = \delta'^*\varepsilon'^* = \varepsilon\delta$. Combining the dual of Proposition 32 (item 1) with Proposition 35 (item 3), we have $(\delta'\varepsilon')^{U*} = \delta'^{U*}\varepsilon'^{U*} = \varepsilon^U\delta^U$. Hence, as illustrated in Fig. 7,

$$[(\delta'\varepsilon')^*]^U = (\varepsilon\delta)^U < \varepsilon^U\delta^U = (\delta'\varepsilon')^{U*}.$$

Other examples (for $V = U$) were discussed in [34], for example the median filter for binary images (which is autodual), whose flat extension is a supremum of flat erosions, and whose dual flat extension is an infimum of flat dilations.

We can interpret Proposition 36 in light of Section 3.3, using the method of the end of the previous subsection. Given $F \in V^E$ and $p \in E$ we have:

$$\psi^{V*}(F)(p) = \bigwedge_{B\in\mathcal{K}(\psi,p)} \bigvee_{q\in B} F(q) \geq \bigvee_{\varphi\in\Phi(\mathcal{K}(\psi,p))} \bigwedge_{B\in\mathcal{K}(\psi,p)} F(\varphi(B)), \tag{57}$$

where $\Phi(\mathcal{K}(\psi, p))$ is the set of choice maps $\varphi : \mathcal{K}(\psi, p) \to \bigcup \mathcal{K}(\psi, p) : B \mapsto \varphi(B) \in B$. Here the inequality is an equality when V is completely distributive, or if F is binary with values $\perp, \top$. Now the equality

$$\bigvee_{\varphi\in\Phi(\mathcal{K}(\psi,p))} \bigwedge_{B\in\mathcal{K}(\psi,p)} F(\varphi(B)) = \bigvee_{B\in\mathcal{K}(\psi^*,p)} \bigwedge_{q\in B} F(q), \tag{58}$$

holds for F binary, hence it is valid for any F. In fact, one can easily show that for $C \in \mathcal{P}(E)$, $C \in \mathcal{K}(\psi^*, p)$ iff $C \cap B \neq \emptyset$ for all $B \in \mathcal{K}(\psi, p)$, that is iff $C \supseteq \{\varphi(B) \mid B \in \mathcal{K}(\psi, p)\}$ for some $\varphi \in \Phi(\mathcal{K}(\psi, p))$; this implies (58). Combining the two equations, we get Proposition 36.

Note that when the operator ψ on sets is autodual ($\psi^* = \psi$), we get $\psi^V \leq \psi^{V*}$. In [34] we gave a method for constructing operators on label images, which are autodual under inversion. We can generalize it to the case where V is modular (5):

Proposition 37. *Assume that V is modular. Let ψ and ζ be operators on sets such that $\psi \subseteq \psi^*$ and $\zeta^V = \zeta^{V*}$ (for example, $\zeta = \mathbf{id}$). Define the operator*

$$\xi = \psi \cup (\zeta \cap \psi^*) = (\psi \cup \zeta) \cap \psi^*. \tag{59}$$

Then $\psi^V \leq \xi^V \leq \xi^{V*} \leq \psi^{V*}$, $\zeta^V \vee \psi^V = \zeta^V \vee \xi^V$, $\zeta^V \wedge \psi^{V*} = \zeta^V \wedge \xi^{V*}$, and

$$\begin{aligned}\psi^V \vee (\zeta^V \wedge \psi^{V*}) &= (\psi^V \vee \zeta^V) \wedge \psi^{V*} \\ &= \xi^V \vee (\zeta^V \wedge \xi^{V*}) \\ &= (\xi^V \vee \zeta^V) \wedge \xi^{V*}.\end{aligned} \tag{60}$$

The latter operator is autodual by complementation, i.e., it commutes with β_E for every dual automorphism β of V.

Proof: The modular equality (5) gives $\psi \cup (\zeta \cap \psi^*) = (\psi \cup \zeta) \cap \psi^*$, so ξ is well-defined. It is clear that $\xi = \xi^*$, and the equality $\xi = \psi \cup (\zeta \cap \psi^*)$ gives $\psi \subseteq \xi$. Corollary 29 implies then that $\psi^V \leq \xi^V$ and dually $\xi^{V*} \leq \psi^{V*}$, while Proposition 36 gives $\xi^V = (\xi^*)^V \leq \xi^{V*}$. Hence $\psi^V \leq \xi^V \leq \xi^{V*} \leq \psi^{V*}$. Now $\zeta \cup \xi = \zeta \cup \psi \cup (\zeta \cap \psi^*) = \zeta \cup \psi$ (since $\zeta \cap \psi^* \subseteq \zeta$), so Proposition 28 gives $\zeta^V \vee \psi^V = (\zeta \cup \psi)^V = (\zeta \cup \xi)^V = \zeta^V \vee \xi^V$, and dually $\zeta^V \wedge \psi^{V*} = (\zeta \cup \psi)^{V*} = (\zeta \cup \xi)^{V*} = \zeta^V \wedge \xi^{V*}$.

As $\psi^V \leq \psi^{V*}$ and $\xi^V \leq \xi^{V*}$, the modular equality gives

$$\psi^V \vee (\zeta^V \wedge \psi^{V*}) = (\psi^V \vee \zeta^V) \wedge \psi^{V*} \quad \text{and}$$
$$\xi^V \vee (\zeta^V \wedge \xi^{V*}) = (\xi^V \vee \zeta^V) \wedge \xi^{V*}.$$

Now

$$\begin{aligned}&\psi^V \vee (\zeta^V \wedge \psi^{V*}) \\ &\quad = \xi^V \vee (\zeta^V \wedge \xi^{V*}) \text{ (since } \zeta^V \wedge \psi^{V*} = \zeta^V \wedge \xi^{V*}), \\ &\quad = (\psi^V \vee \zeta^V) \wedge \xi^{V*} \text{ (since } \psi^V \leq \xi^{V*}), \\ &\quad = (\xi^V \vee \zeta^V) \wedge \xi^{V*} \text{ (since } \psi^V \vee \zeta^V = \xi^V \vee \zeta^V).\end{aligned}$$

For a dual automorphism β of V, item 2 of Proposition 35 gives

$$\begin{aligned}&\beta_E[\psi^V \vee (\zeta^V \wedge \psi^{V*})]\beta_E^{-1} \\ &\quad = \left[\beta_E \psi^V \beta_E^{-1}\right] \wedge \left(\left[\beta_E \zeta^V \beta_E^{-1}\right] \vee \left[\beta_E \psi^{V*} \beta_E^{-1}\right]\right) \\ &\quad = \psi^{V*} \wedge (\zeta^{V*} \vee \psi^V) \\ &\quad = (\psi^V \vee \zeta^V) \wedge \psi^{V*},\end{aligned}$$

so the operator commutes with β_E. □

By the modular equality (5), we can remove the parentheses in (59, 60), and write $\psi \cup \zeta \cap \psi^*$, $\psi^V \vee \zeta^V \wedge \psi^{V*}$ and $\xi^V \vee \zeta^V \wedge \xi^{V*}$. In [34], the above result was used to build some autodual operators on label images, whose behaviour was described in detail:

- From a median filter μ on sets (which is autodual), build the autodual median filter $\mu^U \vee \mathbf{id}^U \wedge \mu^{U*}$.

- In $\mathcal{P}(E)$ ($E = \mathbf{Z}^n$ or $\mathbf{R}^n$), consider the dilation δ and erosion ε by a nonvoid symmetrical structuring element; then $\delta = \varepsilon^*$ and $\varepsilon \subseteq \delta$, and we can consider the autodual set operator $\psi = \varepsilon \cup \mathbf{id} \cap \delta$; it is called an *annular filter* [21]; now take $\psi^U \vee \mathbf{id}^U \wedge \psi^{U*} = \varepsilon^U \vee \mathbf{id}^U \wedge \delta^U$, it is an autodual annular filter on label images.

4. Conclusion and Perpectives

We have given here a general theory of flat increasing morphological operators on images taking their values in an arbitrary complete lattice. It can be applied on multivalued images, images with the reference order [19, 23, 24] (provided that we close the grey-level reference inf-lattice with a greatest element ∞), label images [2, 3, 34, 35], etc. Our theory is "natural", in the sense that it relies only on the usual thresholding, stack processing and superposition techniques (cf. Definitions 4 and 9).

We have shown that many results known in the case of numerical functions extend to this general framework: possibility to restrict thresholds to a sup-generating subset of the lattice of values, preservation of any complete sublattice of values, same behaviour on two-valued images as the underlying set operator, formulas for flat dilation and erosion, flat extension of a union of set operators as the join of their individual flat extensions, and flat extension of a composition of a set operator followed by a dilation or preceded by an erosion, as the composition of their respective flat extensions. However complete distributivity is necessary for the dual flat extension of an operator to coincide with the flat extension of its dual, for the flat extension of a composition of two arbitrary set operators to be the composition of their respective flat extensions, as well as for the extension of an intersection of set operators to be the meet of their individual flat extensions; (but only infinite supremum distributivity is required for a finite intersection).

Due to limitations on space, we have not dealt with the commutation with thresholding, and with increasing mappings $V \to V$ (also called anamorphoses or contrast functions). In a forthcoming paper, we will give a form of "continuity" requirement for increasing maps $V \to V$ (anamorphoses), which guarantees commutation with flat operators, then we will see that the commutation with thresholding requires on the operator a form of "upper semi-continuity" which depends on the lattice of values, and we will indeed describe the form that it takes for an arbitrary lattice of values, in particular for discrete or analog greylevels or vector values (e.g., RGB colours).

We have also left out some generalizations of flat operators. In [18] one defines a *semi-flat* operator by the following modification of Definitions 4 and 9. Let Ψ be a *stack of increasing operators*, in other words for each $v \in V$, $\Psi(v)$ is an increasing operator $\mathcal{P}(E) \to \mathcal{P}(E)$, which decreases as v increases (for $v < w$, $\Psi(w)(X) \subseteq \Psi(v)(X)$ for all $X \in \mathcal{P}(E)$). The action $\Psi_V : \Xi \mapsto \Psi_V \Xi$ of Ψ on stacks is then defined by the action of each operator $\Psi(v)$ on the corresponding set $\Xi(v)$, that is $\Psi_V \Xi(v) = \Psi(v)(\Xi(v))$, and $\Psi_V \Xi$ will indeed be a stack, as for $v < w$ we have

$$\Psi_V \Xi(w) = \Psi(w)(\Xi(w)) \leq \Psi(v)(\Xi(w)) \leq \Psi(v)(\Xi(v)) = \Psi_V \Xi(v).$$

Then the *semi-flat operator corresponding to* Ψ is the operator $\Psi^V : V^E \to V^E$ on V-images, defined by setting for any V-image F:

$$\Psi^V(F) = \Sigma \Psi_V \Theta F,$$

that is

$$\Psi^V(F) = \bigvee_{v \in V} C_{\Psi(v)(X_v(F)),v},$$

so that for every point $p \in E$ we have

$$\psi^V(F)(p) = \bigvee\{v \in V \mid p \in \Psi(v)\,(X_v(F))\}.$$

Generally speaking, the results of Sections 3.2 and 3.3 do not extend to semi-flat operators, while most of Sections 3.4 and 3.5 remains valid for them, except Proposition 24.

Another possible extension of flat operators is given by flat operators in several variables. From an increasing operator $\psi : \mathcal{P}(E_1) \times \cdots \times \mathcal{P}(E_n) \to \mathcal{P}(E)$ one derives the flat operator $\psi^V : V^{E_1} \times \cdots \times V^{E_n}$ by

$$\psi^V(F_1, \ldots, F_n) = \bigvee_{v \in V} C_{\psi(X_v(F_1),\ldots,X_v(F_n)),v},$$

which is indeed the extension of (27) to several variables. One can also consider semi-flat operators in several variables. In fact the two extensions (semi-flat operator and several variables) are linked: given an increasing operator $\psi : \mathcal{P}(E_1) \times \cdots \times \mathcal{P}(E_n) \to \mathcal{P}(E)$, $0 < m < n$, and fixed parameters $A_i \in V^{E_i}$ for $i = 1, \ldots, m$, the operator

$$V^{E_{m+1}} \times \cdots \times V^{E_n} \to V^E : (F_{m+1}, \ldots, F_n) \mapsto \psi^V(A_1, \ldots, A_m, F_{m+1}, \ldots, F_n)$$

is semi-flat. Conversely, given a stack Ψ of increasing operators $\mathcal{P}(E_1) \times \cdots \times \mathcal{P}(E_n) \to \mathcal{P}(E)$, define $\psi : \mathcal{P}(V) \times \mathcal{P}(E_1) \times \cdots \times \mathcal{P}(E_n) \to \mathcal{P}(E)$ by

$$\psi(A, X_1, \ldots, X_n) = \Psi(\bigwedge A)(X_1, \ldots, X_n);$$

then ψ is increasing, and for every $(F_1, \ldots, F_n) \in V^{E_1} \times \cdots \times V^{E_n}$ we have

$$\Psi^V(F_1, \ldots, F_n) = \psi^V(Id, F_1, \ldots, F_n)$$

for the identity function $Id : V \to V : v \mapsto v$.

Let us give two concrete examples. The first one consists in geodesical reconstruction for grey-level images. Given a *mask* $S \in \mathcal{P}(E)$ and a *marker* $R \in \mathcal{P}(S)$, write $\rho(S, R)$ for the *geodesical reconstruction by dilation from the marker R inside the mask S*, in other words the union of connected components of S having an nonvoid intersection with R; when R is not a subset of S, we set set $\rho(S, R) = \rho(S, R \cap S)$. The classical extension of this operation ρ to grey-level images $E \to T$ is the grey-level reconstruction [43] obtained by applying ρ at each threshold level, in other words the operator ρ^T defined by

$$\rho^T(S, R) = \bigvee_{v \in V} C_{\rho(X_v(S), X_v(R)), v}$$

for $R, S \in T^E$. As written here, it is a flat operator in two variables S and R. However, as pointed out in [7], for a fixed mask S, $\rho^T(S, \cdot)$ is a semi-flat operator on marker functions R, and similarly for a fixed marker R, $\rho^T(\cdot, R)$ is a semi-flat operator on mask functions S.

A second example is the *fuzzy Minkowski addition* of functions $E \to V$ (where $E = \mathbf{R}^n$ or $\mathbf{Z}^n$), defined in [22] by

$$F \oplus G = \bigvee_{v \in V} C_{X_v(F) \oplus X_v(G)},$$

which gives at every point $p \in E$:

$$(F \oplus G)(p) = \bigvee_{q \in E} \big(F(p - q) \wedge G(q)\big),$$

and it is under this form that the operator is defined in [5, 9]. Clearly it is a flat operator $V^E \times V^E \to V^E$, but for a fixed $G \in V^E$, $\delta_G : F \mapsto F \oplus G$ is a semi-flat operator on V^E (in fact, when V is ISD, it is a dilation).

Another interesting topic is the theory of flat extensions of *non-increasing* operators. For example the top-hat operators $X \mapsto \varphi(X) \setminus X$ (for a closing φ) and $X \mapsto X \setminus \gamma(X)$ (for an opening γ) are usually extended to grey-level images by the maps $F \mapsto \varphi^T(F) - F$ and $F \mapsto F - \gamma^T(F)$. Some ideas towards such an extension are briefly given at the end of [31], notably by using the threshold decomposition technique of [44].

It is thus clear that the topic of building operators on non-binary images with the help of set operators, thresholding and various forms of "stacking", is still open to further research.

Appendix: Heijmans' Admissibility Conditions

There are works which wait, and which one does not understand for a long time; the reason is that they bring answers to questions which have not yet been raised; for the question often arrives a terribly long time after the answer. — Oscar Wilde

Recall (14) the relation $\lhd$ on L: for $w, x \in L$,

$$w \lhd x \Leftrightarrow \Big[\forall Y \subseteq L,\ x \leq \bigvee Y \Rightarrow \exists y \in Y,\ w \leq y\Big].$$

Heijmans [18] wrote $w \ll x$ and said that w *is below* x. This terminology was probably inspired by [12] (see the new edition [13], pp. 49, 50); there $w \ll x$ is said w is *way below* x, and this corresponds to the weaker condition that $x \leq \bigvee Y$ implies that $w \leq \bigvee A$ for some finite $A \subseteq Y$.

Let us give further properties of the relation $\lhd$; again, our analysis follows the works of Bruns [8] and Papert [28].

Lemma 38. *In a complete lattice L:*

1. *For $w, x \in L$, $w \not\lhd x \Leftrightarrow x \leq \bigvee\{y \in L \mid w \not\leq y\}$.*
2. *For $w \in L$ and $X \subseteq L$, $w \lhd \bigvee X$ implies that $w \lhd x$ for some $x \in X$.*

Proof: Let $w \in L$ and $T = \{y \in L \mid w \not\leq y\}$.

1. If $x \leq \bigvee T$, $w \lhd x$ would give $w \leq y$ for some $y \in T$, which is impossible; hence $w \not\lhd x$. Conversely, if $w \not\lhd x$, then there is some $Y \subseteq L$ such that $x \leq \bigvee Y$ but $w \not\leq y$ for all $y \in Y$; then $Y \subseteq T$, and as $x \leq \bigvee Y$, we get $x \leq \bigvee T$. Therefore $w \not\lhd x \Leftrightarrow x \leq \bigvee T$.
2. Suppose that $w \not\lhd x$ for all $x \in X$; then by item 1 we have $x \leq \bigvee T$ for all $x \in X$, so $\bigvee X \leq \bigvee T$, which means by item 1 again that $w \not\lhd \bigvee X$, a contradiction.

□

Corollary 39. *Let the complete lattice L be completely distributive. Then for $w, x \in L$, $w \lhd x$ implies that there exists $v \in L$ such that $w \lhd v$ and $v \lhd x$.*

Proof: Let $X = \{v \in L \mid v \lhd x\}$. By Lemma 2, we have $x = \bigvee X$, and as $w \lhd \bigvee X$, item 2 of the previous Lemma gives $w \lhd v$ for some $v \in X$. □

We can now introduce Heijmans' admissibility condition (Definition 10.4 of [18]). Write $\rhd$ for the dual of the relation $\lhd$ defined in (14), that is:

$$w \rhd x \iff \left[\forall Y \subseteq L,\ x \geq \bigwedge Y \Rightarrow \exists y \in Y, w \geq y\right]. \quad (61)$$

Definition 40 ([18]). The complete lattice L is called *admissible* if the following four conditions are all satisfied:

1. For every $x \in L$, $x = \bigvee\{w \in L \mid w \lhd x\}$.
2. For every $w, x \in L$, $w \lhd x$ implies that there exists $v \in L$ such that $w \lhd v$ and $v \lhd x$.
3. For every $x \in L$, $x = \bigwedge\{w \in L \mid w \rhd x\}$.
4. For every $w, x \in L$, $w \rhd x$ implies that there exists $v \in L$ such that $w \rhd v$ and $v \rhd x$.

Clearly item 1 is equivalent to the complete distributivity of L (Lemma 2), and item 2 follows from that complete distributivity (Corollary 39). Now items 3 and 4 are the duals of items 1 and 2; as complete distributivity is autodual (extended supremum distributivity (10) is equivalent to extended infimum distributivity (11)), item 3 is equivalent to complete distributivity, and item 4 follows from it. To summarize, Heijmans' admissibility condition is a redundant formulation of complete distributivity.

Heijmans [18] showed that with the admissibility (complete distributivity) of the lattice V, one has the identity

$$X_v(\psi^V(F)) = \bigcap_{w \lhd v} \psi(X_w(F)), \quad (62)$$

which was then used to prove the properties of flat operators (cf. the ones given in Sections 3.5 and 3.6).

Now Heijmans [18] says that L is *strongly admissible* if L is admissible and for every $x, y \in L$ with $x \neq \bot$ and $y \neq \top$, we have $x \lhd y \iff y \rhd x$ (note that in Definition 10.4 of [18], he writes the condition with $x \neq \top$ and $y \neq \bot$, but that is probably a misprint).

Given a complete chain T, for every $x \in T$, we have two cases:

- $x = \bigvee\{y \in T \mid y < x\}$ and for $w \in T$, we have $w \lhd x \iff w < x$;
- $x > \bigvee\{y \in T \mid y < x\}$ and for $w \in T$, we have $w \lhd x \iff w \leq x$.

Hence T is completely admissible in the following two cases:

- $\forall x \in T \setminus \{\bot, \top\}$, $\bigvee\{y \in T \mid y < x\} = x = \bigwedge\{y \in T \mid y > x\}$;
- $\forall x \in T \setminus \{\bot, \top\}$, $\bigvee\{y \in T \mid y < x\} < x < \bigwedge\{y \in T \mid y > x\}$.

For example $\bar{\mathbf{R}}$ and $\bar{\mathbf{Z}}$ are strongly admissible.

On the other hand, for $|T| \geq 4$ and $n \geq 2$, T^n is not strongly admissible, as for $\bot < a < b < \top$ we have $(a, \bot, \ldots, \bot) \lhd (b, b, \ldots, b)$, but $(b, b, \ldots, b) \not\rhd (a, \bot, \ldots, \bot)$ (take $Y = \{(a, \top, \ldots, \top), (\top, \bot, \ldots, \bot)\}$, then (61) will fail). Thus a lattice of analog or discrete vector values (such as RGB colours) will not be strongly admissible. In practice, strong admissibility will be restricted to grey-level images.

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