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# Viscous Lattices

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**Abstract.** Let E be an arbitrary space, and $\delta$ an extensive dilation of $\mathcal{P}(E)$ into itself, with an adjoint erosion $\varepsilon$. Then, the image $\delta[\mathcal{P}(E)]$ of $\mathcal{P}(E)$ by $\delta$ is a complete lattice $\mathcal{L}$ where the sup is the union and the inf the opening of the intersection according to $\delta\varepsilon$. The lattice $\mathcal{L}$, named viscous, is not distributive, nor complemented. Any dilation $\alpha$ on $\mathcal{P}(E)$ admits the same expression in $\mathcal{L}$. However, the erosion in $\mathcal{L}$ is the opening according to $\delta\varepsilon$ of the erosion in $\mathcal{P}(E)$. Given a connection $\mathcal{C}$ on $\mathcal{P}(E)$ the image of $\mathcal{C}$ under $\delta$ turns out to be a connection $\mathcal{C}'$ on $\mathcal{L}$ as soon as $\varepsilon\delta(\mathcal{C}) \subseteq \mathcal{C}$. Moreover, the elementary connected openings $\gamma_x$ of $\mathcal{C}$ and $\gamma'_{\delta(x)}$ are linked by the relation $\gamma'_{\delta(x)} = \delta\gamma_x\varepsilon$. A comprehensive class of connection preverving closings $\varepsilon\delta$ is constructed. Two examples, binary and numerical (the latter comes from the heart imaging), prove the relevance of viscous lattices in interpolation and in segmentation problems.

**Keywords:** lattices, mathematical morphology, connections, image interpolation

## 1. Introduction

This paper is a modified version of my communication [20] at ISMM 2002. On the one hand, Ch.Ronse provided me with a counter example proving that Theorem 8 of this communication needed a supplementary condition, and noticed also that some assumption was useless in Proposition 16. On the other hand, I. Terol Villalobos asked me several questions about the text, showing that some developments were uselessly complicated. I thank them both for their comments, and also because that urged me to review the whole paper. I made several changes and simplifications which finally yield what follows.

The present study stems from three origins. First of all the viscous lattices, also called lattices of dilates, appeared at CMM (Centre de Morphologie Mathématique) during the nineties as a typical counter example of bad properties (those listed in Proposition 3), which is always useful when lecturing courses. It is for this sake of pedagogy that they are introduced in [9] p. 101 and in [18], Section 2.1. But in both cases, the properties of Propositions 1 and 3 are stated without proofs. It was tempting to approach the structure more systematically, and to discover the consequences of these geometrical *quanta of operation* that are the structuring elements $\delta(x)$. Second, in 1998 C. Vachier presented her first results on watershed regularization, where she was applying a viscosity algorithm due to Meyer [10]. Their viscous propagations (see [22], and [23]) suggested me to replace the usual working lattice $\mathcal{P}(E)$ by the more convenient framework that is developed below. My third motivation comes from morphological connections. In this theory, one can easily derive a connection from another by means of extensive dilations (e.g. by discs in $\mathbb{R}^2$) that cluster particles [17], but there are less possibilities to split objects, and the viscous lattice approach seemed to reach this goal.

## 2. Reminder on Lattices, Atoms, Co-Primes and Sup-Generators

In this paper, we compare the lattice $\mathcal{P}(E)$ of the subsets of an arbitrary set $E$ with the viscous set lattice $\mathcal{L}$, which is provided with the same set ordering as $\mathcal{P}(E)$, but where the infimum is no longer the set intersection. This provokes various changes whose description requires the reminder of a few definitions and of some classical results (for the definitions, see [1] and for the results associated with operators [5–7, 11, 14, 16]).

1. A non zero element $A$ of a complete lattice $\mathcal{T}$ is an *atom* if $X \leq A$ implies $X = 0$ or $X = A$. For example the sets $\{x\}, x \in E$ are atoms in $\mathcal{P}(E)$, where they are called singletons.
2. An element $A \in \mathcal{T}$, $A \neq 0$ is said to be *co-prime* when $A \leq X \vee Y$ implies $A \leq X$ or $A \leq Y$, in a non exclusive manner.
3. A class $\mathcal{B}$ is said to be a *sup-generator* when every element $X \in \mathcal{T}$ is the supremum of the elements of $\mathcal{B}$ smaller than it

$$X = \vee\{B \leq X, B \in \mathcal{B}\}.$$

4. Lattice $\mathcal{T}$ is said to be *atomic* (resp. *co-primary*) when it is generated by a class of atoms (resp. co-primes). Clearly, every atom belongs to every sup-generating family.
5. Let 0 and $M$ be the extreme elements of lattice $\mathcal{T}$. If $X, Y \in \mathcal{T}$ are such that

$$X \wedge Y = 0 \quad \text{and} \quad X \vee Y = M,$$

   then $Y$ is called a *complement* of $X$ (and vice versa). The lattice $\mathcal{T}$ is said to be *uniquely complemented* when each of its elements has a unique complement.
6. Several useful properties involve distributivity. Remember that a lattice $\mathcal{T}$ is *distributive* if for all $A$, $B$,$C \in \mathcal{T}$

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C).$$

   The two equalities are equivalent. In a distributive lattice, every atom is co-prime ( [6], Proposition 2.37). A *complemented* and *distributive* lattice is said to be *Boolean* (e.g. $\mathcal{P}(E)$).
7. Lattice $\mathcal{T}$ is modular when for any A, B, C in $\mathcal{T}$

$$B < A \Rightarrow A \wedge (B \vee C) = B \vee (A \wedge C).$$

   Clearly, distributivty implies modularity [1].
8. Let $\mathcal{L}$, $\mathcal{M}$ be two complete lattices. The mappings from $\mathcal{L}$ into $\mathcal{M}$ which commute with the sup (resp. the inf) are called dilations $\delta$ (resp. erosions $\varepsilon$)

$$\delta(\vee X_i) = \vee\delta(X_i), \quad \varepsilon(\wedge X_i) = \wedge\varepsilon(X_i), \quad X_i \in \mathcal{L} \tag{1}$$

   with in particular $\delta(0_{\mathcal{L}}) = 0_{\mathcal{M}}$ and $\varepsilon(M_{\mathcal{L}}) = M_{\mathcal{M}}$.
9. The dilations $\delta : \mathcal{L} \to \mathcal{M}$ and of the erosions $\varepsilon : \mathcal{M} \to \mathcal{L}$ correspond to one another via the duality relation

$$\delta(X) \leq Y \Leftrightarrow X \leq \varepsilon(Y), \quad X, Y \in \mathcal{L}\,, \tag{2}$$

   called "*adjunction*". It is due to E.Galois, and occurs if and only if $\delta$ is a dilation and $\varepsilon$ an erosion [5, 16].
10. Given the adjunction $(\delta, \varepsilon)$, the composition product $\gamma = \delta\varepsilon$ (resp.$\varphi = \varepsilon\delta$) product is an *opening* (resp. a *closing*) i.e. an increasing, idempotent and anti-extensive (resp. extensive) operation [7, 11, 16].
11. The family $\mathcal{L} = \{\delta(X), X \in \mathcal{P}(E)\}$ is both the image of $\mathcal{P}(E)$ under dilation $\delta$ and under the opening $\gamma = \delta\varepsilon$, adjoint to the dilation $\delta$ (Theorem 1.3 in [15], see also [11]).
12. Let $\gamma$ be an opening on $\mathcal{P}(E)$, and $\mathcal{I} \subseteq \mathcal{P}$ be the class of those sets that are invariant under $\gamma$. It is *closed under union* and $\gamma(X)$ is the union of all sets $Y \in \mathcal{I}$ that are smaller than $X$ (Proposition 7.1.1, p. 188, in [7], see also [11]), (dual statement for the closings).
13. Complete lattices are the concern of several dualities. First, we can reverse the sense of the ordering, and invert sup and inf operations, which generates the *dual lattice* $(\mathcal{T}, \leq)$ of $(\mathcal{T}, \geq)$. Atoms, co-primes and sup-generators on $(\mathcal{T}, \leq)$ define, on $(\mathcal{T}, \geq)$, dual anti-atoms, primes, and inf-generators respectively. Moreover, when lattice $(\mathcal{T}, \geq)$ is boolean, the complement operator induces another duality on $(\mathcal{T}, \geq)$. Finally, Eq. (2) of Galois adjunction provides us with a third duality which, just as the first one, applies to any lattice, complemented or not.

## 3. Viscous Lattices

Let $E$ be an arbitrary set, $\mathcal{P}(E)$ be the lattice of its subsets, and let $\delta$ be an extensive dilation: $\mathcal{P}(E) \to \mathcal{P}(E)$, with adjoint erosion $\varepsilon$. The operator $\delta$ is determined by the images of singletons $\{x\}$ of $\mathcal{P}(E)$, since the dilate $\delta(X)$ of any $X \in \mathcal{P}(E)$ is the union of the dilates of the points it contains:

$$\delta(X) = \cup\{\delta(x), x \in X\} \quad X \in P(E) \tag{3}$$

According to the above point 11, the image $\mathcal{L} = \delta(\mathcal{P}(E))$ of the dilation $\delta$ coincides with that of the adjoint opening $\delta\varepsilon$. In fact, family $\mathcal{L}$ turns out to be a complete lattice:

**Proposition 1.** *Given an extensive dilation $\delta$ on $\mathcal{P}(E)$, the set $\mathcal{L} = \delta(\mathcal{P}(E))$ is a complete lattice regarding the inclusion ordering. In this lattice, the supremum of a family $\{X_i, i \in I\}$ coincides with its set union, whereas the infimum $\wedge$ is given the opening according to $\gamma = \delta\varepsilon$ of the intersection $\cap X_i$*

$$\wedge\{X_i, i \in I\} = \gamma(\cap\{X_i, i \in I\}) \quad \{X_i, i \in I\} \in \mathcal{L} \tag{4}$$

*The extreme elements of $\mathcal{L}$ are $E$ and the empty set $\emptyset$. $\mathcal{L}$ is said to be the* viscous lattice of dilation $\delta$.

**Proof:** Consider a family $\{X_i, i \in I\} \in \mathcal{L}$. We draw from Relation (3) that the set union is the smallest element of $\mathcal{L}$ greater than each $X_i$. On the other hand, according to the above points 11 and 12, $\gamma(\cap X_i)$ belongs to $\mathcal{L}$, and is the largest element of $\mathcal{L}$ included in $\cap X_i$, therefore in each $X_i$ which achieves the proof. □

Proposition 1 can be extended to complete lattices (indeed, it is already implicitly contained in [5, 15]); but we keep here the level of generality of the $\mathcal{P}(E)$ lattices. Dually, the set $\mathcal{L}^*$ of all $\varepsilon(x)$, $X \in \mathcal{P}(E)$ is equal to the set of $\varepsilon\delta(X)$. We have

**Proposition 2.** *The two families $\mathcal{L}$ and $\mathcal{L}^*$ turn out to be two isomorphic lattices.*

**Proof:** We draw from [15] (p. 21), or from [5], the two equalities $\delta\varepsilon\delta = \delta$ and $\varepsilon\delta\varepsilon = \varepsilon$, which imply that $\delta\varepsilon$ is the identity on $\mathcal{L}^*$ and $\varepsilon\delta$ is the identity on $\mathcal{L}$, so that the two increasing mappings $\delta : \mathcal{L}^* \to \mathcal{L}$ and $\varepsilon : \mathcal{L} \to \mathcal{L}^*$ are each other's inverse. □

If a lattice is instructive by its bad properties, as well as by its nice ones, then the following result teaches us a lot.

**Proposition 3.** *The viscous lattice $\mathcal{L}$ of dilation $\delta$ is generally neither modular (hence neither distributive) nor co-primary, nor admits any unique complement.*

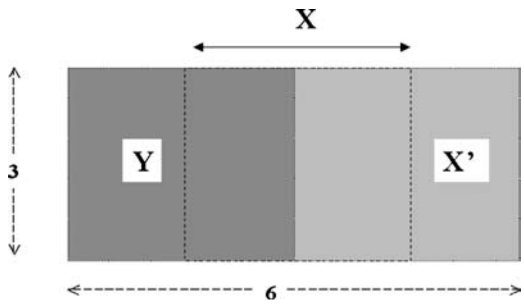



*Figure 1.* Counter example for the lack of distributivity in $L$.

**Proof:** Exhibiting a counter-example of each property is enough. Take for dilation $\delta$ the Minkowski's addition by the structuring element $B$, in $\mathbb{R}^2$ or in $\mathbb{Z}^2$ where $B$ is the compact square of side 3. Consider the three squares $X$, $X'$, and $Y$ of size three depicted in Fig. 1. These three elements are atoms of $\mathcal{L}$. However, a $3 \times 3$ square included in the union $(X \cup X')$ is not necessarily included in either $X$, or $X'$: therefore, atoms are not co-prime. Moreover, setting $Z = X \cup Y$, we have $Y < Z < X' \cup Y$

$$[Z \wedge X'] \cup Y = \emptyset \cup Y = Y < Z = Z \wedge (X' \cup Y)$$

hence lattice $\mathcal{L}$ is not modular. As it is not modular, it is not distributive [1]. Finally, the complements are multiple because when $X \in \mathcal{L}$, any set $Y \in \mathcal{L}$ such that

$$Y \supseteq X^c \quad \text{and} \quad \gamma(Y \cap X) = \emptyset.$$

is a complement of X in $\mathcal{L}$. □

It is remarkable that as soon as $\mathcal{P}(E)$ is regularized by any $\delta$, as small as we want, it looses its basic properties to be distributive and uniquely complemented. The above counter-properties are not independent: they illustrate *a contrario* a well known result in lattice theory (see for example [9], p. 101) according to which in a distributive lattice, any atom is co-prime. The main consequence of such a statement concerns operators, as it is not possible to base ourselves on the complement for defining pairs of dual operations. Fortunately, Galois's adjunctions (2) between dilations and erosions remain defined (as on any complete lattice), which will be useful.

Dilation $\delta$ may be the composition product of two dilations, i.e. $\delta_1 = \delta_2\delta_3$. Then every $\delta_1(x)$ is a union of $\delta_2(y)'s$ so $\mathcal{L}_{\delta_1} \subseteq \mathcal{L}_{\delta_2}$. In particular, $\delta$ may depend on a positive parameter. Suppose for example that $E$ is the

Euclidean space $\mathbb{R}^n$ and take for generating dilation $\delta = \delta_r$ the Minkowski addition by the open ball of radius $r$. We have clearly

$$r \leq r' \Rightarrow \mathcal{L}_r \supseteq \mathcal{L}_{r'}$$

As $r$ varies, the union of all lattices $\mathcal{L}_r$ generates the complete lattice

$$\mathcal{L} = \cup\{\mathcal{L}_r, r > 0\}$$

which is still non-modular, non-distributive, non-uniquely complemented and non co-primary. Lattice $\mathcal{L}$ is composed of all sets that are equal to their interiors, i.e. of all open sets of the Euclidean space $\mathbb{R}^n$. And marvellously, when we add to $\mathcal{L}$ the points, the lines and the various fine sets of the space, in order to obtain the complete family $\mathcal{P}(\mathbb{R}^n)$, then we recover all nice properties of complement, of distributivity, and of co-primarity that seem so natural to us...

## 4. Increasing Operations in $\mathcal{L}$

Consider a viscous lattice $\mathcal{L}$, as defined in Proposition 1. We will distinguish between the increasing operations on $\mathcal{L}$ which do not involve a connection from those which require some. The present section is devoted to the first category; the second one is studied in the next sections.

As every increasing operation in $\mathcal{L}$ is a supremum of erosions and an infimum of dilations of $\mathcal{L}$ into itself ([15], p. 20), we can focus the study on these two operations only. Moreover, as $\mathcal{L}$ is a subset of $\mathcal{P}(E)$, we can wonder to which extent a dilation or an erosion on $\mathcal{P}(E)$ generates homologs on $\mathcal{L}$, and, if so, which ones? In order to avoid any confusion, we denote by $\alpha$ the current dilation on $\mathcal{P}(E)$ in itself and we keep symbol $\delta$ for referring to the fixed dilation of $\mathcal{P}(E)$ that generates $\mathcal{L}$, and whose adjoint erosion is $\varepsilon$.

**Proposition 4.** *1/ Any dilation $\alpha : \mathcal{P}(E) \to \mathcal{P}(E)$ that commutes with $\delta$ is also a dilation of $\mathcal{L}$ into itself.*

*2/ If $\alpha^{-1}$ stands for the erosion adjoint to $\alpha$ in $\mathcal{P}(E)$, and $\beta$ for the erosion adjoint to $\alpha$ in $\mathcal{L}$, then we have*

$$\beta = \delta\varepsilon\alpha^{-1} = \gamma\alpha^{-1} \tag{5}$$

**Proof:** 1/ As dilations $\alpha$ and $\delta$ commute, we get

$$\alpha[\mathcal{L}] = \alpha[\delta[\mathcal{P}(E)]] = \delta[\alpha[\mathcal{P}(E)]] \subseteq \mathcal{L}$$

which means that $\alpha$ maps $\mathcal{L}$ into itself, and as the suprema in $\mathcal{L}$ and in $\mathcal{P}(E)$ are the same, $\alpha$ is also a dilation on $\mathcal{L}$;

2/ The eroded $\beta(X)$, in $\mathcal{L}$, is the greatest $Y \in \mathcal{L}$ such that $\alpha(Y) \subseteq X$, or by adjunction in $\mathcal{P}(E)$, such that $Y \subseteq \alpha^{-1}(X)$. By Proposition (1) this must be $\gamma(\alpha^{-1}(X))$, which is nothing else than relation (5). □

Therefore, each eroded element in $\mathcal{L}$ is equal to the opening by $\delta\varepsilon$ of the homologous eroded element (i.e. of the same adjoint dilation) in $\mathcal{P}(E)$.

## 5. Connections on Viscous Lattices

In Mathematical Morphology a *connection,* or *connected class,* on $\mathcal{P}(E)$ is a set family $\mathcal{C} \subseteq \mathcal{P}(E)$ that satisfies the three following axioms ([17])

$$\begin{aligned} &i/\ \emptyset \in \mathcal{C} \\ &ii/\ x \in E \Rightarrow \{x\} \in \mathcal{C} \\ &iii/\{X_i, i \in I\} \subseteq \mathcal{C} \quad \text{and} \quad \cap X_i \neq \emptyset \ \Rightarrow \cup X_i \in \mathcal{C}. \end{aligned} \tag{6}$$

The second axiom means that class $\mathcal{C}$ is sup generating, and the third one that it is conditionally closed under union. the following theorem [17] provides us with an operating way to act on connections:

**Theorem 5.** *The datum of a connected class $\mathcal{C}$ on $\mathcal{P}(E)$ is equivalent to the family $\{\gamma_x, x \in E\}$ of the so called "point connected openings" such that*

(iii) *for all $x \in E$, we have $\gamma_x(x) = \{x\}$*
(iv) *for all $A \subseteq E$, $x, y \in E$, $\gamma_x(A)$ and $\gamma_y(A)$ are equal or disjoint*
(v) *for all $A \subseteq E$, and all $x \in E$, we have $x \notin A \Rightarrow \gamma_x(A) = \emptyset$.*

Clearly, the invariant sets of the $\gamma's$ coincide with the connected class $\mathcal{C}$. The definition of a connection, as well as the equivalence theorem, extend to a complete sup-generated lattice $\mathcal{T}$ (cf. [18]) by changing, in the first axiom, $\emptyset$ by the zero of the lattice and, in the third one, the intersection and the union by the supremum and the infimum. The second axiom then consists in stating that $\mathcal{C}$ is a sup-generating class. In particular, when there exists in $\mathcal{T}$ a sup-generating family $\mathcal{S}$ that belongs to all the sup-generating classes, the second axiom can be replaced by $\mathcal{S} \subseteq \mathcal{C}$ [13].

In the case of viscous lattices, we are almost in this last situation, for the class $\{\delta(x), x \in E\}$ of the structuring elements plays a specific role. Not only it is sup-generating, but it also belongs to all the sup-generating classes of $\mathcal{L}$ that are not too pathological. It is in particular the case when we have $\varepsilon\delta(x) = \{x\}$ for all points $x \in E$. By adjunction, this condition is equivalent to

$$\delta(x) \subseteq \delta(y) \Leftrightarrow x = y \tag{7}$$

whose meaning is that of a separation axiom. Equivalence 7 is fulfilled for example for the translation invariant dilations on $\mathcal{P}(\mathbb{R}^n)$ or $\mathcal{P}(\mathbb{Z}^n)$, as soon as the transform $\delta(o)$ of the origin is bounded. When Relation 7 is satisfied, then the $\delta(x)$'s become atoms, which ensures us that they belong to all connected classes. Therefore, we can restrict ourselves to such *separated dilations,* which yields the following definition of a connection on $\mathcal{L}$.

*Definition 6.* Let $\mathcal{L}$ be a viscous lattice on $\mathcal{P}(E)$, based on a dilation $\delta$ satisfying Equivalence (7). A class $\mathcal{C}'$ of $\mathcal{L}$ defines a connection on $\mathcal{L}$ when

$$\begin{aligned} &\text{i/ } \emptyset \in \mathcal{C}' \\ &\text{ii/ } x \in E \Rightarrow \delta\{x\} \in \mathcal{C}' \\ &\text{iii/ } \{X_i, i \in I\} \subseteq \mathcal{C}' \text{ and } \wedge X_i \neq \emptyset \Rightarrow \cup X_i \in \mathcal{C}' \end{aligned} \tag{8}$$

Consider now a connection $\mathcal{C}$ on $\mathcal{P}(E)$ (and no longer on $\mathcal{L}$). An interesting feature of dilation $\delta$ is that it may preserve *also* the $\mathcal{C}$-connectivity of the singletons, i.e.

$$x \in E \Rightarrow \delta(x) \in \mathcal{C} \tag{9}$$

Then, since dilation $\delta$ is extensive by definition (see Proposition 1), it preserves the whole class $\mathcal{C}$ (Proposition 8 in [18]) and the adjoint erosion $\varepsilon$ treats the connected components independently of each other (Proposition 11 in [18]), i.e.

$$X = \cup\{X_i, X_i \in \mathcal{C}\} \Rightarrow \varepsilon(X) = \cup\varepsilon(X_i)$$

The correspondence between the two systems of axioms (6) and (8) is so direct that we can wonder whether the restriction to $\mathcal{L}$ of any connection on $\mathcal{P}(E)$ does not induce a connection on $\mathcal{L}$ itself. Indeed that is the case, as we will see now, but the discussion below about Fig. 2(a) indicates that the connections reached in this way may not be the most pertinent ones.

**Proposition 7.** *Let $\mathcal{L}$ be the viscous lattice on $\mathcal{P}(E)$ of (extensive) dilation $\delta$, and let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$. If $\delta$ preserves the connectivity of the singletons of $\mathcal{P}(E)$, then the restriction $\mathcal{C}' = (\mathcal{C} \cap \mathcal{L})$ of connection $\mathcal{C}$ to $\mathcal{L}$ is itself a connection over lattice $\mathcal{L}$. If $\gamma_x$ stands for the point connected opening at point x w.r.t. $\mathcal{C}$, we have*

$$\delta\gamma_x(A) = \gamma_x\delta\gamma_x(A) \subseteq \gamma_x\delta(A), \quad A \in \mathcal{P}(E) \tag{10}$$

**Proof:** We observe firstly that the empty set is a $\delta$ dilate, and all $\delta(x)$, $x \in E$ belong to both $\mathcal{C}$ and $\mathcal{L}$, hence class $\mathcal{C}'$ satisfies the first two axioms of Definition 6. As regarding the third one, consider a family $\{X_i, i \in I\}$ in class $\mathcal{C}'$. When $\wedge X_i \neq \emptyset$, we have $\wedge X_i \neq \emptyset \Rightarrow \cap X_i \neq \emptyset \Rightarrow \cup X_i \in \mathcal{C}$, but $\cup X_i$ also belongs to $\mathcal{L}$ which is closed under union, hence $\cup X_i \in \mathcal{C}'$, which is therefore a connection. As for rel. (10), we observe that $\delta\gamma_x(A), x \in E, A \in \mathcal{P}(E)$, is connected because $\delta$ preserves class $\mathcal{C}$ (by Proposition 11 in [18]), and that $\delta\gamma_x(A) = \gamma_x\delta\gamma_x(A)$ because by extensivity of $\delta$, the set $\delta\gamma_x(A)$ contains the point $x$. The last inequality of

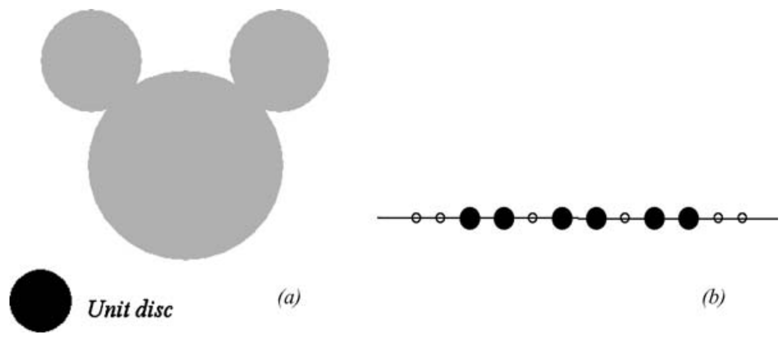



*Figure 2.* (a) Am I connected? (b) The two left handside segments (of two points each), as well as the two right handside ones are $\mathcal{C}'$-connected sets, but the union of the three segments is not.

rel. (10) follows from the fact that $\gamma_x\delta$ is increasing and $\gamma_x(A) \subseteq A$. □

A simpler form of Proposition 7 appears in U. Braga-Neto's Ph.D. thesis [3], pp. 85–86. Conversely, let $\mathcal{C}'$ be a connection on $\mathcal{L}$. Associate with it the class $\mathcal{C}'_1$ conditionally closed under union generated by $\mathcal{C}'$, i.e. the class in $\mathcal{P}(E)$ that is composed of all the unions of elements of $\mathcal{C}'$whose intersections are not empty. We have $\mathcal{C}' \subseteq \mathcal{C}'_1$, since some inf-empty families may have a non empty intersection. Add to $\mathcal{C}'_1$ the class $\mathcal{S}$ of the singletons of $\mathcal{P}(E)$, by putting

$$\mathcal{C} = \mathcal{C}'_1 \cup \mathcal{S} \tag{11}$$

Class $\mathcal{C}$ is, by construction, a connection on $\mathcal{P}(E)$. Here we find one of the set connections mentioned by Ch. Ronse in [12], namely one that is formed by the connected invariant sets of opening $\delta\varepsilon$, plus the singletons. More precisely, it is the smallest extension of $\mathcal{C}'$ to the lattice of the connections on $\mathcal{P}(E)$, but obviously, it is not the only possible one: the maximum connection on $\mathcal{P}(E)$, that is to say $\mathcal{P}(E)$ itself, also includes $\mathcal{C}'$. Now, the restriction of $\mathcal{C}$ to $\mathcal{L}$, where $\mathcal{C}$ is defined by rel.(11), does not restore the connection $\mathcal{C}'$ where we started from to construct $\mathcal{C}$. In other words, Proposition 7 does not draw up a complete inventory of the connections on $\mathcal{L}$, but rather describes those which do not really bring into play the infimum of lattice $\mathcal{L}$.

On the other hand, the class $\mathcal{C}\cap\mathcal{L}$ connects too much. For instance, if we take for $\mathcal{C}$ the arcwise connection in $\mathbb{R}^2$ or in $\mathbb{Z}^2$, and for $\mathcal{L}$ the lattice of the sets open by the unit disc $B$, the union of the three discs in Fig. 2(a) is an element of both $\mathcal{C}$ and $\mathcal{L}$, hence of $\mathcal{C}'$. However, in the lattice $\mathcal{L}$ of the open sets, these three discs are two by two disjoint. And they are disjoint because their *eroded versions* by $B$ are disjoint in $\mathcal{P}(E)$. Therefore, Fig. 2(a) will be considered as composed of three separated particles for the connections on $\mathcal{L}$ which involve connections on $\mathcal{P}(E)$ *before* the dilation $\delta$. Such connections, which will be more restrictive, can be carried out in the following way:

**Theorem 8.** *Let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$ and $\delta$: $\mathcal{P}(E) \to \mathcal{P}(E)$ be an extensive dilation, of adjoint erosion $\varepsilon$, that generates the lattice $\mathcal{L} = \delta(\mathcal{P})$. If the closing $\varepsilon\delta$ preserves the connection $\mathcal{C}$, i.e. $\varepsilon\delta(\mathcal{C}) \subseteq \mathcal{C}$, then the image $\mathcal{C}' = \delta[\mathcal{C}]$ of the connected sets under $\delta$ is a connection on lattice $\mathcal{L}$.*

**Proof:** As connection $\mathcal{C}$ contains the singletons and the empty set, and as $\delta(\emptyset) = \emptyset$, the first two axioms of Definition 8 are satisfied. For proving the third one, we consider a family $\{X'_i, i \in I\}$ of elements of $\mathcal{C}'$, i.e. such that for each $i \in I$ there exists a $X_i \in \mathcal{C}$ with $X'_i = \delta(X_i)$. Suppose that $\{\wedge X'_i, i \in I\} \neq \emptyset$, hence $\emptyset \neq \wedge X'_i = \delta\varepsilon(\cap X'_i) = \delta[\cap\varepsilon\delta(X_i)]$. As $\delta$ is a dilation, it satisfies the implication $A = \emptyset \Rightarrow \delta(A) = \emptyset$, therefore $\cap\varepsilon\delta(X_i) \neq \emptyset$. As $X_i$ is $\mathcal{C}$-connected and $\varepsilon\delta(\mathcal{C}) \subseteq \mathcal{C}$, the union $\cup\varepsilon\delta(X_i)$ turns out to be $\mathcal{C}$-connected and

$$\begin{aligned}\cup\{X'_i, i \in I\} &= \cup\{\delta X_i, i \in I\} = \cup\{\delta\varepsilon\delta X_i, i \in I\}\\ &= \delta[\cup\{\varepsilon\delta(X_i), i \in I\}] \in \mathcal{C}',\end{aligned}$$

which achieves the proof. □

Theorem 8 opens the way to new connections on $\mathcal{L}$, noticeably more restrictive than those of Proposition 7. For example, in Fig. 2(a), when we take the arcwise connection and the dilation $\delta$ by the unit disc, the three lobes of Fig. 2(a) become three separated connected components.

One may notice a curious feature of Theorem 8, which does not assume that the $\delta$-images of the elements of $\mathcal{C}$ be themselves $\mathcal{C}$-connected. For instance, in $\mathbb{Z}^1$ equipped with the arcwise connection, if one takes for $\delta$ the dilation by the doublet $\{0, +3\}$, i.e. $\delta(x) = \{x, x + 3\}$, then any segment is clearly invariant under the closing $\varepsilon\delta$. The conditions of Theorem 8 are then fulfilled; this implies, for example, that the two left-hand side segments of Fig. 2(b) form a $\mathcal{C}'$-connected set (since they are the dilate of a segment by $\delta$), as well as the two right-hand side ones, although their union is not anymore $\mathcal{C}'$-connected. However neither the left two segments, nor the right two ones, are $\mathcal{C}$ -connected sets.

## 6. Key-Dilations

In Theorem 8 there appears the assumption that the adjunction closing $\varepsilon\delta$ preserves connection $\mathcal{C}$, i.e. $\varepsilon\delta(\mathcal{C}) \subseteq \mathcal{C}$. Such a condition is obviously not general. For example, take in $Z^1$ the centred structuring element $\delta(x)$ made of seven points, when point $x$ is at the left of $\Delta$, and of three points only, when not (see Fig. 3).

We observe on the figure that the corresponding closing $\varepsilon\delta$ does not preserve connectivity (the same drawback still occurs when we replace $\delta(x)$ by its extremities). Is the condition $\varepsilon\delta(\mathcal{C}) \subseteq \mathcal{C}$ very demanding? Are there many dilations $\delta$ whose associated closing $\varepsilon\delta$ satisfy such a property, and if so, can connection $\mathcal{C}$ be

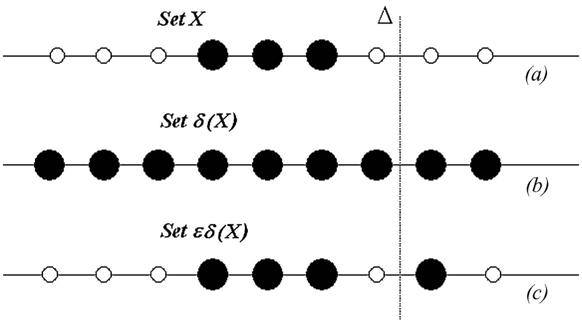



*Figure 3.* An example of a dilation $\delta$ that does not preserve connectivity. (a) initial set $X$ in $Z^1$; (b) dilate $\delta(X)$; (c) closed set $\varepsilon\delta(X)$.

completely arbitrary? That are the questions we treat in this section by constructing a class of convenient pairs $(\delta, \mathcal{C})$. This class (the key-dilations) is probably not the most general one. The following reminder, which generalizes the role of the arcs in the arcwise connections, will be useful.

**Proposition 9.** *A set X is connected if and only if when $(x, y) \in X$, one can find a connected set included in X and containing x and y* ([17], *p.* 54).

In pipe organs, the vertical pipes are put on a parallelepiped box with compressed air inside. Their arrangement forms a 2-D matrix where all pipes of each row belong to a same rank (i.e. have the same timbre: oboe, bourdon, etc.) and all those of a same column play a same key, so that the row /column intersection labels a unique rank/key combination. As the dilations we propose below will be decomposed in a same way, we call them, by analogy, "key-dilations."

One usually says that a dilation $\delta$ in the Euclidean space is *linear of direction* $\alpha$ when the transform of each point $x \in \mathbb{R}^n$ is contained in a straight line $\Delta_\alpha$ of direction $\alpha$. Consider a given family $\{\alpha_1..\alpha_j..\alpha_k\}$ of directions in $\mathbb{R}^n$ (with $k < \infty$), and let $L_j(x, l)$ stand for the closed segment of length $l$ and direction $\alpha_j$ centred at point $x$. The length itself may be a function $l(x, \alpha_j)$ of the point $x$ and the direction $\alpha_j$.

*Definition 10.* Given a finite set $\{\alpha_j, 1 \le j \le k\}$ of directions in $\mathbb{R}^n$, let $\{\delta_j, 1 \le j \le k\}$ be a family of linear dilations on $\delta_j$: $\mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$, of adjoint erosions $\{\varepsilon_j, 1 \le j \le k\}$ and such that for all $x \in \mathbb{R}^n$ and of all directions $\alpha_j$, $1 \le j \le k$, we have

1. $\delta_j(x) = L_j(x, l(x, \alpha_j))$;
2. $\varepsilon_j\delta_j(x) = \{x\}$;
3. $\delta(x) = \delta_k \ldots \delta_2\delta_1(x) = L_k(x, l(x, \alpha_k)) \cdots \oplus L_2(x, l(x, \alpha_2) \oplus L_1(x, l(x, \alpha_1))$.

Then the composition product $\delta$ is said to be a *key-dilation.*

The first axiom of the definition is a condition of 1-D convexity for the mapping $x \to \delta_j(x)$. In addition, by placing $x$ at the center of the transform, we make the mapping $\delta_j$ extensive. But it is not translation invariant and the length $l(x, \alpha_j)$ of the segment $L_j(x, l(x, \alpha_j))$ may vary from point to point. The second axiom repeats the separation axiom already introduced by Rel. (7). Finally, the third one generalizes the convexity condition of the first one. When a second direction, $\alpha_2$ say, is introduced, this third axiom establishes some dependence between the two primitive dilations. The $\delta_2(y)$ dilates of all points $y$ of $L_1(x, l(x, \alpha_1))$ must be translated of each other in order to obtain

$$\delta(x) = \delta_1(x)\delta_2(x) = L_1(x, l(x, \alpha_1)) \oplus L_2(x, l(x, \alpha_2)) \tag{12}$$

so that when $x$ describes a straight line $\Delta_1$ of direction $\alpha_1$, then all the $\delta_2(x)$'s are equipollent segments. By duality between the directions $\alpha_1$ and $\alpha_2$, the same occurs for the $\delta_1(x)$'s when $x$ describes a straight line $\Delta_2$. In addition, Rel. (6) shows that the composition product $\delta = \delta_2\delta_1$ is commutative.

By adding a third direction $\alpha_3$ we generate the key-dilation $\delta = \delta_3\delta_2\delta_1$. If the three directions are coplanar, the segments $L_3(x, l(x, \alpha_3))$ must have the same length along the straight lines $\Delta_2$, and also along the $\Delta_1$ lines, so that their length is the same everywhere, i.e. dilation $\delta_3$ is translation invariant. The same comment applies, of course, for a fourth, fifth, etc. direction, and finally extends to $\mathbb{R}^n$ as follows.

**Proposition 11.** *Let $\delta = \delta_k \ldots \delta_2\delta_1$ be a key-dilation of order k in $\mathbb{R}^n$. If $k \le n$, then for every $j_0 \in [1, k]$ the dilations $\{\delta_j, j \in [1, k], j \ne j_0\}$ are translation invariant along the straight lines of direction $\alpha_{j_0}$. If $k > n$, then there exists n directions for which the previous statement is true, and $k - n$ other ones for which the $\delta_j$'s are translation invariant. Moreover, the product $\delta = \delta_k \ldots \delta_2\delta_1$ is commutative with respect to all its factors.*

In $\mathbb{R}^n$, the key-dilations $\delta_j$, $1 \le j \le k$, operate on the *closed sets* $\mathcal{F}(\mathbb{R}^n)$ of $\mathbb{R}^n$ via the cross sections by

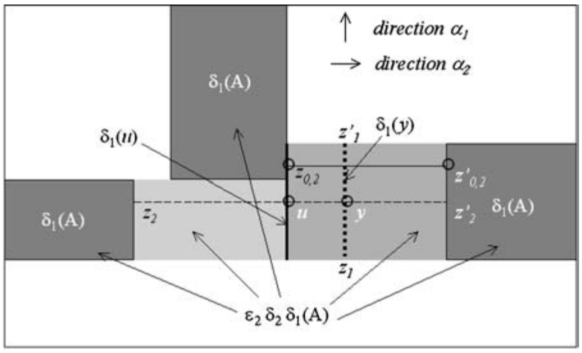



*Figure 4*. The three rectangles that form $\delta_1(A)$ (in dark grey) are supposed to reconnect ouside the frame. The $\alpha_2$ stripe that contains point $y$ is the union of the two rectangles in light and medium grey. The rhombus $R$ is the medium grey rectangle.

straight lines $\Delta_{\alpha_j}$ that span the space by translations. The closing $\varepsilon_j\delta_j(A)$, $A \in \mathcal{F}(\mathbb{R}^n)$ can only remove open segments of pores in direction $\alpha_j$ (they are for example the two light and medium grey rectangles of Fig. 4), and each segment is treated independently of the others. In order to ensure that $\varepsilon_j\delta_j$ maps $\mathcal{F}(\mathbb{R}^n)$ into itself, we will suppose that the structuring element $x \to \delta_j(x)$ is a continuous mapping. Then it follows from Theorem 3.11 in [8] that $\varepsilon_j\delta_j$ is an upper semi-continuous mapping from $\mathcal{F}(\mathbb{R}^n)$ into itself.

Before entering such a multidirectional approach, we will focus on the case when one direction only, $\alpha_1$ say, is involved. It is classically known that in $\mathbb{R}^1$ and in $\mathbb{Z}^1$ the translation invariant closings by segments preserve the arcwise connectivity [3] . We will generalize this result in three ways, by weakening the translation invariance, and by increasing the possible connections and the dimensions of the space.

Below, the expression $\{z, z'\}$ indicates the pair of the two points $z$ and $z'$, and $[z, z']$ the closed segment from $z$ to $z'$. In the notation, the space $\mathbb{R}^1$ is assimilated to an axis, and we keep the same symbols ($x, z$..) for denoting the points *and* their abscissae. The dilate of $z$ is the segment from $z_0$ at the left, to $z_1$ at the right, with $z_1 - z = z - z_0$ , and the notation with subscripts 0 and 1 is the same for any point dilate. Finally, in what follows, "segment" means always "closed segment."

**Proposition 12.** *Let $\delta$ be a key-dilation in $\mathbb{R}^1$. Then*

1. *the dilates of segments are segments*;
2. *the eroded of segments are segments or the empty set*;
3. *Given $\{z, z'\} \in \mathbb{R}^1$, with $z < z'$, and $a \in ]z, z'[$, the following equivalence holds*

$$\{\delta(z) \cup \delta(z') \textit{ is a segment}\} \Leftrightarrow \delta(a) \subseteq \delta(z) \cup \delta(z') \tag{13}$$

4. *The segments are invariant under the closing $\varepsilon\delta$.*

**Proof:**

1. The first point is a direct consequence of Proposition 8 in [18], namely that an extensive dilation with connected point-dilates preserves connectivity.
2. By anti-extensivity of $\varepsilon$, we have $\varepsilon([z, z']) \subseteq [z, z']$. Suppose that $\varepsilon([z, z'])$ is not empty and comprises more than one segment, one can find point $a \in \varepsilon([z, z'])^c$ between two segments of $\varepsilon([z, z'])$. To say $a \notin \varepsilon([z, z'])$ is equivalent to saying $\delta(a) \nsubseteq [z, z']$, therefore either point $a_0$ is in the left outside of $[z, z']$, or $a_1$ in the right outside. Suppose it is $a_0$, and take a point $b$ in a segment of $\varepsilon([z, z'])$ at the left of $a$. We have $a_0 < z \le b_0 \le b < a$, which implies, by symmetry that $a_0 < b_0 \le b_1 < a_1$, i.e. $\delta(b) \subseteq \delta(a)$. Then, according to Rel. (7), or the second axiom of Definition 10, we have $a = b$, which is in contradiction with the assumption $a \notin \varepsilon([z, z'])$. Hence $\varepsilon([z, z'])$ is a segment.
3. We first prove Equivalence (13) in the $\Rightarrow$ sense. Consider a point $a \in ]z, z'[$ and suppose that $\delta(a)$ hits the background, that there exists $b \in \delta(a) \cap [\delta(z) \cup \delta(z')]^c$. As $\delta(z) \cup \delta(z')$ is a segment, point $b$ is located either at the left of $z_0$ or at the right of $z'_1$. In the first case, for example, the following inequalities hold

$$a_0 \le b \le z_0 \le z \le a$$

so that, by symmetry, $a_0 \le z_0 \le z_1 \le a_1$, i.e. $\delta(z) \subseteq \delta(a)$, hence $a = z$, which is incompatible with $a \in ]z, z'[$. The second case is treated in the same way, which results in implication $\Rightarrow$.

Conversely, suppose that $\delta(a) \subseteq \delta(z) \cup \delta(z')$ for some $a \in ]z, z'[$. If the right member of this inclusion is composed of two segments, then $\delta(a)$, as a segment, is included in one of them, $\delta(z)$ say. But $\delta(a) \subseteq \delta(z)$ implies $a = z$, which is a contradiction. Hence $\delta(z) \cup \delta(z')$ is a unique segment.
4. By extensivity of the closing, we have $[z, z'] \subseteq \varepsilon\delta$ $[z, z']$. For proving the inverse inclusion, consider a point $a \in \varepsilon\delta\ [z, z']$, i.e. by adjunction $\delta(a) \subseteq \delta$

$[z, z']$. If $a$ is exterior to $[z, z']$, at the left side for example, we obtain the sequence $z_0 \leq a_0 \leq a \leq z$, hence $\delta(a) \subseteq \delta(z)$, and $a = z$. Therefore $a \in [z, z']$, which achieves the proof. □

As a direct consequence of point 3, note that for all $\{z, z'\} \in \mathbb{R}^1$ the dilate $\delta(z) \cup \delta(z')$ is a segment if and only if

$$\delta(z) \cup \delta(z') = \delta([z, z']) \tag{14}$$

Indeed, when $\delta(z) \cup \delta(z')$ is a segment, then Equivalence (13) gives $\cup\{\delta(a), a \in ]z, z'[\} \subseteq \delta(z) \cup \delta(z')$, hence $\delta([z, z']) \subseteq \delta(z) \cup \delta(z')$. The reverse inclusion derives from the increasingness of $\delta$, which results in Eq. (14). Conversely, this equation implies that $\delta(z) \cup \delta(z')$ is a segment, as the dilate of a segment.

The above properties do not bring any connection into play. We will now introduce a connection $\mathcal{C}$ on $\mathcal{F}(\mathbb{R}^n)$ and assume that all segments whose direction belongs to the family $\{\alpha_j\}$ are $\mathcal{C}$-connected. Coming back first to the one directional case, we can state the following result.

**Proposition 13.** *Let $\delta_1 : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ be a key-dilation of (unique) direction $\alpha_1$, and let $\mathcal{C}$ be a connection on $\mathcal{F}(\mathbb{R}^n)$ that contains all segments of direction $\alpha_1$. If $A \in \mathcal{F}(\mathbb{R}^n)$ is $\mathcal{C}$-connected, then both $\delta_1(A)$ and $\varepsilon_1\delta_1(A)$ are $\mathcal{C}$-connected.*

**Proof:** $\delta_1(A)$ is connected because an extensive dilation with connected point-dilates preserves connectivity (Proposition 8 in [18]). Consider now the closing $\varepsilon_1\delta_1(A)$ and distinguish between the points of $A$ and those of $\varepsilon_1\delta_1(A)\backslash A$, i.e. the added points. If point $x$ is added to $A$ then it is located on a open segment $]z, z'[ \subseteq A^c$ whose extremities lean on $A$. Then $[z, z'] \cup A$ is connected, and the proof is achieved by applying Proposition 9 as previously. □

More generally, we can state the following

**Proposition 14.** *Let $\{\alpha_1, \alpha_2\}$ be two directions in $\mathbb{R}^n$, and let $\mathcal{C}$ be a connection on $\mathcal{F}(\mathbb{R}^n)$ that contains all segments of directions $\{\alpha_1, \alpha_2\}$. If $\delta = \delta_2\delta_1$ is a key-dilation of directions $\{\alpha_1, \alpha_2\}$, then the closing $\varepsilon\delta$ preserves connection $\mathcal{C}$.*

**Proof:** By commutativity of $\delta$, we can write $\varepsilon\delta = \varepsilon_1\varepsilon_2\delta_2\delta_1$. Let $A \in \mathcal{F}(\mathbb{R}^n)$ be connected. According to Proposition 13 both $\delta_1(A)$ and $\varepsilon_2\delta_2\delta_1(A)$ are connected, and $\varepsilon_2\delta_2$ extends $\delta_1(A)$ uniquely by adding open segments $]z_2, z_2'[ \subseteq [\delta_1(A)]^c$ whose extremities $z_2$ and $z_2'$ lean on $\delta_1(A)$. These segments form $\alpha_2$ stripes of variable thicknesses, that may be suppress or not by the later erosion $\varepsilon_2$ (see Fig. 4).

Take an "added point" $y$, i.e. such that $y \in \varepsilon_2\delta_2\delta_1(A)\backslash\delta_1(A)$. The $\alpha_1$-segment $[z_1, z_1']$ going through $y$ is either connected to $\delta_1(A)$ or not. If so, then $[z_1, z_1']$ extends an existing $\alpha_1$-dilated segment of $\delta_1(A)$, so that if the later erosion $\varepsilon_1$ preserves point $y$, then $y$ belongs to a segment that hits $\varepsilon_1\delta_1(A)$.

Consider now the case when the whole segment $[z_1, z_1']$ misses $\delta_1(A)$, and suppose that the later erosion $\varepsilon_1$ preserves point $y$, i.e. that $\delta_1(y) \subseteq [z_1, z_1']$. This case, depicted in Fig. 4, means that every $y_i \in \delta_1(y)$ belongs to a new $\alpha_2$-segment that bridges $[\delta_1(A)]^c$ and whose two extremities $(z_{i,2}, z_{i,2}') \in \delta_1(A)$. Put

$$l_2 = \inf\{| z_{i,2} - y_i | + | z_{i,2}' - y_i |, \quad y_i \in \delta_1(y)\}.$$

As the segments $[z_{i,2}, z_{i,2}']$ are closed, there exist among them a shortest one, $[z_{0,2}, z_{0,2}']$ say, of length $l_2$, and whose one extremity, $z_{0,2}$ for example, belongs to $\delta_1(A)$. The $\alpha_1$-stripe of section $[z_{0,2}, z_{0,2}']$ intersects the $\alpha_2$-stripe going through $y$ and of thickness $\delta_1(y)$ by forming a rhomb $R(y)$ of directions $\{\alpha_1, \alpha_2\}$. Denote by $u$ the intersection point between the $\alpha_2$-chord going through $y$ and the $\alpha_1$-side of rhomb $R$ starting from $z_{0,2}$. By invariance of the $\alpha_1$-dilations along the $\Delta_2$ lines, we have $z_{0,2} \in \delta_1(u)$. Therefore $\delta_1(u)$ extends a $\alpha_1$-segment of $\delta_1(A)$, so that the segment $[u, z_{0,2}] \in \varepsilon_1\varepsilon_2\delta_2\delta_1(A)$ and hits $\varepsilon_1\delta_1(A)$. On the other hand, and still by $\alpha_1$-invariance, the later erosion $\varepsilon_1$ reduces rhomb $R$ to its central segment, that contains $[u, y]$.

Finally, both connected components $[u, y]$ and $[u, z_{0,2}]$ hit the component $\varepsilon_1\delta_1(A)$, therefore the union $[u, y] \cup [u, z_{0,2}] \cup \varepsilon_1\delta_1(A)$ is connected, and also included in $\varepsilon_1\varepsilon_2\delta_2\delta_1(A)$ . As point $y$ was arbitrarily chosen in $\varepsilon_1\varepsilon_2\delta_2\delta_1(A)$, we derive that each pair of points $(x, y) \in \varepsilon_1\varepsilon_2\delta_2\delta_1(A)$ belongs to a connected component included in $\varepsilon_1\varepsilon_2\delta_2\delta_1(A)$ and containing $\varepsilon_1\delta_1(A)$, hence $\varepsilon_1\varepsilon_2\delta_2\delta_1(A)$ is $\mathcal{C}$-connected (Proposition 9), which achieves the proof. □

For extending this result to $k$ directions, we must write

$$\varepsilon\delta = (\varepsilon_{k-1} \ldots \varepsilon_2\varepsilon_1)\varepsilon_k\delta_k(\delta_{k-1} \ldots \delta_2\delta_1).$$

The above proof remains true when we substitute $(\varepsilon_{k-1}\ldots\varepsilon_2\varepsilon_1), \varepsilon_k, \delta_k, (\delta_{k-1}\ldots\delta_2\delta_1)$ for $\varepsilon_1, \varepsilon_2, \delta_2, \delta_1$ respectively. We can state.

**Corollary 15.** *Let $\{\alpha_j, 1 \leq j \leq k\}$ be a finite set of directions in $\mathbb{R}^n$, and $\{\delta_j, 1 \leq j \leq k\}$ be a family of associated linear key-dilations. Denote by $\delta = \delta_k \ldots \delta_2\delta_1$ the composition product of the $\delta_j$'s and by $\varepsilon$ its adjoint erosion. If a connection $\mathcal{C}$ on $\mathcal{F}(\mathbb{R}^n)$ contains all segments of directions $\{\alpha_j\}$, then the closing $\varepsilon\delta$ preserves connection $\mathcal{C}$.*

In two dimensions, Proposition 14 and its corollary open the door to squares, hexagons, octagons, etc. of possible variable sizes, but not to triangles (which is not dramatic); in three dimensions they enable us to use cubes, rhombo-dodecahedra (i.e. dilates of cube diagonals), but not cube-octahedra (which is slightly more dramatic). One may conjecture that the proposition extends to infinite elementary segment dilations, which should include the discs, and all symmetrical compact convex sets of $\mathbb{R}^2$. In practice, the fact that the proposition is partly independent of the translation invariance can be useful when some perspective is involved. When studying traffic control on highways, for example, S. Beucher and M. Bilodeau are led to structuring elements whose sizes reduce with the distance [2].

In $\mathbb{Z}^2$, one has to pay attention that the chosen connection $\mathcal{C}$ must accept the segments of successive 1's in each direction $\alpha_j$. The squares and the octagons, for the 8-connectivity of the square grid, and the hexagons, for the arcwise connection of the hexagonal grid, fulfill such a condition, but neither the octagons for the hexagonal grid connection, nor the dodecagons for the 8-connectivity of the square grid, nor *a fortiori* dilates of Brezenham segments. A similar comment applies to $\mathbb{Z}^3$.

## 7. Geodesic Reconstruction

It remains to express the $\mathcal{C}'$ components of a given set $A$ as a function of its $\mathcal{C}$ components. The following proposition answers the question

**Proposition 16.** *Let $\gamma_x$ (resp. $\gamma'_{\delta(x)}$) be the point connected opening at point $x \in E$ (resp. at $\delta(x) \in \mathcal{L}$) for the connection $\mathcal{C}$ (resp. $\mathcal{C}'$) of Theorem 8. Then the two openings $\gamma'_{\delta(x)}$ and $\gamma_x$ are linked by the relationship*

$$\gamma'_{\delta(x)} = \delta\gamma_x\varepsilon \tag{15}$$

**Proof:** Let $Z \in \mathcal{L}$. For any $x \in E$, by the adjunction $(\varepsilon, \delta)$ we have $x \in \varepsilon(Z)$ iff $\delta\{x\} \subseteq Z$.

1/ If $x \notin \varepsilon(Z)$, then $\gamma_x\varepsilon(Z) = \emptyset$, so $\delta\gamma_x\varepsilon(Z) = \emptyset$. Also $\delta(\{x\}) \not\subseteq Z$, so $\gamma'_{\delta(x)}(Z) = \emptyset$ and $\gamma'_{\delta(x)}(Z) = \delta\gamma_x\varepsilon(Z)$.

2/ Suppose now that $x \in \varepsilon(Z)$. From $\gamma_x\varepsilon(Z) \in \mathcal{C}$ we draw $\delta\gamma_x\varepsilon(Z) \in \mathcal{C}'$. Also $x \in \gamma_x\varepsilon(Z)$, so $\delta(x) \subseteq \delta\gamma_x\varepsilon(Z)$. Moreover, $\delta\gamma_x\varepsilon(Z) \subseteq \delta\varepsilon(Z) \subseteq Z$. As $\delta(x) \subseteq \delta\gamma_x\varepsilon(Z) \subseteq Z$ and $\delta\gamma_x\varepsilon(Z) \in \mathcal{C}'$, we deduce that $\delta\gamma_x\varepsilon(Z) \subseteq \gamma'_{\delta(x)}(Z)$. As $\gamma'_{\delta(x)}(Z) \in \mathcal{C}'$, we have $\varepsilon\gamma'_{\delta(x)}(Z) \in \mathcal{C}$, and as $\delta(x) \subseteq Z$, we have $\delta(x) \subseteq \gamma'_{\delta(x)}(Z)$, so by adjunction $(\varepsilon, \delta)$ we obtain $x \in \varepsilon\gamma'_{\delta(x)}(Z)$. From $x \in \varepsilon\gamma'_{\delta(x)}(Z) \in \mathcal{C}$, we draw $\gamma_x\varepsilon\gamma'_{\delta(x)}(Z) = \varepsilon\gamma'_{\delta(x)}(Z)$. As $\gamma'_{\delta(x)}(Z) \in \mathcal{C}' \subseteq \delta\varepsilon(\mathcal{P}(E))$, we have $\gamma'_{\delta(x)}(Z) = \delta\varepsilon\gamma'_{\delta(x)}(Z)$. Therefore

$$\gamma'_{\delta(x)}(Z) = \delta\varepsilon\gamma'_{\delta(x)}(Z) = \delta\gamma_x\varepsilon\gamma'_{\delta(x)}(Z) \subseteq \delta\gamma_x\varepsilon(Z).$$

From the double inclusion

$$\delta\gamma_x\varepsilon(Z) \subseteq \gamma'_{\delta(x)}(Z) \quad \text{and} \quad \gamma'_{\delta(x)}(Z) \subseteq \delta\gamma_x\varepsilon(Z)$$

we deduce the equality, which achieves the proof. □

If we have a mean to compute $\gamma_x\varepsilon(Z)$, then relation (15) supplies a simple algorithm to derive $\gamma'_{\delta(x)}(Z)$. The most popular one, called geodesic reconstruction works under the following assumptions:

- Space $E$ is supposed to be metric, and $\alpha_\lambda(x)$ stands for the closed ball of radius $\lambda$ at point $x \in E$ ; for each $\lambda > 0$ the family $\{\alpha_\lambda(x), x \in E\}$ is interpreted as structuring elements that generate a dilation $\alpha_\lambda$;
- Space $E$ is equipped with a connection $\mathcal{C}$ that is preserved by each dilation $\alpha_\lambda$;
- The set $\varepsilon Z$ under study is compact and has a finite number of $\mathcal{C}$-components;
- Each $\mathcal{C}$-component $Y$ of $\varepsilon Z$ is "*well linked*" (in the sense of [4], I-19, see also [13]) i.e. for every pair $(a, b)$ of points of $Y$ and for every $\eta > 0$, there exists a finite sequence $a = a_1, \ldots a_n = b$ of points of $Y$ such that $d(a_i, a_{i+1}) \leq \eta$ for every $i < n$.

Then according to a classical result, $\gamma_x(\varepsilon Z)$ is reached by finite iteration, for some $\lambda_0 > 0$. The mapping to be iterated is $\psi_{\varepsilon Z}(Y) = \alpha_{\lambda 0}(Y) \cap \varepsilon Z$. Starting from $Y = \{x\}$, we finally obtain, by considering Relation (15)

$$\gamma'_{\delta(x)} = \delta\gamma_x(\varepsilon Z) = \delta\left[\psi^n_{\varepsilon Z}(\{x\})\right] \quad n < \infty \qquad (16)$$

## 8. Two Applications

In the two examples which follow, the space $E$ is the digital plane, the discs $B_\lambda$ are digital approximations of the Euclidean discs by octagons in the four main directions of the square grid, and the connection $\mathcal{C}$ is the usual 8-connectivity. Therefore, the involved Minkowski additions are digital key-dilations. Theorem 8 is satisfied, so that we can use Relation (16). The integer radius $\lambda$ turns out to be a viscosity index, that we make vary in the first example. In the second example, we try and apply the above set oriented approach to contour detection in numerical imagery.

### *8.1. Set Interpolation*

Consider a binary contour, partly identified by a dotted line such as the set of the black points in Fig. 5. In a viscous lattice $\mathcal{L}$, the opening of the complement of this object, if not empty, turns out to be made of one or two $\mathcal{C}'$-particles, depending on the size of $\delta$. Figure 5 depicts two reconstructions of set $Z \in \mathcal{P}(\mathbb{Z}^2)$ from a given white marker $A \in \mathcal{P}(\mathbb{Z}^2)$ , when $\delta$ is the Minkowski's addition by a disc $B_\lambda$ of radius $\lambda$. When parameter $\lambda$ is small, the "fluidity" of the successive dilations allows to go between the pins of $Z^c$. This occurs in Fig. 5(a) (although, for the sake of display, the light grey invasion has been stopped before it reaches the edges of the field). As parameter $\lambda$ increases, these pins stop the reconstruction process, but if $\lambda$ goes on increasing, then no marker $B_\lambda$ can be found anymore in the figure, and the reconstructed set is empty. When $\lambda$ decreases, before situation 5(a), it reaches a minimum radius $\lambda_{\max}$ for which the reconstruction is the largest possible, without touching the field borders (Fig. 5(b)). Besides, remark that the mapping $Z \to \gamma_{\max}(Z)$ is an opening of $\mathcal{L}$ into itself. Therefore, though the approach is non-parametric, it involves a $\lambda_{\max}$ to be detected from the set under study.

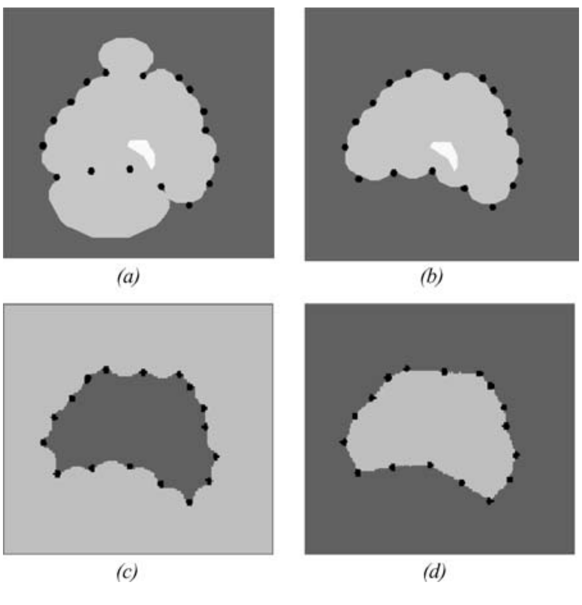


*Figure 5.* (a) In white, marker $A$, in black the complement of set $Z$: geodesic reconstructions by dilations for a radius of viscosity $\lambda = 15$; (b) Case of the optimal radius $\lambda_{\max} = 17$; (c) Optimal reconstruction from the edges of the field; (d) Corresponding median element.

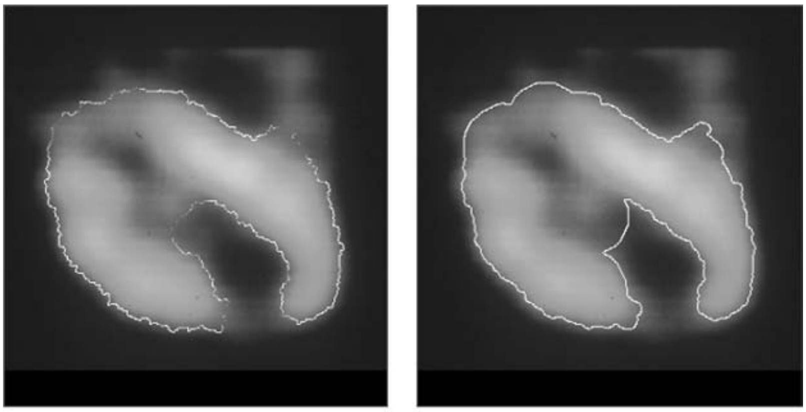
If now the same experiment is carried out again, but with the field border as a marker instead of $A$, we get, as previously, a minimum dilation radius $\lambda'_{\max}$ which does not flood the inner part, with obviously $\lambda'_{\max} = \lambda_{\max}$. The resulting reconstruction is depicted in Fig. 5(c).

For the sake of symmetry, we can interpolate between Fig. 5(b) and (c) by means of their median element (rel. (13) in [19]). If $X$ stands for the light grey set of Fig. 5(b), and $Y$ for the dark grey one of Fig. 5(c), their median element $M(X, Y)$ is given by

$$M(X, Y) = \cup\{\delta_\lambda(X \cap Y) \cap \varepsilon_\lambda(X \cup Y), \lambda \geq 0\}.$$

The interpolator $M$ which we are led to, provides a nice interpolation from a visual point of view (Fig. 5(d)).

Strong relationships link morphological connections and segmentation [21]. They derive from a general result according to which any connection $\mathcal{C}$ on a complete lattice $\mathcal{T}$ partitions each element $A$ of $\mathcal{T}$ into maximum classes that are precisely the $\mathcal{C}$-connected components of $A$ (see Proposition 7 in [18]). The above example illustrates this point. The viscous lattice $\mathcal{L}$ sup-generated by the discs of radius 17, or by their digital approximations, induces a connection $\mathcal{C}'$ in the sense of Theorem 8. For this connection, the two light grey sets of Fig. 5(b) and (c), are the only possible reconstructions of set $Z$ from $\delta(x)$ when the small $\delta(x)$ spans the space. One can observe that the intersection of these two components contains some pixels. However, since their infimum, in the sense of lattice $\mathcal{L}$, is empty, and their union restores the whole set $Z$, they *do partition* $Z$ in the viscous sense.



(a) (b)

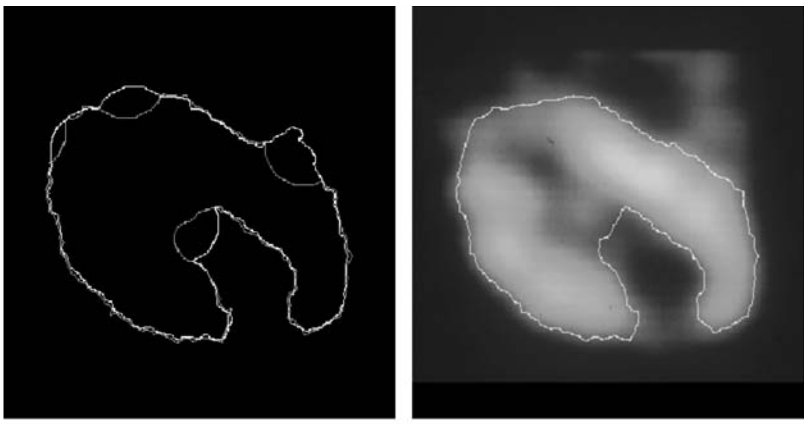
*Figure 6*. (a) Positron image of a heart muscle and its contour by watershed; (b) Optimal reconstruction of the internal $C'$-components.

### 8.2. *Watershed Regularization*

Let us consider the positron image of a heart muscle, in Fig. 6(a). The watershed line of its gradient enables the construction of a representative contour, but whose drawing is quite irregular. We purpose to regularize it by viscous flood. Let us consider the restriction of the gradient to the watershed line. This is a numerical function (looking a bit like the Great Wall of China) whose threshold at the value $t$ gives the watershed points of gradient $\leq t$. Consequently, the successive thresholds appear like dotted lines all the less dense since $t$ is higher. For each value $t$ we determine a radius $\lambda_{\max}$ of the dilation disc which enables the maximum reconstruction of the dotted line from the central marker. Finally, we take the union of the maximum reconstructions as $t$ varies: its represents the largest viscous surface built from the watershed (Fig. 6(b)).

As before, we are able to determine a maximum outer numerical function, from the borders of the field, then to take the support contours of both inner and outer surfaces (Fig. 7(a)) and to calculate their median element. The result, depicted in Fig. 7(b), turns out to be a realistic regularization of the zigzag contour of the initial watershed.

The approach of this second example is different from the one presented in [22], as the algorithm of [22] is not defined in the viscous lattices context. The choice for the markers is different too: in [22], they are purely internal, and the marker at level $n$ is the



(a) (b)

*Figure 7*. (a) Internal and external optimal $C'$-contours, and (b) Median element.

reconstructed image of level $n - 1$. Finally, the third difference deals with the choice of the sizes of dilations $\delta$: our algorithm is non-parametric, therefore general, and could be particularized according to the type of the images investigated.

## 9. Conclusion

From a mathematical point of view, the viscous lattices show the pertinence of Galois adjunctions (i.e. the pair dilation/erosion), and of the morphological connections. Indeed, these two notions remain valid, even when most of the basic useful properties of a lattice, such as distributivity, complementation, atomicity, etc. do vanish.

From a physical point of view, viscous lattices provide a framework without points, lines and fine structures, but where elementary distances such as differentials can still be introduced.

Practically, a combined use of the two remaining notions allows us to generate connected filters and segmentations. The two examples of propagations that are proposed above illustrate the regularization, or fuzziness, effect of the viscosity, where dotted lines turn out to become barriers. The operations that these examples involve are obviously not the only possible ones, and one can imagine PDE's based wave fronts that propagate over a viscous lattice.

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