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Some Results on Minimal Euclidean Reconstruction from Four Points

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Abstract. Methods for reconstruction and camera estimation from miminal data are often used to boot-strap robust (RANSAC and LMS) and optimal (bundle adjustment) structure and motion estimates. Minimal methods are known for projective reconstruction from two or more uncalibrated images, and for "5 point" relative orientation and Euclidean reconstruction from two calibrated parameters, but we know of no efficient minimal method for three or more calibrated cameras except the uniqueness proof by Holt and Netravali. We reformulate the problem of Euclidean reconstruction from minimal data of four points in three or more calibrated images, and develop a random rational simulation method to show some new results on this problem. In addition to an alternative proof of the uniqueness of the solutions in general cases, we further show that unknown coplanar configurations are not singular, but the true solution is a double root. The solution from a known coplanar configuration is also generally unique. Some especially symmetric point-camera configurations lead to multiple solutions, but only symmetry of points or the cameras gives a unique solution.

Keywords: 3D reconstruction, relative orientation, structure from motion, polynomial methods, algebraic geometry

1. Introduction

The estimation of camera motion ("relative orientation") and scene structure from image point correspondences is a common task in computer vision and photogrammetry. Reliable methods for handling *minimal cases*—cases where the omission of one point in one image would give an infinite number of solutions—are important both theoretically and in practice. In particular, they are used to bootstrap robust estimation algorithms such as RANSAC and Least Median Squares [10, 32, 35], and optimal estimation algorithms such as bundle adjustment. Perspective projection is essentially an algebraic model, so minimal reconstruction problems usually reduce to formulating and solving a polynomial system.

The minimal data required for projective reconstruction is well-known [8, 12, 28, 30]. For two views, seven points are needed and Sturm's method [31] (re-introduced into computer vision in [8, 9, 22]) can be used, giving at most three real solutions. For three views, six points suffice and there are again at most 3 solutions [26]. These two problems are actually duals under a formal point/camera centre duality [2].

For Euclidean reconstruction from calibrated cameras, Kruppa's method for 5 points in 2 images was re-introduced and studied in [9], where it was shown that there are in general as many as 10 solutions [5, 6, 9, 13, 33]. A recent robust and real-time implementation of this algorithm is presented in [24]. But the progress on minimal cases for three or more calibrated images is much slower. Longuet-Higgins [21] described an iterative method of finding the solutions to the case of 4 points in 3 perspective images by starting from the solution obtained in the simplifed approximate 3 scaled orthographic images. Holt and Netravali [14] proved that "there is, in general, a unique solution for the relative orientation.... However, multiple solutions are possible, in rare cases, even when the four feature points are not coplanar [14]". They used some results from algebraic geometry to draw general conclusions regarding the number of solutions by considering a single example. This is certainly one step further to show the general uniqueness of the solutions, but still many questions remain unanswered and efficient algorithms do not exist for this problem. We will first reformulate this problem using Euclidean depths of points based on Euclidean invariants to form a polynomial system. By characterising the algebraic variety determined by the polynomials with computer algebra tools on random rational simulations, in addition to the uniqueness of Euclidean reconstruction from 4 corresponding points in $N \geq 3$ views [14], we show that

- Euclidean reconstruction from 4 corresponding points that come from unknown coplanar points has a unique double solution in 3 views, but is generally unique in $N > 3$ views.
- Euclidean reconstruction from 4 corresponding points that come from known coplanar points is generally unique in $N \geq 3$ views.
- Euclidean reconstruction is generally unique for only symmetry of either camera configuration or the four points configuration, but has up to 56 solutions for the simultaneous symmetries of the cameras and points.

Apart from the ability to initialize from fewer scene points, three camera methods are likely to have several practical advantages. Three-image matching is much more discriminant than two-image matching, so there should be fewer outliers in a RANSAC run. Also, it is well known in photogrammetry that a triangle of three widely spaced cameras gives much stabler and more reliable geometry estimates than classical two image stereo, essentially because errors within epipolar planes are uncontrollable under stereo [11].

The paper is organized as follows. Section 2 formulates the minimal data problem for calibrated reconstruction. Section 3 describes our symbolic calculations and Section 4 presents the major theoretical results. Some concluding remarks and future directions are given in Section 5.

Throughout the paper, vectors are denoted in lower case bold and matrices in upper case bold. Scalars are any plain letters or lower case Greek.

2. Problem Formulation

2.1. Minimal Data for Euclidean Reconstruction

For Euclidean reconstruction from image points, each image point gives 2 constraints, each 3D point introduces 3 degrees of freedom (d.o.f.) and each camera pose introduces 6 d.o.f., but there are 7 free d.o.f. in the 3D coordinate system (6 for the Euclidean coordinate frame and 1 for the scene scale).

So a system of P points visible in N calibrated images yields $2NP$ constraints in $3P + 6N - 7$ unknowns. To have at most finitely many solutions, we therefore need:

$$2NP \geq 3P + 6N - 7. \qquad (1)$$

Minimal cases are given by equality here, so we look for integer solutions for N and P satisfying:

$$P = 3 + \frac{2}{2N - 3}.$$

For $N = 2$, $P = 5$, so a minimum of five points is required for a two-image relative orientation and Euclidean reconstruction. Relative orientation from 5 points has been widely studied in photogrammetry, computer vision and applied mathematics [5, 7, 9, 15, 18, 19, 23, 25, 33]. The corresponding polynomial system has at most 20 real algebraic solutions which fall into 10 'twisted pairs': each physical solution has an unphysical 'twisted' partner with negative point depths, corresponding to invisible points behind the camera. So at most 10 (and more often 1–5 [25, 33]) of the 20 solutions are feasible.

For any $N \geq 3$, P is between 3 and 4, so at least 4 points are required for $N \geq 3$-view Euclidean reconstruction from unknown space points. In fact, 4 points in 3 images suffice to fix the 3D structure, after which just 3 points are needed in each subsequent image to fix the camera pose (the standard 3 point pose problem [29]). So at least 4 points are always required for Euclidean reconstruction, and of the minimal $N \geq 3$ cases, the 4 point 3 image problem is the most interesting.

Note that for 4 points in 3 images, (1) becomes $2NP = 24 \geq 3P + 6N - 7 = 23$. So constraint counting suggests that the problem is overspecified. An overspecified polynomial system generically has no solutions, but here (in the noiseless case) we know that there is at least one (the physical one). It is tempting to conclude that the solution is unique. In an appropriate formulation this does in fact turn out to be the case, but it needs to be proved rigorously. For example in the two image case, the 'twisted' partner of the physical solution persists no matter how many points are used, so the system becomes redundant but always has two solutions. The issue is general. Owing to the redundancy, for *arbitrary* image points there is no solution at all. To have at least one solution, the image points must satisfy some (here one, unknown and very complicated) polynomial constraints saying that they are possible projections of a possible 3D geometry. When these constraints on the constraints (i.e. on the image points) are correctly incorporated, the constraint counting argument inevitably gives an exactly specified system not an overspecified one. To find out how many roots actually occur, the only reliable method is detailed polynomial calculations.

Affine camera case. Structure and motion from orthographic or weak perspective views is a well-established topic. 4 noncoplanar points in 3 views suffice to uniquely determine motion and structure, modulo the unrecoverable signs and values of the overall camera-object distances [16, 17, 34]. Many algorithms have been published for this problem, including linear methods in [16, 20], nonlinear algebric methods in [1, 17] and a nonlinear numerical method in [27]. A good review can be found in [27].

2.2. *Parameterizing the Minimal Problem*

Uncalibrated relative orientation from 7 points in 2 images is usually formulated using the fundamental matrix which encodes motion parameters. Similarly, most formulations of relative orientation from 5 points in 2 calibrated images again use motion parameters, either explicitly or via the essential matrix. However, as further images are added, the number of motion parameters increases while (for minimal problems) the number of structure ones remains constant or decreases, and it seems to be the case that for minimal problems with 3 or more images, structure-based parameterizations are simpler than motion-based ones. For 6 points in 3 uncalibrated images, the usual formulation is based on the projective invariants of the 6 point space configuration. And here, for 4 points in 3 calibrated images, our formulation will be based on Euclidean point-point and point-camera distances.

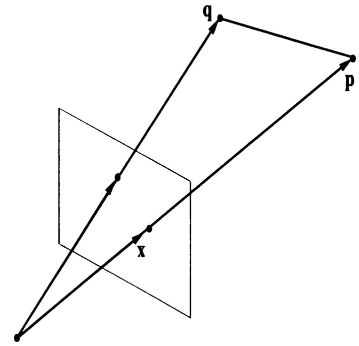

Figure 1. The basic geometric constraint for a pair of points based on Euclidean depths and distance.

We will assume that the calibration matrices

$$\mathbf{K} = \begin{pmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

of the cameras are known, and that the measured image point coordinates $\mathbf{u} = (u, v, 1)^T$ have been normalised by the inverse calibration, $\mathbf{x} = \mathbf{K}^{-1}\mathbf{u}$. This converts them into 3D direction vectors expressed in the 3D camera frame, that point towards the corresponding 3D points. For convenience, we will also assume that they have been normalised to unit vectors, so by image points, we actually mean unit-norm 3D direction vectors $\mathbf{x}$. The 3D point corresponding to an image point/direction $\mathbf{x}$ is determined by a 3D depth (point-camera distance) λ as $\lambda\mathbf{x}$. So the 3D points in the camera centered frame are just properly re-scaled image direction vectors.

Basic Euclidean constraint. The distance between two 3D points represented by 3-vectors $\mathbf{p}$ and $\mathbf{q}$ is given by the cosine rule:

$$\|\mathbf{p} - \mathbf{q}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 - 2\mathbf{p}^T\mathbf{q}$$

Applying this to the normalized direction vectors representing the 3D points in the camera frame, and using the fact that $\|\mathbf{x}_p\| = 1$, gives:

$$\lambda_p^2 + \lambda_q^2 - c_{pq}\lambda_p\lambda_q = \delta_{pq}^2$$

where $c_{pq} = 2\mathbf{x}_i^T\mathbf{x}_j = 2\cos(\theta_{pq})$ is a known constant from the image points, and δ_{pq} is the *unknown* distance between the space points. These cosine-rule constraints are also used in calibrated camera pose from known 3D points [10, 29], except that here the inter-point distances δ_{pq} are not knowns, but unknowns that must be eliminated.

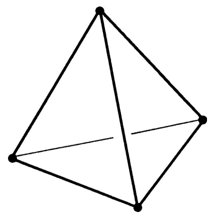

Figure 2. The configuration of 4 non-coplanar points in space.

4 point configuration. A set of four 3D points has 6 independent Euclidean invariants—$3 \times 4 = 12$ d.o.f., modulo the 6 d.o.f. of a Euclidean transformation—and it is convenient to take these to be the $\binom{4}{2} = 6$ inter-point distances δ_{pq}, i.e. the edge lengths of the tetrahedron in Fig. 2. This structure parameterization is very convenient here, as the δ_{pq} appear explicitly in the cosine-rule polynomials. However, it would be less convenient if there were more than 4 points, as the inter-point distances would not all be independent.

Polynomial system for the problem. For N images of 4 points, we obtain a system of $6N$ homogeneous cosine-rule polynomials in $4N$ unknowns λ_{ip} and 6 unknowns δ_{ij}:

$$f(\lambda_{ip}, \lambda_{iq}, \delta_{pq}) = 0, \quad i = 1, \ldots, N, \quad p < q = 1, \ldots, 4$$

The unknown inter-point distances δ_{ij} can be eliminated by equating cosine-rule polynomials from different images

$$\lambda_{ip}^2 + \lambda_{iq}^2 - c_{pq}^i \lambda_{ip} \lambda_{iq} = \lambda_{jp}^2 + \lambda_{jq}^2 - c_{pq}^j \lambda_{jp} \lambda_{jq} (= \delta_{pq}).$$

leaving a system of $6(N-1)$ homogeneous quadratics in $4N$ homogeneous unknowns λ_{ip}.

For $N = 3$ images we obtain 18 homogeneous polynomials $f(\lambda_{ip}, \lambda_{iq}, \delta_{pq}) = 0$ in 18 unknowns λ_{ip}, δ_{ij}, or equivalently 12 homogeneous polynomials in 12 unknowns λ_{ip}. Dehomogenizing (removing the overall 3D scale factor) leaves just 11 inhomogeneous unknowns, so as expected, equation counting suggests that the system is slightly redundant.

3. Random Rational Simulation Method

The most general polynomial solver for $f_i(x_j) = 0$ is to use Gröbner basis to characterize the variety of the ideal generated by the polynomials $\langle f_i(x_j) \rangle$ [3, 4]. The polynomial system has only finitely many solutions if the dimension of the variety is zero, infinite number of solutions for positive dimension, and no consistent solutions for negative dimension. Generally, we expect to have only a finite number of solutions for a well-defined geometric problem when the dimension of the variety is zero. Combining elimination from lexicographic Gröbner bases with numerical root-inding for one-variable polynomials conceptually gives a general polynomial solver. But it is often impossible simply because Gröbner bases could not be computed with limited computer resources. This is true for our polynomial system $g(\lambda_{ip}, \lambda_{iq}, \lambda_{jp}, \lambda_{jq})$ with parametric coefficients c_{pq}^i. We choose the approach offered by Macaulay (http://www.math.uiuc.edu/Macaulay2) among other computer algebra systems, which allows the computation with coefficients in modular arithmetic (a finite prime field $k = \mathbf{Z}/\langle p \rangle$) to speed up computation and minimize memory requirements. We first create projections of random space points with random camera poses.

Random rational simulation. 3D space points $\mathbf{X}_i$, $i = 1,..., 4$ are represented by homogeneous 4-vectors and can be randomly generated in integers modulo a given modulus. A calibrated camera can always be assumed to be a 3×4 image projection matrices $\mathbf{P}_i = (\mathbf{R}_i \mid \mathbf{t}_i)$ where $\mathbf{R}_i$ is a 3×3 rotation, and $\mathbf{t}_i$ is a 3D translation. Random rational 3×3 rotation matrix can be generated using random rational Pythagorean pairs. A random rational Pythagorean pair $a^2 + b^2 = 1$ for rotation can be generated from a pair of random integers l and m by $\frac{2lm}{l^2+m^2}, \frac{l^2-m^2}{l^2+m^2}$. The 3-vector image points in homogeneous coordinates are obtained by $\mathbf{x}_{ip} = \mathbf{P}_i \mathbf{X}_p$ up to scales. Each homogeneous image point $\mathbf{x}_{ip}$ represents the 3D direction vector to its 3D point at its camera center.

The method is summarized as follows:

- Creating random configuration of 4 points and N cameras in integer coordinates and rational rotations;
- Applying the cosine rule to image points to build the system of $6(N-1)$ polynomials;
- Dehomogenizing the system and mapping the rationals into modular arithmetic for a given modulus;
- Computing Gröbner bases with Macaulay2 in modular arithmetics;
- Characterizing the variety from the computed Gröbner basis for each random test.

For each problem, the simulation is run on different random configurations and different moduli. If the characterisation of the variety remains stable, it is meant that the computed stable characterisation is generically equivalent to that of the original system.

Solution feasibility. The dehomogenized polynomial systems have relative point depths y_{ip} and relative image scales z_j as unknown variables. It is straightforward to notice that none of the unknowns can be zero. This says that a *feasible solution* of the system must be toric! Toric solutions are roots in $(\mathbf{C}^*)^n$ for n-dimensional solution space, which means that none of the coordinates can be zero.

Quadratic symmetry and symmetric dehomogenisation. As the unknowns are the depths of the points relative to the camera center, if λ_{ip} is a solution depth, so is $-\lambda_{ip}$. The symmetric solution gives a reconstruction behind the camera. The polynomial system can be de-homogenized by introducing relative depths for points of the same image

$$y_{ip} = \frac{\lambda_{ip}}{\lambda_{i1}}, \quad i = 1, \ldots, N, p = 2, \ldots, 4,$$

and relative scales (squared) for images

$$z_j = \frac{\lambda_{j1}^2}{\lambda_{11}^2}, \quad j = 2, \ldots, N.$$

This amounts to fixing the scales $y_{i1} = 1, i = 1, \ldots, N$, and $z_1 = 1$.

Imposing $\lambda_{11} = 1$ to fix the global reconstruction scale gives the new dehomogenized ideal $\langle g(y_{ip}, z_j)\rangle$ with $4N - 1 = 3N + (N - 1)$ inhomogeneous unknowns

$$\{y_{ip}, z_j | i = 1, \ldots, N, p = 2, \ldots, 4, j = 2, \ldots, N\}.$$

This transformation eliminates the quadratic symmetry solutions

$$\lambda_{ip} \mapsto -\lambda_{ip},$$

for $p = 1, \ldots, 4$ in each image.

As the same point is used in different images for the relative depths of points of the same image, we call this symmetric dehomogenisation.

Asymmetric dehomogenisation. The above dehomogenisation is a natural choice of fixing scales, but the following computation will show that this dehomogenisation contains many spurious solutions. It might be inconvenient for further efforts of developing effective algorithms for digging the real solution out of the system. This motivates us to use an alternative asymmetric dehomogenisation, i.e. take different points in different images to define the relative depths for points of the same image:

$$y_{ip} = \frac{\lambda_{ip}}{\lambda_{ii}}, \quad i = 1, 2, 3, p = 1, \ldots, 4,$$

and relative scales (squared) for images

$$z_j = \frac{\lambda_{j1}^2}{\lambda_{11}^2}, \quad j = 2, \ldots, N,$$

this amounts to fixing the scales $y_{ii} = 1$ and $z_1 = 1$.

Imposing $\lambda_{11} = 1$ to fix the global reconstruction scale gives the new dehomogenized ideal $\langle g'(y_{ip}, z_j)\rangle$ with $4N - 1 = 3N + (N - 1)$ inhomogeneous unknowns

$$\{y_{ip}, z_j | i = 1, \ldots, N, p = 1, \ldots, 4,\\ i \neq p, j = 2, \ldots, N\}.$$

4. Results for Different Cases

4.1. The Case of 3 Images

Symmetric dehomogenisation. For the case of 3 images, with the projection from a random set of 4 points in 3 images, the computation on the dehomogenized system of 12 equations in 11 unknowns turns out a Gröbner basis with 66 generators, 57 of them of degree 2 and 9 of degree 3. The variety is zero-dimensional and has 21 points.

As feasible solutions are toric, by computing the quotient ideal by the product of unknown variables, we show that 20 out of 21 solutions are non-toric, therefore infeasible solutions. It can be further verified that $(0, \ldots, 0, 1, 1)$ is a solution of the system and by saturating the ideal with respect to y_{ip}, it can be shown that it is the only trivial solution with the algebraic multiplicity of 20.

Asymmetric dehomogenisation. However the variety defined by the asymmetric dehomogenised system characterized by its Grübner basis is indeed of dimension 0 and has a unique point, this unique point is the true solution of the polynomial system.

We may conclude that

> Euclidean reconstruction from a minimum of 4 points in 3 images is generally unique, but it may have 20 infeasible solutions if the symmetric dehomogenized system is used to characterise the variety.

4.2. *The Case of More than 3 Images*

With the projection from a random set of 4 points in N images, the computation on the symmetric dehomogenized system always turns out a zero dimensional variety which has 25 points for $N = 4$ and 41 points for $N = 5$. Unsurprisingly, all except one are non-feasible solutions. The asymmetric dehomogenized system however turns out the unique feasible solution.

This allows us to establish the following more general result:

> Euclidean reconstruction from a minimum of 4 points in $N \geq 3$ images is in general unique but may have spurious solutions if using the symmetric dehomogenised polynomial systems.

4.3. *The Case of Coplanar Points*

Unknown coplanarity. Four points in space may be coplanar, all the constraints from the cosine rule still hold for coplanar points. We then generate random coplanar points for N images and compute Grübner bases to characterize the variety defined by the ideal $\langle g(y_{ip}, z_j)\rangle$ without other additional constraints. It turns out the variety has always 0 dimension and 22 points for $N = 3$. Similarly we could quotient out 20 non-toric solutions, which are necessarily infeasible. By computing the radical quotient ideal, we can show that it has a double feasible solution. Again with the asymmetric demogenized system, the only double solution is obtained. In summary,

> Euclidean reconstruction from a minimum of 4 unknown coplanar points in 3 images is in general a double solution.

> Euclidean reconstruction from a minimum of 4 unknown coplanar points in more than 3 images is in general unique.

Known coplanarity. Obviously, not only are all the constraints from the cosine rule still valid, but also additional constraints are available for known coplanar points. The explicit coplanarity constraint for 4 points is given by the vanishing determinant of the re-scaled image points in each image, i.e.

$$h(\lambda_{ip}) = \begin{vmatrix} \lambda_{i1}\mathbf{x}_{i1} & \lambda_{i2}\mathbf{x}_{i2} & \lambda_{i3}\mathbf{x}_{i3} & \lambda_{i4}\mathbf{x}_{i4} \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0,$$

for each image $i = 1, 2, 3$.

This turns out three homogeneous polynomials of degree 3 in λ_{ij} or in $y_{ip} = \lambda_{ip}/\lambda_{i1}$.

We generate random coplanar points and compute Grübner bases to characterize both the variety defined by $\langle g(y_{ip}, z_j), h(y_{ip})\rangle$ with the explicit coplanarity constraints. We find that

> Euclidean reconstruction from a minimum of 4 known coplanar points in $N \geq 3$ images is unique.

4.4. *The Case of Symmetric Configurations*

One symmetric configuration of points and cameras is shown in Fig. 3. The 4 points are 4 corners of a cube and the 3 cameras are orthogonal each other and to the cube faces. Such a symmetric configuration considerably increases the number of feasible solutions. The computation shows that

> Euclidean reconstruction from a symmetric camera-point configuration of 4 points in 3 images as illustrated in Fig. 3 may have 56 feasible solutions. The reconstruction becomes unique if only the cameras

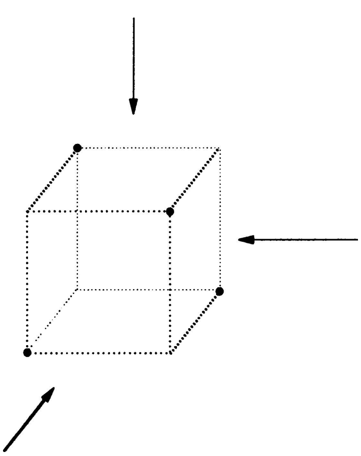

Figure 3. One symmetric configuration of points and cameras.

are symmetric or the points are symmetric, but not simultaneously the two.

5. Discussion and Future Work

This paper gives the first systematic investigation on the difficult topic of Euclidean reconstruction from the minimal data of 4 points in more than 2 calibrated images. The formulation based on Euclidean depths is intrinsic for 4 points. The major results we have established are that the reconstruction from a minimum of 4 points is essentially unique in terms of feasible solutions, even though the underlying algebraic systems may have non-feasible solutions if the system is not properly dehomogenized. We also shown that the coplanar configurations are not in principle singular, but the true solution becomes a double root in the case of three images. Only symmetric camera configuration or symmetric point configuration do not introduce more solutions, but the simultaneous symmetric camera and point configuration has 56 solutions in the case of 3 images. The development was based on modern computer algebra tools in modular arithmetics due to the problem complexity. These results set ground for investigation of more parctical issues of these problems. We are working on the development of efficient numerical algorithms for solving this problem.

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References

1. B.M. Bennet, D.D. Hoffman, J.E. Nicola, and C. Prakash, "Structure from two orthographic views of rigid motion," *Journal of the Optical Society of America*, Vol. 6, No. 7, pp. 1052–1069, 1989.
2. S. Carlsson and D. Weinshall, "Dual computation of projective shape and camera positions from multiple images," *International Journal of Computer Vision*, Vol. 27, No. 3, pp. 227–241, 1998.
3. D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, Springer, 1998.
4. D. Cox, J. Little, and D.O'Shea. *Using Algebraic Geometry*, Springer, 1998.
5. M. Demazure, "Sur deux problèmes de reconstruction," Technical report, INRIA, 1988.
6. I.Z. Emeris, "A general solver based on sparse resultants: Numerical issues and kinematic applications," Technical Report RR-3110, INRIA, 1997.
7. I.Z. Emeris, "A general solver based on sparse resultants: Numerical issues and kinematic applications," Technical Report RR-3110 (http://www.inria.fr/RRRT/RR-3110.html), INRIA, Sophia Antipolis, France, Jan. 1997.
8. O. Faugeras, "What can be seen in three dimensions with an uncalibrated stereo rig?" in *Proceedings of the 2nd European Conference on Computer Vision, Santa Margherita Ligure, Italy*, G. Sandini (Ed.), Springer-Verlag, May 1992, pp. 563–578.
9. O. Faugeras and S. Maybank, "Motion from point matches: Multiplicity of solutions," *International Journal of Computer Vision*, Vol. 3, No. 4, pp. 225–246, 1990.
10. M.A. Fischler and R.C. Bolles, "Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography," *Graphics and Image Processing*, Vol. 24, No. 6, pp. 381–395, 1981.
11. S.I. Granshaw, "Bundle adjustment methods in engineering photogrammetry," *Photogrammetric Record*, Vol. 56, No. 10, pp. 181–207, 1980.
12. R.I. Hartley, R. Gupta, and T. Chang, "Stereo from uncalibrated cameras," in *Proceedings of the Conference on Computer Vision and Pattern Recognition*, Urbana-Champaign, Illinois, USA, 1992, pp. 761–764.
13. A. Heyden and G. Sparr, "Reconstruction from calibrated cameras: a new proof of the kruppa-demazure theorem," *Journal of Mathematical Imaging and Vision*, Vol. 10, pp. 1–20, 1999.
14. R.J. Holt and A. N. Netravali, "Uniqueness of solutions to three perspective views of four points," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Vol. 17, No. 3, pp. 303–307, 1995.
15. B. Horn, "Relative orientation," *Int. J. Computer Vision*, Vol. 4, pp. 59–78, 1990.
16. T.S. Huang and C.H. Lee, "Motion and structure from orthographic projections," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Vol. 11, No. 5, pp. 536–540, 1989.
17. J. Koenderink and A. van Doorn, "Affine structure from motion," *Journal of the Optical Society of America A*, Vol. 8, No. 2, pp. 377–385, 1991.
18. J. Krames, "Zur Ermittlung eines Objektes aus zwei Perspektiven (Ein Beitrag zur Theorie der "gefährlichen Örter")," *Monatshefte für Mathematik und Physik*, Vol. 49, pp. 327–354, 1941.
19. E. Kruppa, "Zur Ermittlung eines Objektes aus zwei Perspektiven mit innerer Orientierung," *Sitzungsberichte Österreichische Akademie der Wissenschaften, Mathematisch-naturwissenschaftliche Klasse, Abteilung II a*, Vol. 122, pp. 1939–1948, 1913.
20. C.H. Lee and T. Huang, "Finding point correspondences and determining motion of a rigid object from two weak perspective views," *Computer Vision, Graphics and Image Processing*, Vol. 52, pp. 309–327, 1990.
21. H.C. Longuet-Higgins, "A method of obtaining the relative positions of 4 points from 3 perspective projections," in *Proceedings of the British Machine Vision Conference, Glasgow, Scotland*, pp. 86–94, 1991.
22. S.J. Maybank and O.D. Faugeras, "A theory of self calibration of a moving camera," *International Journal of Computer Vision*, Vol. 8, No. 2, pp. 123–151, 1992.
23. A.N. Netravali, T.S. Huang, A.S. Krishnakumar, and R.J. Holt, "Algebraic methods in 3D motion estimation from two-view point correspondences," *Int. J. Imaging Systems & Technology*, Vol. 1, pp. 78–99, 1989.

24. D. Nister, "An efficient solution to the five-point relative pose problem," in *Proceedings of the Conference on Computer Vision and Pattern Recognition, Madison, USA*, June 2003, pp. II–195–202.
25. J. Philip, "A non-iterative algorithm for determining all essential matrices corresponding to five point pairs," *Photogrammetric Record*, Vol. 15, No. 88, pp. 589–599, 1996.
26. L. Quan, "Invariants of six points and projective reconstruction from three uncalibrated images," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Vol. 17, No. 1, pp. 34–46, 1995.
27. L.S. Shapiro, A. Zisserman, and M. Brady, "3D motion recovery via affine epipolar geometry," *International Journal of Computer Vision*, Vol. 16, No. 2, pp. 147–182, 1995.
28. A. Shashua, "Algebraic functions for recognition," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Vol. 17, No. 8, pp. 779–789, 1995.
29. C.C. Slama (ed), *Manual of Photogrammetry, 4th edition*. American Society of Photogrammetry and Remote Sensing, Falls Church, Virginia, USA, 1980.
30. M. Spetsakis and J. Aloimonos, "A unified theory of structure from motion," in *Proceedings of DARPA Image Understanding Workshop*, 1990, pp. 271–283.
31. R. Sturm, "Das Problem der Projektivität und seine Anwendung auf die Flächen Zweiten Grades," *Math. Ann.*, Vol. 1, pp. 533–574, 1869.
32. P.H.S. Torr and A. Zisserman, "Robust parameterization and computation of the trifocal tensor," in R.B. Fisher and E. Trucco (Eds.), *Proceedings of the seventh British Machine Vision Conference*, Vol. 2, Edinburgh, Scotland, British Machine Vision Association, September 1996, pp. 655–664.
33. B. Triggs, "Routines for relative pose of two calibrated cameras from 5 points," Technical report, INRIA, 2000.
34. S. Ullman, "*The Interpretation of Visual Motion*," The MIT Press: Cambridge, MA, USA, 1979.
35. Z. Zhang, R. Deriche, O. Faugeras, and Q.T. Luong, "A robust technique for matching two uncalibrated images through the recovery of the unknown epipolar geometry," Rapport de recherche 2273, INRIA, May 1994.