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An Update of Tarski: Two Usages of the Word "True"

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Abstract

This paper is based on Tarski's theory of truth. The purpose of this paper is to solve the liar paradox (and its cousins) and keep both of the deductive power of classical logic and the expressive power of the word "true" in natural language. The key of this paper lies in the distinction between the predicate usage and the operator usage of the word "true". The truth operator is primarily used for characterizing the semantics of the language. Then, we do not need the hierarchy of languages. The truth predicate is mainly used for grammatical function. Tarski's schema of the truth predicate is not necessary in this proposal. The schema of the word "true" is the schema of the truth operator. The liar paradox (and its cousins) can be solved in this way. In the appendix, I show a consistent model for both of the truth predicate and the truth operator.

Keywords The liar paradox · The T-schema · Predicate · Operator

The liar paradox is the paradox connected with the word "true" where we can conclude that a sentence is true if and only if it is not true, but it seems that we can find no problem in the premises and reasoning process. For example, consider the liar sentence:

L : L is not true.

This sentence can yield a contradiction as follows. If the sentence 'L is not true.' is true, then given what it says, L is not true. But L just is the sentence 'L is not true', so we can conclude that if L is true, then L is not true. Conversely, if L is not true, then the sentence 'L is not true' is true. Again, L just is the sentence 'L is not true', so we can conclude that if L is not true, then L is true. We have thus shown that L is not true if and only if L is true.

In order to deal with the liar paradox, multiple solutions have been proposed, among which Tarski's theory of truth is one of the most significant solutions, however, it also

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has some weaknesses. The purpose of this paper is to keep the advantages of Tarski's theory while overcoming the disadvantages.

In the following sections, I will introduce the core ideas of Tarski's theory of truth in Sect. 1. In Sect. 2, I will talk about the main problems facing Tarski's theory. In Sect. 3, I will analyze the two usages of the word "true" in natural language. The differences and connections between the two usages will be talked about. In Sect. 4, I will discuss some philosophical issues connected to my proposal. In the conclusion section, I will explain why this paper is an update of Tarski. In an appendix, I provide a simple model, as a proof of the consistency, to characterize the two usages of the word "true" exactly in a formalized system.

1 Core Ideas of Tarski's Theory of Truth

Tarski's theory of truth is recognized as one of the most important theories of truth. This is generally because of the following reasons. First, classical logic is taken as immutable in his approach. Second, Tarski introduces many thoughts which deeply influence the following studies of logic and philosophy. For instance, the T-schema, semantically closed languages and the distinction between object language and meta-language. In Tarski's words:

"Consider the sentence "snow is white." We ask the question under what conditions this sentence is true or false. It seems clear that if we base ourselves on the classical conception of truth, we shall say that the sentence is true if snow is white, and that it is false if snow is not white. Thus, if the definition of truth is to conform to our conception, it must imply the following equivalence:

The sentence "snow is white" is true if, and only if, snow is white ...We shall now generalize the procedure which we have applied above. Let us consider an arbitrary sentence; we shall replace it by the letter 'p.' We form the name of this sentence and we replace it by another letter, say 'X.' We ask now what is the logical relation between the two sentences "X is true" and 'p.' It is clear that from the point of view of our basic conception of truth these sentences are equivalent. In other words, the following equivalence holds:

(T) *X is true if, and only if, p.*

We shall call any such equivalence (with 'p' replaced by any sentence of the language to which the word "true" refers, and 'X' replaced by a name of this sentence) an "equivalence of the form (T)." " (Tarski 1944, pp. 343–344)

This is Tarski's well-known T-schema: X is true if and only if p, where p is a sentence and X is the name of p. However, Tarski proves that a "semantical closed" language plus classical logic can result in the liar paradox. A semantically closed language is defined as follows:

"We have implicitly assumed that the language in which the antinomy is constructed contains, in addition to its expressions, also the names of these expressions, as well as semantic terms such as the term "true" referring to sentences of this language; we have also assumed that all sentences which determine the adequate usage of this

term can be asserted in the language. A language with these properties will be called "semantically closed"."(Tarski 1944, p. 348)

So, for Tarski, a language is semantically closed if and only if:

(1) it contains names for its expressions,
(2) it contains its own truth predicate,
(3) one can assert all the sentences with the form (T) in the language.[1]

In order to solve the liar paradox, one can reject either classical logic or semantically closed languages. But Tarski thinks that it is unadvisable to reject classical logic, so the only possibility is to reject semantically closed languages. In Tarski's words:

"It would be superfluous to stress here the consequences of rejecting the assumption (II), that is, of changing our logic (supposing this were possible) even in its more elementary and fundamental parts. We thus consider only the possibility of rejecting the assumption (I). Accordingly, we decide not to use any language which is semantically closed in the sense given." (Tarski 1944, p. 349)

For Tarski, the way to reject semantically closed languages is to distinguish between object language and meta-language. Therefore, the "truth" of the sentences in the object language can be "expressed" only in the meta-language rather than in the object language itself. And the "truth" of the sentences in the meta-language can be "expressed" only in the meta-meta-language, and so on. Hence, language is divided into a sequence of languages $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots$. Each language $\mathcal{L}_n$ contains a truth$_n$ predicate which can be applied[2] only to sentences whose maximum subscript in the sentence is less than n. Then each language has a T_n-schema which can be applied only to the sentences in lower languages but cannot be applied to itself and sentences in higher languages. Therefore the T-schema is divided into a countable infinity of T_n-schema*s*.

2 Criticisms of Tarski's Theory of Truth

Tarski's approach has been criticized by many logicians and philosophers who take the opinion that there is only one truth predicate in natural language and refuse the hierarchy of languages. The main objections to Tarski's theory are listed below.[3] Consider the following sentence:

(1) "This sentence is not true"

In Tarski's language, we cannot construct a sentence like (1), because there is no word "true" but instead infinitely many "true$_n$" in his hierarchy of languages. What we can construct is a sentence like "This sentence is not true$_n$". And we cannot apply the

[1] In the following parts, a language is called a *rich enough* language if and only if it meets the first two conditions, i.e. if it contains names for its expressions and its own truth predicate.

[2] When I use the word "applied" (or "apply") in this kind of circumstance, I mean every sentence in the language (level) has this form. That is for any sentence *P* of the language (level), we can say "'*P*' is true iff *P*". This is the key to deduce the liar paradox, because if we replace *P* with the liar sentence, a paradox may appear. One purpose of this paper is to avoid the liar paradox, so I put more emphasis on the application of the T-schema which is connected with the inference of the liar paradox. So when I use the word "applied" (or "apply"), I did not take it as a definition of "true". The same below.

[3] The following criticisms come mainly from Kripke (1975).

T_m-schema, $n \geq m$, to this sentence, but only the T_s-schema, $s > n$. However, there is no "true$_n$" in natural language, there is only one truth predicate.[4] The hierarchy of languages is in conflict with our natural language.

Here is another example. What is the subscript of the word "true" in A and what is the subscript of the word "true" in B in the following example?

(2) A: "B is true"
B: "A is not true"

According to Tarski's hierarchy of languages, since both A and B contain the word "true", either the level of A is higher than the level of B or B's level is higher than A's level, but they cannot be both at the same time, hence they cannot be constructed in Tarski's approach. But each sentence in this example is meaningful in natural language.

Consider the following example:

(3) A: "What B said is true"

The construction of the sentence A in Tarski's hierarchy of languages depends on the level of the sentence B. If the largest subscript in B is 10, then the subscript of A should be greater than 10. If the largest subscript in B is 100, then the subscript of A should be greater than 100. So, we cannot assign the subscript of A in advance of assigning the level of B. This means that a statement should be allowed to seek its own level. We cannot fix its level in advance as in Tarski's hierarchy.

In the case when we have fixed B's level, if it happens that the sentence uttered by B is "A is false", then there arises the problem about the levels of two sentences, just like the problem shown in example (2) above. Hence the two sentences, A and B, cannot be constructed in Tarski's hierarchy. But if what B said is only "snow is white", then they can be constructed in Tarski's language and both of them are true. If what B said is "1+1=3", then they can also be constructed in the language and both of them are false. So, the level of a statement and whether a sentence is a well-formed formula are not *a priori*. It, in some sense, depends on the circumstance.

Another problem is about transfinite levels. We can assert the following sentence:

(4) Snow is white

Also, we can assert that (4) is true, that "(4) is true" is true, etc.. The various occurrences of "is true" in the sequence are assigned increasing subscripts. But there are "technical difficulties" in answering the question how far into the transfinite we can iterate.[5]

[4] Later in this paper, I will show that there are two usages of the word "true", but only one of them is a predicate usage.

[5] The "technical difficulty" is pointed out by Kripke (1975), p. 697.

Therefore, lots of different versions of new solutions to the liar paradox have appeared based on revising Tarski's theory. One way retains semantically closed languages. This kind of solution tries to keep the expressive power of language, but the cost is abandoning classical logic, i.e. decreasing deductive power. Another way attempts to preserve classical logic. This type of solution keeps the deductive power of logic, but the price is weakening the expressive power of language, i.e. rejecting semantically closed languages. But is there a third way? Can we keep both classical logic and semantically closed languages in the system simultaneously? Trying to keep both expressive power and deductive power, to some extent, is one purpose of this paper.

3 Two Usages of the Word "True"

I do not agree that giving up classical logic is the only way to keep semantically closed languages. Another way to realize this aim is to distinguish two usages of the word "true": predicate usage and operator usage.

Indeed, the word "true" has a predicate usage, such as the following example:

(1′) "Snow is white" is true.

In this sentence, "'snow is white"' is a noun which is used as a subject, and "is true" is a predicate. In the examples given above, such as (4), "is true" is also used as a predicate.

But the word "true" has another usage: an operator usage. For example:

(2′) It is true that snow is white.

In this sentence, "true" (exactly speaking, "it is true that") is used as an operator while "snow is white" occurs as a sentence instead of a word.

Since the word "true" has two usages, why do we take the word "true" in the T-schema as a predicate? Why not take it as an operator? In the following, I will argue that the word "true" in the T-schema is actually an operator rather than a predicate.

First, natural language is ambiguous. In some situations, what a sentence looks like at first sight is totally different from its real logical form. For example. How to deal with the sentence "'snow is black' is absolutely not the case."? Normally, we take "is absolutely not the case" as a negation operator. Then this sentence will be treated as "it is not the case that snow is black." But, as we can see, "is absolutely not the case" is a predicate in this sentence. Why do we take a predicate as an operator in this example? Why not take "is true" in the sentence "'snow is white' is true" as an operator? Natural language is much more complicated than what it seems to be. The superficial grammar of a sentence in natural language is not always equivalent to its real function. It is not sensible to understand a word on account of its grammar. We should find out its underlying structure.

Second, one may claim that the word "true" is a semantic word. It is used to characterize the semantics of the language. In other words, it is used to describe the sentences whose semantic values are true. But this is not a reason to take the word "true" as a predicate. In the "Appendix", I will show how to characterize the semantic values of sentences by the truth operator without the revenge of the liar.[6]

Third, I think a comparison with the usage of the modal words "possible" and "necessary" is revealing because they are similar to the usage of the word "true". For example:[7]

$$(3')\begin{cases}(3'a)\ \text{“}\textit{Snow is not white" is possible.}\\(3'b)\ \text{“}\textit{Snow is not white" is necessary.}\\(3'c)\ \text{“}\textit{Snow is not white" is true.}\end{cases}$$

In the sentences (3/a) and (3/b), ""snow is not white"" is a noun which is used as the subject, and the words "possible" and "necessary" are used as predicates. This usage of the words "possible" and "necessary" is similar to the predicate usage of the word "true" in (3/c) where "true" is used as a predicate.

"Possible" and "necessary" can also be used in generalizations, for example:

$$(4')\begin{cases}(4'a)\ \textit{All what he said is possible.}\\(4'b)\ \textit{All what he said is necessary.}\\(4'c)\ \textit{All what he said is true.}\end{cases}$$

In the sentences (4/a) and (4/b), the words "possible" and "necessary" are predicates used in generalizations. This usage of "possible" and "necessary" is similar to the usage of the word "true" in (4/c) where "true" is used as a predicate in a generalization.

The word "possible" and "necessary" also have operator usage, such as:

$$(5')\begin{cases}(5'a)\ \textit{It is possible that snow is not white.}\\(5'b)\ \textit{It is necessary that snow is not white.}\\(5'c)\ \textit{It is true that snow is not white.}\end{cases}$$

In the sentences (5/a) and (5/b), "snow is not white" is a sentence and the words "possible" and "necessary" are used as operators. This usage of "possible" and "necessary" is similar to the usage of the word "true" in (5/c) where "true" is used as an operator.[8]

So, the modal words "possible" and "necessary" have both a predicate usage and an operator usage. But that does not mean that the operator usages of the modal words should be abandoned. On the contrary, as we all know, the words "possible" and "necessary" are usually treated as operators rather than predicates in modal logic.

Montague has found that the predicate usage of the modal words can result in paradoxes, which he takes as an argument against the predicate usage of modal words.[9] Valker Halbach provides a feasible way to reduce the operator usage of the modal word to predicate usage by substituting "is necessarily true" for "is necessary" in order to

[6] See Theorem 3 in the "Appendix".

[7] In the predicate usage, we sometimes use a pronoun rather than the name of a sentence as the subject of these words. For example, "this is possible", "that is necessary" and "it is true".

[8] Similar comparisons can be made to the phrase "is known", because we can also say "'snow is white' is known", "all that he said is known" and "it is known that snow is white". Hence the same method, namely distinguishing the predicate usage and the operator usage, can also be used to deal with the Knower Paradox which is deduced from the sentence "this sentence is not known".

[9] Cf. Montague (1974).

overcome the disadvantages, although Halbach's approach also has weaknesses. [10] But few people have researched how to reduce the predicate usage of the word "true" to the operator usage.[11] And no one, before this paper, has studied how to keep both of the two usages of the word "true" simultaneously as a way to solve the liar paradox. This is, however, one purpose of this paper.

So, why do we always take the modal words as operators rather than predicates in logic, but take the word "true", which is similar to the modal words in usages, as a predicate rather than an operator? I can find no necessary reason [12] to take the word "true" as a predicate without considering the operator usage.

One may claim that although the word "true" has two usages, they are equivalent. In order to deal with this question, we need to distinguish two concepts: form translation and value equivalence.

As for the form translation, some sentences with the form "*X* is true" can be translated into corresponding sentences with the form "it is true that *p*". Such as some sentences similar to (3′c) can be translated into sentences similar to (5′c). But, not every sentence with the truth predicate can be translated into a sentence with the truth operator.

First, when we do not know the contents of *p*, we cannot translate the sentence. For example, when we do not know what the first sentence Plato said, we can still claim "the first sentence Plato said is true". But we cannot say "it is true that the first sentence Plato said". In fact, this sentence can be treated as a quantified sentence:

$\exists x$ (x is the first sentence said by Plato and x is true)

and this sentence is equivalent to a universally quantified sentence. As for quantified sentence, we will talk about it next.

Second, we cannot translate the predicate usage of the word "true" in generalizations into an operator usage as above. Transforming, by the same way, the following sentence

(4′c) All that he said is true

will yield an ill-formed sentence:

(4′d) It is true that all that he said

However, if we know all the sentences that he said and they are finite, for example what he said is just the sentence "Socrates is mortal", then we can translate (4'c) into

(4′e) It is true that Socrates is mortal

10 Cf. Halbach and Welch (2009).

11 Susanne Bobzien also talked about the two usages of the word "true" in Susanne Bobzien (2017). Our opinions about how to deal with the two usages of the word "true" are different. For example, Bobzien preserved the operator usages only and claims that the predicate usage is equivalent to the operator usage, while I keep both of the two usages of the word "true" and point out that the two usages are not always equivalent. Besides, Bobzien uses modal semantics to treat the operator usage which is different from my approach.

12 But I find two possible reasons why we always take the word "true" as a predicate rather than an operator. It will be discussed in the next section.

However, if we do not know all the sentences that he said or all that he said contains infinitely many sentences, such as “all the theorems of PA are true”, then we cannot translate this kind of quantified sentence into a finite long sentence with the form “it is true that...”

One may claim that we can take every generalization as an abbreviation of the infinite conjunction of its instances in the following way. For any sentence

$$\forall x(Px \rightarrow x \text{ is true})$$

where P is some property, such as “was said by Plato”, can be translated into an infinite long sentence:

$$(P\ulcorner\phi_1\urcorner \rightarrow \text{it is true that } \phi_1) \wedge (P\ulcorner\phi_2\urcorner \rightarrow \text{it is true that } \phi_2) \wedge \cdots \wedge (P\ulcorner\phi_n\urcorner \rightarrow \text{it is true that } \phi_n) \wedge \cdots,$$

where for any $i \in \mathbb{N}$, ϕ_i is a sentence of the natural language and $\ulcorner\phi_i\urcorner$ is the name of ϕ_i. However, this is an infinite sentence which cannot be expressed by human beings. On the other hand, not every general claim is equivalent to the infinite conjunction of its instances. For example, universal generalizations are not always entailed by the class of their instances because of compactness considerations.[13] So the truth predicate is necessary to increase the expressive power of the language.

On the other side, can every sentence with the form “it is true that p” be translated into a sentence with the form “X is true”? Formally we can. Because when we claim that ‘it is true that p” , we get the sentence p. Then we can fix the name of p, such as $\ulcorner p\urcorner$. And then we can claim that “$\ulcorner p\urcorner$ is true”. However, it is worth noting that not every translation of this sort is *equivalent*, namely, not every translation keeps the same truth value. For examples, let “‘L’ is not true” be the liar sentence L. Then “it is true that ‘L’ is not true” and “ ‘L’ is true” (i.e. “‘L’ is not true” is true) may have different truth values.[14]

Therefore, the word “true” has two usages: predicate usage and operator usage. And they are not equivalent. Hence we should not insist that the word “true” is only a predicate (or only an operator). Instead, we should keep both of the two usages. They have different functions. The truth operator is used for characterizing the semantics. And the truth predicate is primarily used for its grammatical function. Namely, the truth predicate is mainly used for generalizations where we purport to characterize the truth status of some sentences but we cannot do that by the truth operator grammatically.

Then, we can have a semantically closed language which: (1) contains names for its expressions, (2) contains its own truth predicate, (3) can assert all the sentences with the form (T) in the language. But the form (T) is not the original “X is true if, and only if, p”, instead, it changed into the following one:

(T_o) It is true that p if, and only if, p, where p is a sentence.[15]

[13] Cf. Picollo and Schindler (2018), “Deflationism and the Function of Truth”, *Philosophy of Language*, Vol 32, Issue 1.

[14] For details of their evaluations see the “Appendix”.

[15] This is an informal expression. Formally speaking, (T_o) will be expressed in the “Appendix” as: (T_o) $T_o(p) \leftrightarrow p$ where p is a sentence, T_o is the truth operator.

So Tarski is correct that there is a schema for the word "true". But he is incorrect to take it as a schema for the truth predicate. Actually, it is a schema for the truth operator.

However, since some sentences with the predicate usage of the word "true" can be translated into sentences with corresponding operator usage of the word "true", there is a corresponding schema for the truth predicate. That is only when "*X* is true" is equivalent to "it is true that *p*", can we have the corresponding schema for the truth predicate and vice versa. i.e.:

(T_{op}) *X* is true if, and only if, it is true that *p*, iff *X* is true if, and only if, *p*, where *X* is the name of the sentence *p*.[16]

Hence, we can claim "'snow is white' is true if and only if snow is white" because "'snow is white' is true" is equivalent to "it is true that snow is white". But we cannot obtain the schema "*X* is true if and only if *p*" from this case because not every sentence with the form "*X* is true" is equivalent to the sentence with the form "it is true that *p*", where *p* is a sentence and *X* is the name of *p*. In other words, the application of the original (T) proposed by Tarski[17], where "true" is a predicate, must be restricted to those sentences which can have *equivalent transformations*. [18] Actually, (T_{op}) is equivalent to (T_o). This will be proved in the "Appendix".[19]

With the help of (T_{op}), we can generalize on sentence such as "Snow is white or snow is not white " . Because "'Snow is white' is true" is equivalent to "It is true that snow is white", according to (T_{op}), "'Snow is white' is true" is equivalent to "Snow is white". Then we can substitute "Snow is white" with "'Snow is white' is true", yielding

"Snow is white" is true or "Snow is white" is not true

Because in the displayed sentence the clause "snow is white" is mentioned rather than used, we can replace "snow is white" with a variable, yielding "x is true or x is not true". Then we can formulate the generalization "For every sentence x, x is true or x is not true". It is worth noting that what we used in the generalization is the schema (T_{op}), which is equivalent to (T_o), rather than Tarski's schema (T_p). So, Tarski's schema of the truth predicate does not need to be taken as primitive or axiomatic, although the truth predicate is still needed as a primitive predicate.

Now let us talk about the liar paradox showed in the introduction again. I will show how to deal with the liar paradox and its cousins by the present approach.

From the construction of the liar paradox, we can find that it depends on three factors: a rich enough language to express the special sentence, i.e. the liar sentence, the classical logic (or some rules of inference in classical logic) and Tarski's schema, i.e. the T_p-schema. However, a rich enough language is a fact. Our natural language and

16 Formally speaking, (T_{op}) can be expressed as: $(T_p(X) \leftrightarrow T_o(p)) \leftrightarrow (T_p(X) \leftrightarrow p)$, where *X* is the name of the sentence *p*, T_o is the truth operator and T_p is the truth predicate.

17 From now on, I will call this Tarski's schema as (T_p) or T_p-schema.

18 This is not a restriction to the expressive power, because I keep the predicate usage in situations where the truth operator cannot be used, such as the generalizations. Therefore, the expressive power of my proposal is strong enough to express every sentence with the word "true" in natural language.

19 See Corollary 2 in the "Appendix".

many artificial language are all rich enough to express this kind of special sentences. The classical logic is the basis of arithmetics. It is a huge price to modify classical logic before we have sufficient reason. Among the three key factors, only the T_p-schema comes from our intuition about natural language. But this intuition is an illusion. There is a schema for the word “true”, but it is a schema for the operator usage of the word “true”. In other words, it is the T_o-schema that is used to characterize the semantics of the language. Hence the steps connected with the T_p-schema in the derivation of the liar paradox do not hold any more. Then the liar paradox is solved.

Actually, for the derivation of every paradox connected with the word “true”, such as Curry’s paradox, Yablo’s paradox, the contingent liar and so on, the T_p-schema (or its variations) is a key step to yield contradiction. Since the genuine schema is the T_o-schema, all the other paradoxes about the word “true” can also be treated in the same way as the liar paradox be dealt with.

Is there a new liar or liar-like paradox, i.e. a revenge problem, derived from the T_o-schema on the basis of a rich enough language and classical logic? No. Because in classical logic, the truth operator can be defined by double negation. According to T_o-schema, “it is true that p” is equivalent to “p”. And “p” is equivalent to “$\neg\neg p$” in classical logic. Therefore, “it is true that p” is equivalent to “$\neg\neg p$”. Hence, if the T_o-schema can yield a paradox, similar derivation can also be applied to double negation, then classical logic is inconsistent. But classical logic is consistent. Contradiction.

Now the main problem of Tarski’s approach is solved in this proposal. However, there are still some other issues that need to be discussed.

4 Issues

As I pointed out above, the word “true” has two usages: a predicate usage and an operator usage. But normally we are used to talking about the predicate “true” while omitting the operator “true”. So the first issue is why we always take the word “true” in the T-schema as a predicate rather than an operator? I can find no other reasons except the following two:

The first reason is due to speaking habit. As I have shown above, the word “true” really has a predicate usage, and, in most cases, the operator usage can be translated into the predicate usage *equivalently*. For example, sentence (1’) and sentence (2’) are equivalent. It seems that in natural language the predicate usage is simpler than the operator usage, so we prefer to say “‘snow is white’ is true” instead of “it is true that snow is white”.

The second reason is a historical reason. It is Tarski, one of the most significantly influential logicians in the 20th century, who first presents (T), and, in his paper, the word “true” in (T) is used as a predicate rather than an operator. Since Tarski put forward (T), and indeed the word “true” really has a predicate usage, this becomes one of the core principles in the study of the theory of truth. To some extend, Tarski’s influence plays a decisive role in making one accept that the word “true” in (T) is a predicate.

Hence, speaking habit and the historical reason are the only possible reasons which I can find why most logicians and philosophers who study the theory of truth take "true" as a predicate without considering the operator usage.

Actually, Tarski is right that there is a predicate usage of the word "true" and a schema about "true". But he is incorrect to take the word "true" in the schema as the predicate usage of "true". And this mistake is the crux of the paradox about the word "true".

The second issue is that the distinction of the two usages of the word "true" can help, in some sense, explain the deflationary theory of truth. One of the basic ideas of the deflationary theory of truth is that:

"to assert that a statement is true is just to assert the statement itself. For example, to say that 'snow is white' is true, or that it is true that snow is white, is equivalent to saying simply that snow is white, and this, according to the deflationary theory, is all that can be said significantly about the truth of 'snow is white"'. (Stoljar and Damnjanovic 2010)

Here, deflationism takes the operator usage of "true" as same as the predicate usage of "true". But the predicate usage will generate paradoxes. In order to deal with these paradoxes, the deflationism proposes theories which are so technical that they are far away from our intuition. But if we distinguish the two usages of the word "true" as above and take the usage of the word "true" in the quotation paragraph as an operator usage, then we can avoid the liar paradox without resorting to a much more complicated theory. In other words, the core idea of deflationism should be "to say it is true that p is just to assert p itself." And this is just what (T_o) express.

Beside the core idea mentioned above, the deflationary theory of truth has another essential idea:

"On the contrary, however, advocates of the deflationary theory (particularly those influenced by Ramsey) are at pains to point out that anyone who has the concept of truth in this sense is in possession of a very useful concept indeed; in particular, anyone who has this concept is in a position to form generalizations that would otherwise require logical devices of infinite conjunction" (Stoljar and Damnjanovic 2010)

This means, in some situations, such as example (5), i.e. in generalizations, the predicate "true" is indispensable. And this thesis can also be represented on my approach, because the predicate usage of the word "true" which is kept in my proposal expresses exactly this idea. In other words, one of the two usages of the word "true", i.e. the predicate usage, can explain why the concept of truth is a very useful concept.

Hence, the two usages of the word "true", the operator usage and the predicate usage, can explain perfectly the two essential ideas of the deflationary theory of truth.[20] It is in this sense my proposal can be the foundation of the deflationary theory of truth.

Third, it seems that the function of the operator "true" in the T_o-schema can be defined by other operators. For example, in classical logic, it can be defined by double negation. Therefore, one may claim that the truth operator is cancelable.

However, I do not think this is a real objection to my approach. Even if we found a definition of the truth operator in every logic, we could not find the same definition

[20] Indeed, deflationism has different versions and different claims, but these two ideas are basic and essential for them.

in every logic. For instance, we cannot define the truth operator by double negation in intuitionist logic because the double negation of a sentence is not equivalent to the sentence itself in intuitionist logic.

Therefore, the truth operator is still significant since it is a form for the word "true" and has nothing to do with what logic we take as the basis of the theory of truth. We cannot substitute the truth operator by another operator uniquely in every logic. On the contrary, we can construct a classical theory of truth which is based on classical logic, an intuitionist theory of truth which based on intuitionist logic, or whatever other kind theory of truth which based on other logic, but the T_o-schema must be kept in any theory of truth.

Fourth, is the "schema" a logical truth? If we take Tarski's (T) as the schema where "true" is a predicate, the "schema" is an analytic truth or necessary truth but not a logical truth. This is argued by Roy T. Cook.[21] However, if we take (T_o), which is proposed in this paper, as the real schema, then it is not only an analytical truth, but also a logical truth because the word "true" in this schema is an operator. Hence for any sentence p, it is true that p iff p. We can substitute any sentence for p without changing the truth value of the equivalence. And we can do corresponding logical deduction from this T_o-schema. For example, it is easy to get the truth value of the sentence "it is true that Socrates is red" from the truth value of the sentence "Socrates is red" and vice versa.

5 Conclusion

The present approach is consistent with Tarski's initial idea, i.e. classical logic is the last part that can be revised. For Tarski, it is important for a theory of truth to preserve classical logic, so the way to solve the liar paradox is to reject semantically closed languages. Hence, language is divided into a sequence of languages, and each language of the sequence contains a (T_n) which cannot be applied to sentences in which the maximum subscript of the terms constituting the sentence is greater or equal to n.

In the present proposal, it is not hard to show that we can preserve classical logic if we want to.[22] Hence my proposal keeps what plays an important role in Tarski's mind. So, the basis of the present proposal is also Tarski's basis.

However, it is obvious that my proposal is different from Tarski's because we have different understanding about the schema of the word "true". While Tarski's schema is a schema for the truth predicate, my schema is for the truth operator.

On the other hand, in the present proposal, the truth predicate is also kept. We can construct sentences with the truth predicate. Therefore, the expressive power of my proposal is strong enough. And if a sentence with the truth predicate can be translated *equivalently* into a sentence with the corresponding truth operator, Tarski's schema still holds.

In short, my proposal is called an update of Tarski because:

(1) The deductive power of classical logic can be kept.

[21] For details cf. Cook (2012).

[22] This will be shown in the "Appendix".

(2) The language is semantically closed. The truth operator can be used to characterize the semantics of the language itself by T_o-schema. Hence we do not need the hierarchy of languages.
(3) The truth predicate is also preserved. So the expressivity is strong enough to characterize the predicate usage of the word "true" in natural language.
(4) And there is no liar-like paradox that can be derived from the T_o-schema.[23]

Next, I will sketch a model in a formalized language as a precise characterization, and also a proof of the consistency, of these ideas.

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Appendix: A Model of the Two Usages of the Word "True"

How to define exactly the extensions of the two usages of the word "true"? It is better to talk about them in a formalized language. In this appendix, I use the methods of the revision theory to construct a model to characterize precisely the main ideas of this paper.[24]

In order to express sentences in natural language, let the base language $\mathcal{L}$ be Peano Arithmetic (PA) plus some other terms and predicates which are different from the word "true". Let $\mathcal{L}_\mathcal{T}$ be the language obtained by adding a one place predicate T_p and a unary operator T_o to $\mathcal{L}$, in which T_p is the truth predicate and T_o is the truth operator.

The Syntax of $\mathcal{L}_\mathcal{T}$

The construction of the well-formed formulas of $\mathcal{L}$ is as usual. The construction of the well-formed formulas of the language $\mathcal{L}_\mathcal{T}$ is based on the construction of $\mathcal{L}$:

- If t is a term of $\mathcal{L}$, so it is also a term of $\mathcal{L}_\mathcal{T}$, then $T_p(t)$ is a well-formed formula
- If ϕ is a formula of $\mathcal{L}_\mathcal{T}$, then $T_o(\phi)$ is a formula of $\mathcal{L}_\mathcal{T}$.

Other well-formed formulas of $\mathcal{L}_\mathcal{T}$ are defined as usual, i.e.:

- If $t_1, t_2, t_3 \ldots t_n$ are terms of $\mathcal{L}_\mathcal{T}$ and P_n is a n-ary predicate of $\mathcal{L}_\mathcal{T}$, which is different from T_p, then $P_n(t_1, t_2, t_3 \ldots t_n)$ is a well-formed formula of $\mathcal{L}_\mathcal{T}$.

[23] I am not talking about "update" from the proof-theoretic perspective and therefore not comparing the system with other systems such as TB (For details about TB see Halbach (2011). *Axiomatic Theory of Truth*, Cambridge: Cambridge University Press.) . But it is clear that my system is not a typed theory because I separate the functions of the word "true". Hence self-reference is allowed and can be applied to the truth predicate. Therefore, it is not hard to construct an axiomatic system which contains all axioms of TB, all sentences with the form $\mathrm{T}_o\phi \leftrightarrow \phi$ and some other sentences, such as the truth teller $T_p(\ulcorner\psi\urcorner) \leftrightarrow \psi$ which will be shown in the "Appendix" that it is valid in my system. Obviously, this system is more flexible than TB.

[24] It is worth noting that the model given in this appendix is not the only model for the word "true". We could deal with the two usages of the word "true" in other ways.

- If ϕ is a formula of $\mathcal{L}_{\mathcal{T}}$, then $\neg\phi$, $\forall x\phi$ and $\exists x\phi$, are well-formed formulas of $\mathcal{L}_{\mathcal{T}}$.
- If ϕ and ψ are formulas of $\mathcal{L}_{\mathcal{T}}$, then $\phi\vee\psi, \phi\wedge\psi, \phi \rightarrow \psi, \phi \leftrightarrow \psi$ are well-formed formulas of $\mathcal{L}_{\mathcal{T}}$.

The Semantics of $\mathcal{L}_{\mathcal{T}}$

$\mathcal{M}_0 = \langle \Delta_0, \sigma_0 \rangle$ is the initial model of $\mathcal{L}_{\mathcal{T}}$. Δ_0 is the domain, and σ_0 is the assignment function which contains the standard interpretation of PA.

- For a term t: $\sigma_0(t) \in \Delta_0$
- For an n-ary predicate P_n: $\sigma_0(P_n) \subseteq \Delta_0^n$
- The evaluations of the operators and formulas of $\mathcal{L}_T$ are as usual.

$\mathcal{M}_{\alpha+1} = \langle \Delta_{\alpha+1}, \sigma_{\alpha+1} \rangle$, where $\Delta_{\alpha+1} = \Delta_0$, is the expansion of $\mathcal{M}_\alpha$ by adding the interpretations of T_p and T_o as follows:

- $\sigma_{\alpha+1}(T_p)$=$\{\ulcorner\phi\urcorner : \sigma_\alpha \models \phi\}$, where ϕ is a well-formed sentence of $\mathcal{L}_T$ and $\ulcorner\phi\urcorner$ is the Gödel number of ϕ.
- $\sigma_{\alpha+1}(T_o\phi)$=$\sigma_{\alpha+1}(\phi)$, where ϕ is a sentence of $\mathcal{L}_T$.

For limit ordinal λ, $\mathcal{M}_\lambda = \langle \Delta_\lambda, \sigma_\lambda \rangle$, where $\Delta_\lambda = \Delta_0$, is defined as follows:

- For a term t: $\sigma_\lambda(t) = \sigma_0(t) \in \Delta_0$
- For an n-ary predicate P_n, which is different from T_p, $\sigma_\lambda(P_n) = \sigma_0(P_n) \subseteq \Delta_0^n$
- $\sigma_\lambda(T_p) = \{\ulcorner\phi\urcorner :$ there exists an $\alpha < \lambda$ such that $\mathcal{M}_\alpha \vDash \phi$ and, for any $\alpha < \beta < \lambda, \mathcal{M}_\beta \vDash \phi\}$
- $\sigma_\lambda(T_o\phi)$=$\sigma_\lambda(\phi)$,
- The evaluations of other operators and formulas are as usual.

$\mathcal{M} = \langle \Delta, \sigma \rangle$ is the final model of $\mathcal{L}_T$ where Δ=Δ_0. It is defined as follows:

- For a term t: $\sigma(t) = \sigma_0(t) \in \Delta_0$
- For an n-ary predicate P_n, which is different from T_p, $\sigma(P_n) = \sigma_0(P_n) \subseteq \Delta_0^n$
- $\sigma(T_p) = \{\ulcorner\phi\urcorner :$ there exists a limit ordinal λ such that $\mathcal{M}_\lambda \vDash \phi$ and, for any limit ordinal $\nu > \lambda, \mathcal{M}_\nu \vDash \phi\}$
- $\sigma(T_o\phi)$=$\sigma(\phi)$
- The evaluations of other operators and formulas are as usual.

Case 1. First, let us talk about the liar sentence $\neg T_p(\ulcorner\phi\urcorner)$, where ϕ is the abbreviation of the sentence $\neg T_p(\ulcorner\phi\urcorner)$ itself and $\ulcorner\phi\urcorner$ is the name of ϕ.

Suppose we talk about this example in the final model. Because there is no T_p in $\mathcal{L}$, ϕ is not a sentence of $\mathcal{L}$. Hence it is not true that $\sigma_0 \models \phi$. Then it is not true that $\ulcorner\phi\urcorner \in \sigma_1(T_p)$, so $\sigma_1(T_p(\ulcorner\phi\urcorner)) = 0$. Hence $\sigma_1(\neg T_p(\ulcorner\phi\urcorner)) = 1$. Therefore $\ulcorner\phi\urcorner \in \sigma_2(T_p)$ because $\phi = \neg T_p(\ulcorner\phi\urcorner)$. Then $\sigma_2(T_p(\ulcorner\phi\urcorner)) = 1$. Hence $\sigma_2(\neg T_p(\ulcorner\phi\urcorner)) = 0$.

With the iteration of the models, we can conclude that $\neg T_p(\ulcorner\phi\urcorner)$ is true in the models σ_{2n+1} and false in the models σ_{2n+2}, where $n \in \omega$. So there is no n, such that for any $k > n$, $\sigma_k(\neg T_p(\ulcorner\phi\urcorner))$=1. Hence, $\ulcorner\phi\urcorner \notin \sigma_\omega(T_p)$. And then $\sigma_\omega(T_p(\ulcorner\phi\urcorner)) = 0$, therefore $\sigma_\omega(\neg T_p(\ulcorner\phi\urcorner)) = 1$.

The iteration of the values of the liar sentence can be seen in the following Fig. 1. It is not hard to see that the liar sentence $\neg T_p(\ulcorner\phi\urcorner)$ is true in every limit model. So

$$\neg T_p(\ulcorner\phi\urcorner) = \begin{cases} 1: & \sigma_1 \quad \sigma_3 \quad \dots \quad \sigma_{2n+1} \quad \dots \quad \sigma_\omega \quad \dots \quad \sigma_\lambda \quad \dots \\ & \downarrow \nearrow \downarrow \nearrow \quad \downarrow \quad \nearrow \quad \downarrow \nearrow \quad \downarrow \nearrow \\ 0: & \sigma_2 \quad \sigma_4 \quad \dots \quad \sigma_{2n+2} \quad \dots \quad \sigma_{\omega+1} \quad \dots \quad \sigma_{\lambda+1} \quad \dots \end{cases}$$

Fig. 1 The revision sequence of the liar sentence

$$T_p(\ulcorner\psi\urcorner) = \begin{cases} 1: & \\ & \\ 0: & \sigma_1 \to \dots \to \sigma_n \to \dots \to \sigma_\omega \to \dots \to \sigma_\lambda \to \dots \end{cases}$$

Fig. 2 The revision sequence of the truth teller sentence

in the final model $\mathcal{M} = \langle \Delta, \sigma \rangle$, $\ulcorner\phi\urcorner \in \sigma(T_p)$. Then $\sigma(T_p(\ulcorner\phi\urcorner)) = 1$, therefore $\sigma(\neg T_p(\ulcorner\phi\urcorner)) = 0$.

Case 2. For the truth teller sentence $T_p(\ulcorner\psi\urcorner)$, where ψ is the abbreviation of the sentence $T_p(\ulcorner\psi\urcorner)$ itself and $\ulcorner\psi\urcorner$ is the name of ψ. Since ψ is not a sentence of $\mathcal{L}$, it is not true that $\sigma_0 \models \psi$. Hence $\ulcorner\psi\urcorner \notin \sigma_1(T_p)$, so $\sigma_1(T_p(\ulcorner\psi\urcorner)) = 0$. Then $\ulcorner\psi\urcorner \notin \sigma_2(T_p)$. So $\sigma_2(T_p(\ulcorner\psi\urcorner)) = 0$. It is obvious that with the iteration of the models, the truth teller sentence $T_p(\ulcorner\psi\urcorner)$ is always false.

The evaluations of the truth teller sentence can be seen in the following Fig. 2. So in the final model $\mathcal{M} = \langle \Delta, \sigma \rangle$, $\ulcorner\psi\urcorner \notin \sigma(T_p)$. Then $\sigma(T_p(\ulcorner\psi\urcorner)) = 0$. Therefore, $T_p(\ulcorner\psi\urcorner) \leftrightarrow \psi$.

Definition 1 A sentence ϕ of $\mathcal{L_T}$ is $\mathcal{L_T}$-valid iff for any final model τ of $\mathcal{L_T}$, $\tau \models_{\mathcal{L_T}} \phi$.[25]

Then we can prove that the deductive power of classical logic is kept in $\mathcal{L_T}$:

Theorem 1 *For any sentence ϕ of classical logic, if ϕ is valid in classical logic, ϕ is $\mathcal{L_T}$-valid.*

Proof Suppose there is a sentence ψ that is valid in classical logic but not $\mathcal{L_T}$-valid. Then there is a $\mathcal{L_T}$ model τ such that $\tau(\psi) = 0$. Since, according to the construction of the models for $\mathcal{L_T}$, the evaluations of the normal operators in any $\mathcal{L_T}$ model are as usual, we can construct a model σ of classical logic which has the same evaluations as that of τ for the operators appear in ψ. Then ψ is false in σ, i.e. $\sigma(\psi) = 0$. Contradiction. □

Then we can prove that Tarski's T-schema for the truth predicate is not a $\mathcal{L_T}$-valid schema:

Theorem 2 *$T_p(\ulcorner\phi\urcorner) \leftrightarrow \phi$ is not $\mathcal{L_T}$-valid.*

[25] Hence the truth teller $T_p(\ulcorner\psi\urcorner) \leftrightarrow \psi$ is $\mathcal{L_T}$-valid.

Proof As we can see in the above Case 1, there exists at least one sentence ϕ, e.g. the liar sentence $\neg T_p(\ulcorner \phi \urcorner)$, such that in the final model $\mathcal{M} = \langle \Delta, \sigma \rangle$, $\sigma(\phi){=}\sigma(\neg T_p(\ulcorner \phi \urcorner)) = 0$, but $\sigma(T_p(\ulcorner \phi \urcorner)) = 1$. Hence, $T_p(\ulcorner \phi \urcorner) \leftrightarrow \phi$ is not $\mathcal{L_T}$-valid. □

But we can prove that the schema for the truth operator, i.e. T_o-schema, is $\mathcal{L_T}$-valid:

Theorem 3 *$T_o(\phi) \leftrightarrow \phi$ is $\mathcal{L_T}$-valid, i.e. for any final model τ, $\tau \models_{\mathcal{L_T}} T_o(\phi) \leftrightarrow \phi$.*

Proof Directly from the definition of T_o. □

Corollary 1 *For any sentence ϕ, if we can equivalently translate $T_p(\ulcorner \phi \urcorner)$ to a well-formed sentence $T_o(\phi)$, then $T_p(\ulcorner \phi \urcorner) \leftrightarrow \phi$ is $\mathcal{L_T}$-valid and vice versa i.e. the following T_{op}-schema is $\mathcal{L_T}$-valid:*

$$(T_{op}) \quad (T_p(\ulcorner \phi \urcorner) \leftrightarrow T_o(\phi)) \leftrightarrow (T_p(\ulcorner \phi \urcorner) \leftrightarrow \phi)$$

Proof For any final model τ and for any sentence ϕ, suppose $T_p(\ulcorner \phi \urcorner) \leftrightarrow T_o(\phi)$ is true in τ. According to Theorem 3, $T_o(\phi) \leftrightarrow \phi$ is true in τ. Hence, $\tau(T_p(\ulcorner \phi \urcorner)){=}\tau(T_o(\phi))$ and $\tau(T_o(\phi)){=}\tau(\phi)$. Then $\tau(T_p(\ulcorner \phi \urcorner)){=}\tau(\phi)$. Therefore, $T_p(\ulcorner \phi \urcorner) \leftrightarrow \phi$ is true in τ. The other direction is similar. □

From Theorems 2 and 3, we can conclude that not every instance of an operator usage of the word "true" is *equivalent* to the corresponding instance of the predicate usage. For example, suppose (L) is the liar sentence, i.e. "$(\ulcorner L \urcorner)$ is not true", then the value of the sentence "it is true that $(\ulcorner L \urcorner)$ is not true"[26] is *equivalent* to the value of the sentence (L), namely it is false. But the value of the sentence "'$(\ulcorner L \urcorner)$ is not true' is true" is true.[27] Therefore, not every translation from an operator usage into a predicate usage of the word "true" is *equivalent*, although every operator usage of "true" can be translated into a predicate usage *uniformly*. This is the reason why the operator usage of "true" is not *equivalent* to (part of) the predicate usage of "true". On the other hand, not every truth predicate can be translated *uniformly* into an operator usage of the word "true", let alone an *equivalent* translation. Hence the truth operator and the truth predicate are two different usages of the word "true". They have different functions, although they have some connections. In short, the truth operator is mainly used for characterizing the semantics of the language and the truth predicate is mainly used for generalizations. We cannot reduce one usage to the other.

Definition 2 For any sentence ϕ and ψ of $\mathcal{L_T}$, ϕ and ψ are $\mathcal{L_T}$-equivalent iff for any final model τ of $\mathcal{L_T}$, $\tau \models_{\mathcal{L_T}} \phi \leftrightarrow \psi$.

Corollary 2 *For any sentence ϕ, the T_o-schema and the T_{op}-schema are $\mathcal{L_T}$-equivalent, i.e. for any final model τ of $\mathcal{L_T}$:*

$$\tau \models_{\mathcal{L_T}} (T_o(\phi) \leftrightarrow \phi) \leftrightarrow ((T_p(\ulcorner \phi \urcorner) \leftrightarrow T_o(\phi)) \leftrightarrow (T_p(\ulcorner \phi \urcorner) \leftrightarrow \phi))$$

Proof Straightforward. □

[26] Here, "it is true that" is an operator.

[27] Here, "is true" is a predicate.

Definition 3 A unary logical operator $\odot$ weakly characterize a semantic status ξ iff for any sentence ϕ:

- $\bigodot(\phi)$ receives 1 iff ϕ receives ξ

Definition 4 A unary logical operator $\odot$ strongly characterize a semantic status ξ iff for any sentence ϕ:

- $\bigodot(\phi)$ receives 1 if ϕ receives ξ, and
- $\bigodot(\phi)$ receives 0 otherwise.

It is not hard to see that the weak characterization is equivalent to the strong characterization in bivalent logic. Hence, for $\mathcal{L}_T$, the weak characterization and the strong characterization are equivalent. Then we can prove that:

Theorem 4 *The operator T_o weakly (and strongly) characterize the semantic statuses of the language $\mathcal{L}_T$.*

Proof For any sentence ϕ of $\mathcal{L}_T$, for any final model τ of $\mathcal{L}_T$, according to the definition of the operator T_o, it is obvious that:

$$\tau(T_o(\phi)) = 1 \text{ iff } \tau(\phi) = 1$$

Case 3. It is very interesting to talk about the truth value of the sentences in example (2). Here we formalize example (2) as follows:

$$\text{(2F)}\quad \begin{aligned} \phi &: T_p(\ulcorner\psi\urcorner) \\ \psi &: \neg T_p(\ulcorner\phi\urcorner) \end{aligned}$$

Consider this example in the final model. Since there is no T_p in $\mathcal{L}$, ϕ and ψ are not sentences of $\mathcal{L}$. Then it is not true that $\sigma_0 \models \phi$ and, it is not true that $\sigma_0 \models \psi$. Hence it is not true that $\ulcorner\phi\urcorner \in \sigma_1(T_p)$, and it is not true that $\ulcorner\psi\urcorner \in \sigma_1(T_p)$. Then $\sigma_1(T_p(\ulcorner\phi\urcorner)) = 0$ and $\sigma_1(T_p(\ulcorner\psi\urcorner)) = 0$. Therefore $\sigma_1(\neg T_p(\ulcorner\phi\urcorner)) = 1$, i.e. $\sigma_1(\phi) = 0$ and $\sigma_1(\psi) = 1$. Then it is not true that $\ulcorner\phi\urcorner \in \sigma_2(T_p)$ while it is true that $\ulcorner\psi\urcorner \in \sigma_2(T_p)$. Then $\sigma_2(T_p(\ulcorner\psi\urcorner)) = 1$ and $\sigma_2(T_p(\ulcorner\phi\urcorner)) = 0$. Hence, $\sigma_2(\neg T_p(\ulcorner\phi\urcorner)) = 1$, i.e. $\sigma_2(\phi) = 1$ and $\sigma_2(\psi) = 1$. Then in σ_3, $\sigma_3(\phi) = 1$ and $\sigma_3(\psi) = 0$. And in σ_4, $\sigma_4(\phi) = 0$ and $\sigma_4(\psi) = 0$. In σ_5, $\sigma_5(\phi) = 0$ and $\sigma_5(\psi) = 1$ and so on. The evaluations of the two sentences in example (2) are as follows:

$$\phi = T_p(\ulcorner \psi \urcorner) = \left\{ \begin{array}{lccccccccc} 1: & \sigma_2 & \rightarrow & \sigma_3 & & \sigma_6 & \rightarrow & \sigma_7 & \ldots & \\ & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \\ 0: & \sigma_1 & & \sigma_4 & \rightarrow & \sigma_5 & & \sigma_8 & \ldots & \end{array} \right.$$

$$\psi = \neg T_p(\ulcorner \phi \urcorner) = \left\{ \begin{array}{lccccccccc} 1: & \sigma_1 & \rightarrow & \sigma_2 & & \sigma_5 & \rightarrow & \sigma_6 & \ldots & \\ & & \downarrow & & \uparrow & & \downarrow & & \uparrow & \\ 0: & & \sigma_3 & \rightarrow & \sigma_4 & & \sigma_7 & \rightarrow & \sigma_8 & \ldots \end{array} \right.$$

Fig. 3 The revision sequences of the sentences in (2F) in finite stages

$$\phi = T_p(\ulcorner \psi \urcorner) = \left\{ \begin{array}{lccccccccc} 1: & \sigma_{\omega+1} & \rightarrow & \sigma_{\omega+2} & & \sigma_{\omega+5} & \rightarrow & \sigma_{\omega+6} & \ldots & \\ & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \\ 0: & \sigma_{\omega} & & \sigma_{\omega+3} & \rightarrow & \sigma_{\omega+4} & & \sigma_{\omega+7} & \ldots & \end{array} \right.$$

$$\psi = \neg T_p(\ulcorner \phi \urcorner) = \left\{ \begin{array}{lccccccccc} 1: & \sigma_{\omega} & \rightarrow & \sigma_{\omega+1} & & \sigma_{\omega+4} & \rightarrow & \sigma_{\omega+5} & & \ldots \\ & & \downarrow & & \uparrow & & \downarrow & & \uparrow & \\ 0: & & \sigma_{\omega+2} & \rightarrow & \sigma_{\omega+3} & & \sigma_{\omega+6} & \rightarrow & \sigma_{\omega+7} & \ldots \end{array} \right.$$

Fig. 4 The revision sequences of the sentences in (2F) in infinite stages

As we can see in Fig. 3, there does not exist an n such that for any $k > n, \sigma_k(\phi) = 1$, and there does not exist a m such that for any $s > m$, $\sigma_s(\psi) = 1$. Then $\ulcorner \phi \urcorner \notin \sigma_\omega(T_p)$ and $\ulcorner \psi \urcorner \notin \sigma_\omega(T_p)$. Hence $\sigma_\omega(\phi) = \sigma_\omega(T_p(\ulcorner \psi \urcorner)) = 0$, while $\sigma_\omega(\psi) = \sigma_\omega(\neg T_p(\ulcorner \phi \urcorner)) = 1$. The evaluations of the two sentences will go on as we can see in the following Fig. 4. And it is not hard to find out that the sentence ϕ is false in every limit model and the sentence ψ is true in every limit model. Hence, in the final model, $\sigma(\phi) = \sigma(T_p(\ulcorner \psi \urcorner)) = 1$ and $\sigma(\psi) = \sigma(\neg T_p(\ulcorner \phi \urcorner)) = 1$.

Case 4. Yablo's paradox can also be handled in this proposal. Consider Yablo's paradox in the final model. Because there is no sentence containing the predicate T_p or the operator T_o in $\mathcal{L}$, for any sentence S_n in the sequence of Yablo's paradox, which can be formalized as $\forall x((x > n) \rightarrow \neg T_p(\ulcorner S_x \urcorner))$, $n \in \omega$, it is not true in the model σ_0. Hence for any $n \in \omega$, it is not true that $\ulcorner S_n \urcorner \in \sigma_0(T_p)$. Therefore, for any $n \in \omega$, $\sigma_1(\neg T_p(\ulcorner S_n \urcorner)) = 1$. Hence for any $n \in \omega$, $\sigma_1(S_n) = 1$. So for any $n \in \omega$, $\ulcorner S_n \urcorner \in \sigma_2(T_p)$. Then for any $n \in \omega$, $\sigma_2(T_p(\ulcorner S_n \urcorner)) = 1$. Therefore, for any $n \in \omega$, $\sigma_2(\neg T_p(\ulcorner S_n \urcorner)) = 0$. Then it is not true that for any $x > n$, $\sigma_2(\neg T_p(\ulcorner S_x \urcorner)) = 1$. Hence, for any $n \in \omega$, $\sigma_2(S_n) = 0$. Just like the liar sentence, the evaluations of the sentences in Yablo's paradox alternate with the iteration of models. And hence for any $n \in \omega$, $\ulcorner S_n \urcorner \notin \sigma_\omega(T_p)$. So for any $n \in \omega$, $\sigma_\omega(\neg T_p(\ulcorner S_n \urcorner)) = 1$ i.e. for any

$$S_0 = \forall x((x > 0) \to \neg T_p(\ulcorner S_x \urcorner)) = \left\{ \begin{array}{llllllllll} 1: & \sigma_1 & & \sigma_3 & \ldots & \sigma_\omega & & \ldots & \sigma_\lambda & \ldots \\ & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & & \downarrow & \nearrow \\ 0: & \sigma_2 & & \sigma_4 & \ldots & \sigma_{\omega+1} & & \ldots & \sigma_{\lambda+1} & \ldots \end{array} \right.$$

$$S_1 = \forall x((x > 1) \to \neg T_p(\ulcorner S_x \urcorner)) = \left\{ \begin{array}{llllllllll} 1: & \sigma_1 & & \sigma_3 & \ldots & \sigma_\omega & & \ldots & \sigma_\lambda & \ldots \\ & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & & \downarrow & \nearrow \\ 0: & \sigma_2 & & \sigma_4 & \ldots & \sigma_{\omega+1} & & \ldots & \sigma_{\lambda+1} & \ldots \end{array} \right.$$

$$\vdots$$

Fig. 5 The revision sequences of the Yablo sentences

$n \in \omega$, $\sigma_\omega(S_n) = 1$. It is easy to see that for any $n \in \omega$, for any limit ordinal λ, $\sigma_\lambda(S_n) = 1$. The iterations of the sentences in Yablo's paradox can be seen in Fig. 5. Therefore, in the final model, for any $n \in \omega$, $\ulcorner S_n \urcorner \in \sigma(T_p)$. And so, for any $n \in \omega$, $\sigma(T_p(\ulcorner S_n \urcorner)) = 1$. Hence, for any $n \in \omega$, $\sigma(\neg T_p(\ulcorner S_n \urcorner)) = 0$ i.e. for any $n \in \omega$, $\sigma(S_n) = 0$.

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