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Towards an Algebraic Semantics for Implicatives

R. Zuber[1]

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Abstract
An algebraic semantics, based on factor algebras, for one-way and two-way implicative verbs is proposed. Implicative verbs denote elements of filters or of ideals generated by identity functions in factor algebras. This semantics explains in particular the problem of implicational equivalence raised by two-way implicative verbs, and shows that the negation necessary to establish the implicativity of these verbs is the negation which preserves the presuppositions of sentences with implicative verbs. In addition, it follows from the proposed semantics that any two implicative verbs denoting in the same algebra but belonging to different categories, are semantically related.

Keywords Implicative verb · Factor algebra · Filters and ideals · Implicativity

1 Introduction

Implicatives, as understood here, is the class of implicative verbs studied by Karttunen (cf. Karttunen 1971, 1973, 2012) and giving rise to specific inference patterns. Syntactically implicative verbs are functional expressions which take infinitivals or verb phrases as arguments and give verb phrases as result. For simplicity, which in fact does not alter my proposal, I will consider that implicatives are VP modifiers, that is they apply to VP and give VPs as a result. Semantically, roughly speaking, verb phrases formed by implicative verbs entail the finite positive form of their argument or its negative form (i.e. the negation of the argument). Karttunen distinguishes various classes of implicatives according to whether only their positive form, only their negative form or both positive and negative forms give rise to such entailments. One way implicatives are the implicatives which give rise to entailments only when they are taken in positive (affirmative) context or only when they are taken in negative

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✉ R. Zuber
Richard.Zuber@linguist.univ-paris-Diderot.fr

1 CNRS, Paris, France

contexts (when they are negated) and two-way implicatives gives rise to such entailments in positive (affirmative) as well as in negative contexts, that is when they are negated and not negated. In addition there may be agreement or disagreement between the polarity of the implicative and its argument. Thus positive two-way implicatives entail in the positive context the positive form of their argument and in the negative context they entail the negative form of their argument. Similarly, negative two-way implicatives entail in the positive context the negation of their argument and in the negative context they entail positive form of their argument. Positive one-way implicatives entail positive form of their argument only in the positive or only in the negative context.

Let me recall some examples. In these examples the implicativity will be illustrated only by implicative verbs which may be called *simple implicatives*. Karttunen (2012) indicates that there is (at least in English) a large class of multiword constructions which have similar structures and which are semantically similar to simple implicative verbs.

The verbs *to manage, to bother* and *to happen (to)* are positive two-way implicatives meaning that when such verbs are used in a sentence in affirmative form as in (1a), that is when the sentence is not negated, this sentence entails the corresponding sentence in which the positive form of the argument of the verb occurs, as in (1b) and the negative form of the sentence, as in (2a), entails the negative form of the argument of the verb as in (2b):

(1a) Leo managed to prove the theorem.
(1b) Leo proved the theorem.
(2a) Leo did not manage to prove the theorem.
(2b) Leo did not prove the theorem.

Similarly, the verb *to fail* or *to neglect (to)* is a negative two-way implicative: (3a), which is in affirmative form, entails the sentence with the negated from of the argument of *to fail*, and (4a), which is the negation of (3a), entails the affirmative sentence in (4b), in which the argument of *to fail* occurs:

(3a) Leo failed to prove the theorem.
(3b) Leo did not prove the theorem.
(4a) Leo did not fail to prove the theorem.
(4b) Leo proved the theorem.

The verb *to remember (to)* is another positive two-way positive implicative and the verb *to forget (to)* is another negative two-way implicative. Verbs such as *be unable to, be able to, be forced (to)* or *hesitate* are one-way implicatives. Thus (5a) entails (5b) but the negation of (5a) does not entail the (logical) negation of (5b). Similarly (6a) entails (6b) but the negation of (6a) does not entail the negation of (6b) and (7a) entails (7b) but its negation of (7a) does not entail (7b):

(5a) Dan was not able to come.
(5b) Dan did not come.
(6a) Dan did not hesitate to smile.
(6b) Dan smiled.

(7a) Dan was forced to smile.
(7b) Dan smiled.

As these examples indicate there are also different sub-classes of one-way implicatives. The *if*-implicatives are one-way implicatives which in their positive form entail the positive form of their argument, as in (7a). The class of *only-if*-implicatives is illustrated by the example in (5a): V is *only if*-implicative iff $not - V(A)$ entails $not - A$. In (6a) we have an example of a negative *only if*-implicative verb, that is a verb V such that $not - V(A)$ entails A. Karttunen (1973) indicates that the verb *to hesitate* is probably the only verb representing the class of negative *only if*-implicatives. Observe that *to hesitate* cannot be considered as a two-way implicative because (8) is not contradictory:

(8) Bo wrote a letter that she had hesitated to write.

Finally, Karttunen distinguishes the class of negative *if*-implicatives such as *prevent, discourage* or *dissuade*. One can notice that the syntax of these verbs (in English but not necessarily in other languages) is different from the syntax of other implicatives; informally they take the gerundive as argument:

(9) Leo prevented Lea from smiling.

Notice that strictly speaking (positive) two-way implicatives are *if*-implicatives and *only-if*-implicatives at the same time. Similarly, negative two-way implicatives are in particular negative *if*-implicatives and negative *only if*-implicatives. Thus we will consider that there are four types of implicative verbs: positive *if* and *only if* implicatives and negative *if* and *only if* implicatives and that two-way implicatives represent a particular case of one-way implicatives.

Implicative verbs should be distinguished from factive verbs that is verbs presupposing their (sentential) argument (Kiparsky and Kiparsky 1970). The difference between these two classes of verbs is not only syntactic (in general factive verbs take sentences as arguments) but essentially semantic since, roughly speaking, the negation giving rise to entailments in the case of factives does not apply to their arguments but to the verbs themselves. In addition it seems that presuppositions of factive verbs are essentially related to the intensionality of these verbs (cf. Zuber 2011) which is not the case with implicative verbs.

The proposal which I am going to make in this paper uses in an important way the notion of presupposition of specific verb phrases and in particular of "implicative" verb phrases. The notion of presupposition of VPs will be formally represented and, as we will see, it will play essentially a theoretical role since it is based on the logical mechanism devised to explain the inferential capacities of implicative verbs. However, many empirical details related to presuppositions of particular implicative verbs will not be discussed.

2 Implicational Equivalence

The existence of two-way implicatives poses various logical problems concerning in particular the logical status of the entailment and of logical equivalence to which they give rise. I describe briefly these problems in order to better understand the semantics of implicatives that will be proposed.

First, concerning (logical) entailment recall that for classical entailment the principle of contraposition holds: if sentence S_1 entails sentence S_2 then sentence *not*-S_2 entails sentence *not*-S_1. One can check that if the negation *not* S_1 is interpreted in the same way as the negation used in negated implicative sentences, the contraposition law may not hold. For instance the negation of (6b), that is the fact that Dan did not smile, does not entail the negation of (6a) if we suppose that the negation of (6a) is equivalent to *Dan hesitated to smile*.

The second, related, problem to which the existence of two-way implicatives leads is the problem of what may be called *implicational equivalence*: if a sentence S_1 with a two-way implicative entails a sentence S_2 and the negation of S_1 entails the negation of S_2 then, *a priori* S_1 and S_2 should be logically equivalent. Similarly, if the sentence S_3 entails the negation of the sentence S_4 and the negation of the sentence S_3 entails S_4 then, *a priori*, S_3 is equivalent to the negation of S_4. Thus (1a) should be considered as equivalent to (1b) and (2a) equivalent to (2b). This consequence seems to be illogical since clearly (1a) and (1b) do not have exactly the same truth-conditions. However, this consequence is not quite counter-intuitive and deserves to be discussed if we consider the possibility of equivalence *modulo presuppositions*, that is when the scope of the negation is restricted by presuppositions and the equivalence is taken as holding in situations restricted precisely by the truth of presuppositions.

It is clear what is at stake in the above problems concerning the contraposition and the implicational equivalence: this is the problem of the interpretation of negation: one can check that in (1a) and (2a) for instance, the negation is not interpreted as the complement in the sentential Boolean algebra and, moreover, the negation leading to the implicational equivalence of (2b) is the negation which preserves (does not cancel) presuppositions. In order to understand better this fact I propose a tentative definition of implicational equivalence and indicate three other constructions which also give rise to the implicational equivalence.

An informal definition of implicational equivalence is given in D0:

D0: Sentences S_1 and S_2 are implicatively equivalent iff they have the same truth values in all models in which their presuppositions are satisfied and consistent.

This definition should be considered as provisory since it uses the notion of presupposition which is not formally defined in this paper.

The examples of implicatively equivalent sentences that I am going to present contain various rather well-known cases of presupposing constructions. Consider first pairs of common nouns which are in the so-called privative opposition. These are pairs such as *poet-poetess, author-authoress, actor-actress, prince-princess*, etc. One can assume that elements of such pairs differ just by a presupposition that the so-called marked member of the pairs has: *poetess, authoress* and *princess* presuppose (the property of being) *female* whereas the corresponding unmarked term does not carry a similar presupposition concerning the gender of the involved persons (cf. Zuber 1980). Consider now the following examples:

(10a) Robin is a poetess.
(10b) Robin is a poet.
(11a) Robin is not a poetess.
(11b) Robin is not a poet.

If we assume that to be a poet means, roughly, *to write poetry*, then (10a) entails (10b), because poetesses are also supposed to write poetry. If, in addition we consider that the negation in (11a) is the presupposition preserving negation, then (11a) entails (11b). Indeed, in this case (11a) roughly means that Robin is a woman but does not write a poetry. Consequently she is not a poet. We can thus say that (10a) is implicationally equivalent to (10b).

A similar example can be constructed with some kinship terms:

(12a) Robin is a brother of Bo.
(12b) Robin is a sibling of Bo.
(13a) Robin is not a brother of Bo.
(13b) Robin is not a sibling of Bo.

One can consider that (12a) entails (12b). Moreover, given that in (13a), by assumption, we have the presupposition preserving negation and that (13a) presupposes that Robin is a male, the truth of (13a) entails the truth of (13b) and consequently (12a) is implicatively equivalent to (12b) and (13a) is implicatively equivalent to (13b).

The above examples may appear problematic since the phenomenon on which they are based is marginal in English and for many speakers the negation in (11a) and in (13a) does not preserve the presupposition of "marked terms". It seems to me, however, that these examples do illustrate the basic idea behind the notion of implicative equivalence.

The last example of constructions which illustrates the implicative equivalence uses the fact many factive non-emotive verbs can take two complements, *that* and *whether* and constructions with such verbs differing just by the complementizer used, are in specific semantic relations to each other. Moreover constructions in which the compementizer *that* occurs, give rise to presuppositions.

Take for instance *know that* and *know whether*. The semantic relation between these two "verbs" is indicated in (14). Then, given that we accept bivalence and excluded middle, *to know whether* can be defined by *to know that* in the way indicated in (15):

(14) A KNOWS whether P iff if P is true then A KNOWS that P and if P is false then A KNOWS that not P.
(15) A KNOWS whether P iff A KNOWS that P or A KNOWS that not P.

It follows from (14) and (15) and from the fact that *know that* is a factive verb, that *know that* is implicationally equivalent to *know whether*. To see this consider the following example:

(16a) Leo knows that life is sad.
(16b) Leo knows whether life is sad.
(17a) Leo does not know that life is sad.
(17b) Leo does not know whether life is sad.

One observes that (16a) entails (16b) and (17a) entails (17b). Indeed, if (17a) is true then by supposition its complement is also true and thus, given (15) the sentence (17b) cannot be false.

It is important to keep in mind that the entailment from (17a) to (17b) holds only if the negation in (17a) is taken as the presupposition preserving negation. Interestingly this is also the case with the negation in examples given in (11a) and (13a) in which implicative verbs are involved.

Research on the inferential behaviour of implicative verbs shows that there is no consensus about the theoretical status of inferences in play and even the descriptive side of the phenomenon may be controversial (cf. van Leusen 2012; Baglini and Francez 2016). In addition one cannot exclude that there may be cross-linguistic difference due to peculiarities of tense or aspect systems among languages (cf. Homer 2011; Zuber 1979). In this paper I am not interested in the description and empirical aspects of implicative verbs but in the logical aspects of the mechanism underlying inferential patterns proper to implicatives. The basic question addressed here is what denotations of implicative verbs are in general and how the mechanism allowing for double non-trivial entailments found in constructions with two-way implicative verbs is related to them.

3 Factor Algebras

I will assume, following Keenan and Faltz (1985) that expressions of natural languages have as denotations elements of particular denotational algebras specified by the grammatical category of denoting expressions. Such algebras are in general Boolean algebras. They will be noted D_C, with the meaning that elements of D_C are possible denotations of expressions of category C. We will say that the expression e_1 of category C (cross-categorially) *entails* the expression e_2 of category C iff e_1 denotes α_1 (in D_C), e_2 denotes α_2 (in D_C), and $\alpha_1 \leq \alpha_2$ (where $\leq$ is the Boolean partial order proper to D_C). In this way we can speak about the entailment holding between expressions which are not necessarily sentences but which denote in Boolean algebras D_C, where C is any major grammatical category.

In this article we are basically interested in denotations of modifiers of verb phrases (VPs). Modifiers are functional expressions of category C/C, for any grammatical category C and consequently they denote functions from D_C into D_C. For my proposal it is enough to suppose that VPs denote sets, sub-sets of a given universe E of entities and that implicatives are modifiers of VPs, that is that they denote functions from sets to sets. The functions denoted by implicatives will get their name from the the name of corresponding implicative verbs. Thus two-way positive implicative verbs will denote functions that will be called two-way positive implicative functions and two-way negative implicatives will denote two-way negative implicative functions. Similarly, *if*-implicatives will denote functions that will be called *if*-implicative functions, etc.

As we will see, some definitions and formal properties that will be discussed in this section are more general in the sense that they apply to Boolean algebras in general, independently of the fact that expressions of a specific category can take their denotations there. Informally, this means that we will often consider Boolean algebras

B in general and not only denotational algebras D_C, whose elements are possible denotations of expressions of category C.

To provide the semantics of implicatives and explain some of their logical properties I will use the notion of a factor algebra (of an arbitrarily given Boolean algebra), that is a Boolean algebra in which the unit element is (almost arbitrarily) chosen from a given algebra.

The definition of a factor algebra of a given algebra that we will use is as follows:

D1: Let B be a Boolean algebra $b \in B$ and $0_B < b$. Then B_b is the factor algebra of B, generated by b, such that $B_b = \{x : x \leq b\}$, the zero element of B_b equals 0_B, that is the zero element of B, the unit element is b, the meet and join operations are as in B and the complement c of $x \in B_b$ is defined as $c(x) = x' \wedge b$, where x' is the complement of x in B.

According to D1 the Boolean operations of join and of meet in B_b are defined pointwise by the corresponding operations in B. This is not the case with the operation of complementation which is relativised to some given element b of B, such that $0_B < b$. One can check that definition D1 defines indeed a Boolean algebra. This means in particular that the "new" complement "c" satisfies the complementation ("negation") axioms. For instance it is easy to see that in the algebra B_b we have $c(x) \wedge x = O_B$ and $c(x) \vee x = b$.

Keenan and Faltz (1985) give various arguments purporting to show that (extensional) adjectives denote in the specific factor algebra of sets (denotations of CNs), that they call algebras of restricting functions, that is functions f from the algebra B into itself such that for any $x \in B$, $f(x) \leq x$. This means that some factor algebras have already an application in the semantics of natural languages.

Restrictive algebras can be defined for any Boolean algebra in the following way:

D2: Let B be a Boolean algebra, B/B the Boolean algebra of functions from B into B (with Boolean operations defined pointwise by operations in B), and $RESTR$ be the set of restricting functions, elements of B/B. Then R_B, or *the restricting algebra* on B, is the set $R_B = \{f : f \in B/B \wedge f \in RESTR\}$ regarded as a Boolean algebra where the zero element is the function $\mathbf{0}$ such that for all $x \in B$, $\mathbf{0}(x) = 0_B$, the unit element is the function $\mathbf{1}$ such that $\mathbf{1}(x) = x$, $(f \wedge g)(x) = f(x) \wedge g(x)$, $(f \vee g)(x) = f(x) \vee g(x)$ and $c(f)(x) = x \wedge (f(x))'$.

Given definitions D1, D2 and the definition of the identity function id we have obviously the following:

Fact 1: R_B is the factor algebra of B/B generated by the identity function id (where id is the element of B/B such that for any $x \in B$, $id(x) = x$).

Thus the algebra R_B is the algebra of restricting functions, that is, functions f such that $f(x) \leq x$ for any $x \in B$. If B is the algebra of sets, subsets of a given universe, and if we suppose that common nouns denote sets, then clearly adjectives, which modify common nouns, can be interpreted by restricting functions in this algebra since, informally, the set denoted by *ADJ CN* is always included in the set denoted by CN, e.g *a tall student* is *a student*, etc.

In order to account for the semantic difference between adjectives like *tall* on the one hand and adjectives like *japanese* on the other hand, one can additionally distinguish a sub-class of restricting functions called *intersective* functions (cf. Keenan and Faltz 1985). Intersective functions over the algebra B are functions $f \in B/B$ such that $f(x) = x \wedge f(1_B)$, for any $x \in B$. It follows in particular from this description that intersective functions are monotone (increasing) functions, that is functions f such that if $x \leq y$ then $f(x) \leq f(y)$, for any $x, y \in B$. Restricting (non-intersective) functions are not monotone in general. For instance the restricting function denoted by the adjective *tall* cannot be monotone increasing because *jockeys are human (beings)* but *tall jockeys* may fail to be *tall human (beings)*. This fact will be used in our analysis of some properties of implicative verbs.

The difference between *tall* and *Japanese* can be described as follows: *tall student* does not entail *tall being/existent* but *Japanese student* does entail *Japanese being*. Since the common noun *being/existent* denotes the unit element of the algebra of sets the adjective *japanese* denotes an intersective function whereas the adjective *tall* denotes a non-intersective restrictive function.

Other, non-adjectival, modifiers can also be interpreted by intersective functions. This is the case of various adverbials, in particular of locative adverbials: *sing in the garden* means *sing and be in the garden* and thus the adverb *in the garden* denotes an intersective function, because *to be/to exist* is interpreted as the value 1_B for B the algebra of properties.

By analogy with restricting algebras we can also define negatively restricting algebras NR_B as factor algebras generated by the complement of the identity function. More precisely we have the following definition D3 and the corresponding Fact 2 (cf. Zuber 1997):

D3: Let B be a Boolean algebra and B/B the Boolean algebra of functions from B into B. Then NR_B, the algebra of negatively restricting functions over B is the factor algebra of B/B generated by id' (where $id'(x) = x'$, for any $x \in B$). Fact 2: $NR_B = \{f : f \in B/B, f(x) \leq x'\}$ and for any $x \in B$, $0(x) = 0_B$, $1(x) = x'$, $(f \wedge g)(x) = f(x) \wedge g(x)$, $(f \vee g)(x) = f(x) \vee g(x)$ and $c(f)(x) = (id' \wedge f')(x) = id'(x) \wedge f'(x) = x' \wedge (f(x))'$.

Thus negatively restricting functions are functions f such that $f(x) \leq x'$.

Restricting and negatively restricting functions are good candidates for denotations of one-way implicatives. Two-way implicatives cannot, however, denote in restricting or negatively restricting algebras. The reason is that, roughly speaking, the negation of restricting functions does not have in its scope the argument of these functions but just the function itself. In other words it is not true that for f restricting we have $c(f)(x) \leq x'$, where $c(f)$ is the complement of f in R_B. In the next section I show that factor algebras which are not generated by the identity function (or its complement) can be used to give the semantics of two-way implicatives and in particular to explain the behaviour of negated two-way implicatives.

4 Towards the Semantics of Implicatives

We have seen that functions denoted by implicatives behave in many respects like restricting and negatively restricting functions. However, implicatives cannot denote in restrictive algebras since their negations do not correspond to complements of restricting functions. Thus in order to describe denotations of implicatives, in particular of two-way implicatives, we have to take into consideration the fact that semantics of implicatives essentially involves (non-trivial) presuppositions since the entailments to which they give rise are due to the presupposition preserving negation. Though, as it is recognised by now, presupposition is a very disparate phenomenon in general, it is possible to define formally and in a natural way the presupposition of predicates which is involved in the implicatives and privatively marked common nouns since they are assumed to denote sets.

Recall that the unit element of a Boolean algebra is an element which is entailed by every element of the algebra. In particular it is entailed by any element and its complement (negation). So the unit element can be considered as a presupposition of any element of the algebra in the same way as logical truth can be considered as a presupposition of any proposition (expressed by a declarative sentence). In general these are trivial presuppositions but not in the case of factor algebras, since the negation in this case is relativised to a particular non-trivial element. For instance to obtain the presupposition carried by formal marks which distinguish the marked element from the unmarked one in the privative opposition we take the corresponding semantic property, expressing the presupposition of the marked, as the unit element (or the generator) of the factor algebra. Thus marked terms such as *poetess* and *authoress* will denote in the factor algebra P_F, where P is the algebra of properties (subsets of the universe E) and F is the set of female existents. Indeed, in this case for any $\alpha \in B$ we have $\alpha \leq F$ and $c(\alpha) \leq F$ (cf. Zuber 1982). This means that the property F is presupposed by every member of P_F.

One can thus see that the use of factor algebra enables us to represent the fact that something is implied by a predicate and its negation (its Boolean complement in a factor algebra). This represents the idea that the presuppositions of a predicate constitute the information it contains that is not affected by negation.

We will suppose that in the case of implicatives the presupposition has the type of functions from sets to sets since we consider that implicatives are VP modifiers. This means that presuppositions of implicatives will correspond to functions from sets to sets. These functions are generators of the factor algebra of the algebra P/P, where P is the algebra of properties (sets) since, for simplicity, we consider that properties are denotations of VPs. However, strictly speaking, the notion of implicativity can be considered as an algebraic property which can be defined in any Boolean algebra. For this reason the definition D4 below of implicative functions applies to any Boolean algebra:

D4: Let B be a Boolean algebra. Then the set $\{p \wedge id : p \in B/B\}$ is the set of positive two-way implicative functions in $(B/B)_p$.

Definition D4 allows us to prove:

Proposition 1 *Let B be a Boolean algebra. If f is a positive two-way implicative function in $(B/B)_p$, for some $p \in B/B$, then for any $x \in B$ we have $f(x) \leq x$ and $c(f)(x) \leq x'$, where $c(f)$ is the complement of f in the factor algebra $(B/B)_p$ and x' - the complement of x in B.*

For proof observe that if f is a two-way positive implicative function in $(B/B)_p$, then f is a restricting function on B since $f = p \wedge id$ and consequently $f(x) \leq x$. To get the "negative" part of the entailment we have, by definition of the complement in a factor algebra, $c(f) = f' \wedge p = (p \wedge id)' \wedge p = (p' \vee id') \wedge p = p \wedge id'$. Thus since $c(f)$ is a negatively restricting function on B, we have that $c(f)(x) = (id' \wedge f')(x) = x' \wedge (f(x))' \leq x'$ which is what we desired to show.

In a quite similar way we define two-way negative implicative function with its "negative implicativity" indicated in Proposition 2:

D5: Let B be a Boolean algebra and let $p \in B/B$. Then the function f such that $f = p \wedge id'$ is a negative two-way implicative function in $(B/B)_p$.

Proposition 2 *If f is a negative two-way implicative function in $(B/B)_p$ then for any $x \in B$ we have $f(x) \leq x'$ and $c(f)(x) \leq x$*

Let me illustrate definitions D4 and D5. Consider the positive two-way implicative *to manage (to)*. It is a modifier which applies to a *VP* and gives a "new" *VP to manage to VP*. Now, *to manage to VP* means, roughly speaking, *to try to VP and to VP*. Thus, if *to manage* denotes $MANAGE$, *to try* denotes TRY and *VP* denotes $[VP]$, we have $MANAGE([VP]) = TRY([VP]) \wedge id([VP])$, since $id([VP]) = [VP]$. This means that $MANAGE$ is a positive two-way implicative function in B_{TRY}, where $B = D_{VP/VP}$.

Consider now the negative two-way implicaive *to fail*: *to fail to VP* means *to try to VP and not VP* and $[not VP] = id'([VP])$. Thus $FAIL = TRY \wedge id'$. Consequently $FAIL$ is also an element of $B - TRY$, where $B = D_{VP/VP}$.

Observe that the negations used in the two-way implicatives that is negations which give rise to the entailment of the negated argument of the implicative correspond to the complement $c(f)$ in the factor algebra $(B/B)_p$.

Obviously functions defined in D3 and D4 and in Propositions 1 and 2 are the only two-way implicative functions in $(B/B)_p$ since we have:

Proposition 3 *The function $f = id \wedge p$ is the only two-way positive function on $(B/B)_p$.*

Proposition 4 *The function $f = id' \wedge p$ is the only two-way negative function on $(B/B)_p$.*

One-way implicative functions (that is denotations of one-way implicatives) are functions which are strictly included in two-way implicative functions or which strictly include two-way implicative functions. By definition for $f, g \in B/B$, f is strictly included in g iff $f \leq g$ and $f \neq g$, where $\leq$ is the partial order proper to B/B. Consequently we have the following definition:

D6: Let f be the positive two-way implicative function in the algebra $(B/B)_p$, for some $p \in B/B$. Then any function g such that $g < f$ is an *if*-implicative function in $(B/B)_p$.

D6 is obvious because if $g < f$ then, by definition of f, $g < id \wedge p$ and thus $g < id$ which means that g is an *if*-implicative function because in this case $g(x) \leq x$ for any $x \in B$.

Functions denoted by *only if*-implicatives are the functions which strictly include two-way positive implicative functions:

D7: Let f be the positive two-way implicative function in the algebra $(B/B)_p$ for some $p \in B/B$. Then any function g such that $f < g$ is an *only if*-implicative function in $(B/B)_p$.

To understand definition D7 suppose that $f < g$ and thus that $id \wedge p < g$. Hence, by contraposition, we get (i): $g' < id' \vee p'$. From (i) by set-theoretical operations we get (ii): $g' \wedge p < id' \wedge p$. But $g' \wedge p = c(g)$ and consequently from (ii) we get $c(g) < id'$ which means that g is an *only if*-implicative function because in this case $c(g)(x) \leq x'$.

Other types of one-way implicative functions are obtained by taking functions which are strictly included in two-way negative implicative functions. This is indicated in D8 and in D9:

D8: Let f be the negative two-way implicative function in the algebra $(B/B)_p$. Then any function g such that $g < f$ is a negative *if*-implicative function in $(B/B)_p$.

This definition is obvious because if $g < f$ then, by definition of f, $g < id' \wedge p$ and thus $g < id'$ which means that g is a negative *if*-implicative function since in this case $g(x) \leq x'$.

Negative *only if*-implicative functions are the functions which strictly include the negative two-way implicative function:

D9: Let f be the negative two-way implicative function in the algebra $(B/B)_p$. Then any function g such that $f < g$ is a negative *only if*-implicative function in $(B/B)_p$.

To grasp the content of D9 it is enough to notice that the contraposition law (valid in any Boolean algebra) inverses the polarity of expressions involved in the implicative relation.

If we ignore the distinction between strict and "ordinary" inclusion, it follows from the above definitions that there are four types of implicative functions, corresponding to implicative verbs: positive *if* and *only if* implicative functions and negative *if* and *only if* implicative functions.

Recall that the set $\{x : a \leq x\}$ of elements of a Boolean B, for $a \in B$, is called a (principal) filter (of the algebra B) generated by the given element a and the set $\{x : x \leq a\}$ of elements of B is called an ideal (generated by the given element a). This means that the above definitions can be summed up in algebraic terminology in the following way: *if*-implicative functions form a principal ideal generated by the identity function, *only if*-implicatives form a principal filter generated by the identity function, negative *if*-implicative functions form the principal ideal generated by the complement of the identity function and negative *only if*-implicative functions form the principal filter generated by the complement of the identity function.

The above definitions and propositions formally describe implicative functions, which are taken as possible denotations of implicative verbs. They also indicate various semantic relations between different implicative functions (of the same factor algebra) and thus between different implicative verbs. To make this more precise consider the following definition:

D10: For $p \in B/B$ and $f, g \in (B/B)_p$, f is incompatible with g iff $f \wedge g = \emptyset_B$ and f is strongly compatible with g iff $f \leq g$ or $g \leq f$ or $f \leq c(g)$ or $g \leq c(f)$.

Given the equivalence $f \wedge g = 0_B$ iff $f \leq g'$, definitions D7–D10 lead directly to

Fact 3: Any two implicative functions in $(B/B)_p$ of different types are strongly compatible.

According to Fact 3 any two implicative functions, relative to the same presupposing function, are always in a non-trivial semantic relation: one entails the other or its negation: *to be forced* entails very likely *to be able, to manage* and *to hesitate*. However, the implicative *to happen (to)* does not entail *to manage* or *to hesitate* because *to happen* presupposes probably *not to plan to/to do accidentally* which are not presupposed by *to manage* or *to hesitate*.

Notice that Fact 3 corresponds to the algebraic property that elements of a filter and of the corresponding dual ideal are strongly compatible.

The fact that one-way implicative functions are included in or include two-way implicative functions explains also some properties related to the semantic projection of implicatives. Karttunen (1971) indicates that various semantic properties of the entailing sentence containing an implicative verb are inherited by the corresponding implied sentence containing the argument of the implicative. For instance, as indicated in (18) and (19), in contradistinction to non-implicative verbs, the tense of the implicative verb necessarily matches the tense the complement verb in the entailed sentence. Thus (18b) is necessarily false, whereas (19a) where a non-implicative verb *to hope* occurs, is not necessarily false. A similar observation holds for locative adverbs: (20a) entails (20b) and (21), in which a non-implicative verb occurs, is not necessarily false:

(18a) Leo managed to prove the theorem last week.
(18b) Leo managed to prove the theorem next week.
(19a) Leo hoped to prove the theorem last week.
(19b) Leo hoped to prove the theorem next week.
(20a) In the garden Leo was not able to solve problem number 7.
(20b) Leo did not solve problem number 7 in the garden.
(21) Leo wanted in Oxford to solve the problem in Paris.

The above facts are easy to understand if we recall that functions denoted by locative and temporal adverbials are intersective functions and thus that they are monotone increasing: if A is monotone increasing and F is a positive implicative function, then, by monotonicity of A we have $A(F)(V) \leq A(V)$ because in this case, given the definition of a positive implicative function F, we have $F(V) \leq V$.

5 Conclusion

Implicative verbs are VP modifiers and consequently they denote functions from VP denotations to VP denotations. Given the semantic properties of implicatives there are four types of functions that they can denote and which are determined by the corresponding "restricting" property: (i) $f(x) \leq x$ (*if*-implicatives), (ii) $x \leq f(x)$ (*only-if*-implicatives, (iii) $f'(x) \leq x$ (negative *if*-implicatives) and (iv) $x' \leq f(x)$ (negative *only-if*-implicatives. Two-way implicatives denote functions which have two of the above properties at the same time: functions which have properties (i) and (ii) are denotations of positive two-way implicatives and functions which have properties (ii) and (iv) are denotations of negative two-way implicatives. Observe now that if the complement f' of f is interpreted pointwise in the Boolean algebra of all functions from the denotations of VP into denotations of VP (and $x = id(x)$ and $x' = id'(x)$), then on the one hand, the function which satisfies properties (i) and (ii) is just the identity function and, on the other hand, conditions (ii) and (iv) coincide. This means that any two-way implicative denotes the identity function and, furthermore, that there is no diffference between negative *if*-implicatives and negative *only-if*-implicatives. But, as we have seen, this conclusion cannot be accepted for empirical reasons. To solve this problem I have proposed that implicative verbs denote in factor algebras in which the complement of the function is not defined pointwise by the corresponding algebra to which belongs the argument of the function.

According to my proposal positive (strict) one-way implicatives denote functions which (strictly) include the identity function or are included in the identity function of the factor algebra. Negative one-way implicatives denote functions which (strictly) include, or are strictly included in the unique complement of the identity function of the factor algebra.

Denotations of implicatives are consequently similar in particular to restricting functions or negatively restricting functions but they do not denote in restrictive or negatively restrictive algebras. Moreover, the set of these functions does not form a Boolean algebra since the factor algebra $(D_{VP}/D_{VP})_p$ in which they denote also contains non-implicative functions, which are functions that neither include nor are included in $p \wedge id$ or $p \wedge id'$. These functions are possible denotations of non-implicative verbs such as *to want, to try, to expect*, etc., which are also modifiers of VPs and thus denote functions from denotations of VPs into denotations of VPs.

The proposal made here concerning denotations of implicatives allows us to understand the impression of quasi-logical equivalence or implicational equivalence one has with two-way implicative verbs and their arguments because it offers a direct explanation of the mechanism of double implication to which they give rise. In addition by interpreting implicatives in factor algebras we thereby automatically relativise the interpretation of negation to one that allows both x and its complement ("negation") to entail a non-trivial property as it is the case with implicatives.

We have thus distinguished families of implicatives functions, possible denotations of implicative verbs: for any function $p \in D_{VP}/D_{VP}$ the factor algebra $(D_{VP}/D_{VP})_p$ contains a family of implicative functions. This family is constituted by the join of the filter generated by the identity function with the ideal generated by the complement of the identity function. The verbs *to be forced, to be able, to manage* and *to hesitate* denote in the same family of implicative functions and the verbs *to happen* and *to manage* do not denote in the same family. Every family of implicative functions is thus determined by the function p which generates the algebra $(D_{VP}/D_{VP})_p$ and this function corresponds, roughly speaking, to the presupposition of implicative functions belonging to the same family.

Though the proposal made in this article explains the behaviour of the negation in implicative verbs and thus explains the mechanism of implicativity, it does not offer, strictly speaking, a full semantic description of any particular implicative verb. To obtain such a description, more should be said about the status of generators of factor algebras and their exact relationship with presuppositions. As far as I can see, this should be done in the context of a more general discussion concerning negations and the derivability of presuppositions in natural languages.

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