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A Class of Conceptual Spaces Consisting of Boundaries of Infinite *p*-Ary Trees

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Abstract

A new construction of a certain conceptual space is presented. Elements of this conceptual space correspond to (and serve as code for) concept elements of reality, which potentially comprise an infinite number of qualities. This construction of a conceptual space solves a problem stated by Dietz and his co-authors in 2013 in the context of Voronoi diagrams. The fractal construction of the conceptual space is that this problem simply does not pose itself. The concept of convexity is discussed in this new conceptual space. Moreover, the meaning of convexity is discussed in full generality, for example when space is deprived of it, its substitutes for concept domains are considered.

Keywords Conceptual space · Concepts · Qulities · p-adic integers · p-ary tree · Gromov boundary · Cantor-type set

1 Introduction

Ferdinard de Sauserre (1916) introduced the paradigm of structuralism into linguistics. Almost immediately structuralism was not only restricted to linguistics but swiftly became the research methodology of choice for many other fields of science (such as literary science, philosophy, anthropology, sociology).

With its help many important, long-standing results were achieved.

However, around 1950 structuralist methods began to wear out, and a shift took place, paradigm change towards a new methodology emerged: the cognitivistic paradigm was born.

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The underlying idea of the paradigm of cognitivism is quite simple. It is stipulated, assumed that by observation of (outward) human behavior patterns (such as communication and speech acts, psychological reactions to chosen stimuli, etc.) an analogy can be deuced as to how the (inward) human brain works.

Indeed, this new viewpoint marked a milestone and began a new era in many fields of science. Hence, now we began to speak of cognitive psychology, cognitive sociology, cognitive biology and many more.

This paper may be situated between cognitive and mathematical linguistics. The problems tackled in the paper stem from cognitive linguistics, but the methods employed are from mathematical linguistics.

The groundwork of cognitive linguistics was subsequently laid down around 1980 by Langacker (1986, 2008), Mark Johnson (1987), George Lakoff (1987) and others.

The basic assumption is that the brain encodes reality constructing geometric pictures. (Consequently different people encode differently. The prior knowledge of the world directly influences the choice of geometric picture used in building a given context.)

These already imprinted pictures (concepts) in turn influence the building of the next layer of concepts.

Of course the used code must be sufficiently simple to be efficient.

In cognitive linguistics, when we investigate language as such, we replace natural (spoken) language by the imprinted pictures (symbols) in the brain which constitute the vocabulary of a metalanguage.[1]

One of the central notions in cognitive linguistics is that of conceptual space.

Methodologically abstract conceptual spaces correspond to the imprinted concrete pictures in the brain.

The notion of conceptual space was first introduced and investigated by Gärdenfors (2000, 2017).

Conceptual space is defined as a metric space[2] (X, d), that is a set X of abstract objects on which a so called distance function $d(x, y)$, $x, y \in X$, is given.

Usually X is of higher dimension, e.g. $\mathbb{R}^D$, $D \in \mathbb{N}$, in which each dimension corresponds to quality of a concept.

Moreover, the space X is endowed with a (dis) similarity measure $s = s(x, y)$, $x, y \in X$ which is obtained as a function of the metric d.

As a simple example let, for $x \neq y$,

$$s(x, y) = \frac{1}{d(x, y)}.$$

Then, a small value of $s(x, y)$ indicates a small degree of similarity between x and y, whereas conversely a large value of $s(x, y)$ signifies strong similarity between objects (concepts) x and y.

In writing this paper we have two main hypotheses in mind.

[1] For a different than ours application of metalanguage in linguistic semantics see Wierzbicka (1972).

[2] All mathematical terminology will be defined and illustrated by elementary examples in the respective paragraphs (in which they appear).

Aim (1) We give a new construction of a certain conceptual space whose elements correspond to (and serve as code for) concept elements of reality, which potentially comprise an infinite number of qualities.

Very interestingly this newly defined conceptual space which is a subset of one-dimensional real line $\mathbb{R}$ (such that has a fractional Hausdorff dimension less than 1), proves adequately describe a multi-dimensional space of an infinite amount of qualities. One point in the conceptual space corresponds to an infinite vector of qualities.

The starting point of our construction are integer p-adic numbers.

It is known that one can identify p-adic integer numbers with p-ary weighted trees, where the set of weights is $\{0, 1, \ldots, p-1\}$.

In consequence the conceptual space associated with this coding turns out to be a Gromov boundary (1987, 1993) of an appropriate p-ary tree where p is a prime number.

This special form of conceptual space (the Gromov space) allows for the generalization of our coding methods from trees to hyperbolic groups.

As the most elementary example let $p = 2$, i.e., we are coding using a binary tree.

Here we identify a single real object (concept) with a specified *branch* (not a single *vertex*) of a binary tree.

Each edge starting at the ℓth level of the tree may carry a weight w of either 1 or 0 (the concept has or does not have quality ℓ).

It is quiet natural to stipulate that qualities on a deeper (further down the branch) level (larger ℓ) be less determinant for the description of a given concept.

In a p-ary trees with $p > 2$ a weight $w \in \{0, \ldots, p-1\}$ stands for the intensity with which a given quality is represented in a given concept.

Aim (2) Our construction of a conceptual space solves a problem stated by Dietz (2013) and his co-authors: Douven et al. (2013).

Namely, Dietz and his co-authors used Voronoi diagrams (Okabe and Boots 2000) as a conceptual space. For the reader's convenience we recall the construction of a conceptual space by means of Voronoi diagrams.

This is a quasi partition of the Euclidean space $X = \mathbb{R}^D$ into polytopes according to the following rules. First k prototypes $p_1, \ldots, p_k$ of k concept domains $v(p_1), \ldots, v(p_k)$ are defined by:

$$v(p_i) = \{x \in X : \; d(x, p_i) < d(x, p_j) \text{ for all } j \neq i\},$$

where $d(x, y) = \|x - y\|_{\ell^2(\mathbb{R}^D)}$ is a standard metric in $\mathbb{R}^D$.

The concept domains are open sets in X. Then the problem is that the boundary points of the domains are not themselves contained in any concept domain. Equivalently

$$\bigcup_{i=1}^{k} v(p_i) \neq X.$$

This problem was partially solved using different methods in the original framework of Dietz (2013).

One advantage of our new, fractal construction is that this problem simply does not pose itself. There exist partitions for which each element of the conceptual space is contained in exactly one concept domain.

The idea to consider p-ary trees and p-adic numbers originally comes from Physics. Khrennikov (2016) and Anashin and Khrennikov (2009) suggest that the real numbers are not adequate for description of the physical world and that the p-adic numbers are more suitable; see Schwabl (2007).

The reason being that real numbers have continuous structure, whereas the Universe is built of from a finite number of particles between which there are a lot of “holes”. This property is well reflected by the nature of the Gromov boundary due to its fractal structure.

Moreover, according to the Quantum Mechanics physical action cannot have an arbitrary value but must be an integer multiply of the Planck constant $\hbar$; see Schwabl (2007).

Many important physical theories have been formulated in p-adic language. See e.g. a review of the development of p-adic physical theories for the last 30 years written by Dragovich et al. (2017). This new p-adic trend in the approach to the problems of the material world that we are confronted with has shed a new light on many physical problems and not only physical. For example, Khrennikov and Iurato are supporter of applying methods of p-adic analysis in cognitivism; see the papers by Khrennikov (2014), Iurato and Khrennikov (2015), and the review article by Iurato et al. (2016).

Structure of the paper

The work structure is as follows.

In Sect. 2, starting with the definition of abstract metric space, we recall the definition of conceptual space introduced into cognitive science by Gärdenfors and give some examples illustrating this notion.

In Sect. 3, we remind the reader of the necessary concepts from graph theory and, in particular, trees, which will play an important role in our deliberations.

In Sect. 4 we we introduce a field of p-adic numbers $\mathbb{Q}_p$ and its subring of p-adic integer numbers to show in Sect. 5 how to identify p-adic integers $\mathbb{Z}_p$ with p-ary trees.

The most technical is Sect. 6, in which we define the Gromov boundary, we introduce topology and a metric on it, which is compatible with the topology introduced earlier. In this way, the Gromov boundary is a metric space. The presentation contained in Sect. 6 is mainly based on the second section of Kapovich’s and Benkali’s work (2002).

In Sect. 7 we construct our (fractal, see Falconer (2003)) conceptual space and prove its properties. In particular, we show that it is possible to partition our space into a sum of concept domains. Concept domains can be made as small as we want. (What can often be a useful property.) Furthermore, every point of our conceptual space belongs to exactly one concept domain.

Section 7.4 is devoted to the problem of convexity, which according to Gärdenfors is one of the most important postulates which should be fulfilled by the conceptual domain. There are different points of view on this assumption; see e.g. Jaġer (2006), Zhu et al. (2006) and Mendel (2007). It seems that the concept of convexity must be adapted to the space we are studying and replaced by an appropriate other notion, which in some way is analogous to convexity. An example of the work of Urban and Grzelińska (2017) shows that in the context they are working with, Euclidean convexity is not adequate and is replaced with geodesic convexity.

Finally, in Sect. 8 we will include some remarks and plans for our further research on conceptual spaces.

2 Conceptual Spaces

Let's start with the definition of terms that are important for further discussion.

We would like to point out that the terms we use are slightly different from the terminology used by other authors. This is because our formal geometric constructions are better suited to the terminology we use to describe human knowledge, concepts and notions. All terms used by us are defined in detail at the moment they appear in the text.

Moreover, since Gärdenfors introduced conceptual space to the cognitive sciences, the generalisation of conceptual space as well as the reformulating of the original structure has appeared in literature; see e.g. Rickard (2006) and Rickard et al. (2007).

It is therefore difficult to say that there is some canonical terminology about conceptual space.

Let's start with the concept domain. As an example consider a concept domain, which is associated with the means of transport. It may contain such objects as {crops, horse, bike, scooter, rollers, train, plane, car}. Not all of the above mentioned means of transport are currently used equally by language users. By taking a test and asking randomly selected people for one mode of transport, it is expected that the majority will give the answer {car}. This object, which appears most often by the users of the language is called a prototype. Thus, in the case described above, {car} is a prototype of the concept domain (semantic field) of the means of transport.

The conceptual space consists of conceptual domains.

Let us now consider an example in which objects can be described by means of $D = 3$ real parameters. Each real parameter (that is, each dimension) corresponds to a single quality of the object.

The simplest example, quoted in almost every work on conceptual spaces, of such concepts are colors, which can be defined by three parameters - namely, through the wavelength, saturation and hue.

It is clear that, for example, the concept of 'red' is not only its prototype (one single point x in 3-dimensional space) but also colours in the close vicinity of the prototype (i.e. set in 3-dimensional space called a concept domain, which contains a point-prototype x).

In this example, conceptual space will be a union of all concept domains corresponding to different colours.

Conceptual space objects can be, for example, events or physical items.

From a linguistic point of view, concepts usually correspond to the grammatical category of a noun or verb if time is one of the dimensions of conceptual space. The qualities correspond to the adjective descriptions.

Now we are ready to formalise the necessary terms.

We start with a definition of a metric space (X, d).

Definition 2.1 A metric space is an abstract set X with a function $d : X \times X \to \mathbb{R}^+$ with the following properties:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ (symmetry),
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Example 2.2 The simplest example of a metric space is the set of real numbers $\mathbb{R}$ endowed with the metric $d(x, y) = |x - y|$, where $| \cdot |$ stands for the absolute value.

One can easily generalize this example into

$$\mathbb{R}^D = \{x = (x_1, \ldots, x_D) : \ x_i \in \mathbb{R} \quad \text{for } i = 1, \ldots, D\}$$

with

$$d^{(s)}(x, y) = \left(\sum_{i=1}^{D} |x_i - x_j|^s \right)^{1/s},$$

where $s \geq 1$ is a real number. For $s = 2$ we get the classic Euclidean distance.

One can easily check that the axioms (i)-(iii) of Definition 2.1 are satisfied.

For more details on analysis on metric spaces see Heinonen (2001).

After the notion of metric space has been introduced, we are prepared to define our main object, which is conceptual space $\mathcal{C}$.

Definition 2.3 By (D-*dimensional*) conceptual space $\mathcal{C}$ we mean the product of the metric spaces (X_i, d_i), $i = 1, \ldots, D$, equipped with an appropriate metric d. An element $x \in \mathcal{C}$ is written as a vector $(x_1, \ldots, x_D)$, where $x_i \in X_i$. Each dimension $i = 1, \ldots, D$ describes the ith quality of an object x assigning the value x_i.

Example 2.4 As in Example 2.2, it is easy to check that the functions defined below are metrics in conceptual space $\mathcal{C}$ as defined in Definition 2.3,

$$d^{(s)}(x, y) = \left(\sum_{i=1}^{D} d_i(x_i, x_j)^s \right)^{1/s}.$$

With conceptual space and prototypes, we need an appropriate algorithm to classify the conceptual space element into one of the concept domains determined by these prototypes.

The algorithm that produces the Voronoi diagram is quite often used for this purpose. We described this algorithm in the introduction to this work.

Gärdenforse presents the justification for using Voronoi diagrams as follows:

> A Voronoi tessellation based on a set of prototypes is a simple way of classifying a continuous space of stimuli. The partitioning results in a *discretization*[3] of the space. The prime cognitive effect is that the discretization speeds up *learning*. The reason for this is that remembering the finite prototypes, which is sufficient to *compute* the tessellation once the metric is given, puts considerably less burden on *memory* than remembering the categorization of each single point in the space. In other words, a Voronoi tessellation is a cognitively *economical* way of representing information about concepts. Furthermore, having a space partitioned into a finite number of classes means that it is possible to give *names* to the classes. Gärdenfors (2000, p. 89).

It seems that the statement that the division of space in the human mind is based on knowledge of prototypes is quite likely. However, we cannot agree with the second part of Gärdenford's statement that the brain uses Voronoi diagrams for economic reasons. We do not think that the human brain is limited to merely operating the concept of linearity.

Therefore, in the paper by Urban and Grzelińska (2017) we proposed a different, non-linear partitioning algorithm.

In short, in our opinion the brain is equally easy to operate on non-linear images that are imprinted in it.

Here we will present another construction, which is also far from linear.

We believe that it can be useful for classifying objects. Moreover, it does not seem to be so complex that the human brain cannot effectively cope with it. Especially in this case when he is able to cope with Voronoi diagrams.

3 Graphs and Trees

Here we present only the necessary definitions for the graphs that will be used in our work. More information about graphs and trees can be found in the following: Diestel (2010) and Harris et al. (2008) handbooks.

Definition 3.1 An undirected graph, or simply graph, is a pair $(\mathcal{V}, \mathcal{E})$ of points $v \in \mathcal{V}$, called vertices and a set $\mathcal{E}$ whose elements, called edges, are two element subsets of $\mathcal{V}$. If $\{x, y\} \in \mathcal{E}$ then we say that vertices x and y are adjacent.

Definition 3.2 A path in a graph $(\mathcal{V}, \mathcal{E})$ is a (finite or infinite) sequence of vertices $x_n \in \mathcal{V}$ such that $\{x_i, x_{i+1}\} \in \mathcal{E}$.

Definition 3.3 A graph $(\mathcal{V}, \mathcal{E})$ is called connected if for all $x, y \in \mathcal{V}$ there exists a finite path $x_0, x_1, \ldots, x_n$ such that $x_0 = x$ and $x_n = y$. In other words, a graph $(\mathcal{V}, \mathcal{E})$ is connected if every two points x and y can be joint by a finite path.

Definition 3.4 A tree is a graph $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ in which any two vertices $x, y \in \mathcal{V}$ are connected by exactly one path.

[3] The emphasis in the text in italic font was made by the author of the statement.

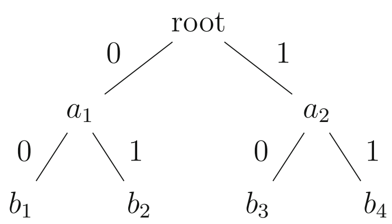

Fig. 1 The first 2 levels (root is sometimes said to be on level 0) of the 2-ary quasi-homogeneous weighted tree $\mathcal{T}_{2,3}$

Definition 3.5 A tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ is a weighted tree if there is a function $w : \mathcal{E} \to [0, +\infty)$. The function w is called the weight function. The value of w on a given edge $e \in \mathcal{E}$, i.e., $v(e)$ is called the weight of the edge e.

In this paper we are interested in locally finite trees.

Definition 3.6 A locally finite tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ is a tree such that every vertex $v \in \mathcal{V}$ belongs to finite number of edges $e \in \mathcal{E}$. This number $\deg(v)$ is called the degree of v.

Definition 3.7 If a given tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ has the function $\deg(v)$ which does not depend on the vertex v and is equal to q then we say we say that $\mathcal{T}$ is homogeneous of degree q and we write $\mathcal{T} = \mathcal{T}_q$.

All trees in this paper are rooted tree. This means that there is one vertex v_o of a tree which is distinguished and called the root of the tree.

Definition 3.8 A tree $\mathcal{T}$ is called quasi-homogeneous if the degree of its root vertex is $\deg(v_o) = q$ and all other vertices $v \neq v_o$ have degree $\deg(v) = q + 1$. In this case we write $\mathcal{T} = \mathcal{T}_{q,q+1}$ (see Fig. 1.)

It is important that an arbitrary graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a metric space.

Theorem 3.9 *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an arbitrary graph. If, for every $v_1, v_2 \in \mathcal{V}$ we define*

$$d(v_1, v_2) = \{\textit{the length of the shortest path joining } v_1 \textit{ and } v_2\},$$

where the length of a given path is defined as the number of its edges.

The proof of Theorem 3.9 is easy and is left for the reader.

In the rest of the paper we consider quasi-homogeneous weighted trees $\mathcal{T}_{p,p+1}$ with the set of weights $\{0, \ldots, p-1\}$, where p is a prime integer.

4 *p*-Adic Fields

Now we are in position to define the field of p-adic numbers. Mostly we follow the presentation included in Sally (1998). The reader may also consult Robert (2000).

Let p be a prime number in $\mathbb{Z}$, and let $x = \frac{a}{b} \in \mathbb{Q}$ be a rational number. We write $x = p^s \frac{k}{l}$. Define the p-adic absolute value on $\mathbb{Q}$ by

$$|x|_p = \begin{cases} p^{-k} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The p-adic absolute value gives a metric on $\mathbb{Q}$ by

$$d_p(x, y) = |x - y|_p, \quad \text{for } x, y \in \mathbb{Q}.$$

The completion[4] of $\mathbb{Q}$ with respect to the metric d_p is denoted by $\mathbb{Q}_p$ and called a p-adic field. An element $x \in \mathbb{Q}_p$ is called the p-adic number. It turns out that the metric space $(\mathbb{Q}_p, d_p)$ is an ultrametric space (or a non-Archimedean space), that is the triangle inequality for $\mathbb{Q}_p$ reads:

$$d_p(x, z) \leq \max\left(d(x, y), d(y, z)\right).$$

This is a stronger inequality than (iii) in Definition 2.1.

Definition 4.1 The set of element $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is called the ring of integers in $\mathbb{Q}_p$. An element $x \in \mathbb{Z}_p$ is called the p-adic integer.

Definition 4.2 Let $x \in \mathbb{Q}_p$. Define a map $\nu : \mathbb{Q}_p \to \mathbb{Z} \cup \{+\infty\}$ by equation

$$p^{-\nu(x)} = |x|_p \quad \text{for } x \neq 0$$

and for $x = 0$ put

$$\nu(0) = +\infty.$$

The function ν is called the valuation and $\nu(x)$ is the valuation of $x \in \mathbb{Q}_p$

Theorem 4.3 *Any nonzero p-adic number $x \in \mathbb{Q}_p$ can be expressed as*

$$x = \sum_{k=\nu(x)}^{\infty} a_k p^k, \tag{4.1}$$

where, for every k, $a_k \in \{0, \ldots, p-1\}$ and $a_{\nu(x)} \neq 0$.

Definition 4.4 Let $0 \neq x \in \mathbb{Q}_p$ be written as the series (4.1). Then the fractional part of $x \in \mathbb{Q}_p$, denoted by $\{x\}$, is defined as follows:

$$\{x\} = \begin{cases} \sum_{k=\nu(x)}^{-1} a_k p^k, & \text{for } \nu(x) < 0, \\ 0 & \text{for } \nu(x) \geq 0. \end{cases}$$

Remark 4.5 Thus, $x \in \mathbb{Z}_p$ (i.e., x is an integer p-adic number) if and only if $\{x\} = 0$.

[4] The procedure for completing metric space is described in the Rudin textbook (1976).

5 Trees and *p*-Adic Integers

It follows from Theorem 4.1 and Remark 4.5 that every $x \in \mathbb{Z}_p$ can be written as a convergent (in p-adic metric d_p) sequence of the form

$$x = \sum_{k=0}^{\infty} a_k p^k, \tag{5.1}$$

where each a_k is from the set $\{0, \ldots, p-1\}$. (We do not assume that $a_0 \neq 0$.) Hence, one can identify every p-adic integer x with a sequence of coefficients a_k, i.e.,

$$x = (a_0, a_1, \ldots) \text{ with } a_k \in \{0, \ldots, p-1\}.$$

Example 5.1 If $p = 2$, then $a_k \in \{0, 1\}$ and every sequence x can be identified with an infinite path starting from the root of the 2-ary weighted tree. In the notation of Sect. 3, $\mathcal{T}_{2,3}$ (i.e. root has degree 2 and other vertices 3).

6 Gromov Boundary

The Gromov boundary of a hyperbolic group or a tree my be thought of as a set of points at infinity. Gromov's theory of boundaries which originates from Geometric Group Theory has found many application (Thurston's geometrization program for three dimensional manifolds, automatic groups, lattices in Lie group etc.).

In different branches of mathematics there are many concepts of boundaries depending on the studied space and the type of problem under consideration. Thus, mentioning just a few examples, we have the Poisson boundary, the Martin boundary, the Furstenberg boundary, the Satake boundary, and many more; see Borel and Ji (2006).

The usefulness of the boundary results from the fact that their structure is usually simpler and that very often the analysed space acts on the boundary with the help of morphisms, thus giving in a way a representation of the studied space on its boundary.

From a philosophical point of view, the boundary defined for such mathematical structures as topological spaces or spaces with (probability) measure (Riemannian manifolds, Lie groups, symmetric spaces etc.) are undoubtedly interesting objects. They allow to penetrate through the boundary of the examined structure (topological or measurable) as with the use of magnifying glass.

When studying analytical objects defined in different spaces, it turns out that their behaviour is not arbitrary but strictly determined by what happens on the boundary of a given space.

For example; see Axler et al. (2001); a harmonic function which belongs to the space $\mathcal{H}_b(\Omega)$ of bounded harmonic functions defined in the domain Ω of the Euclidean space $\mathbb{R}^D$ is uniquely determined by their boundary values on the topological boundary $\partial\Omega$ of Ω. This gives a 1 to 1 correspondence between elements of the space $L^{\infty}(\partial\Omega)$ of bounded measurable functions on the topological boundary $\partial\Omega$ and the space $\mathcal{H}_b(\Omega)$.

Our presentation of the Gromov boundary is closely based on Sect. 2 of the paper by Kapovich and Benkali (2002), an excellent survey of the known results about the Gromov boundaries of hyperbolic groups (and trees).

We start with definition of geodesic metric spaces, next we introduce the notion of δ-hyperbolic space. It turns out that trees belong to this class. Hence, the Gromov boundary can be defined for trees. What is very important is that the Gromov boundary is a metric space.

We want to emphasise that in case of tree the notion of the Gromov boundary simplifies essentially.

However, we chose to present more general theory, not necessarily limited to trees, since we are working on a sequel to this paper - where more general conceptual spaces will be constructed.

6.1 Boundaries

Definition 6.1 We say that a metric space (X, d) is geodesic if for every two points $x, y \in X$, there exist a geodesic segment $[x, y]$ that is a naturally parameterized path from x to y whose length is equal to $d(x, y)$.

Let A be a subset of a metric space and $\delta \geq 0$ be a real parameter. A δ-neighbourhood of A, $\mathcal{N}_\delta(A)$ is defined as the set

$$\mathcal{N}_\delta(A) = \{x \in X : \ d(x, A) := \inf_{y \in A} d(x, y) \leq \delta\}.$$

A δ-neighbourhood of A is a set of all points in X which are within distance less than or equal δ from A.

Definition 6.2 Let (X, d) be a geodesic metric space and $\mathbb{R} \ni \delta \geq 0$. We say that (X, d) is δ-hyperbolic space if for any triangle with geodesic sides in X each side s_1, s_2, s_3 of the triangle is contained in the δ-neighborhood of the union of two other sides, for example $s_1 \subset \mathcal{N}_\delta(s_2 \cup s_3)$. A geodesic metric space is said to be hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.

Definition 6.3 For three points x, y, z belonging a metric space (X, d) the Gromov product of y and z with respect to x is defined as follows:

$$(y, z)_x = (d(x, y) + d(x, z) - d(y, z))\,/2.$$

The Gromov product can be used to give an equivalent definition of δ-hyperbolic metric space (X, d).

Definition 6.4 A metric spece (X, d) is δ-hyperbolic if and only if, for every $x, y, z, v \in X$, we have

$$(x, z)_v \geq \min\{(x, y)_v, (y, z)_v\} - \delta.$$

It is not difficult to check that Definitions 6.2 and 6.4 are equivalent.

Directly from the Definition 6.3 of the Gromov product it follows that the product has the following properties.

Lemma 6.5 *For every* $x, y, z \in X$,

(i) $(y, z)_x = (z, y)_x$,
(ii) $d(x, y) = (x, z)_y + (y, z)_x$,
(iii) $0 \le (y, z)_x \le \min\{d(y, x), d(z, x)\}$,

Definition 6.6 Let (X, d) be a hyperbolic metric space. Fix a point o - the origin of X. A geodesic ray is a path $\gamma : [0, +\infty) \to X$ such that, for every $t \ge 0$, the image by γ of $[0, t]$, $\gamma([0, t])$ is a segment of shortest length from o to $\gamma(t)$.

Remark 6.7 The Gromov product in the δ-hyperbolic space X measures how long geodesics remain close together. Namely, if $x, y, z \in X$ then the initial segments of length $(y, z)_x$ of two geodesics from x to y and x to z are less than or equal to 2δ-close to each other (in the Hausdorff metric).

Now we define an equivalent relation on geodesic rays.

Definition 6.8 Two geodesic rays $\gamma_1 : [0, +\infty) \to X$ and $\gamma_2 : [0, +\infty) \to X$ are are said to be equivalent, $\gamma_1 \sim \gamma_2$, if there is a constant $C > 0$ such that, for every $t \ge 0$, $d(\gamma_1(t), \gamma_2(t)) \le C$.

After this long chain of definitions we are finally able to define the main object of this section.

The relation $\sim$ introduced above is an equivalence relation.[5] A class of abstraction of a ray γ is denoted by $[\gamma] = \{\alpha : \alpha \sim \gamma\}$.

First we define the geodesic boundary.

Definition 6.9 Consider a δ-hyperbolic metric space (X, d) with a base point $o \in X$. The relative geodesic boundary of X with respect to the base-point $o \in X$ is the set

$$\partial_o^g X = \{[\gamma] : \gamma : [0, +\infty) \to X \text{ is a geodesic ray such that } \gamma(0) = o\}.$$

The geodesic boundary of X is defined as the following set

$$\partial^g X = \{[\gamma] : \gamma : [0, \infty) \to X \text{ is a geodesic ray in } X\}.$$

Definition 6.10 Consider a hyperbolic metric space (X, d) with a base point $o \in X$. Let x_n be a sequence in X. The sequence x_n converges to infinity if

$$\liminf_{i,j\to+\infty} (x_i, x_j)_o = +\infty.$$

[5] A relation ρ on a set S is an equivalence relation if

(i) for every $a \in S$, $a \rho a$ (reflexivity),
(ii) for every $a, b \in S$ if $a \rho b$ then $b \rho a$ (symmetry),
(iii) for eery $a, b, c \in S$ if $a \rho b$ and $b \rho c$ then $a \rho c$ (transitivity).

Clearly, the above definition does not depend on a base-point o.

Definition 6.11 We say that two sequences x_n and y_n which converge to infinity are equivalent, $x_n \sim y_n$, if

$$\liminf_{i,j\to+\infty} (x_i, x_j)_o = +\infty. \tag{6.1}$$

It is easy to see that the above definition does not depend on a base-point. The relation $\sim$ defined by (6.1) is an equivalence relation.

Definition 6.12 For a δ-hyperbolic space the (sequential) boundary of X is defined as the following set

$$\partial X = \{[x_n] : \ x_n \text{ converges to infinity}\}.$$

For $x \in X$ and a geodesic ray starting from $x \in X$, define the following map:

$$\begin{aligned} i_x : \partial^g_x X &\to \partial^g X, \\ [\gamma] &\mapsto [\gamma]. \end{aligned}$$

Next, for a geodesic ray[6] γ define the following map For $x \in X$ and a geodesic ray starting from x, define the following map:

$$\begin{aligned} i : \partial^g X &\to \partial X, \\ [\gamma] &\mapsto [\gamma(n)]. \end{aligned}$$

Theorem 6.13 [Proposition 2.10 in Kapovich and Benkali (2002)] *Let (X, d) be a proper[7] δ-hyperbolic metric space. Then*

(1) *for every $x \in X$ the map i_x is a bijection,*
(2) *the map i is a bijection,*
(3) *for every two non-equivalent rays $\gamma_1, \gamma_2 : [0, +\infty) \to X$ there is a bi-infinite geodesic $\gamma : \mathbb{R} \to X$ such that $\gamma|_{[0,+\infty)} \sim \gamma_1$ and, after re-parametrization $\gamma(0)$, $\gamma|_{(-\infty,0]} \sim \gamma_2$.*

Definition 6.14 We will say that a geodesic ray γ in a δ-hyperbolic metric space (X, d) connects the point $\gamma(0) \in X$ to a point $p \in \partial X$ if $p = [\gamma(n)]$.

A bi-infinite geodesic γ connects a point $p \in \partial X$ to a point $q \in \partial X$ if $p = [\gamma(-n)]$ and $q = [\gamma(n)]$.

It follows from Theorem 6.13 that we have the following.

Corollary 6.15 *Let (X, d) be a proper hyperbolic metric space. For every point $x \in X$ and every point in $p \in \partial X$ there is a geodesic ray which joins x and p. Further, for every $p, q \in \partial X$ such that $p \neq q$ there is a bi-infinite geodesic in X joining p and q.*

6 Clearly, if γ is a geodesic ray in X then $\gamma(n)$ converges to infinity.

7 A metric space is said to be proper if all closed balls with a centre x and radius r $B(x, r) = \{y \in X : d(x, y) \leq r\}$ are compact.

6.2 Topologies on Boundaries

It turns out that on the Gromov boundary can be introduced in a natural way topology. In addition, the topological space obtained is metrizable, using a so-called visual metrics.

Recall, that the topological space is a pair $(X, \mathcal{O})$ of two elements. X is any collection of elements (any set) and $\mathcal{O}$ is a family of the subsets of X, such that X and the empty set $\emptyset$ belong to $\mathcal{O}$ and the family $\mathcal{O}$ is closed with respect to the finite intersection and any (infinite) union of elements from $\mathcal{O}$. The elements of the family $\mathcal{O}$ are called open sets.

Very often, rather than the topology $\mathcal{O}$, a smaller collection $\mathcal{B}$ of open sets is given which generates the topology $\mathcal{O}$.

Specifically, a basis $\mathcal{B}$ for a topological space X with topology $\mathcal{O}$ is a collection of open sets such that every element of $\mathcal{O}$ can be obtained by union of some elements of $\mathcal{B}$. We say that $\mathcal{B}$ generates $\mathcal{O}$; see Kelley (2017).

So we are now introducing topology in the hyperbolic space.

We continue the presentation given in the work by Kapovich and Benkali (2002).

We begin with the topology on the geodesic relative boundary $\partial_o^g X$; see Definition 6.9.

Let (X, d) be a δ-hyperbolic metric space and let $o \in X$ be a base-point. For every $p \in \partial_x^g X$ and every $r \geq 0$ we define the set $V(p, r) \subset \partial_o^g X$ by the following formula

$$\begin{aligned} V(p, r) = \{q \in \partial_o^g X :\ \exists\, \gamma_1, \gamma_2 \text{ with } \gamma_1(0) = \gamma_2(0) = o, \\ [\gamma_1] = p,\ [\gamma_2] = q \text{ and } \liminf_{t \to +\infty} (\gamma_1(t), \gamma_2(t))_o \geq r\}. \end{aligned}$$

Family of sets $\{V(p, r) :\ p \in \partial_o^g X,\ r \geq 0\}$ is taken as a topology basis on $\partial_x^g X$.

In the same way we introduce topology on the sequential boundary ∂X of δ-hyperbolic space with a base-point o.

For every $p \in \partial X$ and every $r \geq 0$ we define the set $U(p, r) \subset \partial X$ by the following formula

$$\begin{aligned} U(p, r) = \{q \in \partial X :\ \exists\, x_n, y_n \text{ with } \\ [x_n] = p,\ [y_n] = q \text{ and } \liminf_{i, j \to +\infty} (x_i, y_j)_o \geq r\}. \end{aligned}$$

Family of sets $\{U(p, r) :\ p \in \partial_o^g X,\ r \geq 0\}$ constitutes a basis of a topology on ∂_X. This topology does not depend on o.

One can introduce natural topology on $X \cup \partial X$; for details see Kapovich and Benkali (2002, p. 6). In this topology X and $\overline{X} = X \cup \partial X$ are compact and X is dense in $\overline{X}$.

If X is (not necessary proper) δ-hyperbolic space then the geodesic boundary and the sequential topology coincide. Moreover they have the same topology.

6.3 The Gromov Boundary ∂X as a Metric Space

As we mentioned in the previous section, the Gromov boundary is a metric space. In this section we will define the appropriate metric (visual metric) in accordance with the previously introduced topology. This metric greatly simplifies itself in the case of a tree boundary, which is particularly important in the context we are study.

Definition 6.16 Let (X, d) be a δ-hyperbolic proper metric space with a base-point $o \in X$. Let $\mathbb{R} \ni a > 1$. A metric d_a on the sequential boundary ∂X is called a visual metric with respect to the base-point o and the visual parameter a if there exists a constant $C > 0$ such that:

(i) the metric d_a induces the topology (defined above) on ∂X,
(ii) for every $p \neq q \in \partial X$, for any bi-infinite geodesic γ joining element p with q, and for every $y \in \gamma$ such that $d(o, y) = d(o, \gamma)$,

$$C^{-1} a^{-d(o,y)} \leq d_a(p, q) \leq C a^{-d(o,y)}.$$

The following important theorem gives the existence of a visual metric on the sequential boundary ∂X and also shows relationship between metrics for various visual parameters and different base-points of X.

Theorem 6.17 *Let (X, d) be a proper (i.e., every ball is compact) δ-hyperbolic metric space. Then we have that*

(i) *There exists $a_0 > 1$ such that for every base-point $o \in X$ and for every $a \in (1, a_0)$ there is a visual metric d_a with respect to $o \in X$ on the sequential boundary ∂X.*
(ii) *Suppose that d_1, d_2 are visual metrics on ∂X with respect to the same visual parameter a and the base-points o and o', respectively. Then d_1, d_2 are Lipschitz equivalent, i.e., there is a real constant $L > 0$ such that, for all $p, q \in \partial X$,*

$$L^{-1} d_1(p, q) \leq d_2(p, q) \leq L d_1(p, q).$$

(iii) *Suppose d_1, d_2 are visual metrics on ∂X with respect to the visual parameters a_1, a_2 and the base-points o_1, o_2, respectively. Let $\alpha = \ln a_2 / \ln a_1$. Then d_1, d_2 are Hölder-equivalent. That is, there exists a real constant $H > 0$ such that, for all $p, q \in \partial X$*

$$H^{-1} d_1(p, q)^{\alpha} \leq d_2(p, q) \leq H d_1(p, q)^{\alpha}.$$

For the proof see e.g. Ghys and de la Harpe (1990), Coornaert et al. (1990) or Bridson and Haefliger (1999).

7 Fractional Conceptual Space

In this section we show how to obtain the conceptual space associated with the ring of p-adic integers using the Gromov theory.

We also identify the abstract Gromov boundary with a specific mathematical object. Namely, starting from the space containing objects that can be described (potentially) with infinite number of qualities with the intensity from the set of weights $\{0, \ldots, p-1\}$, we show that conceptual space is (homeomorphic with) the $(p+1)$-ary Cantor set.[8]

The transition from the collection of abstract objects of reality to the conceptual space, which is a (fractal) Cantor set, see Falconer (2003), is accomplished by using the p-ary tree.

On the other hand, we also show how the construction of the $(p+1)$-ary Cantor tree generates a the p-ary tree.

7.1 Construction by the Gromov Theory

To construct our fractional concept space we start with the field of p-adic integers $\mathbb{Z}_p$. A given concept x is encoded by a p-adic integer $x \in \mathbb{Z}_p$, $x = \sum_{k=0}^{\infty} a_k p^k$. Consequently, the concept x is described by an infinite dimensional vector $x = (a_0, a_1, \ldots)$. Each dimension of this vector gives one quality of the concept x. Thus, a quality can be encoded by p integer numbers $\{0, 1, \ldots, p-1\}$ that give the intensity gradation of it. As it was explain in Sect. 5 each sequence $x = (a_0, a_1, \ldots)$ corresponds to exactly one branch of a rooted p-ary tree $\mathcal{T}_{p,p+1}$ (in other words, to exactly one path γ_o which starts from the root of the tree and goes to infinity, i.e. the sequence of vertices $\gamma(n)$ converges to infinity; see Definition 6.10.)

If we now consider a set of all paths γ that are equivalent to pathway γ_o, by means of relation $\sim$ defined in Definition 6.11, we get equivalence class $[\gamma_o]$. This class of equivalence is an element of the Gromov boundary of the quasi-homogeneous tree $\mathcal{T}_{p,p+1}$, out of which we started our construction.

The visual metric d_a defined in Definition 6.16 on the boundary ∂X is very simple when a hyperbolic space (X, d) is a tree with a metric d defined in Theorem 3.9. Clearly, $(\mathcal{T}, d)$ is a proper metric space.

As can be seen from Definition 6.4 and Lemma 6.5, a tree is an example of 0-hyperbolic space. In this case ∂X is called the space of ends of X and Theorem 6.17 gives the existence of the visual metric d_a on ∂X.

The parameter a_0 from Theorem 6.17 is equal to $+\infty$. Thus, on a tree $\mathcal{T}$ for any base-point o (o does not have to be the root of $\mathcal{T}$) and every visual parameter $a > 1$ the visual metric d_a is given, for every $p, q \in \partial\mathcal{T}$, by the following formula

$$d_a(p, q) = a^{-d(o,y)},$$

where $[o, y] = [o, p) \cap [o, q)$.

If in the tree $\mathcal{T}$ its base-point o is at the same time its root then the metric d_a can be interpreted as follows.

[8] It seems that the construction of the $p+1$-ary Cantor set, which is a generalization of the standard construction of the 3-ary Cantor set does not appear in literature, or at least the author did not encounter such generalization.

Let us consider two branches b and b' of a rooted tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ (with a root r) We can identify these branches with two infinite sequences of their vertices:

$$b = (v_0, v_1, \ldots) \text{ and } b' = (v'_0, v'_1, \ldots), \text{ where } v_0 = v'_0 = r \text{ and } v_i, v'_i \in \mathcal{V}.$$

Then

$$d_a([b], [b']) = a^{-\min\{i \geq 0 \colon v_i \neq v'_i\}}, \tag{7.1}$$

where $[b]$ and $[b']$ belong to $\partial\mathcal{T}$. Thus, the longer the two branches b, b' follow the same vertices of the tree $\mathcal{T}$, the smaller the distance between the tree ends $[b], [b'] \in \partial T$ determined by these branches becomes.

7.2 Fractal Conceptual Spaces of Cantor-Type Set

Here we will identify the abstract Gromov boundary of a tree with the corresponding Cantor's set.

We will start with binary trees and then generalise the construction into any p-ary tree (with prime integer p). We believe that this division into $p = 2$ and $p > 2$ cases will be clearer for the reader.

7.2.1 The Case of the Binary Tree ($p = 2$)

Let $p = 2$. We will start by defining the classical 3-ary ((p+1)-ary) Cantor set.

We start the construction from the interval $[0, 1]$. The first approximation of the Cantor set is obtained by dividing the interval $[0, 1]$ into three equal parts and removing the interior of the central part. The remaining set

$$C_3^{(1)} = [0, 1/3] \cup [2/3, 1]$$

is the sum of two intervals of length $1/3$ each.

Next from each of these two intervals we do the same, i.e. we divide each of them into 3 equal parts and remove the interiors of the middle parts. We get a union of 4 intervals of lengths $1/9$:

$$C_3^{(2)} = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

Then we repeat the procedure in a recursive way and on the n-th step we get a set $C_3^{(n)}$ of 2^n intervals of lengths $1/3^n$.

As a result we get a descending sequence of closed sets

$$C_3^{(1)} \supset C_3^{(2)} \supset \ldots C_3^{(n)} \supset C_3^{(n+1)} \ldots.$$

Definition 7.1 We define the 3-ary Cantor set as an intersection of all its approximations

$$C_3 := \bigcap_{i=1}^{\infty} C_3^{(i)}.$$

It is known that the 3-ary Cantor set C_3 is nonempty, compact and nowhere dense[9] in [0, 1].

The construction of the Cantor set C_3 proves the following lemma.

Lemma 7.2 *The real number $x \in [0, 1]$ belongs to C_3 if and only if*

$$x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}, \quad \text{where } x_i \in \{0, 2\}.$$

Thus, every element $x \in C_3$ can be identify with a sequence of $x = (x_1, x_2, \ldots)$, where $x_i \in \{0, 2\}$ but this sequence can be identify with a sequence $x' = (x'_1, x'_2, \ldots)$, where

$$x'_i = \begin{cases} 0 & \text{if } x_i = 0, \\ 1 & \text{if } x_1 = 2. \end{cases}$$

Then we can identify the sequence x' with a branch of binary tree $\mathcal{T}_{2,3}$. As we have already shown equivalence class $[x']$ is an element of the boundary of the tree $\mathcal{T}_{2,3}$.

Now, in turn, we will geometrically show how the construction of the ternary Cantor sets generates a binary tree, thus avoiding an intermediate arithmetic step (Lemma 7.2 and an argument following it).

On the other hand, this geometric approach transfers automatically for $p > 2$.

The construction of the binary tree starts with the root r, which we put on level 0. Next, the vertices at level 1 are elements of the first approximation $C_3^{(1)}$ of the ternary Cantor set. Then we connect the root to the vertices of level 1 with the edges, giving them a weight of 0 and 1 from the left.

In the second step, as tree vertices at level 2 we take the elements of the second approximation $C^{(2)}$. This approximation contains 4 elements. Each level 1 vertex is connected by two edges with two level 2 vertices and given a weight of 0 and 1 as before.

The procedure is repeated inductively receiving an infinite binary tree.

Summing up what we have done so far, we have shown the following identification of sets:

$$\{\text{concepts}\} \leftrightarrow \mathbb{Z}_2 \leftrightarrow \partial\mathcal{T}_{2,3} \leftrightarrow C_3.$$

7.2.2 Generalization to Prime Number $p > 2$

We can generalize the above procedure. Take a prime number $p \geq 2$ and divide the interval [0, 1] into $2p - 1$ subintervals of equal length. Then remove the interiors of every second subinterval starting from the second subinterval on the left-hand side. Thus we remove $p - 1$ subintervals.

In this way, we get the first approximation $C_{p+1}^{(1)}$ of the $p + 1$-ary Cantor set C_{p+1}. The set $C_{p+1}^{(1)}$ consists of p subintervals of length $1/(2p - 1)$ each.

[9] A set in a topological space $(X, \mathcal{O})$ is nowhere dense if its closure has empty interior. Equivalently, a nowhere dense set in X is a set that is not dense in any nonempty open set from $\mathcal{O}$.

Then we repeat the procedure in a recursive way and on the n-th step we get a set $C_{p+1}^{(n)}$ of p^n intervals of lengths $1/(2p-1)^n$.

As before we get a descending sequence of closed sets

$$C_{p+1}^{(1)} \supset C_{p+1}^{(2)} \supset \dots C_{p+1}^{(n)} \supset C_{p+1}^{(n+1)} \dots .$$

Finally, we define the $(p+1)$-ary Cantor set.

Definition 7.3 The $(p+1)$-ary Cantor set is the intersection of all its approximation:

$$C_{p+1} = \bigcap_{i=1}^{\infty} C_{p+1}^{(i)}.$$

Geometric identification of the $p+1$-ary Cantor set and p-ary tree obviously transfers to the case of $p > 2$ from the construction for $p = 2$. The modifications are cosmetic only.

Thus we get the following general identification:

$$\{\text{concepts}\} \leftrightarrow \mathbb{Z}_p \leftrightarrow \mathcal{T}_{p,p+1} \leftrightarrow C_{p+1}. \tag{7.2}$$

For every prime p, the $p+1$-ary Cantor set C_{p+1} is nonempty, compact and nowhere dense.

7.3 Partitioning of Fractal Conceptual Space

In this section we will show how to construct a partition of the fractal conceptual space C_{p+1} in practice.

For this purpose we will use the identity (7.2). In particular, the equivalence of a p-ary tree $\mathcal{T}_{p,p+1}$ and fractal space C_{p+1}.

Suppose that we are given a prototype P. Each prototype is described by an infinitely dimensional vector of qualities. Hence

$$P = (p_1, p_2, p_3, \dots), \quad p_j \in \{0, 1, \dots, p-1\}, \quad j = 0, 1, \dots .$$

Of course, in reality we will operate only on the finite number of qualities, which boils down to the fact that almost all elements of vectors P are equal to 0.

As we have previously noted, P can be treated as an element of the ring of p-adic integers $\mathbb{Z}_p$. Moreover, by (7.2),

$$P \text{ is an infinite branch of the } p\text{-ary tree } \mathcal{T}_{p,p+1}.$$

We select the positive integer ℓ. It will be the number of qualities we take into account. Thus, the equivalence class $[P]$ of element P shall consist of all branches $R = (r_1, r_2, r_3, \dots)$ of the p-ary tree $\mathcal{T}_{p,p+1}$ which have the same weight as P on the first ℓ edges, i.e.

$$[P] = \{R : \; p_j = r_j \quad \text{for } j = 1, \dots, \ell\}.$$

With the selected ℓ let us consider the tree $\mathcal{T}_{p,p+1}$ cut to level ℓ (level ℓ belongs to the truncated tree). Denote this truncated tree by $\mathcal{T}^{(\ell)}_{p,p+1}$. It has p^ℓ different branches. (Since there are p^ℓ different strings with elements from the set $\{0, 1, \ldots, p-1\}$, which are of length of ℓ.) These p^ℓ different branches of the finite tree $\mathcal{T}^{(\ell)}_{p,p+1}$ can be extended in an arbitrary way up to infinity, so we get p^ℓ different branches of the infinite tree $\mathcal{T}_{p,p+1}$. Let's mark them by $P^{(i)}$, $i = 1, \ldots, p^\ell$.

Therefore we have p^ℓ prototypes $P^{(i)}$. For $i = 1, \ldots p^\ell$, the equivalence class

$$[P^{(i)}] = \{R^{(i)} :\ p^{(i)}_j = r^{(i)}_j \quad \text{for } j = 1, \ldots \ell\}.$$

The ends of branches $R^{(i)}$ belonging to the equivalence class $[P^{(i)}]$ shall form a subset $\mathcal{C}^{(i)}$ of the $p+1$-ary Cantor set C_{p+1}.

If we equip the Cantor set C_{p+1} with the visual metric d_a, defined in (7.1) (here the most convenient choice of $a > 1$ is p) the elements of the set $\mathcal{C}^{(i)}$ are distant from each other for a distance not exceeding $p^{-\ell}$, that is[10] diam $\mathcal{C}^{(i)} = p^{-\ell}$.

It is clear that the family of sets $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(p^\ell)}$ is a partition of the Cantor set C_{p+1}, i.e. the sets $\mathcal{C}^{(i)}$, $i = 1, \ldots, p^\ell$ are mutually disjoint ($\mathcal{C}^{(i)} \cap \mathcal{C}^{(j)} = \emptyset$, for $i \neq j$) and

$$C_{p+1} = \bigcup_{i=1}^{p^\ell} \mathcal{C}^{(i)}.$$

Thus, the problem of the inability to classify certain points into concept domains, as was the case with the use of Vronoi diagrams by Dietz (2013) and Douven et al. (2013), does not exist in the case of a fractal conceptual space C_{p+1}.

7.4 Convexity

Gärdenfors (2000) insists that the partitioning of conceptual space should be composed of convex sets. The reason for this is that convexity is probably the simplest example of a more general concept, which is called betweennes and which reflects the intuition that if two elements belong to a given concept domain, then all the elements between them also belong to this concept domain.

In Urban and Grzelińska (2017) we postulated geodesic convexity as more appropriate for our partitioning of conceptual space. In that work, the concept domains were determined by prototypes, which were masses distributed in the Euclidean space. The concept domain was naturally a differential manifold. Hence the concept of geodesics and geodesic convexity has obviously emerged.

A controversy about convexity of concepts is also expresses in Hernandez-Conde (2017). As the example of the Gromov boundary (which is not convex) seems to show,

[10] Let us remind that for a subset A of a metric space (X, d) the diameter of A is defined as diam $A = \sup_{x,y \in A} d(x, y)$.

in the case of spaces that are not connected[11] (as the Gromov boundary), the role of convexity will be played by ordering the set, which creates a conceptual space.

Let us recall that the partial order in the set S is a relation $\leq$ on a set S that meets the following conditions:

(i) $\forall\, a \in S,\ a \leq a$ (reflexivity),
(ii) $\forall\, a, b \in S$, if $a \leq b$ and $b \leq a \Rightarrow a = b$ (antisymmetry),
(iii) $\forall\, a, b, \in S$, if $a \leq b$ and $b \leq c \Rightarrow a \leq c$.

If the partial order additionally satisfies condition

(iv) $\forall\, a, b \in S,\ a \leq b$ or $b \leq a$ (totality),

then we say that $\leq$ is a linear order (or total order) and S is called linearly ordered set (or totaly ordered set).

Because, as we will show in a moment, the boundary of the tree can be identified with a fractal subset of the interval [0, 1]. Thus, our conceptual space inherits the linear order from the real line. If we are dealing with a p-ary tree $\mathcal{T}_{p,p+1}$, then the boundary is the $p+1$-ary Cantor set C_{p+1}.

We believe that in our conceptual space, which is the boundary of a tree $\mathcal{T}_{p,p+1}$, it is the linear order inherited from the real line that seems to be the ideal equivalent of the term betweennes.

We will now show that the boundary of the tree $\partial\mathcal{T}$ can be identified with the Cantor set. As a result we show that we can talk about beetwennes on the metric space $\partial\mathcal{T}$.

We will start with the construction of the classic Cantor set corresponding to the boundary of $\partial\mathcal{T}_{2,3}$.

To sum up, in our opinion for conceptual spaces other than the Euclidean ones, the notion of convexity of concept domains must be replaced by another notion reflecting the essence of being a given element of a concept domain between two other elements of this domain. This undoubtedly depends very much on the problem under consideration, for which we are constructing a conceptual space.

8 Conclusion

In this work we defined the conceptual space as the boundary (set of ends) of a p-ary tree $\mathcal{T}_{p,p+1}$, where p is a prime number. In fact, the whole construction can be carried out without any limitation that p is the prime number. However, consistently, we have kept to this assumption for philosophical and physical rather than mathematical reasons.

The p-ary trees are connected with p-adic numbers which occupy a special place in the description of the surrounding reality, in the description of the universe, as we mentioned in the introduction to this article.

The next step we are planning is to use more general proper hyperbolic spaces to construct conceptual spaces. We hope that these more general conceptual spaces will prove to be a more precise tool for distinguishing and classifying concepts on the basis of their qualities.

[11] The topological space is called connected if it cannot be represented as a sum of two disjoint closed.

To sum up, starting from the given set of concepts, our construction was three-stage. This is illustrated in the diagram below, where the arrows indicate one to one correspondence between the given structures.

$$\text{(set of concepts)} \leftrightarrow \mathbb{Z}_p \leftrightarrow \mathcal{T}_{p,p+1} \leftrightarrow C_{p+1}.$$

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