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Equivalential Structures for Binary and Ternary Syllogistics

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Abstract The aim of this paper is to provide a contribution to the natural logic program which explores logics in natural language. The paper offers two logics called $\mathcal{R}(\forall, \exists)$ and $\mathcal{G}(\forall, \exists)$ for dealing with inference involving simple sentences with transitive verbs and ditransitive verbs and quantified noun phrases in subject and object position. With this purpose, the relational logics (without Boolean connectives) are introduced and a model-theoretic proof of decidability for they are presented. In the present paper we develop algebraic semantics (bounded meet semi-lattice) of the logics using congruence theory.

Keywords Logic of natural languages · Relational syllogistics · Binary syllogistics · Ternary syllogistics · Conguerences · Lattices

1 Introduction

Syllogistic theories have been found wide roles in different areas such as in formal logic (Corcoran 1972; Moss 2008, 2010, 2011; Pratt-Hartmann and Moss 2009; Łukasiewicz 1957), in natural language theory and generalized quantifiers (Benthem 1984; Van Eijck 1985, 2005a, b; Westerståhl 2005; D'Alfonso 2012; Van Benthem 1985), in information and lattice theory (Schroeder 2012; Schumann 2006), in algebraic structures (Sotirov 1999; Bocharov 1986; Peirce 1880; Black 1945). However, the Aristotelian syllogistic did not account for the validity of neither sentences containing transitive or ditransitive verbs. De Morgan (1847) offered some extensions of traditional syllogism with relational facts. The presentations of De Morgan did not

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include syllogisms with binary and ternary relations. Pratt-Hartmann and Moss (2009) extended syllogism with binary relations. Pratt-Hartmann and Third (2006) gave some complexity classes of deciding whether any given set of sentences including ditransitive verbs. Ivanov and Vakarelov (2012) presented a system of relational syllogistic, based on classical propositional logic. The relational variables were interpreted as transitive verbs, and the set variables-as count-nouns in Ivanov and Vakarelov (2012) and Pratt-Hartmann and Moss (2009) with linguistic perspective. The canonical model construction of Ivanov and Vakarelov (2012) is based on the Stone representation theorem for Boolean algebra.

There are two logics called binary syllogistic $\mathcal{R}(\forall, \exists)$ and ternary syllogistic $\mathcal{G}(\forall, \exists)$ in this paper. Sentences of the language of $\mathcal{R}(\forall, \exists)$ consist of two quantifiers "for all" ($\forall$) and "exists" ($\exists$), and plural nous and also transitive verbs. Sentences of the language in natural English are such as "all students see some teachers", "some television channels show all realities". On the other hand, sentences of the language of $\mathcal{G}(\forall, \exists)$ consist of two quantifiers "for all" ($\forall$) and "exists" ($\exists$), and plural nous and also ditransitive verbs. Sentences of the language in natural English are such as "all students give all pencils to all teachers", "some logicians bring all books to all libraries". Sentences like "all students see some teachers" and "all logicians bring some books to all libraries" in English are ambiguous. We use them in meaning of "there is at least one teacher who is seen by all students" and "all of logicians bring at least one book to every library". Our usage of the sentences reflects binary and ternary relational perspective directly. Formal languages of the logics are not closed in Boolean operations and does not have recursion and all models of the logics are *finite*.

1.1 A Linguistic Point that Affects the Semantics

This paper considers just the so-called informative verbs. In its atomic propositions "$QS + verb + QP$" and "$QS + verb + QP_1 +$ to $+QP_2$" where $Q \in \{some, all\}$. These verbs designate actions which can be observed and are not depended on their utterances ('to run', 'to take', etc.). However, there are also the so-called performative verbs. They are carried out only by means of uttering them aloud ('to love', 'to hate', etc.). The syllogistic for performative propositions is first introduced in Schumann (2013). In this system there are examined concepts which have no denotations at all like 'love', 'hate', etc. For these concepts, therefore, we can not define an including relation and we need a novel formal system. Some applications of that new syllogistic are proposed in Schumann and Akimova (2015).

1.2 Explanations on Algebraic Semantics

Algebraic methods for logics based upon correspondence between theorems on logical systems were fertilized by J. Lukasiewicz and A. Tarski, and were irrigated by G. Boole. In order to define an algebra for logics in the traditional sense is followed such a way: two formulas φ and ψ are equivalent if and only if $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are provable in such deductive system, and the deductive equivalence relation, usually denoted by $\varphi \equiv \psi$, is clearly an equivalence relation, and the set of equivalence classes modulo $\equiv$ can be formed naturally. Then one builds the quotient algebra up, based on the equivalent relation that is also a congruence, where the operations of this algebra are induced by the connectives of the logic. On the one hand, one of the names that has spent a considerable amount of time working on algebraic semantics (lattice-based) of logics and presented valuable works is Vakarelov with co-authors. His traces can be found at many points of this article, especially are left by his papers (Orłowska and VaKarelov 2005; Vakarelov 1977; MacCaull and Vakarelov 2005; Font and Verdu 1991) which contains boolean based sentences. Our intention differs by the methods above because our logics does not contains the connectives. As we will explain below, the congruential approach in this paper works on model-theoretic models rather than proof-theoretic semantics. The languages of our logics are not closed under and in Boolean connectives and we would like to provide a new model construction technique for the logics with a new semantic perspective. Our languages do not include the traditional Aristotelian sentences. If we considered Aristotle's with binary and ternary syllogistics together, we could make inferences as follows:

$$\frac{All\ x\ see\ all\ y\quad All\ a\ are\ y}{All\ x\ see\ all\ a}\ (i)\qquad \frac{All\ x\ see\ all\ y\quad All\ a\ are\ x}{All\ a\ see\ all\ y}\ (ii)$$

$$\frac{All\ x\ give\ all\ y\ to\ z\quad All\ a\ are\ x}{All\ a\ give\ all\ y\ to\ z}\ (iii)\qquad \frac{All\ x\ give\ all\ y\ to\ z\quad All\ a\ are\ y}{All\ x\ give\ all\ a\ to\ z}\ (iv)$$

$$\frac{All\ x\ give\ all\ y\ to\ z\quad All\ a\ are\ z}{All\ x\ give\ all\ y\ to\ a}\ (v)$$

Those kinds of inferences allow sentences in binary and ternary syllogistics to obtain nouns in their conclusions from different ones in their premises. Turning to binary and ternary syllogistics without Aristotle's sentences, the conclusions must have the same nous (in the same orders) and relations in a valid inferences. For example, although the sentences with binary relations in (i) and (ii) includes the nouns x and y, the sentences in the conclusions have the nouns x and a allowing Aristotle's sentences in the premisses [it works for ternary syllogistics as well, see (iii), (iv), (v)]. As Corcoran's syllogistic system (Corcoran 1972), the interpretation of nous does not allow to be empty set in this paper. Naturally, this situation leads up universal sentences and quantifiers entails existential sentences and quantifiers. Under the circumstances, the changes must take place on quantifiers in derivations of the syllogistics but no changes for nouns and relations. The changes for quantifiers $\forall$ and $\exists$ in the syllogistics will be represented by elements 1 and 0. Of course, we could use the quantifiers themselves instead of 1 and 0 such a way for the new representation. We did not prefer to do that way not to lead to confusion. For this purpose, we will define a translation called F^R from $\{\forall,\ \exists\}$ to $L^\wedge$. $L^\wedge$ will be a bounded meet semi-lattice so that it reflects that the existential variables follows the universal variables. On the other hand, as we will see in Sect. 2, an inference (or derivations $\vdash$) from a sentence to another

Table 1 Syntax and their natural readings

Syntax	Reading of syntax
$\forall(x, r(\forall, y))$	All x r all y
$\forall(x, r(\exists, y))$	All x r some y
$\exists(x, r(\forall, y))$	Some x r all y
$\exists(x, r(\exists, y))$	Some x r some y

sentence (the same sentence case is obvious) in logical sense shall be correspond to a kind of comparison of the representations of the two sentences under a binary relation-like $\leq$. The unchangeability of nouns and relations will force the structures to have equivalence classes. For that reason, it will be necessary to obtain restricted sets (B_{α}^{Res}) from general sets (B^{Res}) to use $\leq$ in valid inferences while doing translations and obtaining sets.

2 Binary Syllogistic $\mathcal{R}(\forall, \exists)$

In this section, we will construct the new proposed semantics with helpful definitions and translations after we present the logic $\mathcal{R}(\forall, \exists)$ and review the syntax and semantics given by Pratt-Hartmann and Moss (2009) and Ivanov and Vakarelov (2012).

Syntax The language of $\mathcal{R}(\forall, \exists)$ consists of one collection P_R of *unary atoms* (for nouns) and another collection, R_R of *binary atoms* (for transitive verbs) (Table 1).

Semantics A model $\mathcal{M}_R$ is a set M_R, together with interpretation functions

$$[[\ ]]:\ P_R \longrightarrow \mathcal{P}_{\mathcal{R}}(M_R)$$
$$[[\ ]]:\ R_R \longrightarrow \mathcal{P}(M_R \times M_R)$$

This means that for each unary atom $p \in P_R$, $[[p]] \subseteq M_R$, and for each binary atom r, $[[r]] \subseteq M_R \times M_R$. We interpret set terms by subsets of M_R in the following way:

$$[[\forall(r, y)]] = \{\, x \in M_R :\ \text{for all }\ v \in [[y]],\ (x, v) \in [[r]]\,\}$$
$$[[\exists(r, y)]] = \{\, x \in M_R :\ \text{some } v \in [[y]],\ (x, v) \in [[r]]\,\}$$

Here is how set terms are read:

$\forall(r, y)$	those who r all y
$\exists(r, y)$	those who r some y

Finally, we have the definition of truth in a model (Fig. 1):

$$\mathcal{M}_R \models \forall(p, r(\forall, y)) \text{ iff } [[p]] \subseteq [[r(\forall, y)]]$$
$$\mathcal{M}_R \models \forall(p, r(\exists, y)) \text{ iff } [[p]] \subseteq [[r(\exists, y)]]$$

$$\frac{\forall(x, \forall(r, y))}{\exists(x, \forall(r, y))}\ (1) \qquad \frac{\forall(x, \forall(r, y))}{\forall(x, \exists(r, y))}\ (2) \qquad \frac{\exists(x, \forall(r, y))}{\exists(x, \exists(r, y))}\ (3) \qquad \frac{\forall(x, \exists(r, y))}{\exists(x, \exists(r, y))}\ (4)$$

Fig. 1 Rules for binary syllogistic

$$\mathcal{M}_R \models \exists(p, r(\exists, y) \text{ iff } [[p]] \cap [[r(\exists, y)]] \neq \emptyset$$
$$\mathcal{M}_R \models \exists(p, r(\forall, y)) \text{ iff } [[p]] \cap [[r(\forall, y)]] \neq \emptyset$$

Remark 2.1 The following rule is not sound:

$$\frac{\forall(x, \exists(r, y)) \quad \forall(x, \forall(r, y))}{\exists(x, \forall(r, y))}$$

To see the unsoundness, we construct a counter-model. Suppose that $[[r]] = \{(x_1,\ y_1), (x_1,\ y_2), (x_1,\ y_3), (x_2,\ y_1), (x_2,\ y_2)\}$ and $[[y]] = \{y_1,\ y_2,\ y_3\}$ and also $[[x]] = \{x_1,\ x_2\}$. Whereas the premises are true in the model, the conclusion $\forall(x, \forall(r, y))$ is false.

Observation 2.2 *If the rule in Remark 2.1 was sound, a sentence with full universal quantifiers such as* $\forall(x, \forall(r, y))$ *would might be followed by* $\exists(x, \forall(r, y))$ *and* $\forall(x, \exists(r, y)$. *If this case hold and the logic had boolean incorporating, we would have to add an additive operation to apply this derivation to the model construction we will make and also would prefer to use* $L^{\vee}$ *rather than* $L^{\wedge}$.

Lemma 2.3 *Let* Γ *be a set of sentences. The followings are true for the logic.*

(1) $\Gamma \vdash \forall(x, \forall(r, y))$ *iff* $\forall(x, \forall(r, y)) \in \Gamma$.
(2) *If* $\Gamma \nvdash \forall(x, \forall(r, y))$ *and* $\Gamma \vdash \exists(x, \forall(r, y))$*, then* $\exists(x, \forall(r, y)) \in \Gamma$.
(3) *If* $\Gamma \nvdash \forall(x, \forall(r, y))$ *and* $\Gamma \vdash \forall(x, \exists(r, y))$*, then* $\forall(x, \exists(r, y)) \in \Gamma$.
(4) *If* $\Gamma \nvdash \forall(x, \forall(r, y))$ *and* $\Gamma \nvdash \forall(x, \exists(r, y))$ *and* $\Gamma \nvdash \exists(x, \forall(r, y))$ *and* $\Gamma \vdash \exists(x, \exists(r, y))$*, then* $\exists(x, \exists(r, y)) \in \Gamma$.

2.1 Definitions and Translations

In this subsection, we get ready to construct semantics for the logic. For this purpose, we will give some definitions and translations step by step.

Definition 2.4 $L^{\wedge} = (\{0, 1\},\ \wedge)$ such that $L^{\wedge}$ is a bounded meet-semi-lattice where $1 \wedge 0 = 0, 0 \wedge 0 = 0, 1 \wedge 1 = 0, 0 \wedge 1 = 0$ and $x \wedge y = x \ \Leftrightarrow x \leq y$ for an order theoretic binary relation $\leq$ on $\{0, 1\}$. We will use $L^{\wedge}$ in many places such Definitions 2.5, Translation 1, Translation 2, the next section and so on in order to reflect the fact that "existential clauses and sentences are derived from the universal ones".

Definition 2.5 A five tuple B^{Res} is a subset of $L^{\wedge} \times L^{\wedge} \times V \times V \times R$ where $V = \{p_0, p_1, p_2, \ldots\}$, $L^{\wedge} = (\{0, 1\}, \wedge)$, $R = \{r_0, r_1, r_2, \ldots\}$.

Fig. 2 A derivation diagram for rules (1), (2), (3) and (4)

$$\begin{tikzcd} & \forall(x,\forall(r,y)) \arrow[dl] \arrow[dr] & \\ \exists(x,\forall(r,y)) \arrow[dr] & & \exists(x,\forall(r,y)) \arrow[dl] \\ & \exists(x,\exists(r,y)) & \end{tikzcd}$$

Definition 2.6 A quotient set $[(x, y, r)]_R$ is $\{(a, b, x, y, r) : (a, b, x, y, r) \in B^{Res}\}$. A quotient space B^{Res}_α is obtained by sets of $[(x, y, r)]_R$ such $\{u \in B^{Res} : u \in \alpha\}$ where $\alpha = [(x, y, r)]_R$. It is easy to see that $B^{Res}_\alpha \subseteq B^{Res}$ for all $[\alpha]_R$. Hence, two elements of B^{Res} are in an equivalence class if and only if their third, fourth and fifth elements are the same.

Remark 2.7 The intuition underlying the definition of B^{Res}_α, valid derivations in the logic requires sentences in premises and conclusions having the same nouns and relations (verbs). In other words, there is no such valid derivation which premises and their conclusion have different nouns and relations.

Example 2.8 We illustrate Definitions 2.5 and 2.6.

For a give $B^{Res} = \{(1, 0, x, y, r_0), (1, 1, x, y, r_0), (1, 0, k, m, r_1), (0, 1, m, n, r_2), (1, 1, m, l, r_2)\}$,

Quotient sets: $[(x, y, r_0)]_R = \{(1, 0, x, y, r_0),\ (1, 1, x, y, r_0)\}$, $[(k, m, r_1)]_R = \{(1, 0, k, m, r_1)\}$,

$[(m, n, r_2)]_R = \{(0, 1, m, n, r_2)\}$, $[(m, l, r_2)]_R = \{(1, 1, m, l, r_2)\}$.

Some quotient spaces: $B^{Res}_{[(x,y,r_0)]} = \{(1, 0, x, y, r_0),\ (1, 1, x, y, r_0), (0, 0, x, y, r_0),\ (0, 1, x, y, r_0)\}$

$B^{Res}_{[(k,m,r_1)]} = \{(1, 0, k, m, r_1),\ (1, 1, k, m, r_1),\ (0, 0, k, m, r_1),\ (0, 1, k, m, r_1)\}$.

Definition 2.9 We define a binary relation $\leq^r_{res}$ on B^{Res}_α such that $x = (a, b, x, y, r_1) \leq^r_{res} y = (c, d, x, y, r_1)$ if and only if $a \leq c$ and $b \leq d$. Note that two elements of B^{Res} are comparable by the relation if and only if they belong to the same quotient set.

Observation 2.10 *$(T^{Res}_\alpha, \leq^r_{res})$ is a bounded meet semi-lattice.*

After this point, we will make restrictions and translations to show the above bounded meet semi-lattice definition and semantics of the logic we will give are appropriate. The arrows in Fig. 2 indicates "if $\Gamma \vdash \varphi$, then $\Gamma \vdash \psi$" and the arrows do not guarantee the reverse direction.

Translation 1 Let Γ be a set of sentences of binary syllogistic, P_R be a set of nouns of sentences in Γ and R_R is set of binary relations in Γ. We define a translation from quantifiers $\forall$ and $\exists$ in the language of binary syllogistic to $L^\wedge$.

$$\begin{aligned} F^R : \{\forall, \exists\} &\longrightarrow L^{\wedge} \\ \forall &\longrightarrow 1 \\ \exists &\longrightarrow 0 \end{aligned}$$

Please note that the translation is an one to one correspondence.

Translation 2 A $B^{Res}(\Gamma)$ is a restriction of B^{Res} with Γ where $B^{Res}(\Gamma) \subseteq \{1, 0\} \times \{1, 0\} \times P_\Gamma^R \times P_\Gamma^R \times R_\Gamma$. We define a mapping $[\ ]^R$ from Γ to $B^{Res}(\Gamma)$:

$$\begin{aligned} [\,]^R : \Gamma &\longrightarrow B^{Res}(\Gamma) \\ \beta &\longrightarrow [\beta]^R \\ \gamma(x, \alpha(r, y)) &\longrightarrow (F^R(\gamma), F^R(\alpha), x, y, r) \end{aligned}$$

Example 2.11 $B^{Res}(\Gamma) = \{(0, 1, x, y, r), (1, 0, u, w, h)\}$ for a given $\Gamma = \{\exists(x, \forall(r, y)), \forall(u, \exists(h, w))\}$.

Definition 2.12 $[(x, y, r)]_\Gamma = \{(a, b, x, y, r) : (a, b, x, y, r) \in B^{Res}(\Gamma)$ such that $(x, y, r) \in P_\Gamma^R \times P_\Gamma^R \times R_\Gamma\}$.

Definition 2.13 $B^{Res}_{[(x,y,r)]_\Gamma}(\Gamma)$ is a restriction of $B^{Res}(\Gamma)$ by $[(x, y, r)]_\Gamma$. We will abbreviate $B^{Res}_{[(x,y,r)]_\Gamma}(\Gamma)$ by $B^{Res}_\alpha(\Gamma)$ where $[(x, y, r)]_\Gamma$ is to be α.

Remark 2.14 We defined $B^{Res}_\alpha(\Gamma)$ in the previous definition because the binary relation $\leq^R$ that will define in the right next definition is definable on only one restriction. For example, suppose we have $B^{Res}_\alpha(\Gamma)$ and $B^{Res}_\beta(\Gamma)$ be two restrictions on for a given $B^{Res}(\Gamma)$ where $\alpha \neq \beta$. Then $\leq^R$ can not compare i and j such that $i \in B^{Res}_\alpha(\Gamma)$ and $j \in B^{Res}_\beta(\Gamma)$. On the other hand, this definition will help us to make frequent explanations for comparable elements.

Definition 2.15 We define a binary relation $\leq^R$ on $B^{Res}_\alpha(\Gamma)$ such that $(a, b, x, y, g) \leq^R (k, l, x, y, g)$ if and only if $a \leq k$ and $b \leq l$.

Example 2.16 $(0, 1, x, y, r) \leq^R (1, 1, x, y, g)$, $(0, 0, x, y, r) \leq^R (0, 1, x, y, r)$, $(0, 0, x, y, r) \leq^R (1, 0, x, y, r)$. Note that two elements of $B^{Res}(\Gamma)$ are not in same equivalence class iff the elements can not be comparable each other.

Figure 3 is a summarial illustrative demonstration of the definitions given so far. Down arrows at the top of the figure indicate that the existential nouns or fragments of sentences in the language follow the universal quantified ones. Inferences and the semantical comparisons and the mapping between them are at the bottom of the figure.

Observation 2.17 *Every $(B^{Res}_\alpha(\Gamma), \leq^R)$ is a bounded meet semi-lattice.*

Definition 2.18 A down-set of an element (a, b, x, y, r) of $B^{Res}(\Gamma)$ is a set $D^R_{Res}[(a, b, x, y, r)] = \{(k, l, x, y, r) : (k, l, x, y, r) \leq^R (a, b, x, y, r)\}$.

$$\begin{array}{c} \forall \\ \downarrow \\ \exists \end{array} \quad \xmapsto{\;F^{R}\;} \quad \begin{array}{c} 1 \\ \downarrow \\ 0 \end{array}$$

$$\begin{array}{lcl}
\forall(x,\forall(r,y)) \vdash \forall(x,\exists(r,y)) & & (1,0,x,y,r) \leq^{R} (1,1,x,y,r) \\
\forall(x,\forall(r,y)) \vdash \exists(x,\forall(r,y)) & \xmapsto{\;[\ ]^{R}\;} & (0,1,x,y,r) \leq^{R} (1,1,x,y,r) \\
\forall(x,\exists(r,y)) \vdash \exists(x,\exists(r,y)) & & (0,0,x,y,r) \leq^{R} (1,0,x,y,r) \\
\exists(x,\forall(r,y)) \vdash \exists(x,\exists(r,y)) & & (0,0,x,y,r) \leq^{R} (0,1,x,y,r)
\end{array}$$

Fig. 3 Translations F^R and $[\ \ ]^R$

Example 2.19 $D^R_{Res}[(0, 1, x, y, r)] = \{(0, 0, x, y, r), (0, 1, x, y, r)\}$, $D^R_{Res}[(0, 0, x, y, r)] = \{(0, 0, x, y, r)\}$.

Definition 2.20 $D^R_{Res}[B^{Res}(\Gamma)] = \{D^R_{Res}[i] : i \in B^{Res}(\Gamma)\}$

Lemma 2.21 *For a given $B^{Res}(\Gamma)$, the followings hold:*

(1) *If* $(0, 0, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$, *then neither of* $(1, 1, x, y, r)$, $(0, 1, x, y, r)$, $(1, 0, x, y, r)$ *in* $B^{Res}(\Gamma)$.
(2) *If* $(0, 1, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$, *then* $(1, 1, x, y, r) \notin B^{Res}(\Gamma)$.
(3) *If* $(1, 0, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$, *then* $(1, 1, x, y, r) \notin B^{Res}(\Gamma)$.

Observation 2.22 *Lemma* 2.21 *provides important information about decisions on derivations and will be used in the next lemma.*

Lemma 2.23 *For a given $B^{Res}(\Gamma)$, the followings hold:*

(1) $(1, 1, x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$ *iff* $(1, 1, x, y, r) \in B^{Res}(\Gamma)$.
(2) *If* $(1, 0, x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$ *and* $(1, 1, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$, *then* $(1, 0, x, y, r) \in B^{Res}(\Gamma)$.
(3) *If* $(0, 1, x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$ *and* $(1, 1, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$, *then* $(0, 1, x, y, r) \in B^{Res}(\Gamma)$.
(4) *If* $(0, 0, x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$ *and* $(1, 1, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$ *and* $(0, 1, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$ *and* $(1, 0, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$ *then* $(0, 0, x, y, r) \in B^{Res}(\Gamma)$.

Theorem 2.24 $\Gamma \vdash \alpha(x, \theta(r, y))$ *iff* $(F^R(\alpha), F^R(\theta), x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$, *in other words,* $\mathcal{M}_R = (M_R, [[\,]]) :\Leftrightarrow \mathcal{M}^{res}_R = (D^R_{Res}[B^{Res}(\Gamma)], \in)$.

Proof 2.24. If $\alpha(x, \theta(r, y)) \in \Gamma$, then $(F^R(\alpha), F^R(\theta), x, y, r) \in B^{Res}(\Gamma)$ and furthermore $(F^R(\alpha), F^R(\theta), x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$ by the construction. We will

Table 2 Syntax and their natural readings

Syntax	Reading of syntax
$\forall(x, g(\forall, y; \forall, z))$	All x g all y to all z
$\forall(x, g(\forall, y; \exists, z))$	All x g all y to some z
$\forall(x, g(\exists, y; \forall, z))$	All x g some y to all z
$\forall(x, g(\exists, y; \exists, z))$	All x g some y to some z
$\exists(x, g(\forall, y; \forall, z))$	Some x g all y to all z
$\exists(x, g(\exists, y; \forall, z))$	Some x g some y to all z
$\exists(x, g(\forall, y; \exists, z))$	Some x g all y to some z
$\exists(x, g(\exists, y; \exists, z))$	Some x g some y to some z

prove the theorem by considering $\alpha(x, \theta(r, y)) \notin \Gamma$ for all cases (except Case 1 due to Lemma 2.3.)

$(\Rightarrow)$:

Case 1: If $\Gamma \vdash \forall(x, \forall(r, y))$, then the proof is trivial by Lemma 2.3 (1).

Case 2: If $\Gamma \vdash \exists(x, \forall(r, y))$ and $\Gamma \nvdash \forall(x, \forall(r, y))$, then $\exists(x, \forall(r, y)) \in \Gamma$ by Lemma 2.3. Then $(0, 1, x, y, r) \in B^{Res}(\Gamma)$. Hence, $\exists(x, \forall(r, y)) \in D^R_{Res}[B^{Res}(\Gamma)]$ directly.

Case 3: If $\Gamma \vdash \forall(x, \exists(r, y))$ and $\Gamma \nvdash \forall(x, \forall(r, y))$, then $\forall(x, \exists(r, y)) \in \Gamma$ by Lemma 2.3. So, $(1, 0, x, y, r) \in B^{Res}(\Gamma)$. Therefore, $(1, 0, x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$.

Case 4: If $\Gamma \vdash \exists(x, \exists(r, y))$ and $\Gamma \nvdash \forall(x, \forall(r, y))$ and $\Gamma \nvdash \exists(x, \forall(r, y))$ and $\Gamma \nvdash \forall(x, \exists(r, y))$, then $\exists(x, \exists(r, y)) \in \Gamma$ by Lemma 2.3. We conclude $\exists(x, \exists(r, y)) \in D^R_{Res}[B^{Res}(\Gamma)]$ directly.

$(\Leftarrow)$:

Case i: $(1, 1, x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$ iff $\forall(x, \forall(r, y)) \in \Gamma$ by Lemma 2.23.

Case ii: Assume that $(1, 0, x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$ but $(1, 1, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$. Then $(1, 0, x, y, r) \in B^{Res}(\Gamma)$ by Lemma 2.23 (1). Finally, $\forall(x, \exists(r, y)) \in \Gamma$.

Case iii: Assume that $(0, 1, x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$ but $(1, 1, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$. Then $(0, 1, x, y, r) \in B^{Res}(\Gamma)$ by Lemma 2.23 (2). Finally, $\exists(x, \forall(r, y)) \in \Gamma$.

Case iv: Assume that $(0, 0, x, y, r) \in D^R_{Res}[B^{Res}(\Gamma)]$ but $(1, 0, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$ and $(0, 1, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$ and $(1, 1, x, y, r) \notin D^R_{Res}[B^{Res}(\Gamma)]$. Then $(0, 1, x, y, r) \in B^{Res}(\Gamma)$ by Lemma 2.23 (3). Finally, $\exists(x, \exists(r, y)) \in \Gamma$.

3 Ternary Syllogistic: $\mathcal{G}(\forall, \exists)$

Syntax Language of $\mathcal{G}(\forall, \exists)$ starts with one collection P_G of *unary atoms* (for nouns) and another collection, G_G of *ternary atoms* (for ditransitive verbs) (Table 2).

Semantics A model $\mathcal{M}_G$ is a set M_G, together with interpretation functions

$$[[\ ]]:\ P_G \longrightarrow \mathcal{P}(M_G)$$
$$[[\ ]]:\ G_G \longrightarrow \mathcal{P}(M_G \times M_G \times M_G)$$

This means that for each unary atom $p \in P_G$, $[[p]] \subseteq M_G$, and for each ternary atom g, $[[g]] \subseteq M_G \times M_G \times M_G$. We then interpret set terms by subsets of M_G in the following way:

$$[[g(\exists, y; \forall, z)]] = \{\, x \in M_G : \text{ for all } v \in [[Y]], \text{ for all } w \in [[z]],\ (x, v, w) \in [[g]]\,\}$$
$$[[g(\exists, y; \forall, z)]] = \{\, x \in M_G : \text{ for some } v \in [[y]], \text{ for all } w \in [[z]],\ (x, v, w) \in [[g]]\,\}$$
$$[[g(\forall, y; \exists, z)]] = \{\, x \in M_G : \text{ for all } v \in [[y]], \text{ for some } w \in [[z]],\ (x, v, w) \in [[g]]\,\}$$
$$[[g(\exists, y; \exists, z)]] = \{\, x \in M_G : \text{ for some } v \in [[y]], \text{ for some } w \in [[z]],\ (x, v, w) \in [[g]]\,\}$$

Here is how set terms are read:

$g(\forall, y; \forall, z)$	those who g all y to all z
$g(\forall, y; \exists, z)$	those who g all y to some z
$g(\exists, y; \forall, z)$	those who g some y to all z
$g(\exists, y; \exists, z)$	those who g some y to some z

Finally, we have the definition of truth in a model:

$$\mathcal{M}_G \models \forall(p, g(\forall, y; \forall, z)) \text{ iff } [[p]] \subseteq [[g(\forall, y; \forall, z)]]$$
$$\mathcal{M}_G \models \forall(p, g(\forall, y; \exists, z)) \text{ iff } [[p]] \subseteq [[g(\forall, y; \exists, z)]]$$
$$\mathcal{M}_G \models \forall(p, g(\exists, y; \forall, z)) \text{ iff } [[p]] \subseteq [[g(\exists, y; \forall, z)]]$$
$$\mathcal{M}_G \models \forall(p, g(\exists, y; \exists, z)) \text{ iff } [[p]] \subseteq [[g(\exists, y; \exists, z)]]$$
$$\mathcal{M}_G \models \exists(p, g(\forall, y; \forall, z)) \text{ iff } [[p]] \cap [[g(\forall, y; \forall, z)]] \neq \emptyset$$
$$\mathcal{M}_G \models \exists(p, g(\forall, y; \exists, z)) \text{ iff } [[p]] \cap [[g(\forall, y; \exists, z)]] \neq \emptyset$$
$$\mathcal{M}_G \models \exists(p, g(\exists, y; \forall, z)) \text{ iff } [[p]] \cap [[g(\exists, y; \forall, z)]] \neq \emptyset$$
$$\mathcal{M}_G \models \exists(p, g(\exists, y; \exists, z)) \text{ iff } [[p]] \cap [[g(\exists, y; \exists, z)]] \neq \emptyset$$

4 Proof System

It's worth reminding again, the arrows in Fig. 4 indicates "if $\Gamma \vdash \varphi$, then $\Gamma \vdash \psi$" and the arrows do not guarantee the reverse direction (Table 3).

Lemma 4.1 *Let Γ be a set of sentences. The followings are true for the logic.*

(1) $\Gamma \vdash \forall(x, g(\forall, y; \forall, z))$ *iff* $\forall(x, g(\forall, y; \forall, z)) \in \Gamma$.
(2) *If* $\Gamma \nvdash \forall(x, g(\forall, y; \forall, z))$ *and* $\Gamma \vdash \exists(x, g(\forall, y; \forall, z))$, *then* $\exists(x, g(\forall, y; \forall, z)) \in \Gamma$.
(3) *If* $\Gamma \nvdash \forall(x, g(\forall, y; \forall, z))$ *and* $\Gamma \nvdash \exists(x, g(\forall, y; \forall, z))$ *and* $\Gamma \vdash \exists(x, g(\exists, y; \forall, z))$, *then* $\exists(x, g(\exists, y; \forall, z)) \in \Gamma$.
(4) *If* $\Gamma \nvdash \forall(x, g(\forall, y; \forall, z))$ *and* $\Gamma \nvdash \exists(x, g(\forall, y; \forall, z))$ *and* $\Gamma \vdash \exists(x, g(\forall, y; \exists, z))$, *then* $\exists(x, g(\forall, y; \exists, z)) \in \Gamma$.

Table 3 Rules for ternary syllogistic

$\frac{\forall(x,g(\forall,y;\forall,z))}{\exists(x,g(\forall,y;\forall,z))}$ (1)	$\frac{\forall(x,g(\forall,y;\forall,z))}{\forall(x,g(\exists,y;\forall,z))}$ (2)	$\frac{\forall(x,g(\forall,y;\forall,z))}{\forall(x,g(\forall,y;\exists,z))}$ (3)	$\frac{\exists(x,g(\forall,y;\forall,z))}{\exists(x,g(\exists,y;\forall,z))}$ (4)
$\frac{\exists(x,g(\forall,y;\forall,z))}{\exists(x,g(\forall,y;\exists,z))}$ (5)	$\frac{\forall(x,g(\exists,y;\forall,z))}{\exists(x,g(\exists,y;\forall,z))}$ (6)	$\frac{\forall(x,g(\exists,y;\forall,z))}{\forall(x,g(\exists,y;\exists,z))}$ (7)	$\frac{\forall(x,g(\forall,y;\exists,z))}{\exists(x,g(\forall,y;\exists,z))}$ (8)
$\frac{\forall(x,g(\forall,y;\exists,z))}{\forall(x,g(\exists,y;\exists,z))}$ (9)	$\frac{\forall(x,g(\exists,y;\exists,z))}{\exists(x,g(\exists,y;\exists,z))}$ (10)	$\frac{\exists(x,g(\forall,y;\exists,z))}{\exists(x,g(\exists,y;\exists,z))}$ (11)	$\frac{\exists(x,g(\exists,y;\forall,z))}{\exists(x,g(\exists,y;\exists,z))}$ (12)

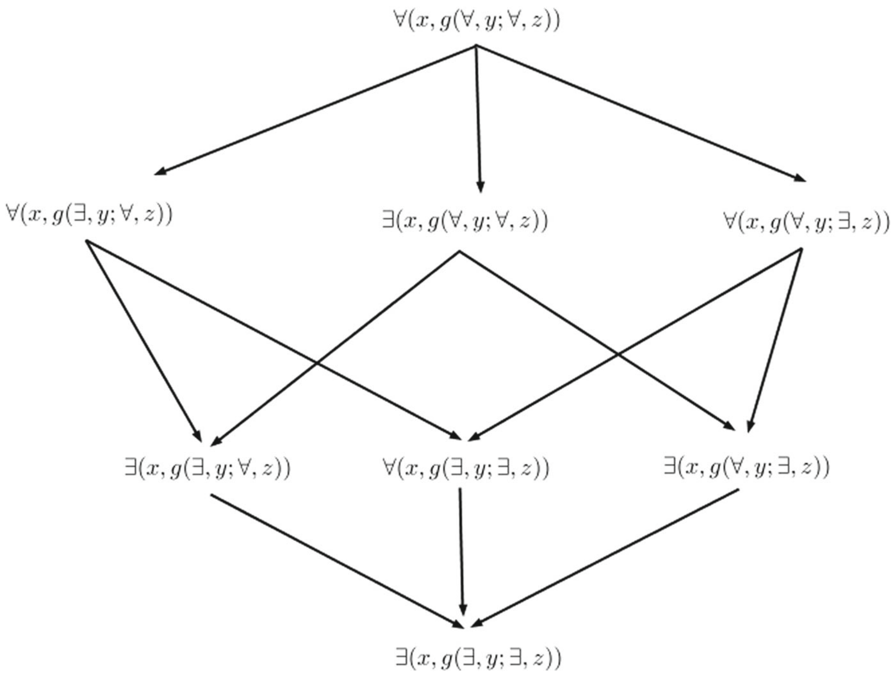

Fig. 4 A derivation diagram for rules from (1) to (12)

(5) *If* $\Gamma \nvdash \forall(x, g(\forall, y; \forall, z))$ *and* $\Gamma \nvdash \forall(x, g(\exists, y; \forall, z))$, *then* $\forall(x, g(\exists, y; \forall, z)) \in \Gamma$.
(6) *If* $\Gamma \nvdash \forall(x, g(\forall, y; \forall, z))$ *and* $\Gamma \nvdash \forall(x, g(\exists, y; \forall, z))$ *and* $\Gamma \vdash \forall(x, g(\exists, y; \exists, z))$, *then* $\forall(x, g(\exists, y; \exists, z)) \in \Gamma$.
(7) *If* $\Gamma \nvdash \forall(x, g(\forall, y; \forall, z))$ *and* $\Gamma \nvdash \forall(x, g(\exists, y; \forall, z))$ *and* $\Gamma \vdash \exists(x, g(\exists, y; \forall, z))$, *then* $\exists(x, g(\exists, y; \forall, z)) \in \Gamma$.
(8) *If* $\Gamma \nvdash \forall(x, g(\forall, y; \forall, z))$ *and* $\Gamma \nvdash \exists(x, g(\forall, y; \forall, z))$ *and* $\Gamma \nvdash \forall(x, g(\exists, y; \forall, z))$ *and* $\Gamma \nvdash \forall(x, g(\exists, y; \exists, z))$ *and* $\Gamma \nvdash \exists(x, g(\forall, y; \exists, z))$ *and* $\Gamma \nvdash \exists(x, g(\exists, y; \forall, z))$ *and* $\Gamma \vdash \exists(x, g(\exists, y; \exists, z))$ *then* $\exists(x, g(\exists, y; \exists, z)) \in \Gamma$.

4.1 Definitions and Translations

Definition 4.2 A seven tuple T^{Res} is a subset of $\{0, 1\} \times \{0, 1\} \times \{0, 1\} \times V \times V \times V \times G$ where $V = \{p_0, p_1, p_2, \ldots\}$, $L^{\wedge} = (\{0, 1\}, \wedge)$, $G = \{g_0, g_1, g_2, \ldots\}$.

Definition 4.3 If two elements of T^{Res} are in the same equivalence class, then their fourth, fifth, sixth, seventh elements are the same. $[(x, y, z, g)]_G = \{(a, b, c, x, y, z, g) : (a, b, c, x, y, z, g) \in T^{Res}$ and $(x, y, z, g) \in V \times V \times V \times G\}$. Other definition is quotient sets $T^{Res}_{\beta} = \{x \in T^{Res} : x \in [\beta]_G\}$. Note that $T^{Res}_{\beta} \subseteq T^{Res}$ for each $[\beta]_G$. Moreover, $T^{Res} = \uplus T^{Res}_{\beta}$ for all $[\beta]_G$.

Example 4.4 For a given $T^{Res} = \{(1, 0, 0, x, y, z, g_0), (1, 1, 0, x, y, z, g_0), (1, 0, 1, k, m, l, g_1), (0, 1, 0, m, n, z, g_2), (1, 1, 1, m, l, x, g_2)\}$, quotient sets of elements are $[(x, y, z, g_0)]_G = \{(1, 0, 0, x, y, z, g_0), (1, 1, 0, x, y, z, g_0)\}$, $[(k, m, l, g_1)]_G = \{(1, 0, 1, k, m, l, g_1)\}$, $[(m, n, z, g_2)]_G = \{(0, 1, 0, m, n, z, g_2)\}$, $[(m, l, x, g_2)]_G = \{(1, 1, 1, m, l, x, g_2)\}$.

Definition 4.5 $\leq^g_{res}$ is a binary relation on T^{Res}_{β} such that $x = (a, b, c, x, y, z, g_1) \leq^g_{res} y = (d, e, f, x, y, z, g_1)$ if and only if $a \leq d$ and $b \leq e$ and $c \leq f$. Note that two elements of T^{Res} are comparable if the elements belong to the same quotient set. The comparison issue forced us to make the definition T^{Res}_{β}.

Translation 1 We define a mapping from quantifiers of the language of ternary syllogistic to $L^{\wedge}$ as the following.

$$
\begin{aligned}
F^G : \{\forall, \exists\} &\longrightarrow L^{\wedge} \\
\forall &\longrightarrow 1 \\
\exists &\longrightarrow 0
\end{aligned}
$$

Translation 2 A seven tuple $T^{Res}(\Gamma)$ is a set which is T^{Res} restricted to Γ. $T^{Res}(\Gamma)$ is a subset of $\{1, 0\} \times \{1, 0\} \times \{1, 0\} \times P^G_{\Gamma} \times P^G_{\Gamma} \times P^G_{\Gamma} \times G_{\Gamma}$. We define a bijective mapping $[\ ]^G$ from Γ to $T^{Res}(\Gamma)$:

$$
\begin{aligned}
[\ ]^G : \Gamma &\longrightarrow T^{Res}(\Gamma) \\
\gamma(x, g(\alpha, y; \beta, z)) &\longrightarrow (F^G(\gamma), F^G(\alpha), F^G(\beta), x, y, z, g)
\end{aligned}
$$

Example 4.6 $T^{Res}(\Gamma) = \{(0, 1, 1, x, y, z, g), (0, 0, 1, u, v, w, h)\}$ for given $\Gamma = \{\exists(x, g(\forall, y; \forall, z)), \exists(u, h(\exists, v; \forall, w))\}$.

Definition 4.7 We define a binary relation $\leq^G_{Res}$ on $T^{Res}(\Gamma)_{\beta}$ such that $(a, b, c, x, y, z, g) \leq^G_{Res} (k, l, m, x, y, z, g)$ if $a \leq k$ and $b \leq l$ and $c \leq m$. In other words, two elements of $T^{Res}(\Gamma)$ are comparable iff they are belong to the same equivalence class.

Definition 4.8 A down-set is $D^G_{Res}[(a, b, c, x, y, z, g)] = \{(k, l, m, x, y, z, g) : (k, l, m, x, y, z, g) \leq^G_{Res} (a, b, c, x, y, z, g)\}$.

Example 4.9

$$D^G_{Res}[(0,1,0,x,y,z,g)] = \{(0,1,0,x,y,z,g), (0,0,0,x,y,z,g)\}$$
$$D^G_{Res}[(0,0,0,x,y,z,g)] = \{(0,0,0,x,y,z,g)\}$$

Definition 4.10

$$D^G_{Res}[T^G_{Res}(\Gamma)] = \{D^G_{Res}[i] : i \in T^G_{Res}(\Gamma)\}$$

Lemma 4.11 *For a given $D^R_{Res}[T^{Res}(\Gamma)]$, the followings hold:*

(1) *If* $(0,0,0,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then neither of* $(1,1,1,x,y,z,g)$, $(0,1,1,x,y,z,g)$, $(1,0,1,x,y,z,g)$, $(1,1,0,x,y,z,g)$, $(1,0,0,x,y,z,g)$, $(0,0,1,x,y,z,g)$, $(0,1,0,x,y,z,g)$ *in* $T^{Res}(\Gamma)$.
(2) *If* $(0,0,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then neither of* $(1,1,1,x,y,z,g)$, $(0,1,1,x,y,z,g)$, $(1,0,1,x,y,z,g)$ *in* $T^{Res}(\Gamma)$.
(3) *If* $(0,1,0,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then neither of* $(1,1,1,x,y,z,g)$, $(0,1,1,x,y,z,g)$, $(1,1,0,x,y,z,g)$ *in* $T^{Res}(\Gamma)$.
(4) *If* $(0,0,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then neither of* $(1,1,1,x,y,z,g)$, $(0,1,1,x,y,z,g)$, $(1,0,1,x,y,z,g)$ *in* $T^{Res}(\Gamma)$.
(5) *If* $(0,1,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then* $(1,1,1,x,y,z,g) \notin T^{Res}(\Gamma)$.
(6) *If* $(1,1,0,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then* $(1,1,1,x,y,z,g) \notin T^{Res}(\Gamma)$.
(7) *If* $(1,0,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then* $(1,1,1,x,y,z,g) \notin T^{Res}(\Gamma)$.

Lemma 4.12 *For a given $D^R_{Res}[T^{Res}(\Gamma)]$, the followings hold:*

(1) $(1,1,1,x,y,z,g) \in D^R_{Res}[T^{Res}(\Gamma)]$ *iff* $(1,1,1,x,y,z,g) \in T^{Res}(\Gamma)$.
(2) *If* $(1,1,0,x,y,z,g) \in D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(1,1,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then* $(1,1,0,x,y,z,g) \in T^{Res}(\Gamma)$.
(3) *If* $(1,0,1,x,y,z,g) \in D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(1,1,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then* $(1,0,1,x,y,z,g) \in T^{Res}(\Gamma)$.
(4) *If* $(0,1,1,x,y,z,g) \in D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(1,1,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then* $(0,1,1,x,y,z,g) \in T^{Res}(\Gamma)$.
(5) *If* $(1,0,0,x,y,z,g) \in D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(1,0,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(1,1,0,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then* $(1,0,0,x,y,z,g) \in T^{Res}(\Gamma)$.
(6) *If* $(0,1,0,x,y,z,g) \in D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(0,1,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(1,1,0,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then* $(0,1,0,x,y,z,g) \in T^{Res}(\Gamma)$.
(7) *If* $(0,0,1,x,y,z,g) \in D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(0,1,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(1,0,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then* $(0,0,1,x,y,z,g) \in T^{Res}(\Gamma)$.
(8) *If* $(0,0,0,x,y,z,g) \in D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(0,1,0,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(1,0,0,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$ *and* $(0,0,1,x,y,z,g) \notin D^R_{Res}[T^{Res}(\Gamma)]$, *then* $(0,0,0,x,y,z,g) \in T^{Res}(\Gamma)$.

Theorem 4.13 $\Gamma \vdash \alpha(x, g(\theta, y; \delta, z))$ *iff* $(F^G(\alpha), F^G(\theta), F^G(\delta), x, y, z, g) \in D^G_{Res}[T^G_{Res}(\Gamma)]$, *in other words,* $\mathcal{M}_G = (M_G, [[]]) :\Leftrightarrow \mathcal{M}^{res}_G = (D^G_{Res}[T^G_{Res}(\Gamma)], \in)$.

Proof 4.15. The proof is very similar to Theorem 2.24 and routine by using Lemmas 4.1 and 4.12. We omit the proof to save space.

5 Conclusion

This paper offers some new algebraic structures for binary and ternary syllogistics. Mappings, translations and definitions to obtain the algebraic structures show that binary, ternary and probably n-ary syllogistics can be represented with bounded meet semi-lattice structures on quotient sets. We do not account more than three-arity syllogistics because they do not take place in natural languages.

We hope that linguists, computer scientists and logicians might be interested in results in this paper and the results will help with other results in several areas. We also hope that this paper contributes to the natural logic program initiated by Van Benthem, and have been being continued by Moss and others. We would like to continue the study adding intersective adjectives to sentences with transitive and ditransitive verbs.

One may extend the syllogistic structures with boolean connectives and restricted to relative clauses without more change to find new algebraic structures.

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