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A Diagrammatic Inference System with Euler Circles

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Abstract Proof-theory has traditionally been developed based on linguistic (symbolic) representations of logical proofs. Recently, however, logical reasoning based on diagrammatic or graphical representations has been investigated by logicians. Euler diagrams were introduced in the eighteenth century. But it is quite recent (more precisely, in the 1990s) that logicians started to study them from a formal logical viewpoint. We propose a novel approach to the formalization of Euler diagrammatic reasoning, in which diagrams are defined not in terms of regions as in the standard approach, but in terms of topological relations between diagrammatic objects. We formalize the unification rule, which plays a central role in Euler diagrammatic reasoning, in a style of natural deduction. We prove the soundness and completeness theorems with respect to a formal set-theoretical semantics. We also investigate structure of diagrammatic proofs and prove a normal form theorem.

Keywords Proof-theory · Diagrammatic reasoning · Euler diagram

1 Introduction

Euler diagrams were introduced by Euler (1768) to illustrate syllogistic reasoning. In Euler diagrams, logical relations among the terms of a syllogism are simply represented by topological relations among circles. For example, the universal categorical statements of the forms *All A are B* and *No A are B* are represented by the inclusion

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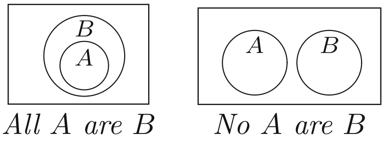


Fig. 1 Universal statements

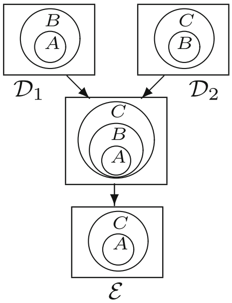


Fig. 2 Barbara with Euler diagrams

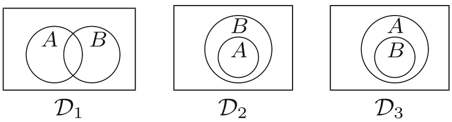


Fig. 3

and the exclusion relations between circles, respectively, as seen in Fig. 1. Given two Euler diagrams which represent the premises of a syllogism, the syllogistic inference can be naturally replaced by the task of manipulating the diagrams, in particular of unifying the diagrams and extracting information from them. For example, the well-known syllogism named "Barbara," i.e., *All A are B and All B are C; therefore All A are C*, can be represented diagrammatically as in Fig. 2.

However, things become complicated when existential statements come into the picture. In Euler's original system, any minimal region, i.e. region inside of some circles and outside of the rest of the circles (possibly none) in a diagram, is assumed to represent a non-empty set. Thus, in this system, diagram $\mathcal{D}_1$ of Fig. 3 says that three sets $A \cap B$, $A \setminus B$, and $B \setminus A$ are non-empty. This existential import destroys the simple correspondence between categorical statements and Euler diagrams (cf. Hammer and Shin 1998). For instance, *Some A are B* can be expressed by the disjunction of $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ of Fig. 3:

Venn (1881) and Peirce (1933) overcame this difficulty by removing the existential import from regions, and by introducing new syntactic devices. Venn first fixed a so-called "primary diagram" such as $\mathcal{D}_1$ of Fig. 3, which does not convey any specific information about the relation between A and B. Meaningful relations between circles are then expressed by specifying which regions are "empty" using the novel syntactic device of *shading*, which corresponds to logical negation. Observe that *All A are B* is equivalent to *There is nothing which is A but not B*, and the statement is expressed by making use of the shading as in Fig. 4. In Venn diagrams, existential claims are

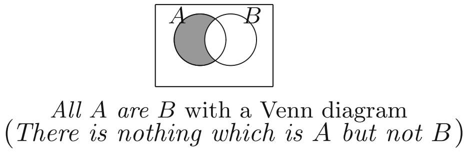


Fig. 4

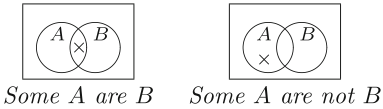


Fig. 5 Existential statements

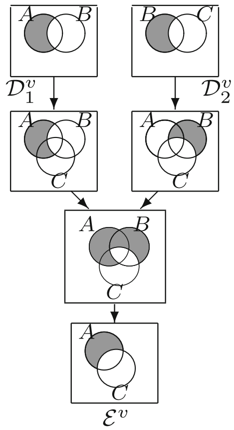


Fig. 6 Barbara with Venn diagrams

expressed by using another syntactic device, "×," which was introduced by Peirce (1933, 4. 359), and which represents non-emptiness of the corresponding region as seen in Fig. 5.

Two Venn diagrams may be combined into another Venn diagram by accommodating the labels of circles and then by superposing the shaded regions, as illustrated in Fig. 6. Because of their expressive power and their uniformity in formalizing the manipulation of combining diagrams (simply as the superposition of shadings), Venn diagrams have been very well studied; formal semantics and inference systems are given, and basic logical properties such as soundness, completeness, and decidability are shown. Cf. Venn-I, -II systems of Shin (1994), Spider diagrams SD1 and SD2 of Howse et al. (2000), Molina (2001), etc. For a recent survey, see Stapleton (2005).

However, the development of systems of Venn diagrams is obtained at the cost of clarity of the representations of Euler diagrams. As Venn (1881) himself already pointed out, when more than three circles are involved, Venn diagrams fail in their main purpose of affording intuitive and sensible illustration. (For some discussions on visual disadvantages of Venn diagrams, see Hammer and Shin 1998.) In order to make up for the shortcoming of Venn diagrams, *Euler diagrams with shading* were introduced by considering some shaded regions of Venn diagrams as "missing" regions.

E.g., Euler/Venn diagrams of Swoboda and Allwein (2004); Spider diagrams ESD2 of Molina (2001) and SD3 of Howse et al. (2005). At the concrete level of representation, the diagrams in these systems are Euler diagrams. However, their abstract syntax and semantics are still defined in terms of regions. Thus we call both Venn diagrams and Euler diagrams with shading *region-based diagrams*.

The region-based framework still fails in capturing the simplicity of representations and inferences of Euler diagrams; it has the following complications:

(1) In region-based diagrams, logical relations among circles are represented by the use of shading or missing regions. This makes the translation of categorical statements uncomfortably complex. Cf. the translation of universal categorical statements in Fig. 4.

(2) The inference rule of *unification*, which plays a central role in Euler diagrammatic reasoning, is defined by way of the superposition of Venn diagrams. For example, when we unify two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ of Fig. 2 to derive the diagram $\mathcal{E}$, they are first transformed into $\mathcal{D}_1^v$ and $\mathcal{D}_2^v$ of Fig. 6, respectively; then, by the derivation of Fig. 6, the diagram $\mathcal{E}^v$ is obtained; finally, $\mathcal{E}^v$ is transformed into $\mathcal{E}$ of Fig. 2. In this way, processes of deriving conclusions are often made complex, and hence less intuitive.

In contrast to the studies in the tradition of region-based diagrams, we propose a novel approach to formalize Euler diagrams in terms of topological relations. Our system has the following features and advantages:

(1) Our diagrammatic syntax and semantics are defined in terms of *topological relations*, inclusion and exclusion relations, between two diagrammatic objects. This formalization makes the translations of categorical sentences natural and intuitive.

(2) We decompose the unification operation into more primitive unification rules, where one of the two unified diagrams is restricted to be a minimal diagram, i.e., a diagram consisting of two objects. This enables us to define the unification directly without making a detour to Venn diagrams, and hence to capture the inference process as illustrated in Fig. 2. Also, by decomposing the unification operation, the validity of the primitive unification rules becomes immediate, and the operational meaning of them is clear. Our completeness theorem ensures that general complex diagrams, which are not necessarily minimal, may be unified by using our unification rules.

(3) We formalize the unification in the style of Gentzen's natural deduction (Gentzen 1934). This makes it possible to compare our Euler diagrammatic inference system directly with linguistic natural deduction systems. Through such a comparison, we can apply well-developed proof-theoretical techniques such as normalization of proofs to diagrammatic reasoning studies.

From a perspective of proof-theory, the contrast between the standpoints of the region-based framework and our topological-relation-based framework can be understood as follows: At the level of representation, the contrast is analogous to the one between disjunctive normal formulas and implicational formulas; at the level of reasoning, the contrast is analogous to the one between resolution calculus style proofs and natural deduction style proofs. See Mineshima et al. (2010) for a formal discussion.

From a perspective of cognitive psychology, our system is designed not just as an alternative of usual linguistic/symbolic representations; we make the best use of advantages of diagrammatic representations so that inherent definiteness or specificity of diagrams can be exploited in actual reasoning. (See Sato et al. 2010 for our cognitive experimental studies.)

In this paper, we start our study by concentrating on the following basic syntactic devices: inclusion and exclusion relations between two circles and points; crossing relations between circles, which say nothing specific about the semantic relationship between the circles as it does in Venn diagrams; named points (constant symbols) to represent the existence of particular objects. Although our basic system is weaker in its expressive power than usual Venn diagrammatic systems (e.g. Shin's Venn-II, which is equivalent to the monadic first order logic), our system is expressive enough to characterize basic logical reasoning such as syllogistic reasoning. (In Mineshima et al. 2009, we discuss natural extensions of our system.)

The rest of this paper is organized as follows. In Sect. 2, we introduce a *topological-relation-based* Euler diagrammatic representation system EUL. We give a definition of an Euler diagrammatic syntax EUL in Sect. 2.1 and a set-theoretical semantics for it in Sect. 2.2. In Sect. 3, we formalize a diagrammatic inference system GDS. We introduce two kinds of inference rules: *unification* and *deletion*. We define in Sect. 3.2 the notion of *diagrammatic proof*, which is considered as a chain of unification and deletion steps. The inference system GDS is shown in Sect. 3.3 to be sound (Theorem 3.5) and complete (Theorem 3.14). In Sect. 3.4, we discuss some consequences of completeness of GDS. In particular, a normal form theorem (Theorem 3.18) of GDS is shown.

2 A Diagrammatic Representation System EUL for Euler Circles

2.1 Diagrammatic Syntax of EUL

Let us start by defining the diagrams of EUL.

Definition 2.1 (EUL-*diagram*) An EUL-*diagram* is a plane ($\mathbb{R}^2$) with a finite number, at least two, of *named simple closed curves* [1] (simply called *named circles*, and denoted by $A, B, C, \ldots$) and *named points* (denoted by $a, b, c, \ldots$), where

- no two named simple closed curves and points are completely concurrent, and
- no two named circles and points have the same name.

Named circles and named points are collectively called *(diagrammatic) objects*, and denoted by $s, t, u, \ldots$. We use a rectangle to represent the plane for an EUL-diagram. EUL-diagrams are denoted by $\mathcal{D}, \mathcal{E}, \mathcal{D}_1, \mathcal{D}_2, \ldots$.

When $\mathcal{D}$ is an EUL-diagram, we denote by $pt(\mathcal{D})$ the set of named points of $\mathcal{D}$, by $cr(\mathcal{D})$ the set of named circles of $\mathcal{D}$, by $ob(\mathcal{D})$ the set of objects of $\mathcal{D}$, i.e., $ob(\mathcal{D}) = pt(\mathcal{D}) \cup cr(\mathcal{D})$.

[1] See Blackett (1983) for a formal definition of simple closed curves on $\mathbb{R}^2$.

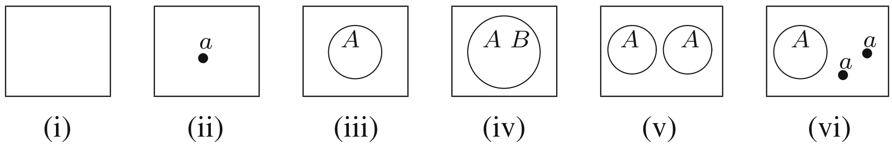


Fig. 7 Non-well-formed diagrams of EUL

Examples of non well-formed diagrams are given in Fig. 7. (i), (ii), (iii) consists of less than two objects; in (iv), named circles A and B are completely concurrent, i.e., located at the same place; in (v) and (vi), two objects have the same name.

Note that any two objects are spatially distinct in a diagram by definition.

Definition 2.2 (*Minimal diagram*) An EUL-diagram consisting of only two objects is called a *minimal diagram*. Minimal diagrams are denoted by $\alpha, \beta, \gamma, \ldots$.

We study mathematical properties of EUL-diagrams in terms of the following topological relations between two diagrammatic objects:

Definition 2.3 (EUL-*relation*) EUL-relations are the following binary relations between distinct diagrammatic objects:

$A \sqsubset B$	"the interior[2] of A is *inside of* the interior of B,"
$A \vdash\!\dashv B$	"the interior of A is *outside of* the interior of B,"
$A \bowtie B$	"there is at least one *crossing* point between A and B,"
$b \sqsubset A$	"b is *inside of* the interior of A,"
$b \vdash\!\dashv A$	"b is *outside of* the interior of A,"
$a \vdash\!\dashv b$	"a is *outside of* b (i.e. a is not equal to b)."

EUL-relations $\vdash\!\dashv$ and $\bowtie$ are symmetric, while $\sqsubset$ is not. Note that all EUL-relations are irreflexive.

Proposition 2.4 *Let $\mathcal{D}$ be an* EUL-*diagram. For any distinct objects s and t of $\mathcal{D}$, exactly one of the* EUL-*relations $s \sqsubset t$, $t \sqsubset s$, $s \vdash\!\dashv t$, $s \bowtie t$ holds.*

Observe that, by Proposition 2.4, for a given EUL-diagram $\mathcal{D}$, the set of EUL-relations holding on $\mathcal{D}$ is uniquely determined. We denote the set by $\mathsf{rel}(\mathcal{D})$.

The following properties, as well as Proposition 2.4, characterize EUL-diagrams.

Lemma 2.5 *Let $\mathcal{D}$ be an* EUL-*diagram. Then for any objects (named circles and points) $s, t, u \in ob(\mathcal{D})$, we have the following:*

1. (Transitivity) *If $s \sqsubset t$, $t \sqsubset u \in \mathsf{rel}(\mathcal{D})$, then $s \sqsubset u \in \mathsf{rel}(\mathcal{D})$.*
2. ($\vdash\!\dashv$-downward closedness) *If $s \vdash\!\dashv t$, $u \sqsubset s \in \mathsf{rel}(\mathcal{D})$, then $u \vdash\!\dashv t \in \mathsf{rel}(\mathcal{D})$.*
3. (Point determinacy) *For any $x \in pt(\mathcal{D})$ other than s, exactly one of $x \sqsubset s$ and $x \vdash\!\dashv s$ is in $\mathsf{rel}(\mathcal{D})$.*
4. (Point minimality) *For any $x \in pt(\mathcal{D})$, $s \sqsubset x \notin \mathsf{rel}(\mathcal{D})$.*

[2] Here, the interior of a named circle A means the region strictly inside of A (cf. Blackett 1983).

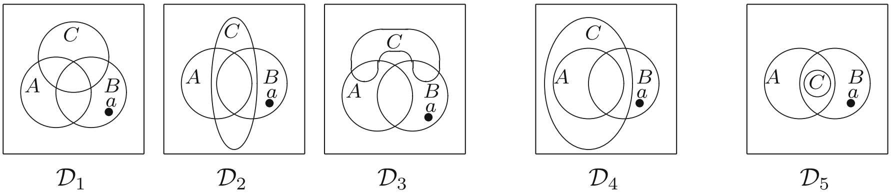


Fig. 8 Equivalence of EUL-diagrams

In order to study mathematical properties of our diagrammatic system, we consider *equivalence classes of diagrams*. Our equivalence relation among EUL-diagrams is defined in terms of EUL-relations as follows.

Definition 2.6 (*Equivalence among* EUL*-diagrams*) Any EUL-diagrams $\mathcal{D}$ and $\mathcal{E}$ are *syntactically equivalent* when $\mathsf{rel}(\mathcal{D}) = \mathsf{rel}(\mathcal{E})$.

For example, diagrams $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ of Fig. 8 are equivalent since exactly the same EUL-relations $A \bowtie B$, $A \bowtie C$, $B \bowtie C$, $a \vdash\!\dashv A$, $a \sqsubset B$, and $a \vdash\!\dashv C$ hold on them. (See Mineshima et al. 2009 for extensions of our representation system EUL, where $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are distinguished by regarding intersection, union, and complement regions respectively as diagrammatic objects.) On the other hand, $\mathcal{D}_1$ and $\mathcal{D}_4$ (resp. $\mathcal{D}_1$ and $\mathcal{D}_5$) are not equivalent since different EUL-relations hold on them: $A \sqsubset C$ holds on $\mathcal{D}_4$ in place of $A \bowtie C$ of $\mathcal{D}_1$ (resp. $C \sqsubset A$ and $C \sqsubset B$ hold on $\mathcal{D}_5$ in place of $A \bowtie C$ and $C \bowtie B$ of $\mathcal{D}_1$).

Our equation of diagrams may be explained in terms of a kind of "continuous transformation (deformation)" of named circles, which does not change any of the EUL-relations in a diagram. The named circle C in $\mathcal{D}_1$ of Fig. 8 can be continuously transformed, without changing the EUL-relations with A, with B and with a in such a way that C covers (resp. is disjoint from) the intersection region of A and B as it does in $\mathcal{D}_2$ (resp. in $\mathcal{D}_3$).

In what follows, the diagrams which are syntactically equivalent are identified, and they are referred to by a single name.

2.2 Set-Theoretical Semantics of EUL

In this section, we give a formal semantics for EUL. We adopt the standard set-theoretical semantics.[3] Intuitively, each circle is interpreted as a set of elements of a given domain, and each point is interpreted as an element of the domain. However, observe that each point of EUL can be considered as a special circle which does not contain, nor cross, any other objects. This observation enables us to interpret the EUL-relations $\sqsubset$ and $\vdash\!\dashv$ uniformly as the subset relation and the disjointness relation, respectively.

Definition 2.7 (*Model*) Let $\mathcal{D}$ be an EUL-diagram. Let M be a pair (U, I), where U is a non-empty set (the domain of M), and $I : ob(\mathcal{D}) \longrightarrow \mathcal{P}(U)$ is an interpretation

[3] For similar set-theoretical approaches to semantics of Euler diagrams, see Howse et al. (2005), Hammer (1995), Swoboda and Allwein (2004) etc. Our semantics is distinct from theirs in that diagrams are interpreted in terms of binary relations, and not every region in a diagram has a meaning.

function which assigns to each diagrammatic object a non-empty subset of U such that

- $I(x)$ is a singleton for any named point x, and
- $I(x) \neq I(y)$ for any points x, y of distinct names.

$M = (U, I)$ is a *model of* $\mathcal{D}$, written as $M \models \mathcal{D}$, if the following truth-conditions (1) and (2) hold: For all objects s, t of $\mathcal{D}$,

(1) $I(s) \subseteq I(t)$ if $s \sqsubset t$ holds on $\mathcal{D}$,
(2) $I(s) \cap I(t) = \emptyset$ if $s \vdash\dashv t$ holds on $\mathcal{D}$.

Note that we assign a non-empty set to each named circle. Note also that when s is a named point a, for some $e \in U$, $I(a) = \{e\}$, and the above $I(a) \subseteq I(t)$ of (1) is equivalent to $e \in I(t)$. Similarly, $I(a) \cap I(t) = \emptyset$ of (2) is equivalent to $e \notin I(t)$.

The well-definedness of the truth-conditions follows from Proposition 2.4.

Remark 2.8 (*Semantic interpretation of* $\bowtie$*-relation*) By Definition 2.7, the EUL-relation $\bowtie$ does not contribute to the truth-condition of EUL-diagrams. Informally speaking, $s \bowtie t$ may be understood as $I(s) \cap I(t) = \emptyset$ *or* $I(s) \cap I(t) \neq \emptyset$, which is true in any model.

Definition 2.9 (*Validity*) An EUL-diagram $\mathcal{E}$ is a *semantically valid consequence* of EUL-diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$, written as $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}$, when the following holds: For any model M, if $M \models \mathcal{D}_1$ and ...and $M \models \mathcal{D}_n$, then $M \models \mathcal{E}$.

3 Diagrammatic Inference System GDS

In this section, we introduce Generalized Diagrammatic Syllogistic inference system GDS for the EUL-diagrams defined in Sect. 2.1. There are two inference rules of GDS: *unification* and *deletion*. We first give an informal explanation of our unification in Sect. 3.1, and we then formalize it in Sect. 3.2. In Sect. 3.3 our GDS is shown to be sound and complete with respect to our set-theoretical semantics. In Sect. 3.4, we discuss some consequences of the completeness theorem of GDS. In particular, we define a class of *normal diagrammatic proofs* of GDS and we show a normal form theorem.

3.1 Introduction to Unification

Before giving a formal description of our diagrammatic inference system, we motivate our inference rule *unification*. Let us consider the following question: Given diagrams $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ of Fig. 9, what diagrammatic information on A, B and c can be obtained by the conjunction of the given diagrams? (In what follows, in order to avoid notational complexity in a diagram, we express each named point, say $\overset{c}{\bullet}$, simply by its name c). Figures 9, 10, and 11 represent the three ways of solving the question.

In Fig. 9, at the first step, $\mathcal{D}_1$ and $\mathcal{D}_2$ are unified to obtain $\mathcal{D}_1 + \mathcal{D}_2$, where the point c in $\mathcal{D}_1$ and $\mathcal{D}_2$ are identified, and B is added to $\mathcal{D}_1$ so that c is inside of B and B overlaps with A without any implication of a relationship between A and B. Then, $\mathcal{D}_1 + \mathcal{D}_2$ is combined with another diagram $\mathcal{D}_3$ to obtain $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$. Note that

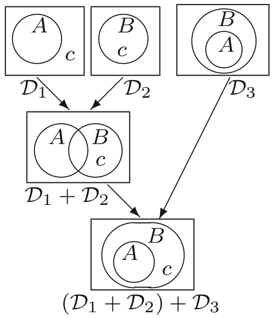


Fig. 9

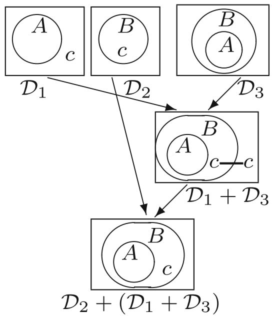


Fig. 10

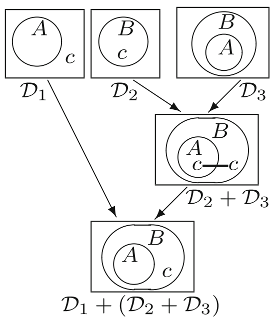


Fig. 11

the diagrams $\mathcal{D}_1 + \mathcal{D}_2$ and $\mathcal{D}_3$ share two circles A and B: $A \bowtie B$ holds on $\mathcal{D}_1 + \mathcal{D}_2$ and $A \sqsubset B$ holds on $\mathcal{D}_3$. Since the semantic information of $A \sqsubset B$ on $\mathcal{D}_3$ is more specific than that of $A \bowtie B$ on $\mathcal{D}_1 + \mathcal{D}_2$, according to our semantics of EUL (recall that $A \bowtie B$ means just "true" in our semantics), one keeps the relation $A \sqsubset B$ in the unified diagram $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$. Observe that the unified diagram represents the information of these diagrams $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, that is, their *conjunction*.

Figures 10 and 11, illustrate other procedures to solve the question. At the first step of unifying $\mathcal{D}_1$ and $\mathcal{D}_3$ in Fig. 10 (and $\mathcal{D}_2$ and $\mathcal{D}_3$ in Fig. 11), there are two possible positions of the point c. Such disjunctive ambiguities may be represented by Peirce's linking of points (cf. Peirce 1933; Shin 1994) as illustrated in Figs. 10 and 11.

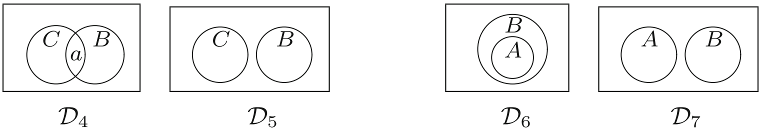


Fig. 12 Inconsistency

However, in order to formalize the most basic diagrammatic system, we keep our diagrams free from such disjunctive ambiguity, and we impose some constraint on unification, called the constraint for *determinacy*: Any two diagrams are not permitted to be unified when the relations between each point and all circles of the two diagrams are not determined. Thus $\mathcal{D}_1$ and $\mathcal{D}_3$ of Fig. 10 (respectively $\mathcal{D}_2$ and $\mathcal{D}_3$ of Fig. 11) are not permitted to be unified.

We impose another constraint on unification called a constraint for *consistency*, in order to avoid complexity due to conflicting graphical information represented in a single diagram.[4] For example, it is not permitted to unify two diagrams $\mathcal{D}_4$ and $\mathcal{D}_5$ when, as is shown in Fig. 12, they share two circles C and B such that $a \sqsubset C$ and $a \sqsubset B$ hold on $\mathcal{D}_4$ and $C \vdash\!\dashv B$ holds on $\mathcal{D}_5$. Note that these relations $a \sqsubset C, a \sqsubset B$, and $C \vdash\!\dashv B$ are incompatible in the same diagram. The diagrams $\mathcal{D}_6$ and $\mathcal{D}_7$ in Fig. 12 are also not permitted to be unified in our system. Recall that each circle is interpreted by a non-empty set in our semantics of Definition 2.7, and hence relations $A \sqsubset B$ and $A \vdash\!\dashv B$ are also incompatible.

3.2 Generalized Diagrammatic Syllogistic Inference System GDS

We formalize our unification of two diagrams by restricting one of them to be a *minimal diagram*, except for one rule called the Point Insertion-rule. Our completeness (Theorem 3.14) ensures that any diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ may be unified, under the constraints for determinacy and consistency, into one diagram whose semantic information is equivalent to the conjunction of that of $\mathcal{D}_1, \ldots, \mathcal{D}_n$. (We will return to this issue in Sect. 3.4.1.)

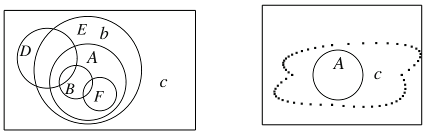
We give a formal description of inference rules in terms of EUL-relations. Given a diagram $\mathcal{D}$ and a minimal diagram α, the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ for the unified diagram $\mathcal{D} + \alpha$ is defined.

The unification rules are divided into three groups, Group (I), (II), and (III). The rules in Group (I) and (II) are classified according to the number and type of objects shared by a diagram $\mathcal{D}$ and a minimal diagram α. In Group (I), $\mathcal{D}$ and α share one object. The rules in this group are further divided into two types: those in which one point is shared (U1–U2 rules) and those in which one circle is shared (U3–U8 rules).

[4] In place of our syntactic constraint, it is possible to allow unification of inconsistent diagrams by introducing an inference rule corresponding to the absurdity rule of Gentzen's natural deduction system: We can infer any diagram from a pair of inconsistent diagrams. Such a rule is introduced in, for example, Howse et al. (2005) for spider diagrams; Hammer and Danner (1996) for Venn diagrams; Swoboda and Allwein (2004) for Euler/Venn diagrams. However, such a rule requires a linguistic symbol, say ⊥, or some arbitrary convention to represent inconsistency, and hence we prefer our syntactic constraint in our framework of a diagrammatic inference system.

Each rule is specified by the relation holding on α, and has a constraint for determinacy. In Group (II), $\mathcal{D}$ and α share two circles (hence α consists of two circles). We distinguish two rules in this group (U9 and U10 rules), depending on whether $A \sqsubset B$ or $A \vdash\!\dashv B$ holds on α. Both rules have a constraint for consistency. The rule in Group (III) is Point Insertion rule, where neither of two premise diagrams is restricted to be minimal.

For a better understanding of the unification rule, we also give a schematic diagrammatic representation and a concrete example of each rule. In the schematic representation of diagrams, to indicate the occurrence of some objects in a context on a diagram, we write the indicated objects explicitly and indicate the context by "dots" as in the diagram to the right below. For example, when we need to indicate only A and c on the left hand diagram, we could write it as shown on the right.



Definition 3.1 (*Inference rules of* GDS) *Axiom*, *unification*, and *deletion* of GDS are defined as follows.

Axiom:

A1: For any circles A and B, any minimal diagram where $A \bowtie B$ holds is an axiom.
A2: Any EUL-diagram which consists only of, at least two, points is an axiom.

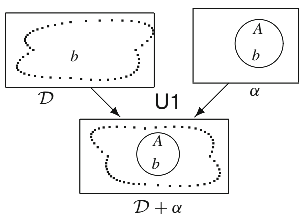
Unification: We denote by $\mathcal{D} + \alpha$ the unified diagram of $\mathcal{D}$ with a minimal diagram α. $\mathcal{D} + \alpha$ is defined when $\mathcal{D}$ and α share one or two objects.

(I) $\mathcal{D}$ and α share one object:

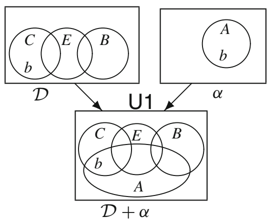
U1 rule **Premises**: $b \sqsubset A$ holds on α, and $b \in pt(\mathcal{D})$.
Constraint for determinacy: $pt(\mathcal{D})$ is the singleton $\{b\}$.
Conclusion: The set $\mathsf{rel}(\mathcal{D} + \alpha)$ of the unified diagram is as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}.$$

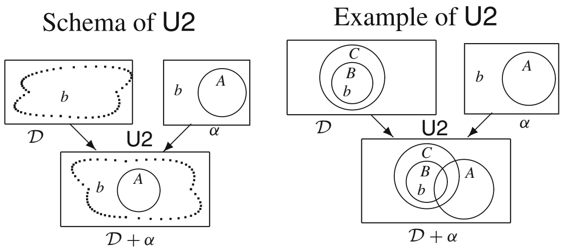
Schema of U1



Example of U1

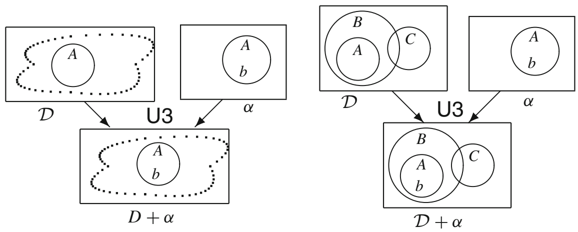


U2 rule **Premises:** $b \vdash\!\dashv A$ holds on α, and $b \in pt(\mathcal{D})$.
Constraint for determinacy: $pt(\mathcal{D})$ is the singleton $\{b\}$.
Conclusion: $\mathsf{rel}(\mathcal{D} + \alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$

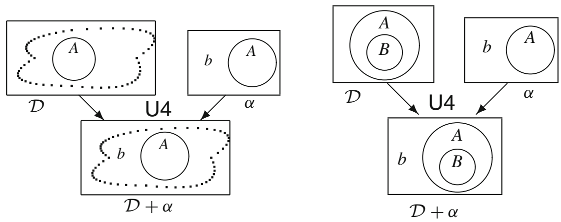


U3 rule **Premises**: $b \sqsubset A$ holds on α, and $A \in cr(\mathcal{D})$.
Constraint for determinacy: $A \sqsubset X$ or $A \vdash\!\dashv X$ holds for all circles X of $\mathcal{D}$.
Conclusion: $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\mathsf{rel}(\mathcal{D}) \ \cup \ \mathsf{rel}(\alpha) \cup \ \{b \sqsubset X \mid A \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \\ \cup \ \{b \vdash\!\dashv X \mid A \vdash\!\dashv X \in \mathsf{rel}(\mathcal{D})\} \cup \ \{b \vdash\!\dashv x \mid x \in pt(\mathcal{D})\}$$

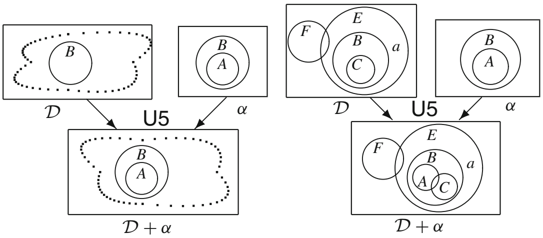


U4 rule **Premises**: $b \vdash\!\dashv A$ holds on α, and $A \in cr(\mathcal{D})$.
Constraint for determinacy: $X \sqsubset A$ holds for all circles X of $\mathcal{D}$.
Conclusion:
$\mathsf{rel}(\mathcal{D}{+}\alpha) {=} \mathsf{rel}(\mathcal{D}) {\cup} \mathsf{rel}(\alpha) {\cup} \{b \vdash\!\dashv X \mid X \sqsubset A {\in} \mathsf{rel}(\mathcal{D})\} {\cup} \{b \vdash\!\dashv x \mid x {\in} pt(\mathcal{D})\}$



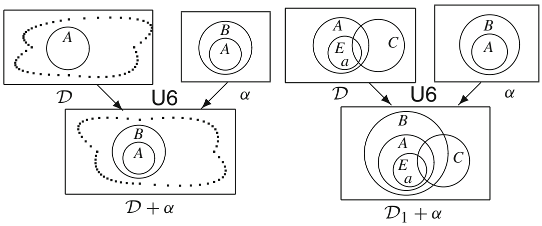
U5 rule **Premises**: $A \sqsubset B$ holds on α, and $B \in cr(\mathcal{D})$.
Constraint for determinacy: $x \vdash\!\dashv B$ holds for all $x \in pt(\mathcal{D})$.
Conclusion: $\mathsf{rel}(\mathcal{D} + \alpha)$ is as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \sqsubset B \text{ or } X \bowtie B \in \mathsf{rel}(\mathcal{D})\} \cup \{A \sqsubset X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \{A \vdash\!\dashv X \mid X \vdash\!\dashv B \in \mathsf{rel}(\mathcal{D})\} \cup \{x \vdash\!\dashv A \mid x \in pt(\mathcal{D})\}$$



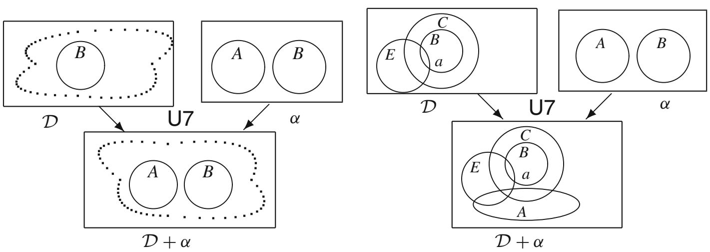
U6 rule **Premises**: $A \sqsubset B$ holds on α, and $A \in cr(\mathcal{D})$.
Constraint for determinacy: $x \sqsubset A$ holds for all $x \in pt(\mathcal{D})$.
Conclusion: $\mathsf{rel}(\mathcal{D} + \alpha)$ is as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{X \bowtie B \mid A \sqsubset X \text{ or } A \vdash\!\dashv X \text{ or } A \bowtie X \in \mathsf{rel}(\mathcal{D})\} \cup \{X \sqsubset B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{x \sqsubset B \mid x \in pt(\mathcal{D})\}$$

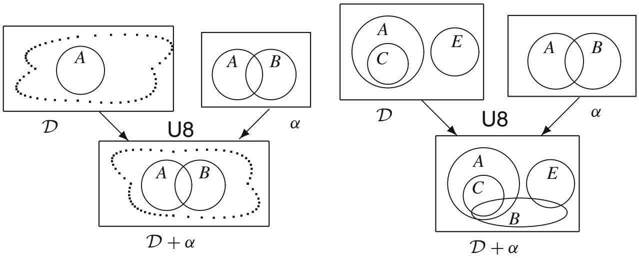


U7 rule **Premises**: $A \vdash\!\dashv B$ holds on α, and $B \in cr(\mathcal{D})$.
Constraint for determinacy: $x \sqsubset B$ holds for all $x \in pt(\mathcal{D})$.
Conclusion: $\mathsf{rel}(\mathcal{D} + \alpha)$ is as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid B \sqsubset X \text{ or } B \vdash\!\dashv X \text{ or } B \bowtie X \in \mathsf{rel}(\mathcal{D})\} \cup \{X \vdash\!\dashv A \mid X \sqsubset B \in \mathsf{rel}(\mathcal{D})\} \cup \{x \vdash\!\dashv A \mid x \in pt(\mathcal{D})\}$$



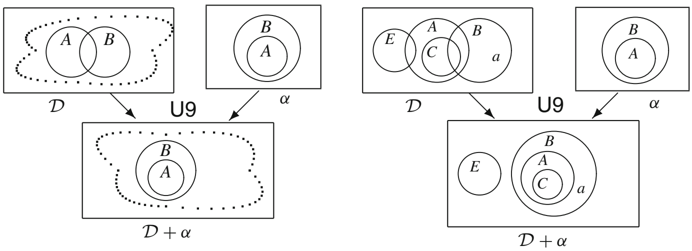
U8 rule **Premises**: $A \bowtie B$ holds on α, and $A \in cr(\mathcal{D})$.
Constraint for determinacy: $pt(\mathcal{D}) = \emptyset$.
Conclusion: $\mathsf{rel}(\mathcal{D} + \alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{B \bowtie X \mid X \in cr(\mathcal{D})\}$



(II) $\mathcal{D}$ and α share two circles:

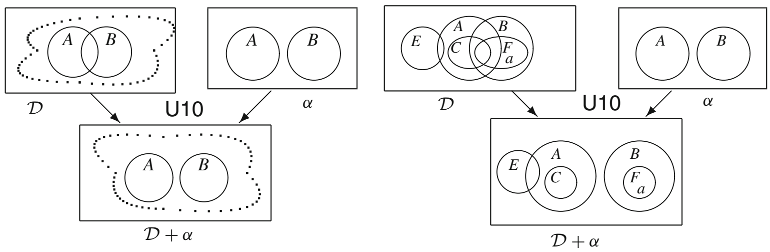
U9 rule **Premises**: $A \sqsubset B$ holds on α, and $A \bowtie B$ holds on $\mathcal{D}$.
Constraint for consistency: There is no object s such that $s \sqsubset A$ and $s \vDash B$ hold on $\mathcal{D}$.
Conclusion: $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\begin{aligned}\big(\mathsf{rel}(\mathcal{D}) \ \setminus \ & \{A \bowtie B\} \setminus \{A \bowtie X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \setminus \{A \bowtie X \mid B \vDash X \in \mathsf{rel}(\mathcal{D})\} \\ \setminus \ & \{X \bowtie B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \setminus \{Y \bowtie X \mid Y \sqsubset A \text{ and } B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \\ \setminus \ & \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \vDash B \in \mathsf{rel}(\mathcal{D})\}\big) \\ \cup \ & \{A \sqsubset B\} \cup \{A \sqsubset X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \{A \vDash X \mid B \vDash X \in \mathsf{rel}(\mathcal{D})\} \\ \cup \ & \{X \sqsubset B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{Y \sqsubset X \mid Y \sqsubset A \text{ and } B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \\ \cup \ & \{X \vDash Y \mid X \sqsubset A \text{ and } Y \vDash B \in \mathsf{rel}(\mathcal{D})\}\end{aligned}$$



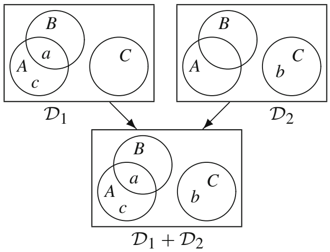
U10 rule **Premises**: $A \vDash B$ holds on α, and $A \bowtie B$ holds on $\mathcal{D}$.
Constraint for consistency: There is no object s such that $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{D}$.
Conclusion: $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\begin{aligned}\big(\mathsf{rel}(\mathcal{D}) \setminus & \{A \bowtie B\} \setminus \{X \bowtie B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \setminus \{X \bowtie A \mid X \sqsubset B \in \mathsf{rel}(\mathcal{D})\} \\ \setminus & \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}\big) \\ \cup & \{A \vDash B\} \cup \{X \vDash B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{X \vDash A \mid X \sqsubset B \in \mathsf{rel}(\mathcal{D})\} \\ \cup & \{X \vDash Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}\end{aligned}$$



(III) Neither of two premise diagrams is restricted to be minimal:

Point Insertion **Premises**: $X\Box Y \in \mathsf{rel}(\mathcal{D}_1)$ *iff* $X\Box Y \in \mathsf{rel}(\mathcal{D}_2)$ holds for any circles X, Y with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$, and $pt(\mathcal{D}_2) = \{b\}$ such that $b \notin pt(\mathcal{D}_1)$.
Conclusion: $\mathsf{rel}(\mathcal{D}_1 + \mathcal{D}_2) = \mathsf{rel}(\mathcal{D}_1) \cup \mathsf{rel}(\mathcal{D}_2) \cup \{b \vdash\!\dashv x \mid x \in pt(\mathcal{D}_1)\}$



Deletion **Premise**: $\mathcal{D}$ contains an object s.
Constraint: $\mathcal{D}$ is not minimal.
Conclusion: $\mathsf{rel}(\mathcal{D} - s) = \mathsf{rel}(\mathcal{D}) \setminus \{s\Box t \mid t \in ob(\mathcal{D}), \Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}\}$

Definition 3.2 (*Diagrammatic proofs of* GDS) A *diagrammatic proof* (or *d-proof*, for short) π of GDS is defined inductively as follows:

1. An axiom is a d-proof of itself.
2. A diagram $\mathcal{D}$ is a d-proof from the premise $\mathcal{D}$ to the conclusion $\mathcal{D}$.
3. Let π_1 be a d-proof from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ to $\mathcal{F}$ and π_2 be a d-proof from $\mathcal{E}_1, \ldots, \mathcal{E}_m$ to $\mathcal{E}$, respectively. If $\mathcal{D}$ is obtained by an application of unification to $\mathcal{F}$ and $\mathcal{E}$, then the following is a d-proof π from $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}_1, \ldots, \mathcal{E}_m$ to $\mathcal{D}$ in GDS.
$$\begin{array}{ccc} \pi_1 & & \pi_2 \\ \mathcal{F} & & \mathcal{E} \\ & \searrow \swarrow & \\ & \mathcal{D} & \end{array}$$
4. Let π_1 be a d-proof from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ to $\mathcal{E}$. If $\mathcal{D}$ is obtained by an application of deletion to $\mathcal{E}$, then the following is a d-proof π from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ to $\mathcal{D}$ in GDS.
$$\begin{array}{c} \pi_1 \\ \mathcal{E} \\ \downarrow \\ \mathcal{D} \end{array}$$

Here $\overset{\pi}{\mathcal{D}}$ means a d-proof π with $\mathcal{D}$ as the conclusion.

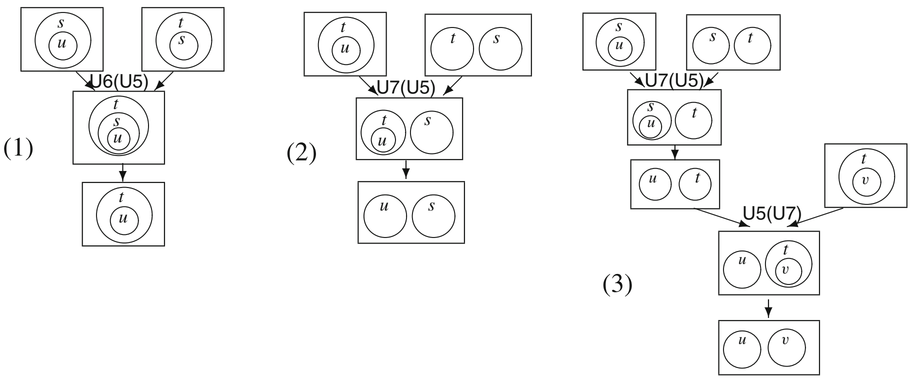
The *height* of a d-proof is defined as the maximum length of the branches in the underlying tree, where the length of a branch is the number of applications of inference rules.

Definition 3.3 (*Provability*) Let Γ be a set of EUL-diagrams. An EUL-diagram $\mathcal{E}$ is *provable* from Γ, written as $\Gamma \vdash \mathcal{E}$, if there is a d-proof of $\mathcal{E}$ in GDS from a sequence $\mathcal{D}_1, \ldots, \mathcal{D}_m$ such that $\mathcal{D}_i \in \Gamma$. We call Γ (resp. $\mathcal{E}$) premise (resp. conclusion) diagrams.

Lemma 3.4 *The following hold in* GDS*:*

1. *If* $\Gamma \vdash u \sqsubset s$ *and* $\Gamma \vdash s \sqsubset t$*, then* $\Gamma \vdash u \sqsubset t$*;*
2. *If* $\Gamma \vdash u \sqsubset t$ *and* $\Gamma \vdash s \vdash\!\dashv t$*, then* $\Gamma \vdash u \vdash\!\dashv s$*;*
3. *If* $\Gamma \vdash u \sqsubset s$ *and* $\Gamma \vdash s \vdash\!\dashv t$ *and* $\Gamma \vdash v \sqsubset t$*, then* $\Gamma \vdash u \vdash\!\dashv v$.

Proof Immediate by the following d-proofs.



□

Our description of unification rules was given in a static way, i.e., in terms of the set of relations. Alternatively, our unification rules can be described operationally. Recall that each unification rule is applied to a diagram $\mathcal{D}$ and a minimal diagram α. From an operational point of view, α may be considered to be an instruction on how to modify the diagram $\mathcal{D}$ into a diagram $\mathcal{D} + \alpha$ by (i) adding an object (U1–U8) or (ii) rearranging the configuration of objects (U9, U10). Although we shall not discuss details of implementation in this paper, let us illustrate with U1 and U9 rules. The other rules can be implemented in similar ways: U2, U5–U8, in which a circle is added, are similar to U1; U10, in which some $\bowtie$-relations are changed, is similar to U9; and for U3–U4, in which a point is added, the location of named point to be added is determined by our constraint for determinacy.

In U1 rule of Definition 3.1, a circle A is added to $\mathcal{D}$ so that $b \sqsubset A$ and $A \bowtie X$ hold for all $X \in cr(\mathcal{D})$. An implementation of such an operation is given by Stapleton et al. (2008), and is further developed in Stapleton et al. (2011). In Fig. 13, we roughly sketch their idea using the example of an application of U1 rule given in Definition 3.1. For details, we refer to the construction given in Stapleton et al. (2008).

In U9 rule, a given diagram $\mathcal{D}$, where $A \bowtie B$ holds, is modified into $\mathcal{D} + \alpha$ so that $A \sqsubset B$ holds. We sketch, in Fig. 14, an implementation of such modification of $\mathcal{D}$.

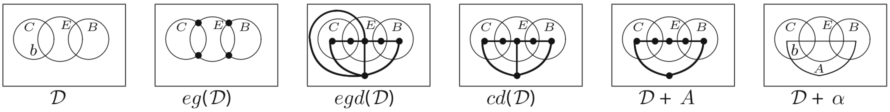


Fig. 13 Implementation of U1 rule. We first delete the point b from a given diagram $\mathcal{D}$. We then regard the diagram as an *Euler graph* $eg(\mathcal{D})$, whose vertexes are the crossing points of curves, and whose edges are the curve segments that connect the vertexes. Then by taking a dual graph of $eg(\mathcal{D})$, we obtain an *Euler graph dual* as in $egd(\mathcal{D})$. A maximal subgraph of the Euler graph dual that contains all vertexes but no multiple edges defines a *concrete dual* as in $cd(\mathcal{D})$. Then, by finding a Hamiltonian cycle of the concrete dual, we obtain a diagram $\mathcal{D}+A$. We finally obtain $\mathcal{D}+\alpha$ by adding the point b to the appropriate region, which is determined due to the constraint for determinacy

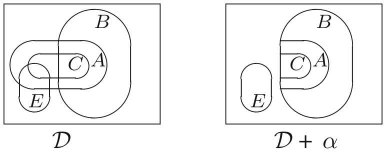


Fig. 14 Implementation of U9 rule. In a given diagram $\mathcal{D}$ where $A \bowtie B$ holds, we take any circle X that is inside A, and eliminate the curve segments of A and X that are outside B. Then we obtain $\mathcal{D}+\alpha$

3.3 Soundness and Completeness of GDS

We prove soundness (Theorem 3.5) and completeness (Theorem 3.14) of GDS with respect to our formal semantics.

In what follows, we sometimes refer to any minimal diagram, say α where $s \sqsubset t$ holds, by the EUL-relation holding on it, as $s \sqsubset t$.

Theorem 3.5 (Soundness of GDS) *Let* $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$ *be* EUL*-diagrams. If* $\mathcal{D}_1, \ldots, \mathcal{D}_n \vdash \mathcal{E}$ *in* GDS*, then* $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}$*.*

Proof By induction on the height of a given d-proof as usual. □

For the completeness, we impose the following condition for premise diagrams:

Definition 3.6 (*Semantic consistency*) A set $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of diagrams is *semantically consistent* if there is a model M such that $M \models \mathcal{D}_i$ for any $1 \leq i \leq n$.

Without this condition, any diagram, say $\mathcal{E}$ where $A \vdash\dashv C$ holds, is a valid consequence of an inconsistent set of premise diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ where $a \sqsubset B$ and $a \vdash\dashv B$ hold, respectively, although there is no d-proof of $\mathcal{E}$ from $\mathcal{D}_1$ and $\mathcal{D}_2$ in GDS. (Cf. also footnote 4.)

It is obvious that the soundness theorem (Theorem 3.5) also holds under the assumption of the semantic consistency of the premise diagrams. The following is an important consequence of semantic consistency:

Lemma 3.7 (Semantic consistency) *Let* $\vec{\alpha}$ *be a set of minimal diagrams which is semantically consistent. Then none of the following holds in* GDS *for any objects* s *and* t*:*

1. $\vec{\alpha} \vdash s \sqsubset t$ *and* $\vec{\alpha} \vdash s \mathrel{\text{⊣⊢}} t$.
2. *There is an object* u *such that* $\vec{\alpha} \vdash s \mathrel{\text{⊣⊢}} t$ *and* $\vec{\alpha} \vdash u \sqsubset s$ *and* $\vec{\alpha} \vdash u \sqsubset t$.

Given an EUL-diagram $\mathcal{E}$ and two objects, say s and t, on $\mathcal{E}$, a minimal diagram is obtained from $\mathcal{E}$ by deleting all objects other than s and t. By Proposition 2.4, the set of such minimal diagrams of $\mathcal{E}$ is uniquely determined. According to our semantics, the set of minimal diagrams is semantically equivalent to the original diagram $\mathcal{E}$. Hence, the premise $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}$ of the completeness is equivalent to saying that $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \beta$ for any minimal diagram β which corresponds to some relation holding on $\mathcal{E}$. Thus we first show atomic completeness (Proposition 3.13), which restrict the conclusion diagram to be minimal. Then using such provable minimal diagrams, we give a canonical way to construct a d-proof of $\mathcal{E}$.

In order to show the completeness theorem of GDS, we construct two kinds of syntactic models, called *canonical models*, in a similar way as the construction of Lindenbaum algebras in the literature of algebraic semantics for various propositional logics. We first define the simpler one.

Definition 3.8 (*Canonical model* $M_{\vec{\alpha}}$) Let $\vec{\alpha}$ be a set of minimal diagrams which is semantically consistent. A canonical model $M_{\vec{\alpha}} = (M_{\vec{\alpha}}, I_{\vec{\alpha}})$ for $\vec{\alpha}$ is defined as follows:

- The domain $M_{\vec{\alpha}}$ is the set of diagrammatic objects (named circles and points) which occur in any minimal diagram $\alpha \in \vec{\alpha}$.
- $I_{\vec{\alpha}}$ is an interpretation function such that, for any object t,

$$I_{\vec{\alpha}}(t) = \{s \mid \vec{\alpha} \vdash s \sqsubset t \text{ in GDS}\} \cup \{t\}.$$

Observe that in the above definition of $I_{\vec{\alpha}}$, when t is a named point, say a, its interpretation $I_{\vec{\alpha}}(a)$ is the singleton $\{a\}$ since $\vec{\alpha} \nvdash s \sqsubset a$ for any object s by soundness (Theorem 3.5).

Lemma 3.9 (Canonical model $M_{\vec{\alpha}}$) *Let* $\vec{\alpha}$ *be a set* $\alpha_1, \ldots, \alpha_n$ *of minimal diagrams which is semantically consistent. Then* $M_{\vec{\alpha}}$ *is a model of* $\vec{\alpha}$.

Proof We show that $M_{\vec{\alpha}} \models \alpha_i$ for each $\alpha_i \in \vec{\alpha}$ $(1 \le i \le n)$. The case $\alpha_i = s \bowtie t$ is trivial. Otherwise, we divide into the following cases according to the form of α_i:

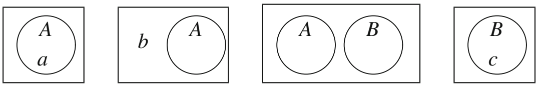
1. When $\alpha_i \in \vec{\alpha}$ is $s \sqsubset t$, we have $\vec{\alpha} \vdash s \sqsubset t$ in GDS. We show $M_{\vec{\alpha}} \models s \sqsubset t$, i.e., $I_{\vec{\alpha}}(s) \subseteq I_{\vec{\alpha}}(t)$. Let $u \in I_{\vec{\alpha}}(s)$.
   (a) When $u \equiv s$, we immediately have $s \in I_{\vec{\alpha}}(t)$ by the fact $\vec{\alpha} \vdash s \sqsubset t$.
   (b) Otherwise, by the definition of $I_{\vec{\alpha}}(s)$, we have $\vec{\alpha} \vdash u \sqsubset s$. By composing it with $\vec{\alpha} \vdash s \sqsubset t$ as seen in Lemma 3.4(1), we have $\vec{\alpha} \vdash u \sqsubset t$ in GDS, that is, $u \in I_{\vec{\alpha}}(t)$.
2. When $\alpha_i \in \vec{\alpha}$ is $s \mathrel{\text{⊣⊢}} t$, we have $\vec{\alpha} \vdash s \mathrel{\text{⊣⊢}} t$ in GDS. We show $M_{\vec{\alpha}} \models s \mathrel{\text{⊣⊢}} t$, i.e., $I_{\vec{\alpha}}(s) \cap I_{\vec{\alpha}}(t) = \emptyset$. When both s and t are points, the claim is trivial. Otherwise, assume to the contrary that some $u \in I_{\vec{\alpha}}(s) \cap I_{\vec{\alpha}}(t)$.
   (a) When $u \equiv s$, we have $s \in I_{\vec{\alpha}}(t)$, i.e., $\vec{\alpha} \vdash s \sqsubset t$. This, together with $\vec{\alpha} \vdash s \mathrel{\text{⊣⊢}} t$, is a contradiction by Lemma 3.7(1).

(b) The same applies to the case $u \equiv t$.
(c) Otherwise, $s \not\equiv u \not\equiv t$, and we have $\vec{\alpha} \vdash u \sqsubset s$ and $\vec{\alpha} \vdash u \sqsubset t$ by the definition of $I_{\vec{\alpha}}(s)$ and $I_{\vec{\alpha}}(t)$. They contradict $\vec{\alpha} \vdash s \mathrel{\vdash\!\dashv} t$ by Lemma 3.7(2).

□

As an illustration of the canonical model, let us consider the following example.

Example 3.10 (*Canonical model $M_{\vec{\alpha}}$*) Let $\vec{\alpha}$ be the following minimal diagrams $\alpha_1, \alpha_2, \alpha_3, \alpha_4$:



α_1 α_2 α_3 α_4

Observe that we have $\vec{\alpha} \nvdash b \sqsubset B$ and $\vec{\alpha} \nvdash b \mathrel{\vdash\!\dashv} B$. In such a case, we say that the point b is indeterminate with respect to the circle B. Let us construct the canonical model for the $\vec{\alpha}$ by defining: $I_{\vec{\alpha}}(A) = \{A, a\}$ and $I_{\vec{\alpha}}(B) = \{B, c\}$. Note that the indeterminate point b w.r.t. B is not contained in the interpretation $I_{\vec{\alpha}}(B)$ of B. With this interpretation, for any named point $x \in I_{\vec{\alpha}}(B)$, we have $\vec{\alpha} \vdash x \sqsubset B$ (i.e., for $c \in I_{\vec{\alpha}}(B), \vec{\alpha} \vdash c \sqsubset B$). In general, validity of $\sqsubset$-relation in the model $M_{\vec{\alpha}}$ imply provability of $\sqsubset$-relation.

In the above model, however, $x \notin I_{\vec{\alpha}}(B)$ does not necessarily imply $\vec{\alpha} \vdash x \mathrel{\vdash\!\dashv} B$; because we do not have $\vec{\alpha} \vdash b \mathrel{\vdash\!\dashv} B$, while $b \notin I_{\vec{\alpha}}(B)$. Thus, in the canonical model $M_{\vec{\alpha}}$ of Definition 3.8, validity of $\mathrel{\vdash\!\dashv}$-relation does not imply provability of $\mathrel{\vdash\!\dashv}$-relation, and hence the model is not enough to establish completeness. Let us try to modify the above model $M_{\vec{\alpha}}$ so that the indeterminate point b w.r.t. B is contained in the interpretation $I'_{\vec{\alpha}}(B)$ of B: $I'_{\vec{\alpha}}(A) = \{A, a\}$ and $I'_{\vec{\alpha}}(B) = \{B, c, b\}$. This definition also provides a model of $\vec{\alpha}$, and we have $\vec{\alpha} \vdash x \mathrel{\vdash\!\dashv} B$ for any named point $x \notin I'_{\vec{\alpha}}(B)$. However, in this model, $x \in I'_{\vec{\alpha}}(B)$ does not necessarily imply $\vec{\alpha} \vdash x \sqsubset B$; because we do not have $\vec{\alpha} \vdash b \sqsubset B$, while $b \in I'_{\vec{\alpha}}(B)$.

Although the above two kinds of models alone are insufficient to establish completeness, we can obtain our completeness result in the following manner: we construct the model $M_{\vec{\alpha}}$ of Definition 3.8 for validity of $\sqsubset$-relation, which implies provability of $\sqsubset$-relation, and the model $M_{\vec{\alpha},B}$ of the following Definition 3.11 for validity of $\mathrel{\vdash\!\dashv}$-relation, which implies provability of $\mathrel{\vdash\!\dashv}$-relation.

Definition 3.11 (*Canonical model $M_{\vec{\alpha},B}$*) Let $\vec{\alpha}$ be a set of minimal diagrams which is semantically consistent. Let B be a fixed named circle. A canonical model $M_{\vec{\alpha},B} = (M_{\vec{\alpha},B}, I_{\vec{\alpha},B})$ for $\vec{\alpha}$ is defined as follows:

- The domain $M_{\vec{\alpha},B}$ is the same set as $M_{\vec{\alpha}}$ of Definition 3.8.
- $I_{\vec{\alpha},B}$ is an interpretation function defined as follows: For any object t, when $t \equiv B$ or $\vec{\alpha} \vdash B \sqsubset t$ holds,
  $I_{\vec{\alpha},B}(t) = I_{\vec{\alpha}}(t) \cup \{s \mid \vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \nvdash s \sqsubset B$ and $\vec{\alpha} \nvdash s \mathrel{\vdash\!\dashv} B\}$;
  otherwise, $I_{\vec{\alpha},B}(t) = I_{\vec{\alpha}}(t)$.

As seen in Definition 3.8, observe that $I_{\vec{\alpha},B}(a) = \{a\}$ when a is a named point. Note also that $I_{\vec{\alpha},B}(t)$ is equal to $I_{\vec{\alpha}}(t)$ of Definition 3.8 when $\vec{\alpha} \nvdash B \sqsubset t$.

Let us show that $M_{\vec{\alpha},B}$ is a model of $\vec{\alpha}$.

Lemma 3.12 (Canonical model $M_{\vec{\alpha},B}$) *Let $\vec{\alpha}$ be a set $\alpha_1, \ldots, \alpha_n$ of minimal diagrams which is semantically consistent. Let B be a fixed named circle. Then $M_{\vec{\alpha},B}$ is a model of $\vec{\alpha}$.*

Proof We show that, for each $\alpha_i \in \vec{\alpha}$ $(1 \le i \le n)$, $M_{\vec{\alpha},B} \models \alpha_i$. The case $\alpha_i = s \bowtie t$ is trivial. Otherwise, we divide into the following cases according to the form of α_i. We sometimes write $\vec{\alpha} \nvdash s \square t$ when none of $\vec{\alpha} \vdash s \sqsubset t, \vec{\alpha} \vdash t \sqsubset s,$ and $\vec{\alpha} \vdash s \vdash\!\dashv t$ holds.

1. When $\alpha_i \in \vec{\alpha}$ is $s \sqsubset t$, we have $\vec{\alpha} \vdash s \sqsubset t$. We show $I_{\vec{\alpha},B}(s) \subseteq I_{\vec{\alpha},B}(t)$. Let $u \in I_{\vec{\alpha},B}(s)$.
   (a) When $u \equiv s$, by the fact $\vec{\alpha} \vdash s \sqsubset t$, we have $s \in I_{\vec{\alpha},B}(t)$ by the definition of $I_{\vec{\alpha},B}(t)$.
   (b) Otherwise ($u \not\equiv s$), we divide into the following two cases according to s and B:
      (i) When $s \equiv B$ or $\vec{\alpha} \vdash B \sqsubset s$ hold, by the definition of $I_{\vec{\alpha},B}(s)$, we have (i-1) $\vec{\alpha} \vdash u \sqsubset s$ or (i-2) $\vec{\alpha} \nvdash u \square B$. (i-1) implies, together with $\vec{\alpha} \vdash s \sqsubset t$, that $\vec{\alpha} \vdash u \sqsubset t$, by Lemma 3.4(1), i.e., $u \in I_{\vec{\alpha},B}(t)$. For (i-2), $\vec{\alpha} \vdash B \sqsubset s$ and $\vec{\alpha} \vdash s \sqsubset t$ imply $\vec{\alpha} \vdash B \sqsubset t$ by Lemma 3.4(1). Hence, in conjunction with $\vec{\alpha} \nvdash u \square B$, we have $u \in I_{\vec{\alpha},B}(t)$ by definition.
      (ii) When $s \not\equiv B$ and $\vec{\alpha} \nvdash B \sqsubset s$, by the definition of $I_{\vec{\alpha},B}(s)$, we have $\vec{\alpha} \vdash u \sqsubset s$. Hence this case is the same as (i-1).
2. When $\alpha_i \in \vec{\alpha}$ is $s \vdash\!\dashv t$, we have $\vec{\alpha} \vdash s \vdash\!\dashv t$. We assume $s \not\equiv B \not\equiv t$ since the other cases are similar. We show that $I_{\vec{\alpha},B}(s) \cap I_{\vec{\alpha},B}(t) = \emptyset$. When both s and t are points, the claim is trivial. Otherwise, assume to the contrary that some $u \in I_{\vec{\alpha},B}(s) \cap I_{\vec{\alpha},B}(t)$.
   (a) When $u \equiv s$, we have $s \in I_{\vec{\alpha},B}(t)$. We divide into the following two cases according to whether or not $\vec{\alpha} \vdash B \sqsubset t$ holds:
      (i) When $\vec{\alpha} \vdash B \sqsubset t$ holds, by the definition of $I_{\vec{\alpha},B}(t)$, we have (i-1) $\vec{\alpha} \vdash s \sqsubset t$ or (i-2) $\vec{\alpha} \nvdash s \square B$. Case (i-1) contradicts $\vec{\alpha} \vdash s \vdash\!\dashv t$. For (i-2), from $\vec{\alpha} \vdash s \vdash\!\dashv t$ and $\vec{\alpha} \vdash B \sqsubset t$, we have, by Lemma 3.4(2), $\vec{\alpha} \vdash s \vdash\!\dashv B$, which contradicts $\vec{\alpha} \nvdash s \square B$.
      (ii) When $\vec{\alpha} \nvdash B \sqsubset t$, we have $\vec{\alpha} \vdash s \sqsubset t$ by the definition of $I_{\vec{\alpha},B}(t)$, which contradicts $\vec{\alpha} \vdash s \vdash\!\dashv t$.
   (b) The same applies to the case $u \equiv t$.
   (c) Otherwise ($s \not\equiv u \not\equiv t$), we divide into the following cases: (i) $\vec{\alpha} \vdash B \sqsubset s$ and $\vec{\alpha} \vdash B \sqsubset t$; (ii) $\vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \nvdash B \sqsubset t$; (iii) $\vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \vdash B \sqsubset t$; (iv) $\vec{\alpha} \vdash B \sqsubset s$ and $\vec{\alpha} \nvdash B \sqsubset t$. (i) contradicts $\vec{\alpha} \vdash s \vdash\!\dashv t$. For (ii), by the definitions of $I_{\vec{\alpha},B}(s)$ and $I_{\vec{\alpha},B}(t)$, we have $\vec{\alpha} \vdash u \sqsubset s$ and $\vec{\alpha} \vdash u \sqsubset t$, which contradict $\vec{\alpha} \vdash s \vdash\!\dashv t$. For (iii), by the definition of $I_{\vec{\alpha},B}(s)$, we have $\vec{\alpha} \vdash u \sqsubset s$. By the definition of $I_{\vec{\alpha},B}(t)$, we have (iii-1) $\vec{\alpha} \vdash u \sqsubset t$ or (iii-2) $\vec{\alpha} \nvdash u \square B$. (iii-1), together with $\vec{\alpha} \vdash u \sqsubset s$, contradicts $\vec{\alpha} \vdash s \vdash\!\dashv t$. For (iii-2), $\vec{\alpha} \vdash u \sqsubset s$, $\vec{\alpha} \vdash s \vdash\!\dashv t$, and $\vec{\alpha} \vdash B \sqsubset t$ imply, by Lemma 3.4(3), that $\vec{\alpha} \vdash u \vdash\!\dashv B$, which contradicts $\vec{\alpha} \nvdash u \square B$. (iv) is similar to (iii). □

Using the two kinds of canonical models introduced so far, we prove the following *atomic completeness*, from which completeness (Theorem 3.14) of GDS is derived.

When Γ is a set $\mathcal{D}_1, \dots, \mathcal{D}_n$ of diagrams, we sometimes write $M \models \Gamma$ for the formula $\forall_{1 \le i \le n}(M \models \mathcal{D}_i)$.

Proposition 3.13 (Atomic completeness) *Let $\mathcal{D}_1, \dots, \mathcal{D}_n$ be a set of* EUL*-diagrams which is semantically consistent. Let β be a minimal diagram. If $\mathcal{D}_1, \dots, \mathcal{D}_n \models \beta$, then $\mathcal{D}_1, \dots, \mathcal{D}_n \vdash \beta$ in* GDS.

Proof We first consider the case where the premise diagrams $\mathcal{D}_1, \dots, \mathcal{D}_n$ are restricted to minimal diagrams $\alpha_1, \dots, \alpha_n$. Then we extend to the general case. We denote by $\vec{\alpha}$ the set of given minimal diagrams. Assume $\vec{\alpha} \models \beta$. When β is $s \bowtie t$, we immediately have $\vec{\alpha} \vdash s \bowtie t$ since it is an axiom. Otherwise, we divide into the following two cases according to the form of β.

(1) When β is of the form $s \sqsubset t$, by the assumption $\vec{\alpha} \models s \sqsubset t$, we have, in particular for the canonical model of Definition 3.8, $M_{\vec{\alpha}} \models \vec{\alpha} \Rightarrow M_{\vec{\alpha}} \models s \sqsubset t$. Then by Lemma 3.9, we have $M_{\vec{\alpha}} \models s \sqsubset t$, i.e., $I_{\vec{\alpha}}(s) \subseteq I_{\vec{\alpha}}(t)$. Since $s \in I_{\vec{\alpha}}(s)$ by Definition 3.8, we have $s \in I_{\vec{\alpha}}(t)$, that is, $\vec{\alpha} \vdash s \sqsubset t$ in GDS.

(2) When β is of the form $s \mapsto t$, observe that if s and t are both points, then the assertion is trivial since β is an axiom in that case. Otherwise, we assume, without loss of generality, that t is a named circle B. By the assumption $\vec{\alpha} \models s \mapsto B$, we have, in particular for the canonical model of Definition 3.11, $M_{\vec{\alpha},B} \models \vec{\alpha} \Rightarrow M_{\vec{\alpha},B} \models s \mapsto B$. Then by Lemma 3.12, we have $M_{\vec{\alpha},B} \models s \mapsto B$, i.e., $I_{\vec{\alpha},B}(s) \cap I_{\vec{\alpha},B}(B) = \emptyset$. Hence we have $s \notin I_{\vec{\alpha},B}(B)$ and $B \notin I_{\vec{\alpha},B}(s)$. Then by the definition of $I_{\vec{\alpha},B}(B)$ and $I_{\vec{\alpha},B}(s)$ of Definition 3.11, we have $\vec{\alpha} \nvdash s \sqsubset B$, and $\vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \vdash s \Box B$ for some $\Box \in \{\sqsubset, \sqsupset, \mapsto\}$. Therefore, we have $\vec{\alpha} \vdash s \mapsto B$ in GDS.

Next, we extend the premises to general diagrams $\mathcal{D}_1, \dots, \mathcal{D}_n$ instead of minimal diagrams $\vec{\alpha}$. Let $\mathcal{D}_1, \dots, \mathcal{D}_n \models \beta$. Then, by the definition of our semantics, it is equivalent to the fact that, for any model M, $M \models \Gamma_1 \wedge \cdots \wedge M \models \Gamma_n \Rightarrow M \models \beta$, where Γ_i is a set of all minimal diagrams whose relations hold on $\mathcal{D}_i$. Thus there is a sequence $\alpha_1, \dots, \alpha_k$ of minimal diagrams such that each relation holding on α_j $(1 \le j \le k)$ holds on some $\mathcal{D}_i$ $(1 \le i \le n)$ and $\alpha_1, \dots, \alpha_k \models \beta$. Then there is a d-proof from $\alpha_1, \dots, \alpha_k$ to β in GDS. Since each α_j is derived from some $\mathcal{D}_i$ by some applications of Deletion rule, we have $\mathcal{D}_1, \dots, \mathcal{D}_n \vdash \beta$. □

By extending the conclusion diagram β of atomic completeness to a general (not restricted to minimal) diagram $\mathcal{E}$, we establish the completeness of GDS.

Theorem 3.14 (Completeness of GDS) *Let $\mathcal{D}_1, \dots, \mathcal{D}_n, \mathcal{E}$ be* EUL*-diagrams. Let $\mathcal{D}_1, \dots, \mathcal{D}_n$ be semantically consistent. If $\mathcal{D}_1, \dots, \mathcal{D}_n \models \mathcal{E}$, then $\mathcal{D}_1, \dots, \mathcal{D}_n \vdash \mathcal{E}$ in* GDS.

Proof Using the atomic completeness theorem, we construct a d-proof of $\mathcal{E}$ from the given premise diagrams $\mathcal{D}_1, \dots, \mathcal{D}_n$ in a canonical way (see also Example 3.15 given after this proof):

(I) From the premise diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$, by using atomic completeness and U1, U2-rules, we first construct EUL-diagrams such that each of them consists of a point and all circles of $\mathcal{E}$, and in each of them $A \bowtie B$ holds for any pair of circles.
A diagram is called a *Venn-like diagram* when $A \bowtie B$ holds for any pair of circles in it.

(II) Then, by unifying all Venn-like diagrams of (I) with the Point Insertion rule, we construct a Venn-like diagram consisting of all points and circles of $\mathcal{E}$.

(III) By using atomic completeness, we construct d-proofs for all point-free minimal diagrams in each of which a relation $A \sqsubset B$ or $A \vdash\!\dashv B$ of $\mathcal{E}$ holds.

(IV) We then construct a diagram $\mathcal{F}$, by unifying the minimal diagrams of (III) and the Venn-like diagram of (II) with U9 and U10-rules.

(V) Finally, we check that the diagram $\mathcal{F}$ of (IV) coincides with the conclusion $\mathcal{E}$.

A diagrammatic proof is called a *canonical diagrammatic proof* when it is constructed in accordance with the above canonical construction.

We now formalize the above (I)–(V). We denote by Γ the set $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of the given premise diagrams.

(I) *For each point $a \in pt(\mathcal{E})$, let $P_a = \{a \square X \mid a \square X \in \mathsf{rel}(\mathcal{E}),\ \square \in \{\sqsubset, \vdash\!\dashv\}\}$. Then the set P_a gives rise to an* EUL*-diagram $\mathcal{P}_a$ such that $\Gamma \vdash \mathcal{P}_a$ in* GDS.

Proof of (I) Let $R_1, \ldots, R_n$ be an enumeration of the elements of P_a, and $\beta_1, \ldots, \beta_n$ be the corresponding minimal diagrams where R_i holds on β_i. Note that all β_i share the same point a and they differ only in their circles. The assumption $\Gamma \models \mathcal{E}$ of completeness implies $\Gamma \models \beta_i$ since $R_i \in \mathsf{rel}(\mathcal{E})$. Hence we have $\Gamma \vdash \beta_i$ in GDS by Proposition 3.13. Then starting from β_1, by successively applying U1-rule (when β_i is $a \sqsubset B_i$ for $1 < i \leq n$) or U2-rule (when β_i is $a \vdash\!\dashv B_i$ for $1 < i \leq n$), we have a d-proof of $\mathcal{P}_a$ from Γ in GDS. □

(II) *Let $\{a_1, \ldots, a_m\} = pt(\mathcal{E})$. Let P be the union of the relations of all P_{a_i} ($1 \leq i \leq m$) of (I), i.e. $P = \bigcup_{1 \leq i \leq m} P_{a_i}$. Then P gives rise to an* EUL*-diagram $\mathcal{P}$ such that $\Gamma \vdash \mathcal{P}$ in* GDS.

Proof of (II) We have $\Gamma \vdash \mathcal{P}$ in GDS by successively applying the Point Insertion rule for all diagrams $\mathcal{P}_{a_i}$ ($1 \leq i \leq m$) of (I). □

Note that when $\mathcal{E}$ does not contain any point, the set $\bigcup_{1 \leq i \leq m} P_{a_i}$ becomes empty. In such a case, we construct a Venn-like diagram $\mathcal{P}$ (without any point) which consists of all circles of $\mathcal{E}$. This is possible by successively applying U8-rule to axioms of the form $X \bowtie Y$ for $X, Y \in cr(\mathcal{E})$.

(III) *Let β be a minimal diagram such that $A \sqsubset B$ or $A \vdash\!\dashv B$ of* $\mathsf{rel}(\mathcal{E})$ *holds. Then we have $\Gamma \vdash \beta$ in* GDS.

Proof of (III) Immediate by atomic completeness (Proposition 3.13). □

(IV) *Let $R_1, \ldots, R_l$ be all relations of the form $A \sqsubset B$ or $A \vdash\!\dashv B$ holding on $\mathcal{E}$, and let $\beta_1, \ldots, \beta_l$ be the corresponding minimal diagrams, where R_i holds on β_i for*

$1 \leq i \leq l$. *Let* P *be the set of relations of (II). Then the set* $P \cup \{R_1, \ldots, R_l\}$ *of relations gives rise to an* EUL*-diagram* $(\cdots(\mathcal{P} + \beta_1) + \cdots) + \beta_l$ *which is provable from* Γ *in* GDS.

Proof of (IV) By induction on l. Let $\mathcal{P} + \mathcal{B}_l$ denote the diagram $(\cdots(\mathcal{P} + \beta_1) + \cdots) + \beta_l$. We show the induction step ($l > 1$) since the same applies to the base step ($l = 1$).
We divide into the following two cases according to whether (1) $A \mapsto\!\!\!\!\dashv B$ or (2) $A \sqsubset B$ holds on β_l.
Case (1): Since $cr(\mathcal{E}) = cr(\mathcal{P} + \mathcal{B}_{l-1})$ by the construction (II) of $\mathcal{P}$ and (III), we have $A, B \in cr(\mathcal{P} + \mathcal{B}_{l-1})$. We claim that $A \bowtie B$ or $A \mapsto\!\!\!\!\dashv B$ holds on the diagram $\mathcal{P} + \mathcal{B}_{l-1}$. Assume to the contrary that $A \sqsubset B$ or $B \sqsubset A$ holds. If $A \sqsubset B$ holds on $\mathcal{P} + \mathcal{B}_{l-1}$, since $\Gamma \vdash \mathcal{P} + \mathcal{B}_{l-1}$ by the induction hypothesis, we have $\Gamma \models \mathcal{P} + \mathcal{B}_{l-1}$, which implies $\Gamma \models A \sqsubset B$. This contradicts the assumption that Γ is semantically consistent because we have $\Gamma \models A \mapsto\!\!\!\!\dashv B$. The same applies in case $B \sqsubset A$. Thus exactly one of $A \bowtie B$ and $A \mapsto\!\!\!\!\dashv B$ holds on the diagram $\mathcal{P} + \mathcal{B}_{l-1}$ by Proposition 2.4.
Now we prove $\Gamma \vdash (\mathcal{P}+\mathcal{B}_{l-1})+\beta_l$. When $A \mapsto\!\!\!\!\dashv B$ holds on $\mathcal{P}+\mathcal{B}_{l-1}$, we obtain the assertion immediately by the induction hypothesis since $(\mathcal{P} + \mathcal{B}_{l-1}) + \beta_l$ is $\mathcal{P} + \mathcal{B}_{l-1}$ itself. When $A \bowtie B$ holds on $\mathcal{P} + \mathcal{B}_{l-1}$, by applying U10-rule to β_l and $\mathcal{P} + \mathcal{B}_{l-1}$, we have $\Gamma \vdash (\mathcal{P} + \mathcal{B}_{l-1}) + \beta_l$ in GDS. The application of U10-rule is possible because there is no object s such that both $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{P} + \mathcal{B}_{l-1}$: If there were such an object s, since $\Gamma \vdash \mathcal{P} + \mathcal{B}_{l-1}$, we have $\Gamma \vdash s \sqsubset A$ and $\Gamma \vdash s \sqsubset B$ by applying a series of Deletion. Then we would have $\Gamma \models s \sqsubset A$ and $\Gamma \models s \sqsubset B$. This contradicts the assumption that Γ is semantically consistent because we have $\Gamma \models A \mapsto\!\!\!\!\dashv B$.
Case (2) where $A \sqsubset B$ holds on β_l is similar. □

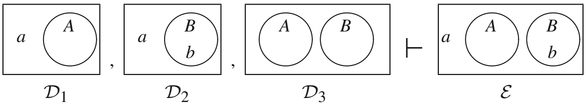
(V) *For any* EUL*-relation* R, $R \in$ rel$((\cdots(\mathcal{P} + \beta_1) + \cdots) + \beta_l)$ *if and only if* $R \in$ rel$(\mathcal{E})$.

Proof of (V) We denote by $\mathcal{P} + \mathcal{B}_l$ the diagram $(\cdots(\mathcal{P} + \beta_1) + \cdots) + \beta_l$.
$\Leftarrow$) rel$(\mathcal{E}) \subseteq$ rel$(\mathcal{P} + \mathcal{B}_l)$ is immediate by the constructions (II) and (IV).
$\Rightarrow$) Let $R \in$ rel$(\mathcal{P} + \mathcal{B}_l)$. We divide into the following two cases depending on whether or not R is of the form $s \bowtie t$:

(1) When $R = s \bowtie t$, assume to the contrary that $s \bowtie t \notin$ rel$(\mathcal{E})$. Since $\mathcal{E}$ is a diagram, for some $\square \in \{\sqsubset, \sqsupset, \mapsto\!\!\!\!\dashv\}$, $s \square t \in$ rel$(\mathcal{E})$ by Proposition 2.4. Then, by definition, for some j, β_j is of the form $s \square t$, which implies that $s \square t \in$ rel$(\mathcal{P} + \mathcal{B}_l)$. This contradicts Proposition 2.4 since $s \bowtie t \in$ rel$(\mathcal{P} + \mathcal{B}_l)$ by the assumption.
(2) In case, $R \neq s \bowtie t$, we show that $R \in$ rel$(\mathcal{P} + \mathcal{B}_l) \Rightarrow R \in$ rel$(\mathcal{E})$ by induction on l. We prove the induction step ($l > 1$) since the same applies to the base step. Assume to the contrary that $R \notin$ rel$(\mathcal{E})$. Then, since rel$(\mathcal{P}) \setminus \{X \bowtie Y \mid X, Y \in cr(\mathcal{P})\} \subseteq$ rel$(\mathcal{E})$ by the construction (II), R should be a relation between circles (not points), and $R \neq \beta_i$ for any i. Hence, there is some $1 \leq i \leq l$ such that $R \notin$ rel$(\mathcal{P} + \mathcal{B}_{i-1})$ but $R \in$ rel$((\mathcal{P}+\mathcal{B}_{i-1})+\beta_i)$. We show the case $(\mathcal{P}+\mathcal{B}_{i-1})+\beta_i$ is obtained

by U10-rule. (The case of U9-rule is shown similarly.) Assume $A \mapsto B$ holds on β_i. By the definition of U10-rule, there are the following three cases according to the form of R: (i) $R = X \mapsto B$ such that $X \sqsubset A \in \mathsf{rel}(\mathcal{P} + \mathcal{B}_{i-1})$; (ii) $R = X \mapsto A$ such that $X \sqsubset B \in \mathsf{rel}(\mathcal{P} + \mathcal{B}_{i-1})$; (iii) $R = X \mapsto Y$ such that $X \sqsubset A, Y \sqsubset B \in \mathsf{rel}(\mathcal{P} + \mathcal{B}_{i-1})$. For case (i), by the induction hypothesis, we have $X \sqsubset A \in \mathsf{rel}(\mathcal{E})$. Then, since $A \mapsto B \in \mathsf{rel}(\mathcal{E})$, we have $X \mapsto B \in \mathsf{rel}(\mathcal{E})$, contrary to the assumption $R = X \mapsto B \notin \mathsf{rel}(\mathcal{E})$. Similarly, cases (ii) and (iii) also lead to contradictions. Therefore, we have $R \in \mathsf{rel}(\mathcal{E})$. □

Example 3.15 (*Canonical d-proof of* GDS) As an illustration of the canonical construction of d-proofs, let us consider the following diagrams $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and $\mathcal{E}$:



We have a canonical d-proof of $\mathcal{E}$ from $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ as in Fig. 15:

We first derive, by using atomic completeness, all pointed minimal diagrams $\mathcal{D}_1, \mathcal{D}_4, \mathcal{D}_5$, and $\mathcal{D}_6$ each of which corresponds to an EUL-relation holding on the conclusion $\mathcal{E}$. Next, following the construction (I) with U1 and U2 rules, we construct Venn-like diagrams $\mathcal{D}_7$ and $\mathcal{D}_8$ each of which consists of a point a (resp. b) and all circles A and B of $\mathcal{E}$. Then, following the construction (II) with Point Insertion rule, we unify them to obtain a Venn-like diagram $\mathcal{D}_9$ consisting of all points a and b and all circles A and B of $\mathcal{E}$. Finally, following the construction (IV) with U10 rule, we obtain the conclusion $\mathcal{E}$.

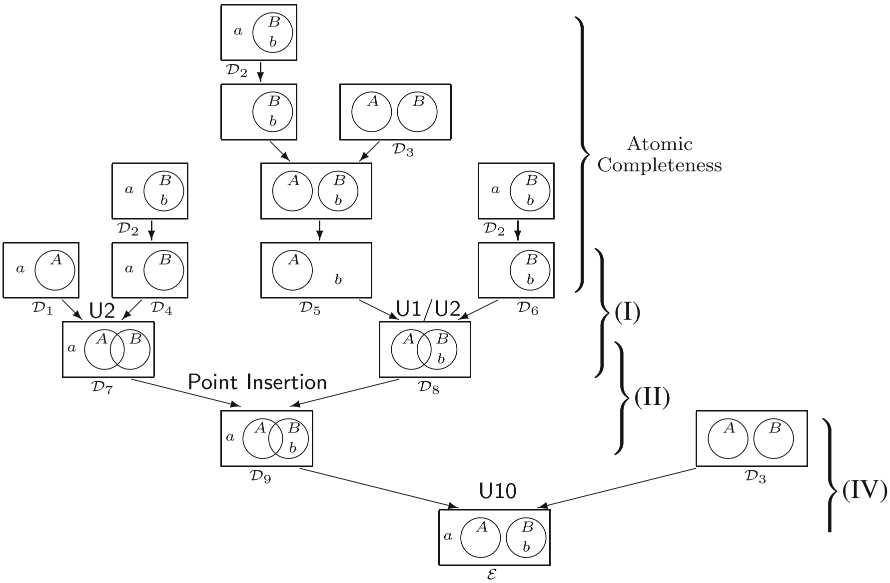


Fig. 15 Canonical d-proof

3.4 Some Consequences of Completeness of GDS

In this section, we discuss some consequences of our completeness (Theorem 3.14) of GDS.

3.4.1 Unification of Any (Two) Diagrams

Let $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{E}$ be EUL-diagrams such that for any model M, $M \models \mathcal{E}$ if and only if $M \models \mathcal{D}_1$ and $M \models \mathcal{D}_2$, that is, $\mathcal{E}$ is semantically equivalent to the conjunction of $\mathcal{D}_1$ and $\mathcal{D}_2$. We may write such $\mathcal{E}$ as $\mathcal{D}_1 + \mathcal{D}_2$. Our completeness (Theorem 3.14) ensures that $\mathcal{D}_1, \mathcal{D}_2 \vdash \mathcal{D}_1 + \mathcal{D}_2$ in GDS. This shows that the general notion of unification of two diagrams (cf. Hammer and Shin 1998) is completely characterized by our formalization of unification of two diagrams, where one of them is restricted to a minimal diagram.

3.4.2 Normal Diagrammatic Proofs

In order to prove a normal form theorem of GDS, we shall modify the semantic method introduced in our completeness proof, by adopting a semantic normal form proof for the linguistic proofs found in, for example, Okada (1999).

Let us define a class of normal diagrammatic proofs of GDS, called the $\pm$-normal d-proofs:

Definition 3.16 ($\pm$-*normal d-proofs*) A d-proof π is in $\pm$-*normal form* if a unification $(+)$ and a deletion $(-)$ appear alternately in π.

In Definition 3.8 and 3.11 of our canonical models, it is possible to modify the interpretation of each object by restricting the provability *with a* $\pm$*-normal d-proof* as follows:

- For Definition 3.8, $I'_{\vec{\alpha}}(t) = \{s \mid \vec{\alpha} \vdash s \sqsubset t$ with a $\pm$-normal d-proof $\} \cup \{t\}$
- For Definition 3.11, when $t \equiv B$ or $\vec{\alpha} \vdash B \sqsubset t$ with a $\pm$-normal d-proof, $I'_{\vec{\alpha},B}(t) = I'_{\vec{\alpha}}(t) \cup \{s \mid \vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \nvdash s \sqsubset B$ and $\vec{\alpha} \nvdash s \vdash\!\dashv B\}$

These slight modifications of canonical models also enable us to prove the essential part of atomic completeness (Proposition 3.13); because any d-proof appearing in our proofs of Lemmas 3.9 and 3.12 is in $\pm$-normal form. Hence we obtain the following version of atomic completeness:

Corollary 3.17 *Let $\vec{\alpha}$ be a set of minimal diagrams which is semantically consistent. Let β be a minimal diagram. If $\vec{\alpha} \models \beta$, then $\vec{\alpha} \vdash \beta$ in* GDS *with a $\pm$-normal d-proof.*

Then, together with soundness of GDS, we obtain the following normal form theorem:

Theorem 3.18 *(*$\pm$-normal form for minimal diagrams*) Let $\vec{\alpha}$ be a set of minimal diagrams which is semantically consistent. Let β be a minimal diagram. If $\vec{\alpha} \vdash \beta$ in* GDS*, then $\vec{\alpha} \vdash \beta$ in* GDS *with a $\pm$-normal d-proof.*

Proof Let $\vec{\alpha} \vdash \beta$ in GDS. Then, by soundness (Theorem 3.5) of GDS, we have $\vec{\alpha} \models \beta$, which implies that $\vec{\alpha} \vdash \beta$ in GDS with a $\pm$-normal d-proof by Corollary 3.17. □

Although the above normal form theorem states only the existence of normal d-proofs, by defining a procedure to rewrite d-proofs, the theorem can be extended to a *normalization theorem*: *Any d-proof is rewritten into a* $\pm$*-normal d-proof in a finite number of steps.*

3.4.3 Structure of Canonical Diagrammatic Proofs

In order to investigate the structure of canonical d-proofs of completeness (Theorem 3.14), we give a proposition, which is proved in a way similar to that of $\pm$-normal form Theorem 3.18.

In our Definitions 3.8 and 3.11 of canonical models, it is possible to modify the interpretation of each object by restricting the provability *using only* U3–U7 *and* Deletion *rules*:

- For Definition 3.8, $I''_{\vec{\alpha}}(t) = \{s \mid \vec{\alpha} \vdash s \sqsubset t$ with U3–U7 and Deletion rules$\} \cup \{t\}$
- For Definition 3.11, when $t \equiv B$ or $\vec{\alpha} \vdash B \sqsubset t$ with U3–U7 and Deletion rules, $I''_{\vec{\alpha},B}(t) = I''_{\vec{\alpha}}(t) \cup \{s \mid \vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \nvdash s \sqsubset B$ and $\vec{\alpha} \nvdash s \vdash\!\dashv B\}$.

Recall that U3–U7 rules are unification where exactly one named circle (not point) is shared between the two premise diagrams.

These slight modifications of canonical models also enables us, in a way similar to that in Corollary 3.17, to prove atomic completeness. Thus we obtain the following slightly stronger version of atomic completeness:

Corollary 3.19 *Let* Γ *be a set of* EUL*-diagrams which is semantically consistent. Let* β *be a minimal diagram. If* $\Gamma \models \beta$*, then* $\Gamma \vdash \beta$ *in* GDS *with* U3–U7 *and* Deletion *rules.*

Thus soundness (Theorem 3.5) and Corollary 3.19 imply that any minimal diagram is provable by using only U3–U7 and Deletion rules:

Proposition 3.20 *(*U3–U7 rules*) Let* Γ *be a set of* EUL*-diagrams which is semantically consistent. Let* β *be a minimal diagram. If* $\Gamma \vdash \beta$ *in* GDS*, then* $\Gamma \vdash \beta$ *in* GDS *with* U3–U7 *and* Deletion *rules.*

Completeness (Theorem 3.14), the $\pm$-normal form theorem (Theorem 3.18), and the above Proposition 3.20 give a more precise classification of inference rules of GDS in terms of proof-construction as follows:

- U3–U7 and Deletion rules for derivation of a minimal diagram.
- U1, U2 (resp. U8) rules for construction of a Venn-like diagram consisting of a single point (resp. no point).
- Point Insertion rule for construction of a Venn-like diagram consisting of multiple points.
- U9, U10 rules for construction of the conclusion.

See also the canonical d-proof given in Example 3.15.

Based on the classification of inference rules and the canonical construction of d-proofs, we showed a correspondence between our Euler diagrammatic proofs and Gentzen's natural deduction proofs. See Mineshima et al. (2010) for a detailed discussion.

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