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Scope Dominance with Monotone Quantifiers over Finite Domains

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Abstract. We characterize pairs of monotone generalized quantifiers Q_1 and Q_2 over finite domains that give rise to an entailment relation between their two relative scope construals. This relation between quantifiers, which is referred to as *scope dominance*, is used for identifying entailment relations between the two scopal interpretations of simple sentences of the form NP_1–V–NP_2. Simple numerical or set-theoretical considerations that follow from our main result are used for characterizing such relations. The variety of examples in which they hold are shown to go far beyond the familiar existential-universal type.

Key words: Ambiguity, inference, monotonicity, natural language, quantifier, scope, underspecification

1. Introduction

Scope ambiguity in simple transitive sentences of the form NP_1–V–NP_2 is one of the well-studied areas in natural language semantics. It has been often observed that whether this kind of ambiguity is manifested in natural language may depend on entailment relations between the readings of such sentences. For instance, Zimmermann (1993) characterizes the class of *scopeless* ("name like") noun phrases – the class of NP_2s for which the two scope construals of the sentence NP_1–V–NP_2 are equivalent for any noun phrase NP_1 and transitive verb V. A more general notion, first addressed by Westerståhl (1986), involves uni-directional entailment between the two analyses, which is referred to here as *scope dominance.* A sentence NP_1–V–NP_2 exhibits scope dominance if one of its two analyses entails the other. A familiar case is when the subject (or object) denotes an existential quantifier (e.g., *some student*) and the object (or subject, respectively) denotes a universal quantifier (e.g., *every teacher*). Westerståhl shows that in the class of non-trivial upward monotone quantifiers over finite domains, scope dominance appears if and only if the subject or object are existential or universal.

Altman et al. (2002) generalize Westerståhl's result, and show a full characterization of scope dominance with *arbitrary* upward monotone quantifiers over *countable* domains. In this paper, we generalize Westerståhl's result in another way, and characterize scope dominance between simple upward *or downward* monotone quantifiers over finite domains. It leads to a general characterization of entailments over finite domains between the semantic analyses of sentences with (potential) scope ambiguity as in the following cases, where both subject and object are monotone.

(1) Less than five referees read each of the abstracts.
(2) Less than five referees read at least one of the abstracts.

In sentence (2), the object narrow scope construal entails the object wide scope construal. In (1), the entailment between the two construals is in the opposite direction. Note that the definite noun phrase *the abstracts* leads in both sentences to the presupposition that there are at least two abstracts, which is crucial for the respective entailments to hold. Similarly to Westerståhl's result about upward monotone quantifiers, in both examples scope dominance is created by the presence of an existential or universal quantifier. However, as we shall see, our extension of Westerståhl's characterization reveals many more cases of scope dominance with monotone quantifiers other than *every* or *some*.

This work is part of a broader enterprise that aims to characterize general entailment patterns between different readings of ambiguous sentences in natural language. One central motivation for studying this question comes from the promise it carries for improving existing techniques for *reasoning under ambiguity.* Towards the end of this paper we describe this new line of research.

The rest of this paper is organized as follows. Section 2 gives some essential background on generalized quantifier theory. Section 3 briefly discusses some previous results on various scope dominance relations. Section 4 proves our characterization of scope dominance relations with monotone quantifiers over finite domains, and exemplifies its relevance for the analysis of scopally ambiguous English sentences. Section 5 concludes the article and elaborates in some detail on its relevance for reasoning under ambiguity.

2. Background

This section reviews some notions from generalized quantifier theory that will be used in our characterization of scope dominance.

A (*generalized*) *quantifier* over a domain E is a set $Q \subseteq \wp(E)$. In this paper, we are particularly interested in *monotone* quantifiers, those quantifiers that are closed under supersets or subsets. Formally, a quantifier Q over E is called *upward (downward) monotone* iff for any set A in Q and A' a superset (subset) of A: A' is in Q as well. In the sequel, we sometimes use the abbreviations "MON↑" and "MON↓" for "upward/downward monotone." Two "degenerate" kinds of monotone quantifiers over a domain E are the two *trivial* quantifiers: the empty quantifier and the quantifier $\wp(E)$. For an upward (downward) monotone quantifier Q, it is sometimes useful to designate the collection of Q's minimal (maximal) sets. Formally, given a quantifier Q, a set $A \in Q$ is *minimal* in Q iff for any $A' \subsetneq A$: $A' \notin Q$. Analogously, given a quantifier Q, a set $A \in Q$ is *maximal* in Q iff for any $A \subsetneq A' : A' \notin Q$.

Given a binary relation $R \subseteq E^2$ and $a \in E$, we write $R_a \stackrel{\text{def}}{=} \{y \in E : R(a, y)\}$ and $R^a \stackrel{\text{def}}{=} \{x \in E : R(x, a)\}$. The *Object Narrow Scope* (ONS) analysis of a simple transitive sentence NP_1–V–NP_2 is naturally interpreted in a domain E as the proposition $Q_1 Q_2 R$ as defined below, where Q_1 and Q_2 are the subject and object quantifiers (NP_1 and NP_2, respectively) over E, and the relation $R \subseteq E^2$ is the denotation of the verb V.

(3) $Q_1 Q_2 R \stackrel{\text{def}}{\Leftrightarrow} \{x \in E : R_x \in Q_2\} \in Q_1.$

The *Object Wide Scope* (OWS) analysis is $Q_2 Q_1 R^{-1}$, which by (3) is equivalent to the requirement $\{y \in E : R^y \in Q_1\} \in Q_2$. The notion of *scope dominance*, which plays a special role in this paper, is defined as follows.

DEFINITION 1 (Scope dominance). Given two quantifiers Q_1 and Q_2 over E we say that Q_1 is *scopally dominant* over Q_2 iff for every $R \subseteq E^2 : Q_1 Q_2 R \Rightarrow Q_2 Q_1 R^{-1}$.

Consider for instance the following familiar type of sentences.

(4) A competent referee read every abstract.

In this case, we say that the ONS reading, with the $\exists\forall$ order of quantifiers, is dominant over the OWS reading, with the opposite order.[1]

For a quantifier Q over E, the following notions of quantifier *negation* will be useful for characterizing scope dominance:

$$\neg Q = \{X \subseteq E : X \notin Q\} \qquad (Q\text{'s complement})$$
$$Q\neg = \{X \subseteq E : E \backslash X \in Q\} \qquad (Q\text{'s post-complement})$$
$$Q^d = \neg Q \neg = \{X \subseteq E : E \backslash X \notin Q\} \qquad (Q\text{'s dual})$$

Some simple properties of quantifier duality are the following, for any quantifier Q over E:

1. $(Q^d)^d = Q$
2. $Q = \emptyset \Leftrightarrow Q^d = \wp(E)$
3. Q is MON$\uparrow$ (MON$\downarrow$) iff Q^d is MON$\uparrow$ (MON$\downarrow$).

The relevance of duality to scope dominance comes from the following simple fact.

FACT 1. *For any two quantifiers Q_1 and Q_2 over E: Q_1 is scopally dominant over Q_2 iff Q_2^d is scopally dominant over Q_1^d.*

A *determiner* over a domain E is a function D that assigns to every $A \subseteq E$ a quantifier $D(A)$. Two important properties of determiners are *conservativity* and *permutation invariance*. A determiner D over E is called *conservative* iff for all $A, B \subseteq E : B \in D(A) \Leftrightarrow B \cap A \in D(A)$. A determiner D over E is called *permutation invariant* iff for every permutation π on E, and for all $A, B \subseteq E : B \in D(A) \Leftrightarrow \pi B \in D(\pi A)$, where for a set $X \subseteq E, \pi X = \{\pi(x) : x \in X\}$.

In part of Section 4, we will concentrate on quantifiers that satisfy $Q = D(A)$ for some $A \subseteq E$ and a conservative and permutation invariant determiner D. In the sequel, we refer to such quantifiers as *CPI-based.*

As pointed out by Väänänen and Westerståhl (2001), every monotone CPI-based quantifier Q over a finite domain E can be represented as follows, for some $A \subseteq E$ and $n \geq 0$.

(5a) $Q = \{X : |A \cap X| \geq n\}$, if Q is MON↑
(5b) $Q = \{X : |A \cap X| < n\}$, if Q is MON↓

The duals of such CPI-based quantifiers can be represented as follows, respectively (note that a dual of a CPI-based quantifier is also CPI-based).

(6a) $Q^d = \{X : |A \cap X| \geq |A| - n + 1\}$
(6b) $Q^d = \{X : |A \cap X| < |A| - n + 1\}$

In Table I we give some examples of monotone CPI-based quantifiers $D(A)$ over a finite domain E for various determiners D and arbitrary sets $A \subseteq E$, together with their presentation according to the scheme in (5). In these examples, for any real number r, the notations $\lfloor r \rfloor$ and $\lceil r \rceil$ standardly stand for the integer value closet to r from below and from above, respectively.

Table I. CPI-based quantifiers.

every′ (A)	$= \{X \subseteq E : |A \cap X| \geq |A|\}$
not_every′ (A)	$= \{X \subseteq E : |A \cap X| < |A|\}$
some′ (A)	$= \{X \subseteq E : |A \cap X| \geq 1\}$
no′ (A)	$= \{X \subseteq E : |A \cap X| < 1\}$
more_than_n′ (A)	$= \{X \subseteq E : |A \cap X| > n + 1\}$
less_than_n′ (A)	$= \{X \subseteq E : |A \cap X| < n\}$
more_than_half′ (A)	$= \{X \subseteq E : |A \cap X| \geq \lfloor \frac{|A|}{2} \rfloor + 1\}$
at_least_half′ (A)	$= \{X \subseteq E : |A \cap X| \geq \lceil \frac{|A|}{2} \rceil\}$
less_than_half′ (A)	$= \{X \subseteq E : |A \cap X| < \lceil \frac{|A|}{2} \rceil\}$

3. Related Works

Westerståhl (1986) characterizes the pairs Q_1 and Q_2 of CPI-based, upward monotone quantifiers over finite domains, for which Q_1 is scopally dominant over Q_2. He shows that if both quantifiers are not trivial, then Q_1 is dominant over Q_2 iff $Q_1 =$ **some**$'$ (A) or $Q_2 =$ **every**$'$ (B), for some $A, B \subseteq E$. Some more results about scope dominance appear in van Benthem (1989). He shows that a quantifier Q is dominant over *any* upward monotone quantifier iff $Q =$ **some**$'$ (A), for some $A \subseteq E$. Furthermore, he shows that a quantifier Q is dominant over *any* (not necessarily monotone) quantifier iff it is a principal ultrafilter, or the empty quantifier.

Altman et al. (2002) extend Westerståhl's result for *all* upward monotone quantifiers over *countable* domains. They show that for such quantifiers, Q_1 is scopally dominant over Q_2 iff one of the following requirements holds:

(i) Q_1 is existential
(ii) Q_2 is universal
(iii) Q_1 satisfies (U), $Q_2 \neq \emptyset$ and Q_2 satisfies (DCC)
(iv) Q_2 is a filter, $Q_1 \neq \wp(E)$ and Q_1 satisfies (FIN)

where (U), (DCC) and (FIN) are defined as follows:

- A quantifier Q satisfies the *union property* (U) if for all $A_1, A_2 \subseteq E$: if $A_1 \cup A_2 \in Q$ then $A_1 \in Q$ or $A_2 \in Q$.
- A quantifier Q satisfies the *Descending Chain Condition* (DCC) if for every descending sequence $A_1 \supseteq A_2 \supseteq \cdots A_n \supseteq \cdots$ in Q, the intersection $\cap A_i$ is in Q as well.
- A quantifier Q satisfies (FIN) if every set in Q contains a finite subset that is also in Q.

Other scope commutativity properties of qunatifiers were studied by Zimmermann (1993) and Westerståhl (1996). Zimmermann characterizes the class of *scopeless* quantifiers: those quantifiers Q that satisfy for all $Q_1 \subseteq \wp(E)$ and $R \subseteq E^2 : QQ_1R \Leftrightarrow Q_1QR^{-1}$. He shows that the scopeless quantifiers over E are precisely the principal ultrafilters over E.[2] Westerståhl (1996) characterizes the class of *self-commuting* quantifiers: those quantifiers Q, such that for every $R \subseteq E^2 : QQR \Leftrightarrow QQR^{-1}$. He shows that $Q \subseteq \wp(E)$ is self-commuting iff Q is either a union or an intersection of atoms, or a finite symmetric difference of atoms, or a negation of such a symmetric difference. Clearly, the notion of scope dominance is more general than scopelessness or self-commutativity: a quantifier Q is scopeless iff Q and Q^d are both scopally dominant over any quantifier Q_1; Q is self-commuting iff it is scopally dominant over itself. However, it should be noted that the actual results of Altman et al., as well as the new results presented in this paper, do not fully subsume the results by Zimmermann and Westerståhl.

4. Scope Dominance with Monotone Quantifiers over Finite Domains

In this section, we introduce a general result that completely characterizes the relations of scope dominance between pairs of upward monotone quantifiers and downward monotone quantifiers over finite domains. We then study the implications of this result for the natural subclass of CPI-based quantifiers, and extend the coverage of our technique to scope dominance over finite domains between pairs of CPI-based *downward* monotone quantifiers. Throughout this section, we exemplify how these results are used for characterizing scope dominance in natural language, which leads to the identification of previously unobserved entailments between wide scope and narrow scope analyses of potentially ambiguous sentences.

4.1. Scope Dominance with Quantifiers of Mixed Monotonicity

The following proposition, the central result in this subsection, characterizes scope dominance between pairs of upward monotone quantifiers and downward monotone quantifiers over finite domains.

PROPOSITION 2. *Let Q_1 and Q_2 be two quantifiers over a finite domain E, s.t. Q_1 is* MON↑ *and Q_2 is* MON↓. *Let the natural number n be defined by*:

$$n = \max\{|Y| : Y \textit{ is minimal in } \neg Q_2\}.$$

Then Q_1 is scopally dominant over Q_2 iff one of the following holds:

(i) *Neither quantifier is trivial, and for every $Q \subseteq Q_1$, if $|Q| = n + 1$ then $\bigcap Q \neq \emptyset$.*
(ii) $Q_1 = \emptyset$
(iii) $Q_2 = \wp(E)$.
(iv) $Q_2 = \emptyset$ *and* $Q_1 \neq \wp(E)$.

Proof. It is easy to verify that if at least one of Q_1 and Q_2 is trivial, then Q_1 is scopally dominant over Q_2 iff one of the clauses (ii)–(iv) holds. We therefore assume that both quantifiers are not trivial, and prove that Q_1 is scopally dominant over Q_2 iff (i) holds.

Only if: In order to obtain a contradiction assume that Q_1 is scopally dominant over Q_2, but there is a subset Q of Q_1 s.t. $|Q| = n + 1$ and $\bigcap Q = \emptyset$. Denote $Q = \{X, X_1, \ldots, X_n\}$. Let Y be any minimal set in $\neg Q_2$ of cardinality n, and denote $Y = \{y_1, \ldots, y_n\}$. Let $R = \bigcup_{i=1}^{n}(X_i \times \{y_i\})$. From $\bigcap Q = \emptyset$ it follows that for every $x \in X$, $R_x \subsetneq Y$, and because Y is minimal in $\neg Q_2$, $R_x \in Q_2$. Hence, $X \subseteq \{x \in E : R_x \in Q_2\}$. Since $X \in Q_1$ and Q_1 is MON↑: $\{x \in E : R_x \in Q_2\} \in Q_1$. However, $\{y \in E : R^y \in Q_1\} = Y \notin Q_2$, in contradiction to the assumption that Q_1 is scopally dominant over Q_2.

If: Assume for obtaining a contradiction that for every $Q \subseteq Q_1$ s.t. $|Q| = n+1 : \bigcap Q \neq \emptyset$, but Q_1 is *not* scopally dominant over Q_2. Then there is $R \subseteq E^2$ s.t. $\{x \in E : R_x \in Q_2\} \in Q_1$ and $\{y \in E : R^y \in Q_1\} \notin Q_2$. Let $Y \subseteq \{y \in E : R^y \in Q_1\}$ be a minimal set in $\neg Q_2$. Since Q_2 is MON↓ and not empty, it follows that $\emptyset \in Q_2$, hence $Y \in \neg Q_2$ is not empty. Since E is finite, $|Y| < \aleph_0$. For every $y \in Y$ let A_y be a minimal set in Q_1 s.t. $A_y \subseteq R^y$. Let $A \subseteq \{x \in E : R_x \in Q_2\}$ be also a minimal set in Q_1. Let $Q = \{A\} \cup \{A_y : y \in Y\}$. If $x \in \bigcap_{y \in Y} A_y$ then $Y \subseteq R_x$, and because Q_2 is MON↓ and $Y \notin Q_2$, also $R_x \notin Q_2$. It follows that for such x, $x \notin A$, and therefore $\bigcap Q = \emptyset$. By definition of n, $|Y| \leq n$ (since Y is minimal in $\neg Q_2$). Thus, $|Q| \leq |Y| + 1 \leq n + 1$. But $|Q| \neq n + 1$ since $\bigcap Q = \emptyset$, and we assumed that if $|Q| = n + 1$ then $\bigcap Q \neq 0$. Thus, $|Q| = k$ for $0 < k < n + 1$, and we show that there is $Q' \subseteq Q_1$ s.t. $|Q'| = n + 1 - k$ and $Q' \cap Q = \emptyset$.

To do that we first claim that $|\{x \in E : E \backslash \{x\} \notin Q\}| \geq n - k$. This is true because $|\{x \in E : E \backslash \{x\} \notin Q\}| = |E \backslash \{x \in E : E \backslash \{x\} \in Q\}| = |E| - |\{x \in E : E \backslash \{x\} \in Q\}|$. But $|E| \geq n$, and $|\{x \in E : E \backslash \{x\} \in Q\}| \leq |Q| = k$. Thus, $|E| - |\{x \in E : E \backslash \{x\} \in Q\}| \geq n - k$.

So let $X \subseteq \{x \in E : E \backslash \{x\} \notin Q\}$ with $|X| = n - k$, and let $Q' = \{E \backslash \{x\} : x \in X\} \cup \{E\}$.$Q' \subseteq Q_1$, because if $x \in X$ then there is $A' \in Q$ s.t. $x \notin A'$ (since $\bigcap Q = \emptyset$), and therefore $A' \subseteq E \backslash \{x\}$; from the upward monotonicity of Q_1, it follows that $E \backslash \{x\} \in Q_1$. To see that $Q' \cap Q = \emptyset$ note that by the definition of X : if $x \in X$ then $E \backslash \{x\} \notin Q$. Furthermore, if $E \in Q$ then, since all the sets in Q are minimal in Q_1, E is the only minimal set in Q_1, which implies that $Q = \{E\}$. But this contradicts the fact that $\bigcap Q = \emptyset$, hence $E \notin Q$. Thus, every set in Q' is not in Q.

From $Q' \cap Q = \emptyset$ it follows that $|Q' \cup Q| = |Q'| + |Q| = n + 1$. Furthermore, from $\bigcap Q = \emptyset$ it follows that $\bigcap (Q' \cup Q) = \emptyset$, in contradiction to the assumption that for every $Q \subseteq Q_1$ s.t. $|Q| = n + 1 : \bigcap Q \neq \emptyset$. □

The dual of the kind of scope dominance that is introduced in Proposition 2 is the case in which Q_1 is MON↓ and Q_2 is MON↑. Corollary 3 below is a direct consequence of Proposition 2.

COROLLARY 3. *Let Q_1 and Q_2 be two quantifiers over a finite domain E, s.t. Q_1 is* MON↓ *and Q_2 is* MON↑. *Let*

$$n = \max \{|Y| : Y \textit{ is minimal in } Q_1\neg\}.$$

Then Q_1 is scopally dominant over Q_2 iff one of the following holds:

(i) *Neither quantifier is trivial, and for every $Q \subseteq Q_2^d$, if $|Q| = n + 1$ then $\bigcap Q \neq \emptyset$.*
(ii) $Q_1 = \emptyset$.
(iii) $Q_2 = \wp(E)$.
(iv) $Q_1 = \wp(E)$ *and* $Q_2 \neq \emptyset$.

4.2. EXAMPLES

Let us now consider some examples for the application of Proposition 2 and Corollary 3 to natural language. First, note that Proposition 2 implies that if $B \neq \emptyset$ then **every**′ (B) is scopally dominant over any MON↓ quantifier. By duality, if $B \neq \emptyset$ then every MON↓ quantifier is scopally dominant over **some**′ (B). Another consequence of Proposition 2 is that if $|B| \geq 2$ then **some**′ (B) is not scopally dominant over any non-trivial MON↓ quantifier. Incidently, these three consequences hold in infinite domains as well. Example 1 below illustrates these facts using CPI-based quantifiers with simple determiners.

EXAMPLE 1. Reconsider sentences (1) and (2), restated below as (7) and (8). Since the set of abstracts B is presupposed to be non-empty,[3] the OWS reading (7b) of sentence (7) entails its ONS reading (7a). Similarly, the ONS analysis (8a) of sentence (8) entails its OWS analysis (8b).

(7) Less than five referees read each of the abstracts.
a. **less_than_5**′ (A) **every**′ (B) R
b. **every**′ (B) **less_than_5**′ (A) R^{-1}

(8) Less than five referees read at least one of the abstracts.
a. **less_than_5**′ (A) **some**′ (B) R
b. **some**′ (B) **less_than_5**′ (A) R^{-1}

Such examples with existential and universal quantifiers do not exhaust the cases of scope dominance with monotone quantifiers, as the following example demonstrates.

EXAMPLE 2. By Proposition 2, **more_than_half**′ (A) is scopally dominant over **no**′ (B) for all $A, B \subseteq E$. By Corollary 3, **not_every**′ (A) (=(**no**′ $(A))^d$) is scopally dominant over **at_least_half**′ (B) (=(**more_than_half**′ $(B))^d$), for all $A, B \subseteq E$. Consider now the following sentences.

(9) More than half of the referees read no abstract.
(10) No abstract was read by more than half of the referees.

To begin with, it is not at all clear that these two sentences are scopally ambiguous. For many speakers both sentences are unambiguous, and have only an ONS reading. Under this unambiguous interpretation, our characterization accounts for the entailment from (the unambiguous) sentence (9) to (the unambiguous) sentence (10). For speakers who may consider these sentences (or their parallels in other languages) ambiguous, our characterization accounts for the entailment from the ONS analysis of (9) to its OWS analysis, and for the entailment in the opposite direction in (10).

Note that the *more than/ at least half of* quantifiers that are involved in Example 2 are not first order definable, so these entailments cannot be derived by any axiom system of the first order Predicate Calculus.

So far we have considered only "simple" natural language quantifiers – quantifiers that are denoted by simple NPs of the form *Determiner-Noun.* However, when NP coordination comes into play, many of the potentially infinite number of quantifiers that are created in this way are not CPI-based – hence, according to standard assumptions, they are not expressible as NPs of the form *Determiner-Noun* in any natural language. For example, the quantifier $\mathbf{every}'\ (A_1) \cup \mathbf{every}'\ (A_2)$ that is denoted by coordinations such as *every author or every teacher* is CPI-based only when either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. The following simple (dual) lemmas help in characterizing scope dominance also with non-CPI-based quantifiers.[4]

LEMMA 4. *Let Q_1 and Q_2 be two quantifiers over a domain E s.t. Q_2 is* MON↓ *and Q_1 is scopally dominant over Q_2. Then every quantifier Q s.t. $Q \subseteq Q_1$ is also scopally dominant over Q_2.*

LEMMA 5. *Let Q_1 and Q_2 be two quantifiers over a domain E s.t. Q_1 is* MON↓ *and Q_1 is scopally dominant over Q_2. Then for every quantifier Q s.t. $Q_2 \subseteq Q$, Q_1 is scopally dominant over Q.*

EXAMPLE 3. Sentences (11), (12) and (13) exhibit the same scope dominance relations as sentences (7), (9) and (10) respectively, due to Lemma 4.

(11) Less than five referees read each of the abstracts and more than three manuscripts.
(12) More than half of the referees and more than three TAs read no abstract.
(13) No abstract was read by more than half of the referees and more than three TAs.

Similarly, sentence (14) exhibits the same scope dominance relations as sentence (8), due to Lemma 5.

(14) Less than five referees read at least one of the abstracts or more than three manuscripts.

Note that the coordinate NPs in this example do not necessarily denote CPI-based quantifiers. For instance, if there are five abstracts in *A* and five manuscripts in *M*, the quantifier $\mathbf{every}'\ (A) \cap \mathbf{more\_than\_3}'\ (M)$, which is denoted by the object of sentence (11), is not CPI-based.

The following example demonstrates how Lemma 4 can be used to identify that a disjunction (union) of two quantifiers is not scopally dominant over a downward monotone quantifier, even when one of the disjuncts is.

EXAMPLE 4. Let $A, B \subseteq E$ s.t. $|A| \geq 2$ and $|B| \geq 2$.
Then $Q_1 =$ **at_least_half**$'$ $(A) = \{X \subseteq E : |A \cap X| \geq |A \backslash X\}$ is not scopally dominant over $Q_2 =$ **less_than_2**$'$ $(B) = \{X \subseteq E : |B \cap X| < 2\}$. This fact follows from Proposition 2, since $2 = \max\{|Y| : Y$ is minimal in $\neg Q_2\}$, and there is $Q \subseteq Q_1$ s.t. $|Q| = 3$ and $\bigcap Q = \emptyset$. By Lemma 4, for every $A' \subseteq E$, $Q_1 \cup$ **every**$'$ (A') is not scopally dominant over Q_2.[5] As a result of these facts, the ONS reading of sentence (15) below does not entail the ONS reading of sentence (16).

(15) At least half of the referees (or each of the TAs) read less than two of the abstracts.
(16) Less than two of the abstracts was read by at least half of the referees (or each of the TAs).

The following examples demonstrate that conjectures analogous to Lemmas 4 and 5, but in which the inclusions are in opposite direction, do not hold.

EXAMPLE 5. Let B be a non-empty subset of E and $b', j', m' \in E$. Let $Q_1' = \{X \subseteq E : m' \in X \vee \{b', j'\} \subseteq X\}$, $Q_1'' = \{X \subseteq E : b' \in X \vee \{m', j'\} \subseteq X\}$. Then neither Q_1' nor Q_1'' is scopally dominant over **no**$'$ (B), since these quantifiers contain disjoint sets. However $Q_1' \cap Q_1'' = \{X \subseteq E : \{m', b'\} \subseteq X \vee \{m', j'\} \subseteq X \vee \{b', j'\} \subseteq X\}$ *is* scopally dominant over **no**$'$ (B). This accounts for the entailment between the ONS and the OWS analyses of sentence (19), as opposed to the lack of similar entailments in sentences (17) and (18).

(17) Mary or [Bill and John] read no paper.
(18) Bill or [Mary and John] read no paper.
(19) [Mary or [Bill and John]] and [Bill or [Mary and John]] read no paper.

EXAMPLE 6. Let A_1, A_2 and B be non-empty subsets of E.
Then **every**$'$ (A_1) and **every**$'$ (A_2) are scopally dominant over **no**$'$ (B), but **every**$'$ $(A_1) \cup$ **every**$'$ (A_2) is scopally dominant over **no**$'$ (B) only if (and if) $A_1 \cap A_2 \neq \emptyset$.

In the examples we have seen thus far, all the downward monotone quantifiers were CPI-based. In the following example this is not necessarily so.

EXAMPLE 7. Let $A, B_1, B_2 \subseteq E$, s.t. $|A| = 4$, $|B_1| \geq 2$, $|B_2| \geq 1$ and $B_1 \cap B_2 = \emptyset$. Let $Q_1 = \{X \subseteq E : |A \cap X| \geq 3\}$ (= **at_least_3**$'$ (A)), $Q_2' = \{Y \subseteq E: |B_1 \cap Y| < 2\}$ (= **at_most_1**$'$ (B_1)) and $Q_2'' = Y \subseteq E : |B_2 \cap Y| < 1\}$ (= **no**$'$ (B_2)). Then,

$$2 = \max\{|Y| : Y \text{ is minimal in } \neg(Q_2' \cap Q_2'')\}$$

and for every $Q \subseteq Q_1$, if $|Q| = 3$, then $\bigcap Q \neq \emptyset$. By Proposition 2, Q_1 is scopally dominant over $Q_2' \cap Q_2''$. Hence the ONS reading of sentence (20) entails the ONS reading of sentence (21).

(20) At least three referees read at most one abstract and no manuscript.
(21) At most one abstract and no manuscript was read by at least three referees.

On the other hand, Q_1 is not scopally dominant over $Q'_2 \cup Q''_2$, since

$$3 = \max\{|Y| : Y \text{ is minimal in } \neg(Q'_2 \cup Q''_2)\}$$

and there is a $Q \subseteq Q_1$ s.t. $|Q| = 4$ and $\bigcap Q = \emptyset$. Thus, the ONS reading of sentence (22) does not entail the ONS reading of sentence (23).

(22) At least three referees read at most one abstract or no manuscript.
(23) At most one abstract or no manuscript was read by at least three referees.

4.3. Scope Dominance with CPI-Based Quantifiers of Mixed Monotonicity

Proposition 2 is a general characterization of scope dominance with quantifiers of mixed monotonicity over finite domains. As we have seen in the previous subsection, checking whether two given quantifiers satisfy the condition in this proposition is not always straightforward. But when both quantifiers are CPI-based, which is the case in simple NPs in natural language, the task of identifying scope dominance can be simplified using Väänänen and Westerståhl's presentation (5) of monotone CPI-based quantifiers Q, which is reproduced below.

(24a) $Q = \{X : |A \cap X| \geq n\}$, if Q is MON↑
(24b) $Q = \{X : |A \cap X| < n\}$, if Q is MON↓

In this presentation, the values of n and $|A|$ characterize Q completely, and it is therefore possible to identify scope dominance using a simple condition on these values of the two quantifiers. To do that, the following simple combinatorial lemma is useful, whose proof is given in an appendix for sake of completeness.

LEMMA 6. *Let $\ell, m, k, n \in IN$ s.t. $\ell, k > 0, m \geq 0$ and $0 < n \leq k$. Let X be a set with $|X| = k$. Then 1 and 2 below are equivalent:*

1. *There is an ℓ-ary sequence of (not necessarily distinct) subsets $X_1, \ldots, X_\ell$ of X, s.t. $|X_i| = n$, $1 \leq i \leq \ell$, and every $x \in X$ is in at most m of the X_is.*
2. *$\ell n \leq mk$.*

Using this lemma, we observe the following corollary of Proposition 2.

COROLLARY 7. *Let Q_1 and Q_2 be two CPI-based quantifiers over a finite domain E s.t. Q_1 is* MON↑ *and Q_2 is* MON↓. *According to the presentation in* (24), *assume that for some* $A, B \subseteq E$ *and* $n, m \geq 0$: $Q_1 = \{X : |A \cap X| \geq n\}$ *and* $Q_2 = \{Y : |B \cap Y| < m\}$. *Then* Q_1 *is scopally dominant over* Q_2 *iff one of the following holds*:

(i) $\frac{|A|}{n} < \frac{m+1}{m}$ *and both* $0 < n \leq |A|$ *and* $0 < m \leq |B|$ (*both quantifiers are not trivial*).
(ii) $n > |A|$ ($Q_1 = \emptyset$).
(iii) $m > |B|$ ($Q_2 = \wp(E)$).
(iv) $n > 0$ *and* $m = 0$ ($Q_2 = \emptyset$ *and* $Q_1 \neq \wp(E)$).

Proof. Note that $m = \max\{|Y| : Y \text{ is minimal in } \neg Q_2\}$. Assume that both Q_1 and Q_2 are non-trivial. According to Proposition 2, Q_1 is scopally dominant over Q_2 iff the following condition holds.

(i) $\forall Q \subseteq Q_1[|Q| = m + 1 \rightharpoonup \bigcap Q \neq \emptyset]$.

Now, (i) is equivalent to the following condition:

(ii) For every sequence $A_1, \ldots, A_{m+1}$ of (not necessarily different) subsets of A : $\bigcap_{i=1}^{m+1} A_i \neq \emptyset$.

To see that, assume first that (i) does not hold, and let $Q \subseteq Q_1$ s.t. $|Q| = m + 1$ and $\bigcap Q = \emptyset$. Let us denote $Q = \{X_1, \ldots, X_{m+1}\}$. For every i s.t. $1 \leq i \leq m+1$, let $A_i \subseteq X_i \cap A$ s.t. $|A_i| = n$. Clearly, $\bigcap_{i=1}^{m+1} A_i = \emptyset$, hence (ii) does not hold. As for the other direction, assume that (ii) does not hold, and let $A_1, \ldots, A_{m+1}$ be a sequence of subsets of A s.t. $\bigcap_{i=1}^{m+1} A_i = \emptyset$. Let $Q = \{A_1, \ldots, A_{m+1}\}$. If $|Q| = m + 1$, then we are done. Otherwise, $|Q| = k$ for $0 < k < m + 1$, and it is left to be shown that there is $Q' \subseteq Q_1$ s.t. $|Q'| = m + 1 - k$ and $Q' \cap Q = \emptyset$. To see that, simply apply the same argument from the "if" direction in the proof of Proposition 2 (with a substitution of m for n.) Thus, (i) does not hold. By Lemma 6, (ii) holds iff $\frac{|A|}{n} < \frac{m+1}{m}$. □

For the dual case, of two CPI-based quantifiers where Q_1 is MON↓ and Q_2 is MON↑, Corollary 7 can be used to prove the following characterization.

COROLLARY 8. *Let* Q_1 *and* Q_2 *be two CPI-based quantifiers over a finite domain* E *s.t.* Q_1 *is* MON↓ *and* Q_2 *is* MON↑. *According to the presentation in (24), assume that for some* $A, B \subseteq E$ *and* $n, m \geq 0$: $Q_1 = \{X : |A \cap X| < n\}$ *and* $Q_2 = \{Y : |B \cap Y| \geq m\}$. *Then* Q_1 *is scopally dominant over* Q_2 *iff one of the following holds*:

(i) $|B| > (m - 1)(|A| - n + 2)$ *and both* $0 < n \leq |A|$ *and* $0 < m \leq |B|$ (*both quantifiers are not trivial.*)
(ii) $n = 0$ ($Q_1 = \emptyset$).

(iii) $m = 0\,(Q_2 = \wp(E))$.
(iv) $n > |A|$ *and* $m \leq |B|$ ($Q_1 = \wp(E)$ *and* $Q_2 \neq \emptyset$).

EXAMPLE 8. Corollaries 7 and 8 allow us to characterize scope dominance with simple NPs using simple numerical considerations. For instance, the scope dominance in sentence (9) is derived from the following numerical consideration, Corollary 7, and the representation in Table I.

- In **more_than_half′** *(A)*: $n = \lfloor \frac{|A|}{2} \rfloor + 1$;
- In **no′** *(B)*: $m = 1$;
- $\frac{|A|}{n} = \frac{|A|}{\lfloor \frac{|A|}{2} \rfloor + 1} < 2 = \frac{m+1}{m}$.

Similar considerations according to these corollaries point to scope dominance also in examples like the following:

(25) At least four of the five referees read less than three of the seven abstracts.
(26) Less than four of the five referees read at least three of the seven abstracts.

4.4. SCOPE DOMINANCE WITH DOWNWARD MONOTONE CPI-BASED QUANTIfiERS

Proposition 10 below characterizes scope dominance with two CPI-based quantifiers that are MON↓. Its proof uses the following consequence of Lemma 6, the proof of which appears in Appendix A.

COROLLARY 9. *Let* $k, n, m > 0$. *Let Y be a set with* $|Y| \geq m$. *Then* 1 *and* 2 *below are equivalent*:

1. *There is a sequence of* (*not necessarily different*) *subsets of Y*: $Y_1, \ldots, Y_k$, *s.t.* $|Y_i| = m$, $1 \leq i \leq k$, *and*

$$|\{y \in Y : y \text{ is in at most } n-1 \text{ of the } Y_i s\}| \geq m.$$

2. $k(2m - |Y|) \leq (n-1)m$.

PROPOSITION 10. *Let* Q_1 *and* Q_2 *be two* MON↓ *CPI-based quantifiers over a finite domain E. According to the presentation in* (24), *assume that for some* $A, B \subseteq E$ *and* $n, m \geq 0$: $Q_1 = \{X : |A \cap X| < n\}$ *and* $Q_2 = \{Y : |B \cap Y| < m\}$. *Then* Q_1 *is scopally dominant over* Q_2 *iff one of the following holds*:

(i) $2 - \frac{|B|}{m} > \frac{n-1}{|A|-n+1}$ *and both* $0 < n \leq |A|$ *and* $0 < m \leq |B|$ (*both quantifiers are not trivial*).
(ii) $n = 0$ ($Q_1 = \emptyset$).
(iii) $m > |B|$ ($Q_2 = \wp(E)$).

Proof. It is easy to verify that if at least one of Q_1 and Q_2 is trivial, then Q_1 is scopally dominant over Q_2 iff one of the clauses (ii)–(iii) holds. We therefore assume that both quantifiers are not trivial, and prove that Q_1 is scopally dominant over Q_2 iff (i) holds.
Q_1 is *not* scopally dominant over Q_2 iff the following condition holds:

C1. There exists $R \subseteq E^2$ such that

- $|\{x \in A : |R_x \cap B| < m\}| < n$, which holds iff: $|\{x \in A : |R_x \cap B| \geq m\}| \geq |A| - n + 1$; and
- $|\{y \in B : |R^y \cap A| < n\}| \geq m$.

We claim that C1 is equivalent to the following condition.

C2. There are $T \subseteq E^2$ and $A' \subseteq A$ with $|A'| = |A| - n + 1$ such that

$$|T_a \cap B| = m \text{ for every } a \in A', \text{ and}$$
$$|\{y \in B : |\{a \in A' : y \in T_a\}| < n\}| \geq m.$$

To see that, assume first that C1 holds, and consider the subset $A' \subseteq \{x \in A : |R_x \cap B| \geq m\}$ s.t. $|A'| = |A| - n + 1$. For each $a \in A'$, let $B_a \subseteq R_a \cap B$ s.t. $|B_a| = m$. Define $T = \bigcup_{a \in A'}(\{a\} \times B_a)$, and note that $\{y \in B : |R^y \cap A| < n\} \subseteq \{y \in B : |\{a \in A' : y \in T_a\}| < n\}$.

As for the other direction, if C2 holds, define $R = T \cap (A' \times B)$.

Now, C2 is equivalent to the requirement that there are $|A| - n + 1$ subsets of $B : B_1, \ldots, B_{|A|-n+1,}$ s.t. $|B_i| = m$, and

$$|\{b \in B : b \text{ is in at most } n - 1 \text{ of the } B_i s\}| \geq m.$$

By Corollary 9, this requirement holds iff $(|A| - n + 1)\ (2m - |B|) \leq (n - 1)m$. □

EXAMPLE 9. As an example in which both quantifiers are MON↓, note that Proposition 10 entails that **less_than_half**′ (A) is scopally dominant over **not_every**′ (B), for any $A \subseteq E$ and any non-empty $B \subseteq E$. Such a case appears in the following sentence, in which the OWS analysis entails the ONS reading.

(27) Not every one of the referees read less than half of the abstracts.

5. Conclusions – Scope Dominance and Reasoning under Ambiguity

In this paper we have introduced results that go beyond previously known facts about scope dominance. We showed a general characterization of scope dominance with upward–downward pairs of monotone quantifiers over finite domains, and gave a simple numerical characterization of scope dominance with all pairs of upward or downward monotone CPI-based quantifiers. One obvious area for further research is the extension of the formal coverage of our results. This includes questions like scope dominance between non-CPI-based downward monotone quantifiers, over *in*finite domains, with non-monotone quantifiers or with scope permutations of more than two quantifiers. Another question for further research is the use of formal results about scope dominance for computing entailment relations in simple fragments of natural language. For instance, given a simple transitive sentence NP_1–V–NP_2, the task is to decide whether the ONS analysis *entails* the OWS analysis. In other words: whether the subject quantifier is scopally dominant over the object quantifier under *any* model. For a recent work that studies this question with upward monotone quantifiers, see Altman and Winter (2003).

One area where answers to this type of questions may be especially useful is *reasoning under ambiguity.* Consider for instance the following two sentences.

(28) At least three referees read Abstract 1 or Abstract 2.
(29) At least two referees read Abstract 1 or Abstract 2.

It is easy to see that each of the two readings of sentence (28) entails each of the two readings of sentence (29). More explicitly, each of the two statements in (30) below entails each of the two statements in (31).

(30a) $\mathbf{at\_least\_3}'\,(A)\,\{B : a_1 \in B \vee a_2 \in B\}R$ (ONS_1)

(30b) $\{B : a_1 \in B \vee a_2 \in B\}\,\mathbf{at\_least\_3}'\,(A)\,R^{-1}$ (OWS_1)

(31a) $\mathbf{at\_least\_2}'\,(A)\,\{B : a_1 \in B \vee a_2 \in B\}R$ (ONS_2)

(31b) $\{B : a_1 \in B \vee a_2 \in B\}\,\mathbf{at\_least\_2}'\,(A)\,R^{-1}$ (OWS_2)

This is an instance of what van Deemteer (1998) calls the ∀∀ *inference relation* between ambiguous sentences: *each* reading of the antecedent entails *each* reading of the consequent. Virtually any system for inference under ambiguity (e.g., Reyle, 1993, 1995, 1996; van Deemter, 1996, 1998, Eijck and Jaspars, 1996) agrees that ∀∀ inferences should be classified as valid when reasoning with natural language ambiguous sentences. Results about scope dominance show that in the case of (28)–(29), the validity of the ∀∀ inference can be decided without taking into account all four readings of the two sentences. Once observing that the ONS_1 reading of the antecedent (28) entails the OWS_2 reading of the consequent (29), the other three entailments between the readings of these sentences follow from the scope dominance of the object over the subject in both of them.

We see that for the purpose of reasoning under ambiguity, we may ignore in some cases weaker or stronger analyses among the analyses of ambiguous sentences. The study of scope dominance allows us to decide whether such weak or strong readings exist in cases of *scopally* ambiguous sentences, and hence to allow more economical underspecified representations, and computation of inferences, for sentences with scope ambiguity of quantifiers. More generally, the study of entailment patterns between different readings of ambiguous sentences, and their implications for inference under ambiguity, is a new and potentially fruitful area for research in this domain, that may improve the design and tractability of underspecified languages for inference and meaning representation. The results that were reported in this paper were obtained with an eye to this line of research, and some of their implications are currently studied.

Appendix A: Combinatorial Proofs

Proof of Lemma 6. Let $X = \{x_0, \ldots, x_{k-1}\}$. Then for every sequence $X_0, \ldots, X_{\ell-1}$ of (not necessarily different) subsets of *X*, for every *i* s.t. $0 \leq i \leq k-1$ let $m_i = |\{X_j : 0 \leq j \leq \ell - 1 \wedge x_i \in X_j\}|$.
(1) ⇒ (2):
Let $X_0, \ldots, X_{\ell-1}$ be a sequence of (not necessarily different) subsets of *X*, such that for every *j* s.t. $0 \leq j \leq \ell-1 : |X_j| = n$, and for every *i* s.t. $0 \leq i \leq k-1 : m_i \leq m$. Thus,

$$\ell n = \sum_{i=0}^{k-1} m_i \leq mk$$

(2) ⇒ (1):
Assume that $\ell n \leq mk$. Construct a sequence $X_0, \ldots, X_{\ell-1}$ of (not necessarily different) subsets of *X* as follows:

$$\begin{aligned} X_0 &= \{x_0, \ldots, x_{n-1}\} \\ &\vdots \\ X_j &= \{x_{((jn) \bmod k}, \ldots, x_{((j+1)n-1) \bmod k}\} \\ &\vdots \\ X_{\ell-1} &= \{x_{((\ell-1)n) \bmod k}, \ldots, x_{(\ell n-1) \bmod k}\} \end{aligned}$$

It is not hard to verify that for all *i, j* s.t. $0 \leq i, j \leq k-1 : m_j - 1 \leq m_i \leq m_j + 1$. Assume for contradiction that for some *i* s.t. $0 \leq i \leq k-1 : m_i = m' > m$. Thus,

$$\ell n = \sum_{i=0}^{k-1} m_i \geq m' + (m'-1)(k-1) = (m'-1)k + 1 > mk$$

in contradiction to the assumption that $\ell n \leq mk$. Hence, for all *i* s.t. $0 \leq i \leq k-1 :$ $m_i \leq m$. □

Proof of Corollary 9. Note, first, that if $|Y| \geq 2m$ then both (1) and (2) hold. So assume $2m - |Y| > 0$.
(1) $\Rightarrow$ (2):
Let $Y_1, \ldots, Y_k$ be a sequence of subsets of Y that satisfy (1). Let $X \subseteq \{y \in Y : y$ is in at most $n-1$ of the $Y_i s\}$ s.t. $|X| = m$. For each i s.t. $1 \leq i \leq k$ let $X_i \subseteq X \cap Y_i$ with $|X_i| = 2m - |Y|$.[6] Since every $x \in X$ is in at most $n-1$ of the $X_i s$, it follows from Lemma 6 that $k(2m - |Y|) \leq (n-1)m$.
(2)$\Rightarrow$ (1):
Let $X \subseteq Y$ with $|X| = m$. By Lemma 6 there is a sequence $X_1, \ldots, X_k$ of (not necessarily different) subsets of X, s.t. $|X_i| = 2m - |Y|$, $1 \leq i \leq k$, and every $x \in X$ is in at most $n-1$ of the $X_i s$. Let $Y' \subseteq Y \backslash X$ with $|Y'| = |Y| - m$. For every i s.t. $1 \leq i \leq k$ define $Y_i = X_i \cup Y'$. □

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Notes

1. Standardly, we henceforth use the term "reading" when referring to a statement that represents an actual interpretation of a sentence. When referring only to a formal derivation of a statement, with no commitment as to its empirical status, we refer to an "analysis" of a sentence.
2. Zimmermann characterizes scopelessness in a more general case, where Q and Q_1 are not necessarily defined over the same domain. The property we mention here is a direct result of his characterization.
3. Plausibly, plurality in sentences (7) and (8) leads to the presupposition that there are at least *two* abstracts. However, for obtaining the entailment between the analyses of these sentences, the weaker non-emptiness assumption is sufficient.
4. Thanks to Ya'acov Peterzil for pointing this out to us.
5. The same argument holds for any quantifier, not only **every**′ (A').
6. Note that $|X \cup Y_i| = |Y_i \backslash (Y \backslash X)| \geq |Y_i| - |Y \backslash X| = 2m - |Y|$.

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