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Globally Stabilizing a Class of Underactuated Mechanical Systems on the Basis of Finite-Time Stabilizing Observer

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Abstract Globally exponentially stabilizing a class of underactuated mechanical systems (UMS) with nonaffine nonlinear dynamics is investigated in this paper. The considered UMS has a nonaffine nonlinear subsystem that can be globally asymptotically stabilized by saturated feedbacks, but the saturated feedback cannot be analytically expressed in closed-form. This obstacle limits the real-time applications of most controllers presented in literatures. In this paper, a hybrid feedback strategy is presented to globally exponentially stabilize the UMS with nonaffine and strict-feedback canonical forms. The hybrid feedback strategy is characterized by the composition of partial states feedback and partial virtual outputs feedback based on a higher-order finite-time stabilizing observer. The presented hybrid feedback controller can be synthesized by applying Lyapunov stability theory. Some numerical simulations associated with two underactuated nonlinear systems, the Acrobot system and the Inertia-Wheel-Pendulum (IWP) system, are employed to demonstrate the effectiveness of the proposed controller. The presented control strategy can be applied in real time, thus providing a new feasible dynamic model other than the differential flatness systems for synthesizing the mechanical systems of general underactuated legged robots.

Keywords Non-affine nonlinear systems · Underactuation · Finite-time · Observer · Controller

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1 Introduction

The control problems of the affine nonlinear systems including both macro and micro mechanical systems have been given much attention over the past three decades [1, 2]. Unfortunately, most dynamically mechanical systems could just be modeled as nonaffine nonlinear systems in practice. In the past couple of decades, a few researches were involved in the control problems for nonaffine systems with some special properties. Some typical examples of the relevant results include the Nonlinear Small Gain Theorem presented by Teel [3] for a class of nonlinear systems with feedforward normal form [4]. Based on the nonaffine passive systems theory—feedback equivalence and the technique of bounded state feedback, Lin [5] presented a systematic method for globally asymptotically stabilizing (GAS) a class of nonaffine systems including the polynomial systems with stable free dynamics (or zero-input stable in [3]). For the UMS [6, 7], Olfati-Saber [7] showed that the dynamics of almost all of this class of mechanical systems could be transformed into a certain kind of nonaffine form. Li and Qian [8] (and some references therein) studied the global finite-time stabilization by dynamic output feedback for a class of power form nonlinear systems with nonsmooth continuous property. Moulay and Perruquent [9] presented a constructive method for stabilizing a class of polynomial nonaffine systems with the control variables up to the third order. Because of the difficulties in finding the analytical feedback for stabilizing the general nonaffine systems, the approximation methods have also been given much attention. For instance, Zhao and Farrell [10] investigated the control method based on the locally weighted online approximation for a class of uncertain nonaffine systems. More recently, Dong et al. [11] presented a self-organizing approximation approach for the purpose of tracking control of the single-input single-output (SISO) nonaffine systems.

Motivated by the potentially significant applications in complex robotic systems [12–16], we are interested in the global stabilization of a class of nonaffine system with strict feedback form

$$\begin{aligned} \dot{z}_1 &= \boldsymbol{\psi}_1(\boldsymbol{\xi}_1)z_2 & \dot{\boldsymbol{\xi}}_1 &= \boldsymbol{\xi}_2 \\ \dot{z}_2 &= \boldsymbol{\psi}_2(z_1, \boldsymbol{\xi}_1) & \dot{\boldsymbol{\xi}}_2 &= \boldsymbol{u} \end{aligned} \quad (1)$$

where $z_1, z_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{u} \in \boldsymbol{R}^r$, $r \geq 1$, and $\boldsymbol{\psi}_1(\boldsymbol{\xi}_1)$ be positive definite matrix in a diagonal form. The nonaffine nonlinear systems (1) represent large classes of robotic systems and UMS. Examples of relevant systems includes the dynamically legged robots [12–16], Acrobot [17], Inertia-Wheel-Pendulum (IWP) [18, 19], and TORA [19] etc. In the relevant researches, most of the presented controllers employed switched-based control methods [15, 17], and some researches only showed a local stabilization effect [20]. Ortega et al. [18] presented the interconnection and damping assignment (IDA) method based on the well-known passivity-based control (PBC) to stabilize a class of UMS, which is roughly limited to a special class of UMS due to the difficulties in solving the matching partial differential equations (PDFs). To solve the global stabilization problem of system (1), Olfati-Saber [19] presented the control scheme composed of saturated feedback followed by a backstepping procedure. However, as it will be shown in the next section, the saturated feedback cannot be generally obtained in the closed-form of the nonaffine subsystem of Eq. 1, so that the followed backstepping procedure using the numerical approximation as it applied in [19] cannot be realized in real time.

Inspired by these relevant works [15–20], a hybrid feedback strategy composed of partial states feedback and partial virtual outputs feedback on the basis of a finite-time stabilizing observer is proposed in this paper. To the best knowledge of the authors, it is the first time that this kind of controller to be presented to stabilize the nonaffine system given by Eq. 1, of which the saturated feedback for the nonaffine subsystem cannot be analytically obtained. The presented novel control strategy can be realized in real time, thus providing a new feasible dynamic model other than the differential flatness systems [16, 32] for synthesizing the UMSs of general legged robots.

The remainder of this paper is organized as follow. In Section 2, the saturated feedback for the nonaffine subsystem of Eq. 1 is concisely reviewed, and some important Lemmas are presented. Section 3 provides a third-order finite-time stabilizing observer for the second order linear subsystem of Eq. 1. In section 4, we present a output feedback finite-time stabilizing controller on the basis of the higher-order finite-time stabilizing observer provided in Section 3 to realize the trajectory tracking tasks. Section 5 contains the simulation results associated with two UMS, Acrobot and IWP, in order to demonstrate the effects of the presented controller. We conclude this paper in Section 6.

2 The Saturated Controller for the Non-Affine Subsystem and Some Key Lemmas

Consider the nonaffine subsystem of Eq. 1

$$\begin{aligned} \dot{z}_1 &= \boldsymbol{\psi}_1(\boldsymbol{\xi}_1)z_2 \\ \dot{z}_2 &= \boldsymbol{\psi}_2(z_1, \boldsymbol{\xi}_1) \end{aligned} \quad (2)$$

Some assumptions are provided as follows to define the range of the addressed problem in this paper. Actually, the UMS in potential field always satisfy these assumptions.

A1. Both the matrix $\boldsymbol{\psi}_1(\boldsymbol{\xi}_1)$ and its inverse matrix $\boldsymbol{\varphi}_1(\boldsymbol{\xi}_1) = \boldsymbol{\psi}_1^{-1}(\boldsymbol{\xi}_1)$ are positive definite and diagonal matrices.

A2. The vector $\boldsymbol{\psi}_2(z_1, \boldsymbol{\xi}_1)$ is smooth, and satisfies $\boldsymbol{\psi}_2(0, 0) = 0$ and $\det\left|\frac{\partial \boldsymbol{\psi}_2(z_1, \boldsymbol{\xi}_1)}{\partial \boldsymbol{\xi}_1}\right| \neq 0$.

A3. There exists an isolated root $\boldsymbol{\xi}_1 = \boldsymbol{\mu}(z_1)$ of $\boldsymbol{\psi}_2(z_1, \boldsymbol{\mu}(z_1)) = 0$ with the property $\boldsymbol{\mu}(0) = 0$.

On the basis of the assumptions A1-A3, one can define the virtual input

$$\boldsymbol{\xi}_1 = \boldsymbol{\mu}(z_1) + v$$

and a local diffeomorphism around a neighborhood of $\boldsymbol{v} = 0$

$$\boldsymbol{w} = \boldsymbol{\Phi}(z_1, \boldsymbol{v}) := \boldsymbol{\varphi}_1(\boldsymbol{\xi}_1)\ \boldsymbol{\psi}_2(z_1, \boldsymbol{\xi}_1)\big|_{\boldsymbol{\xi}_1 = \boldsymbol{\mu}(z_1)+v}$$

Since the assumption A2 states det $\left|\frac{\partial \boldsymbol{\psi}_2(z_1, \boldsymbol{\xi}_1)}{\partial \boldsymbol{\xi}_1}\right| \neq 0$, that means $\boldsymbol{\psi}_2(z_1, \boldsymbol{\xi}_1)\big|_{\boldsymbol{\xi}_1 = \boldsymbol{\mu}(z_1)+\boldsymbol{v}}$ is invertible at $v = 0$, and $\boldsymbol{\varphi}_1(\boldsymbol{\xi}_1)$ is also invertible due to the assumption A1. According to the implicit function theorem, $\mathbf{w} = \boldsymbol{\Phi}(z_1, \boldsymbol{v})$ has unique inverse function $\boldsymbol{v} = \boldsymbol{\beta}(z_1, \mathbf{w})$that satisfies equation $\boldsymbol{w} = \boldsymbol{\Phi}(\mathbf{z}_1, \boldsymbol{\beta}(z_1, \boldsymbol{w}))$, then the dynamics (2) equivalents to

$$\begin{aligned} \dot{z}_1 &= \boldsymbol{\psi}_1(\boldsymbol{\xi}_1) z_2 \\ \dot{z}_2 &= \boldsymbol{\psi}_1(\boldsymbol{\xi}_1)\boldsymbol{w} \end{aligned} \tag{3}$$

for all $z_1 \in R^r$. It is closely related to the literatures [3–5], and the following saturated feedback controller was presented in [19] for globally asymptotically stabilizing the origin of Eq. 3 as well as the system (2).

Lemma 1 *Considering the assumptions A1-A3, the controller*

$$\boldsymbol{\xi}_1 = \boldsymbol{\mu}(z_1) + \boldsymbol{\beta}(z_1, w) \tag{4}$$

where

$$\boldsymbol{w} = -k_1 \frac{z_1 + z_2}{\left(1 + (z_1 + z_2)^{\mathrm{T}}(z_1 + z_2)\right)^{1/2}} \tag{5}$$

and $k_1 = k sign\left(\frac{\partial \boldsymbol{\psi}_2(z_1, \boldsymbol{\xi}_1)}{\partial \boldsymbol{\xi}_1}\right)$, $k > 0$ *is a constant, globally asymptotically stabilize the origins of both equivalent system* (3) *and its original nonaffine system* (2).

Proof Since it will be used in the sequel, the Lemma 1 can be concisely proved as follows.

Define a positive definite function

$$V_1 = k_1\left[\left(1 + (z_1 + z_2)^{\mathrm{T}}(z_1 + z_2)\right)^{1/2} - 1\right] + \frac{1}{2} z_2^{\mathrm{T}} z_2 \tag{6}$$

time derivative of Eq. 6 can be written as

$$\dot{V}_1 = -\boldsymbol{w}^{\mathrm{T}}\left[\boldsymbol{\psi}_1(\boldsymbol{\xi}_1) z_2 + \boldsymbol{\psi}_1(\boldsymbol{\xi}_1)\boldsymbol{w}\right] + z_2^{\mathrm{T}}\boldsymbol{\psi}_1(\boldsymbol{\xi}_1)\boldsymbol{w}$$

where $\boldsymbol{w}$ is given by Eq. 5. Thus it is followed that

$$\dot{V}_1 = -\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\psi}_1(\boldsymbol{\xi}_1)\boldsymbol{w} \tag{7}$$

As $\boldsymbol{\psi}_1(\boldsymbol{\xi}_1)$ is a positive definite and diagonal matrix, then

$$\dot{V}_1 < 0$$

for all $z_1 + z_2 \neq 0$. Based on the LaSalle's invariance principle, all the solutions of the closed-loop system converge to the largest invariant set $\Omega = \left\{(z_1, z_2) : \dot{V}_1 = 0\right\}$. For the system (3), the largest invariant set is given by $\Omega = (0, 0)$. Therefore, the origin $(z_1, z_2) = (0, 0)$is globally asymptotically stable. □

Remark 1 The expression (5) is a vector sigmoidal function, and can be expressed as $\boldsymbol{w} = [w_1(z_1), ..., w_r(z_r)]^{\mathrm{T}}$, where $w_i(z_i), i = 1, 2, \ldots, r$ denote the scalar sigmoidal functions. Besides the saturated function given by Eq. 5, both tanh(·) and atan(·) are additional saturated functions that are commonly used in literatures.

As the solution $\boldsymbol{\xi}_1 = \boldsymbol{\mu}(z_1)$ of $\boldsymbol{\psi}_2(z_1, \boldsymbol{\mu}(\mathbf{z}_1)) = 0$ cannot be obtained in the closed forms of a general nonlinear equation, the backstepping procedure on the basis of the saturated feedback (4) cannot be used directly. In the literature [19], a piece-wise linear approximation method is used to get the solution of equation $\boldsymbol{\psi}_2(z_1, \boldsymbol{\mu}(z_1)) = 0$ by a constructed look up table of pairs $(z_1, \boldsymbol{\mu}(z_1))$ for globally asymptotically stabilizing the Acrobot system. Even though the numerical solution $\boldsymbol{\mu}(z_1)$and the time derivates $\dot{\boldsymbol{\mu}}(z_1)$ and $\ddot{\boldsymbol{\mu}}(z_1)$can be accurately obtained by the look up table method, the calculation efficiency would obstacle the method to be applied in real time. In the next section, a new finite-time stabilizing observer is introduced to get the time-derivates $\dot{\boldsymbol{\mu}}(z_1)$and $\ddot{\boldsymbol{\mu}}(z_1)$that can be realized in real time.

As the last part of this section, several Lemmas are provided as following since they will be used in the next sections.

Lemma 2 [21] *For any real numbers* a_i, $i = 1, 2, \ldots, n$ *and* $0 < \gamma \leq 1$, *the following inequality holds*

$$\left(\sum_{i=1}^{n} |a_i|\right)^{\gamma} \leq \sum_{i=1}^{n} |a_i|^{\gamma}. \tag{8}$$

For $x \in R, y \in R$, when $0 < \gamma = p/q \leq 1$, where $p > 0$ and $q > 0$ are odd integers, then

$$\left|x^{\gamma} - y^{\gamma}\right| \leq 2^{1-\gamma} \left|x - y\right|^{\gamma} \leq 2 \left|x - y\right|^{\gamma}. \tag{9}$$

When $\gamma > 1$ is a constant, then

$$\left|x - y\right|^{\gamma} \leq 2^{\gamma-1} \left|x^{\gamma} - y^{\gamma}\right|. \tag{10}$$

Lemma 3 [21]*: Let a, b be positive real numbers and $\beta(x, y) > 0$ be a real-valued function, then*

$$|x|^{a} |y|^{b} \leq \frac{a\beta(x, y)}{a+b} |x|^{a+b} + \frac{b\beta^{-a/b}(x, y)}{a+b} |y|^{a+b} \tag{11}$$

Remark 2 Lemma 3 can be proved by the Young's inequality $|xy| \leq \frac{|x|^m}{m} + \frac{|y|^n}{n}$, where $\frac{1}{m} + \frac{1}{n} = 1$, and $m > 0, n > 0$.

Lemma 4 [22]*: Given $0 < \gamma = p/q \leq 1$, where $p > 0$ and $q > 0$ are odd integers, and $\xi \neq \alpha$, then the following inequality holds:*

$$\int_{\alpha}^{\xi} (s^{1/\gamma} - \alpha^{1/\gamma})^{2-\gamma} ds > 0 \tag{12}$$

Remark 3 Lemma 4 can be proved by Eq. 10 and equality $(x)^{\gamma} = sign(x) |x|^{\gamma}$. The power-integrator (12) will be used to design the Lyapunov function candidate in the next section.

3 The Third Order Finite-Time Stabilizing Observer

Consider the linear subsystem of Eq. 1

$$\begin{aligned} \dot{\boldsymbol{\xi}}_1 &= \boldsymbol{\xi}_2 \\ \dot{\boldsymbol{\xi}}_2 &= \boldsymbol{u} \end{aligned} \tag{13}$$

where $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{u} \in \boldsymbol{R}^r$. To stabilize the system (1), the control problem for the linear subsystem (13) is to track the trajectory presented by Eq. 4, namely $\boldsymbol{\alpha}_1 = \boldsymbol{\xi}_1 = \boldsymbol{\mu}(\mathbf{z}_1) + \boldsymbol{\beta}(z_1, w)$. On the basis of assumption A1, and in order to simplify the formulations while without lose any generality, the control problem of the system (13) can be simplified to a single input second-order linear system

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \tag{14}$$

where $\xi_1, \xi_2, u \in R$. To obtain the time derivates $\dot{\alpha}_1$ and $\ddot{\alpha}_1$ of the virtual input $\alpha_1(t)$, the third-order finite-time stabilizing observer can be given as follows.

Proposition 1 *Suppose the $\alpha_1(t)$ is aC^2 function, then the observer*

$$\begin{aligned} \dot{\hat{\alpha}}_1 &= \hat{\alpha}_2 - \hat{k}_1(\hat{\alpha}_1 - \alpha_1)^{5/7} \\ \dot{\hat{\alpha}}_2 &= \hat{\alpha}_3 - \hat{k}_2(\hat{\alpha}_2 - \dot{\hat{\alpha}}_1)^{3/7} \\ \dot{\hat{\alpha}}_3 &= -\hat{k}_3(\hat{\alpha}_3 - \dot{\hat{\alpha}}_2)^{1/7} \end{aligned} \tag{15}$$

stabilizes in finite settling time $T^* > 0$. When $t \geq T^*$is satisfied, then the following equations hold

$$\hat{\alpha}_1 = \alpha_1(t), \quad \hat{\alpha}_2 = \dot{\alpha}_1(t), \quad \hat{\alpha}_3 = \ddot{\alpha}_1(t) \tag{16}$$

Proof The main steps are only provided in this proof. For a detailed proving procedure, the readers are encouraged to refer the relevant paper [21] or [22] to which the detailed proof is provided for synthesizing the finite time controllers.

Define $e_1 = \hat{\alpha}_1 - \alpha_1$, $e_2 = \hat{\alpha}_2 - \dot{\alpha}_1$, and $e_3 = \hat{\alpha}_3 - \ddot{\alpha}_1$to be the output errors of the observer, where $\hat{\alpha}_2 = \dot{\hat{\alpha}}_1, \hat{\alpha}_3 = \dot{\hat{\alpha}}_2$. It is obvious that the errors dynamics can be expressed as

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= v \end{aligned} \tag{17}$$

For the subsystem e_1 of Eq. 17, select the positive definite function $U_1 = \frac{1}{2}e_1^2$, then the time derivate of it is given by

$$\dot{U}_1 = e_1 \left(e_2 - \hat{\beta}_1\right) + e_1\hat{\beta}_1 \tag{18}$$

If define

$$\hat{\beta}_1 = -\hat{k}_1 e_1^{5/7} \tag{19}$$

where $\hat{k}_1 > 0$, then Eq. 17 can be written as

$$\dot{U}_1 = -\hat{k}_1 e_1^{12/7} + e_1 \left(e_2 - \hat{\beta}_1\right) \tag{20}$$

For the subsystem (e_1, e_2) of Eq. 17, define a new positive definite function $U_2 = U_1 + W_1$, where

$$W_1 = \int_{\beta_1}^{e_2} (s^{7/5} - \hat{\beta}_1^{7/5})^{9/7} ds > 0$$

due to Lemma 4. Then it follows that

$$\dot{U}_2 = -\hat{k}_1 e_1^{12/7} + e_1 \left(e_2 - \hat{\beta}_1\right) + \frac{\partial W_1}{\partial e_2} e_3 + \frac{\partial W_1}{\partial \hat{\beta}_1} \frac{\partial \hat{\beta}_1}{\partial e_1} e_2 \tag{21}$$

By applying inequalities (9) and (11), it is not difficult to show that the second term of right hand side of Eq. 21 satisfies

$$\left|e_1(e_2-\hat{\beta}_1)\right| \le \delta_1 \left|e_1\right|^{12/7} + \delta_2 \left|e_2^{7/5}-\hat{\beta}_1^{7/5}\right|^{12/7} \quad (22)$$

where $\delta_1 > 0$ and $\delta_2 > 0$ are two constants. The third term of the right hand side of Eq. 21 can be expressed as

$$\frac{\partial W_1}{\partial e_2} e_3 = \left(e_2^{7/5}-\hat{\beta}_1^{7/5}\right)^{9/7}\left(e_3-\hat{\beta}_2+\hat{\beta}_2\right) \quad (23)$$

where $\hat{\beta}_2$ is a virtual input that will be determined later. The fourth term of right hand side of Eq. 21 satisfies

$$\begin{aligned}\left|\frac{\partial W_1}{\partial \hat{\beta}_1}\frac{\partial \hat{\beta}_1}{\partial e_1}e_2\right| &\le \frac{9}{7}k_1^{7/5}\left|\int_{\hat{\beta}_1}^{e_2}\left(s^{7/5}-\hat{\beta}_1^{7/5}\right)^{2/7}ds\right|\left|e_2\right| \\ &\le \frac{9}{7}k_1^{7/5}\left|e_2^{7/5}-\hat{\beta}_1^{7/5}\right|^{2/7}\left|e_2-\hat{\beta}_1\right|\left|e_2\right|\end{aligned} \quad (24)$$

Using the inequalities (9) and (11) once more, it can be shown that Eq. 24 satisfies inequality

$$\left|\frac{\partial W_1}{\partial \hat{\beta}_1}\frac{\partial \hat{\beta}_1}{\partial e_1}e_2\right| \le \delta_3 \left|e_1\right|^{12/7} + \delta_4 \left|e_2^{7/5}-\hat{\beta}_1^{7/5}\right|^{12/7} \quad (25)$$

Substitute (22), (23) and (25) into Eq. 21, then Eq. 21 has the form

$$\begin{aligned}\dot{U}_2 \le\ &\left(-\hat{k}_1+\delta_1+\delta_3\right)e_1^{12/7} \\ &+(\delta_2+\delta_4)\left(e_2^{7/5}-\hat{\beta}_1^{7/5}\right)^{12/7} \\ &+\left(e_2^{7/5}-\hat{\beta}_1^{7/5}\right)^{9/7}\hat{\beta}_2 \\ &+\left(e_2^{7/5}-\hat{\beta}_1^{7/5}\right)^{9/7}\left(e_3-\hat{\beta}_2\right)\end{aligned} \quad (26)$$

It is obvious that there exists a constant

$$\hat{k}_2 = -k_2+\delta_2+\delta_4 > 0$$

where $k_2 > 0$, and another constant $k_1 = \hat{k}_1 - \delta_1 - \delta_3 > 0$, if given the virtual input

$$\hat{\beta}_2 = -\hat{k}_2\left(e_2^{7/5}-\hat{\beta}_1^{7/5}\right)^{3/7} \quad (27)$$

then Eq. 26 follows that

$$\begin{aligned}\dot{U}_2 \le &-k_1 e_1^{12/7} - k_2\left(e_2^{7/5}-\hat{\beta}_1^{7/5}\right)^{12/7} \\ &+\left(e_2^{7/5}-\hat{\beta}_1^{7/5}\right)^{9/7}\left(e_3-\hat{\beta}_2\right)\end{aligned} \quad (28)$$

As to the overall system (17), if one selects $U = U_2 + W_2$ where $W_2 = \int_{\hat{\beta}_2}^{e_3}(s^{7/3}-\hat{\beta}_2^{7/3})^{11/7}ds > 0$, then proceeding along similar lines as the derivations given above, it can be shown that there exists a constant $\hat{k}_3 > 0$, and if one selects the virtual input

$$v = -\hat{k}_3\left(e_3^{7/3}-\hat{\beta}_2^{7/3}\right)^{1/7} \quad (29)$$

then the time derivative of the positive definite function U can be written as

$$\dot{U} \le -\hat{k}e_1^{12/7} - \hat{k}\left(e_2^{7/5}-\hat{\beta}_1^{7/5}\right)^{12/7} - \hat{k}\left(e_3^{7/3}-\hat{\beta}_2^{7/3}\right)^{12/7} \quad (30)$$

where $\hat{k} > 0$ is a constant. On the other hand, according to the definition of the positive definite function U, it can be shown that

$$U \le 2e_1^2 + 2\left(e_2^{7/5}-\hat{\beta}_1^{7/5}\right)^2 + 2\left(e_3^{7/3}-\hat{\beta}_2^{7/3}\right)^2$$

Using the inequality (8), it is not difficult to show that

$$\dot{U} \le -\frac{k}{2}U^{6/7}$$

According to the Theorem 4.2 of [23], if the time satisfies

$$t \ge T^* = \frac{14}{k}U(t_0)^{1/7}$$

then the errors $e_i = 0, i = 1, 2, 3$, hence the equations of Eq. 16 are satisfied. This completes the proof of Proposition 1. □

Remark 4 Although the third order finite-time stabilizing observer is presented by Eq. 15, the $(n+1)$-th order observer can be intuitionally provided as

$$\begin{aligned}\dot{\hat{\alpha}}_1 &= \hat{\alpha}_2 - \hat{k}_1(\hat{\alpha}_1-\alpha_1)^{(2n+1)/(2n+3)} \\ &\cdots \\ \dot{\hat{\alpha}}_i &= \hat{\alpha}_{i+1} - \hat{k}_i(\hat{\alpha}_i-\dot{\hat{\alpha}}_{i-1})^{[2(n-i)+3]/(2n+3)} \\ &\cdots \\ \dot{\hat{\alpha}}_{n+1} &= -\hat{k}_{n+1}(\hat{\alpha}_{n+1}-\dot{\hat{\alpha}}_n)^{1/(2n+3)}\end{aligned} \quad (31)$$

For the n-th order Brunovsky's canonical system

$$\dot{e}_1 = e_2, \ldots, \dot{e}_i = e_{i+1}, \ldots, \dot{e}_n = v$$

Remark 5 As that pointed out by Levant et al. in [24], the accuracy of a nonsmooth feedback observer reduces when $n > 4$. For the linear system with order less than five, the observer (31) will provide an effective method to obtain the time-derivates in real time for constructing a backstepping based closed-loop controller.

For the purpose of clarity, we show the stability of the finite-time stabilizing observer presented in Eq. 15 by an example as follow.

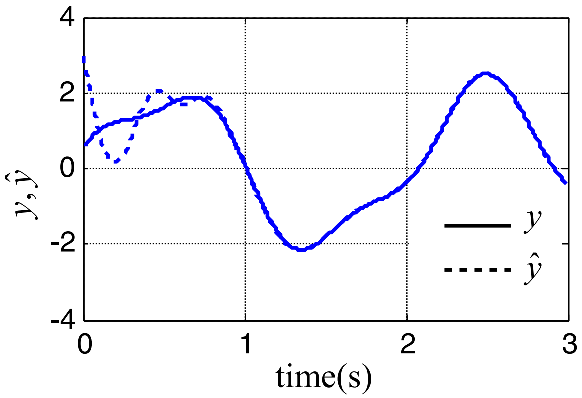

Fig. 1 The function $y(t)$and the output $\hat{y}(t)$ of the observer (15)

Example Consider the smooth function

$y(t) = 2sin\,(\pi t) + 0.5cos(2.4\pi t)$.

Let's solve the time-derivates $\dot{y}(t)$ and $\ddot{y}(t)$ using observer (15).

Given $\alpha_1(t) = y(t)$ and use the observer (15), the outputs $\hat{y}(t)$, $\dot{\hat{y}}(t)$, and $\ddot{\hat{y}}(t)$ are plotted in Figs. 1, 2 and 3, which show the finite-time stability of the presented nonsmooth feedback nonlinear observer.

4 The Finite-Time Stabilizing Controller Based on the Finite-Time Stabilizing Observer

This section considers the globally exponentially stabilization problem of the nonlinear system (1) on the basis of the saturated feedback (4) of the nonaffine subsystem (2) and the finite-time stabilizing observer (15) of the linear subsystem (13). It is intuitional that

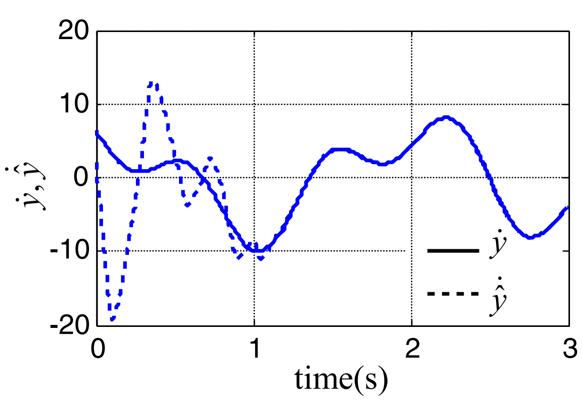

Fig. 2 The first order time-derivate $\dot{y}(t)$ and the output$\dot{\hat{y}}(t)$of the observer (15)

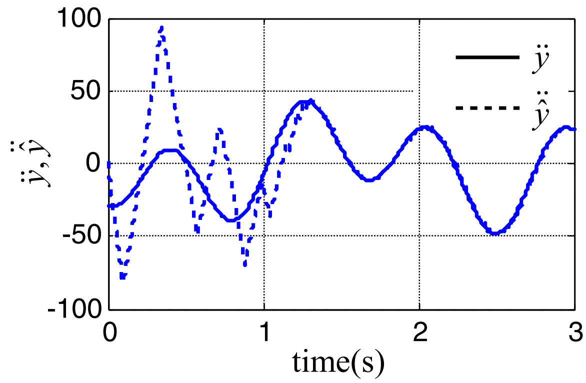

Fig. 3 The second order time-derivate $\ddot{y}(t)$ and the output $\ddot{\hat{y}}(t)$of the observer (15)

the control task of the linear subsystem (13) of Eq. 1 is actually a trajectory-tracking problem. To this end, let's define the error variables

$$\begin{aligned} \zeta_1 &= \xi_1 - \alpha_1 \\ \zeta_2 &= \xi_2 - \dot{\alpha}_1 \end{aligned} \tag{32}$$

where $\alpha_1 = \alpha_1(t)$is the desired trajectory of the virtual control input ξ_1 of the nonaffine subsystem (2), then the dynamics of the errors system can be written as

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= u - \ddot{\alpha}_1 \end{aligned} \tag{33}$$

To stabilize the linear system (33), a finite-time stabilizing controller is provided in proposition 2.

Proposition 2 *There exists constants $k_1 > 0$ and $k_2 > 0$, such that the state feedback controller*

$$u = \ddot{\alpha}_1 - k_2(\zeta_2^{7/5} - \beta_1^{7/5})^{3/7} \tag{34}$$

where $\beta_1 = -k_1\zeta_1^{5/7}$, stabilizes the origin of (33) in finite settling-time.

Proof Using the lemmas 2-4, and referring to the proof of proposition 1, then the proposition 2 follows. □

Remark 6 As the desired trajectory $\alpha_1 = \alpha_1(t)$for the linear system (33) cannot be obtained generally in a closed-form, one has to make use of an observer to get the time derivates of the numerical trajectory $\alpha_1 = \alpha_1(t)$. For a linear system, it is well known that the observers and the controllers can be separately designed due to the "Separation Principle" [1, 25], which is commonly adopted in output feedback control such as [20, 25]. However, in this paper we will directly prove the stability of the observer-based finite-time stabilizing controller, which will be used to stabilize the trajectory-tracking control system (33).

Proposition 3 *Using the finite-time stabilizing observer* (15)*, the finite time stabilizing controller* (34) *stabilizes the origin of the linear system* (33) *in finite settling-time.*

Proof Replace the desired trajectory $(\alpha_1(t), \dot{\alpha}_1(t), \ddot{\alpha}_1(t))$ of the linear system (32)–(33) by the outputs $\left(\hat{\alpha}_1(t), \dot{\hat{\alpha}}_1(t), \ddot{\hat{\alpha}}_1(t)\right)$of the observer (15), and then the errors dynamics (33) can be written as

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= u - (\ddot{\alpha}_1 + e_3) \end{aligned} \tag{35}$$

where the error variables are given by

$$\begin{aligned} \zeta_1 &= \xi_1 - (\alpha_1 + e_1) \\ \zeta_2 &= \xi_2 - (\dot{\alpha}_1 + e_2) \end{aligned} \tag{36}$$

The finite-time stabilizing controller (34) can be rewritten as

$$u = (\ddot{\alpha}_1 + e_3) - k_2(\zeta_2^{7/5} - \beta_1^{7/5})^{3/7} \tag{37}$$

where$\beta_1 = -k_1\zeta_1^{5/7}$. Select the positive definite function

$$\hat{V}(\boldsymbol{\zeta}, \boldsymbol{e}) = V(\boldsymbol{\zeta}) + U(\boldsymbol{e})$$

for the closed-loop system (35)-(37)-(17), and $V(\boldsymbol{\zeta})$is given by

$$\begin{aligned} V(\boldsymbol{\zeta}) &= \frac{1}{2}\zeta_1^2 + \int_{\beta_1}^{\zeta_2} \left(s^{7/5} - \beta_1^{7/5}\right)^{9/7} ds \\ &\leq 2\zeta_1^2 + 2\left(\zeta_2^{7/5} - \beta_1^{7/5}\right)^2 \end{aligned} \tag{38}$$

and the function $U(\boldsymbol{e})$ that was defined in the proof of Proposition 1, is given by

$$\begin{aligned} U(\boldsymbol{e}) &= \frac{1}{2}e_1^2 + \int_{\beta} 1^{e_2}(s^{7/5} - \hat{\beta}_1^{7/5})^{9/7} ds \\ &\quad + \int_{\beta_2}^{e_3} (s^{7/3} - \hat{\beta}_2^{7/3})^{11/7} ds \\ &\leq 2e_1^2 + 2\left(e_2^{7/5} - \hat{\beta}_1^{7/5}\right)^2 + 2\left(e_3^{7/3} - \hat{\beta}_2^{7/3}\right)^2 \end{aligned} \tag{39}$$

Considering the controller (37), and using the Lemma 2-4, it is easy to show that

$$\dot{V} \leq -k\zeta_1^{12/7} - k\left(\zeta_2^{7/5} - \beta_1^{7/5}\right)^{12/7} \tag{40}$$

where $k > 0$is a constant. In the proof of Proposition 1, it is also shown that

$$\dot{U} \leq -\hat{k}e_1^{12/7} - \hat{k}\left(e_2^{7/5} - \hat{\beta}_1^{7/5}\right)^{12/7} - \hat{k}\left(e_3^{7/3} - \hat{\beta}_2^{7/3}\right)^{12/7} \tag{41}$$

where $\hat{k} > 0$ is also a constant. Then it follows that

$$\dot{V} \leq -\frac{k}{2}V^{6/7}, \qquad \dot{U} \leq -\frac{k}{2}U^{6/7} \tag{42}$$

Thus the following inequality holds

$$\dot{V} + \dot{U} \leq -\frac{k}{2}\left(V^{6/7} + U^{6/7}\right) \tag{43}$$

By the inequality (8) of lemma 2, it is easy to show that

$$\dot{V} + \dot{U} \leq -\frac{k}{2}(V + U)^{6/7} \tag{44}$$

According to the Theorem 4.2 in reference [23], the controller (37) renders the origin of the linear system (35) to stabilize in finite settling-time. □

Remark 7 Esfandiari, Khalil and Atassi et al. [25–27] have been given much attention on the robustness analysis of high-gain- observer-based controllers. They also generalized the “Separation Principle” to a class of nonlinear system [27]. For the nonlinear systems, the closed-loop control system based on the high-gain-observer tends to show us the so-called peaking phenomenon [1], which will damage the stability of the observer-based closed-loop control system. However, as to the linear systems, the peaking phenomenon does not happen in the high-gain-observer-based closed-loop control systems.

Remark 8 The finite-time stabilizing observer presented by proposition 1 uses Hölder continuous feedback, which belongs to a class of nonsmooth feedback. The Hölder continuous feedback not only shows a finite settling time but also inherits the robustness of a discontinuous feedback, such as the higher order sliding mode [24]. The literatures [22, 28] devote to the robustness analysis for some kinds of finite-time stabilizing controllers.

As the last part of this section, we will present the main result of the paper.

Proposition 4 *Suppose the nonaffine systems* (1) *satisfy the assumptions A1-A3, and there exists a saturated feedback* (4) *such that the origin of the nonaffine* *subsystem* (2) *globally asymptotically stabilize. then there exists rdimensional third-order finite-time stabilizing observers* (15) *and rdimensional second order finite-time stabilizing controllers* (34) *rendering the origin of the nonaffine system* (1) *to globally asymptotically stabilize.*

Proof Based on the assumption A1, to prove the proposition 4, we only need to prove the stability of the single input system

$$\begin{array}{ll} \dot{z}_1 = \psi_1(\xi_1)z_2 & \dot{\xi}_1 = \xi_2 \\ \dot{z}_2 = \psi_2(z_1,\xi_1) & \dot{\xi}_2 = u \end{array} \tag{45}$$

where $z_1, z_2, \xi_1, \xi_2, u \in R$, controlled by the scalar controller

$$u = \ddot{\alpha}_1 - k_2\left(\zeta_2^{7/5} - \beta_1^{7/5}\right)^{3/7}$$

on the basis of observer (15). According to the assumptions A1-A3 and Lemma 1, there exists saturated feedback

$$\xi_1 = \mu(z_1) + \beta(z_1, w) \tag{46}$$

where

$$w = -k_1\frac{z_1+z_2}{\left(1+(z_1+z_2)^2\right)^{1/2}} \tag{47}$$

globally asymptotically stabilizing the origin of the nonaffine subsystem (z_1, z_2) of Eq. 1. Consider the error system of Eq. 45

$$\begin{array}{l} \dot{z}_1 = \psi_1(\alpha_1)z_2 \\ \dot{z}_2 = \psi_2(z_1,\alpha_1) \\ \dot{\zeta}_1 = \zeta_2 \\ \dot{\zeta}_2 = u - (\ddot{\alpha}_1 + e_3) \end{array} \tag{48}$$

where (ζ_1, ζ_2) is defined by Eq. 36. Let's define a positive definite function for the system (48), $\tilde{V} = V_1 + V + U$, where

$$V_1 = k_1\left[\left(1+(z_1+z_2)^T(z_1+z_2)\right)^{1/2} - 1\right] + \frac{1}{2}z_2^T z_2 \tag{49}$$

and the functions V and U are defined by Eqs. 38 and 39 respectively. Due to the proposition 3, the controller (37) with making use of the observer (15) renders the virtual input to satisfy

$$\alpha_1(t)|_{t>T^*} = \xi_1(t)$$

where $\xi_1(t)$ is given by Eq. 46, and

$$T^* = \frac{14}{k}(V+U)(t_0)^{1/7}$$

where $k > 0$is a constant. Therefore, when $t > T^*$, the motion of the linear subsystem accurately tracks the virtual input $\xi_1(t)$while the virtual input $\xi_1(t)$ globally asymptotically stabilize the nonaffine subsystem (z_1, z_2) of Eq. 45 to its origin. Based on the lemma 1 and proposition 3, for all $(z_1, z_2, \zeta_1, \zeta_2) \neq 0$, it follows that

$$\tilde{V} \leq -w^T\psi_1(\xi_1)w - \frac{k}{2}(V+U)^{6/7} < 0 \tag{50}$$

Thus, the controller (37)-(46)-(15) globally asymptotically stabilize the system (45) to its origin. □

Remark 9 Due to the saturated feedback (47), the virtual input (46) is bounded at the neighborhood of all points$\mu(z_1)$that satisfy $\psi_2(z_1, \mu(z_1)) = 0$. Relatively large errors during the transition process of the observer (15) can not be enlarged by the virtual controller (46). This is helpful for stabilizing the overall system.

Remark 10 Even though the system (1) is a class of strict feedback form system [19], the backstepping-based control method cannot be directly used since the virtual input (46) cannot be obtained in a closed-form. By introducing the higher-order finite-time stabilizing observer for the cascade linear subsystem of Eq. 45, then the standard backstepping procedure can be utilized to stabilize the nonaffine system (45) on the basis of the saturated feedback for the nonaffine subsystem of Eq. 45.

5 Examples of Application

In this section, two benchmark UMS: Acrobot [17, 19] and IWP [18, 19, 30] are employed to verify the feasibility of the presented hybrid feedback controller. The main property of this class of UMS is that the passive coordinates of the system possess the kinetic symmetry [19]. The dynamics of an UMS belonging to this case can be rigorously transformed into the nonaffine system (1) with strict feedback form.

Essentially, under this case, the dynamics of the UMS can be expressed as

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)+\frac{\partial H}{\partial \theta_1}=0 \\ &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}_2}\right)-\frac{\partial T}{\partial \boldsymbol{\theta}_2}+\frac{\partial H}{\partial \boldsymbol{\theta}_2}=\boldsymbol{\tau} \end{aligned} \tag{51}$$

where Tdenotes the kinetic energy, $H$the potential energy, $\theta_1 \in R$the passive generalized coordinate, $\boldsymbol{\theta}_2 \in \boldsymbol{R}^{n-1}$the actuated generalized coordinates, $\boldsymbol{\tau} \in \boldsymbol{R}^{n-1}$the generalized actuation forces, and$n$ the degree of freedoms (DOFs) of the mechanical systems . In the equations (51), $\partial T/\partial\theta_1 = 0$is considered since the kinetic energy is symmetric about the passive generalized coordinate θ_1. For the Acrobot and IWP system, the DOFs satisfy$n = 2$, and then the dynamics (51) can be written as a matrix form

$$\begin{aligned} &m_{11}\ddot{\theta}_1+m_{12}\ddot{\theta}_2+h_1=0 \\ &m_{21}\ddot{\theta}_1+m_{22}\ddot{\theta}_2+h_2=\tau \end{aligned} \tag{52}$$

where $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$is the inertia matrix, h_1 and $h_2$are the speed product terms and potential forces respectively. By the partial feedback linearization [30] and the coordinates transformation

$$\begin{aligned} &z_1=\theta_1+\gamma(\theta_2) \\ &z_2=\partial T/\partial\dot{\theta}_1 \\ &\xi_1=\theta_2 \\ &\xi_2=\dot{\theta}_2 \end{aligned} \tag{53}$$

where

$$\gamma(\theta_2)=\int_0^{\theta_2}\frac{m_{12}(s)}{m_{11}(s)}ds \tag{54}$$

the dynamics (51) can be written as

$$\begin{aligned} &\dot{z}_1=m_{11}^{-1}(\xi_1)z_2=\psi_1(\xi_1)z_2 \qquad \dot{\xi}_1=\xi_2 \\ &\dot{z}_2=-\partial H/\partial\theta_1=\psi_2(z_1,\xi_1) \qquad \dot{\xi}_2=u \end{aligned} \tag{55}$$

where $u = \ddot{\theta}_2$ is defined to be the new input. It is obvious that Eq. 55 is a special case of Eq. 1 and satisfies the assumptions A1-A3 in Section 1.

5.1 Acrobot system

Let $L_i, l_{ci}, m_i, I_i,\ i = 1,2$be the links length, the mass center length, the mass, and the inertia momentum about the axis$z$at the mass center of the two links, respectively. Then the dynamics of the Acrobot system (Fig. 4) can be written as Eq. 52, where

$$\begin{aligned} &m_{11}=a+2b\cos\theta_2 \\ &m_{12}=m_{21}=c+b\cos\theta_2 \\ &m_{22}=c \end{aligned}$$

and the relevant parameters are given by

$$\begin{aligned} &a=m_1l_{c1}^2+m_2(L_1^2+l_{c2}^2)+I_1+I_2 \\ &b=m_2L_1l_{c2} \\ &c=m_2l_{c2}^2+I_2 \end{aligned}$$

$$\begin{aligned} &h_1=-m_2L_1l_{c2}\sin\theta_2\left(2\dot{\theta}_1\dot{\theta}_2+\dot{\theta}_2^2\right)+g_1 \\ &h_2=m_2L_1l_{c2}\sin\theta_2\dot{\theta}_1^2+g_2 \end{aligned}$$

where

$$\begin{aligned} &g_1=-(m_1l_{c1}+m_2L_1)g\sin\theta_1-m_2gl_{c2}\sin(\theta_1+\theta_2) \\ &g_2=-m_2gl_{c2}\sin(\theta_1+\theta_2) \end{aligned}$$

and $g$is the acceleration of gravity. The potential energy for the Acrobot system is given by

$$H=(m_1l_{c1}+m_2L_1)\,g\cos\theta_1+m_2l_{c2}g\cos(\theta_1+\theta_2).$$

Then the partial derivate in Eq. 55 can be written as

$$\partial H/\partial\theta_1=d\sin\theta_1+e\sin(\theta_1+\theta_2)$$

where $d = (m_1l_{c1} + m_2L_1)\,g$, and $e = m_2l_{c2}g$. For the Acrobot system, if one defines

$$y=\tan\frac{x}{2},\quad \cos x=\frac{1-y^2}{1+y^2},\quad dx=\frac{2}{1+y^2}dy \tag{56}$$

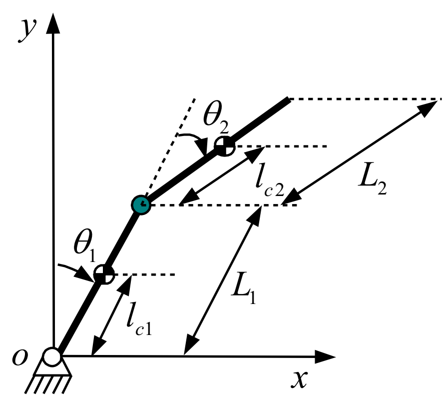

Fig. 4 The Acrobot system

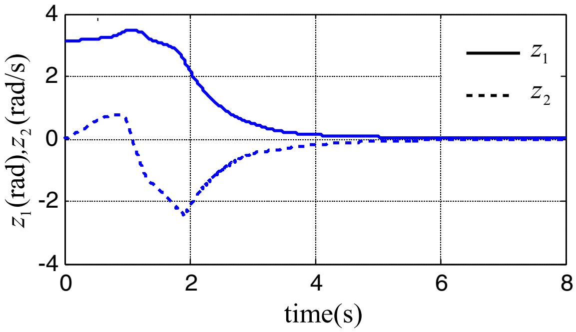

Fig. 5 The motion trajectories of z_1 and z_2 for the Acrobot

then the integration (54) can be analytically expressed as

$$\gamma(\theta_2) = \int_0^{\theta_2} \frac{c + b\cos x}{a + 2b\cos x} dx = \frac{\theta_2}{2} + \frac{2c - a}{\sqrt{a^2 - 4b^2}} \arctan\left(\sqrt{\frac{a-2b}{a+2b}} \tan\left(\frac{\theta_2}{2}\right)\right) \quad (57)$$

for $\theta_2 \in [-\pi, \pi]$. With this in mind, the functions $\psi_1(\xi_1)$ and $\psi_2(z_1, \xi_1)$ of Eq. 55 can be explicitly expressed as

$$\begin{aligned} \psi_1(\xi_1) &= m_{11}^{-1}(\xi_1) \\ \psi_2(z_1, \xi_1) &= d\sin(z_1 - \gamma(\xi_1)) \\ &\quad + e\sin(z_1 - \gamma(\xi_1) + \xi_1) \end{aligned} \quad (58)$$

For the purpose of clarity, we present the controller for the Acrobot system in the following. Let's define the error variables for the observer

$$\begin{aligned} e_1 &= \hat{\alpha}_1 - \alpha_1 \\ e_2 &= \hat{\alpha}_2 - \dot{\alpha}_1 = \dot{\hat{\alpha}}_1 - \dot{\alpha}_1 \\ e_3 &= \hat{\alpha}_3 - \ddot{\alpha}_1 = \ddot{\hat{\alpha}}_1 - \dot{\alpha}_1 \end{aligned} \quad (59)$$

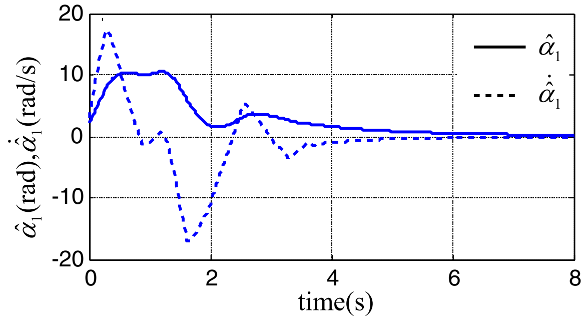

Fig. 7 The observer's output $\hat{\alpha}_1$ and $\dot{\hat{\alpha}}_1$ for the Acrobot

and define the error variables for the linear subsystem of Eq. 55

$$\begin{aligned} \zeta_1 &= \xi_1 - (\alpha_1 + e_1) = \xi_1 - \hat{\alpha}_1 \\ \zeta_2 &= \xi_2 - (\dot{\alpha}_1 + e_2) = \xi_2 - \dot{\hat{\alpha}}_1 \end{aligned} \quad (60)$$

The error dynamics for the Acorobt system can be written as

$$\begin{aligned} \dot{z}_1 &= \psi_1(\alpha_1) z_2 \\ \dot{z}_2 &= \psi_2(z_1, \alpha_1) \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= u - \ddot{\hat{\alpha}}_1 \end{aligned} \quad (61)$$

and the observer (15) for the Acrobot can also be rewritten as

$$\begin{aligned} \dot{\hat{\alpha}}_1 &= \hat{\alpha}_2 - \hat{k}_1(e_1)^{5/7} \\ \dot{\hat{\alpha}}_2 &= \hat{\alpha}_3 - \hat{k}_2(\hat{\alpha}_2 - \dot{\hat{\alpha}}_1)^{3/7} \\ \dot{\hat{\alpha}}_3 &= -\hat{k}_3(\hat{\alpha}_3 - \dot{\hat{\alpha}}_2)^{1/7} \end{aligned} \quad (62)$$

To stabilize the Acorobot system (61)-(62), the following Corollary can be given by the Lemma 1 and Propositions 1-4.

Corollary 1 *(Acrobot) For the observer-based Acorobt system* (61)–(62)*, there exists a group control*

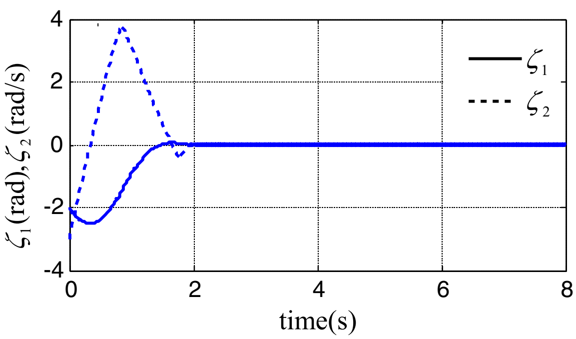

Fig. 6 The motion trajectories of ζ_1 and ζ_2 for the Acrobot

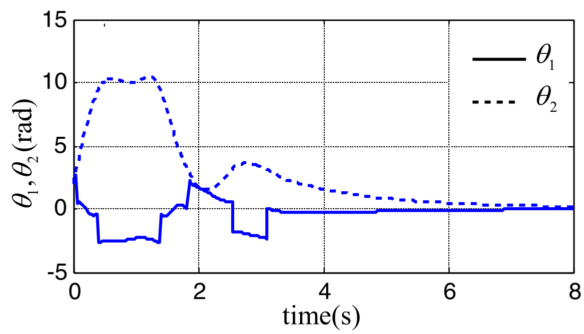

Fig. 8 The angular position of the joints for the Acrobot

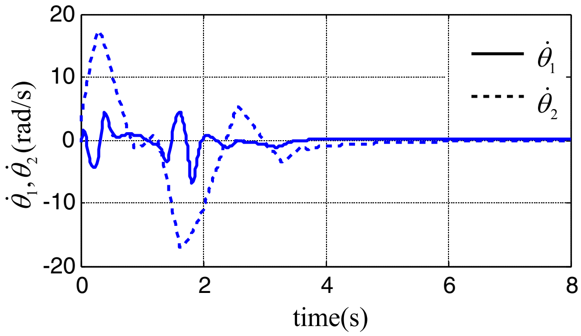

Fig. 9 The angular speed of the joints for the Acrobot

parameters $k_i > 0$ *and* $\hat{k}_i > 0, i = 1, 2, 3,$ *the controller*

$$u = \ddot{\alpha}_1 - k_2 \left(\zeta_2^{7/5} - \beta_1^{7/5}\right)^{3/7} \tag{63}$$

where

$$\beta_1 = -k_2 \zeta_1^{5/7} \tag{64}$$

$$\alpha_1 = \mu(z_1) + w \tag{65}$$

$$w = k_1 \frac{z_1 + z_2}{\left(1 + (z_1 + z_2)^2\right)^{1/2}} \tag{66}$$

and $\mu(z_1)$*satisfies* $\psi_2(z_1, \mu(z_1)) = 0,$ *globally exponentially stabilize the origin* $(z_1, z_2, \zeta_1, \zeta_2, e_1, e_2, e_3) = 0$ *of the dynamic system* (61)–(62).

Given the structure parameters of the Acrobot system to be $L_1 = L_2 = 1(m), l_{c1} = l_{c2} = 0.5(m)$, $m_1 = m_2 = 1(Kg), I_1 = I_2 = 1/3(Kgm^2)$, and the acceleration of gravity is $g = 9.81\ (N/Kg)$, then the numerical simulations results of swinging up the

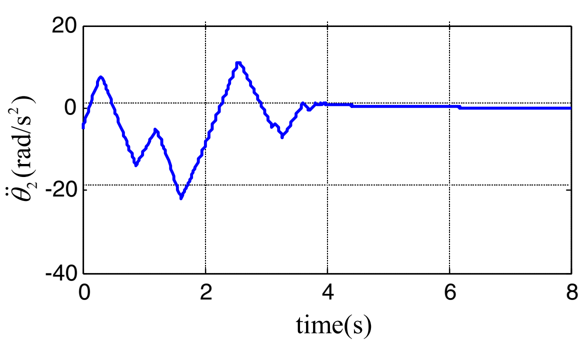

Fig. 10 The angular acceleration of the actuated joint for the Acrobot

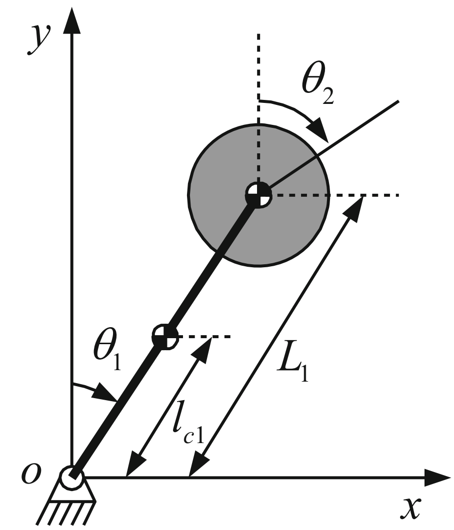

Fig. 11 The IWP system

Acorobt system from the initial state $(\theta_1, \dot\theta_1, \theta_2, \dot\theta_2) = (\pi, 0, 0, 0)$ and then stabilizing the system to its unstable inverted position are illustrated in Figs. 5, 6, 7, 8, 9 and 10.

It is worth pointing out that, to get the root $\mu(z_1)$of $\psi_2(z_1, \mu(z_1)) = 0$ for every given $z_1$in Eq. 65, we use the "fzero" function in the Matlab software, which utilizes the inverse quadratic interpolation methods to get the zero point of a continuous function.

5.2 IWP

The dynamic parameters for the IWP system [30] (Fig. 11) can be given by

$$m_{11} = m_1 l_{c1}^2 + m_2 L_1^2 + I_1 + I_2$$
$$m_{12} = m_{21} = m_{22} = I_2$$

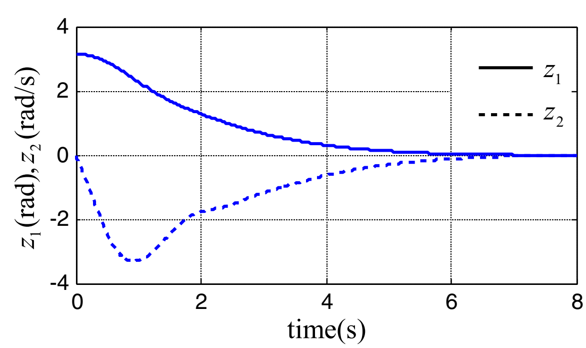

Fig. 12 The motion of variables z_1 and z_2 for the IWP system

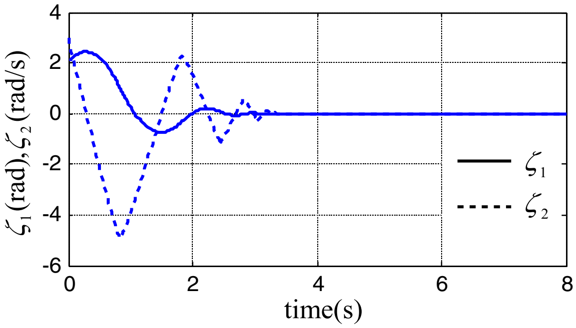

Fig. 13 The motion of error variables ζ_1 and ζ_2 for IWP system

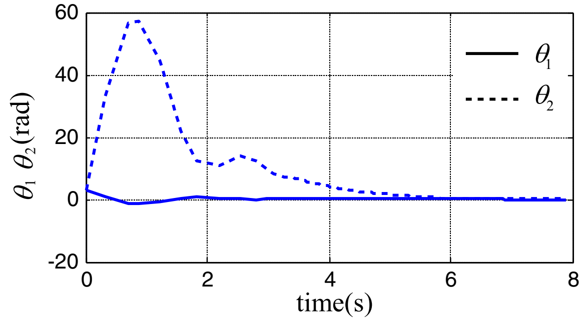

Fig. 15 The angular position of the joints for the IWP system

and the potential forces can be written as

$$h_1 = -(m_1 l_{c1} + m_2 L_1) g \sin\theta_1$$
$$h_2 = 0$$

The potential energy of the IWP system can be expressed as

$$H = (m_1 l_{c1} + m_2 L_1)\, g \cos\theta_1$$

By the coordinates transformations (53), the control equations for the IWP system are given by

$$\begin{array}{ll} \dot{z}_1 = \psi_1 z_2 & \dot{\xi}_1 = \xi_2 \\ \dot{z}_2 = \psi_2(z_1, \xi_1), & \dot{\xi}_2 = u \end{array} \tag{67}$$

where,

$$\psi_1) = 1/m_{11}$$
$$\psi_2(z_1, \xi_1) = (m_1 l_{c1} + m_2 L_1) g \sin(z_1 - \gamma(\xi_1)) \tag{68}$$

$$\gamma(\xi_1) = \int_0^{\xi_1} \frac{m_{12}}{m_{11}} dx = \frac{m_{12}}{m_{11}} \xi_1 \tag{69}$$

Because of the special simple structure, it has been shown that the IWP is a kind of differentially flat UMS [29–32], which can be globally stabilized by output feedback controller. Whereas, we apply the hybrid feedback method presented in this paper to verify the effectiveness of the presented new controller.

It is different from that of the Acrobot system, the zero solution of the smooth function $\psi_2(z_1, \xi_1)$for the IWP can be obtained in a closed-form. For all givenz_1, the solution of $\psi_2(z_1, \mu(z_1)) = 0$ can be expressed as

$$\mu(z_1) = \frac{m_{11}}{m_{12}} (z_1 - n_s \pi) \tag{70}$$

for the IWP system, where $n_s$is a constant. We consider the absolute zero position control for both the generalized coordinates θ_1 and θ_2 of the IWP system. Towards this end, let $n_s = 0$, then Eq. 70 is simplified to

$$\mu(z_1) = \frac{m_{11}}{m_{12}} z_1 \tag{71}$$

Using Eq. 71 and the corollary 1, the numerical simulations results of swinging up the IWP system from the initial state $(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = (\pi, 0, 0, 0)$ and then stabilizing the system to its unstable inverted position are illustrated in Figs. 12, 13, 14, 15, 16 and 17, where

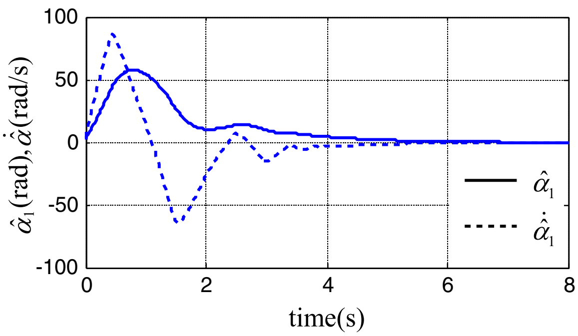

Fig. 14 The observer's output $\hat{\alpha}_1$ and $\dot{\hat{\alpha}}_1$ for the IWP system

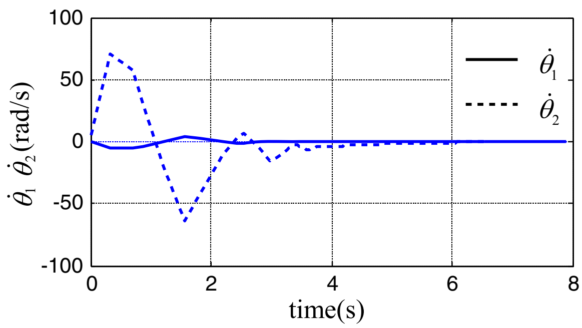

Fig. 16 The angular speed of the joints for the IWP system

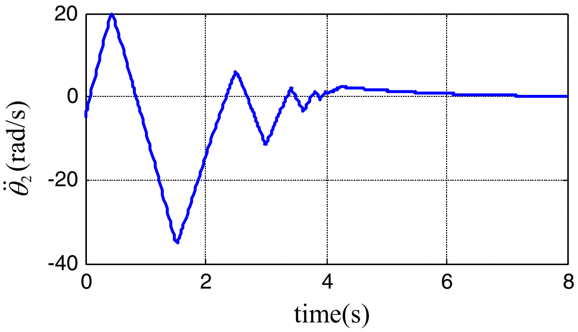

Fig. 17 The angular acceleration of the actuated joint for the IWP system

the structure parameters of the IWP are given by $L_1 = 1(m)$, $l_{c1} = 0.5(m)$, $m_1 = 1(\mathrm{Kg})$, $m_2 = 2(\mathrm{Kg})$, $I_1 = 1/3(\mathrm{Kgm}^2)$, $I_2 = 2/3(\mathrm{Kgm}^2)$, and the acceleration of gravity is also supposed to be$g = 9.81$ (N/Kg).

6 Conclusion

For the purpose of providing more selections to design an UMS or legged robotic system, globally exponentially stabilizing a class of nonaffine nonlinear systems with strict feedback forms is investigated in this paper. A finite-time stabilizing observer is presented to get the time-derivates of the virtual input so that the backstepping–based controller can be applied to stabilize the considered nonaffine nonlinear system in real time. It is also shown that a finite-time controller on the basis of a finite-time observer can be realized for the high-order linear systems. By constructing a partial states feedback for the nonaffine subsystem and a finite-time stabilizing virtual outputs feedback for the cascade linear subsystem, it is shown that the considered class of nonaffine systems with strict feedback form can be globally exponentially stabilized to the origin.

On the basis of the methodology presented in this paper, some dynamically legged robots, such as the hopping robots, biped robots and other dynamically mechanical systems, will have more selections than a differentially flat system in designing the UMS [16, 32], such that the robotic systems can show both better controllability and larger design space [15] which will greatly improve the energy-efficiency of the UMS.

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