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Distributed Time-Varying Formation Tracking Analysis and Design for Second-Order Multi-Agent Systems

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Abstract Distributed time-varying formation tracking analysis and design problems for second-order multi-agent systems with one leader are studied respectively, where the states of followers form a predefined time-varying formation while tracking the state of the leader. Different from the previous results on formation tracking control, the formation for the followers can be described by specified time-varying vectors and the trajectory of the leader can also be time-varying. A distributed formation tracking protocol is constructed using only neighboring relative information. Necessary and sufficient conditions for second-order multi-agent systems with one leader to achieve time-varying formation tracking are proposed by utilizing the properties of the Laplacian matrix, where the formation tracking feasibility constraint is also given. An approach to design the formation tracking protocol is proposed by solving an algebraic Riccati equation. The presented results can be applied to deal with the target enclosing problems and consensus tracking problems for second-order multi-agent systems with one target/leader. An application in the target enclosing of multiple vehicles is provided to demonstrate the effectiveness of the theoretical results.

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1 Introduction

Formation control of multi-agent systems has attracted considerable research interest during the past decades, and has found applications in a variety of areas, such as cooperative localization [1, 2], source seeking [3, 4], drag reduction [5], target enclosing [6, 7], and load transportation [8], etc. Although several approaches which include leader-follower, virtual structure and behavior-based ones have been proposed to propel the states of all agents to form the desired configuration in the state space, increasing attention is being devoted to designing the formation control protocol using local neighboring information [9, 10].

Based on the consensus strategy, Ren [11] constructed a series of distributed formation control protocols using local neighboring information for second-order multi-agent systems, and showed that leader-follower, virtual structure and behavior-based formation control approaches can be unified in the framework of consensus-based ones. Oh and Ahn [12] proposed a formation control strategy using neighboring displacements for first-order multi-agent systems to achieve formation under time-varying topologies. Necessary and sufficient conditions for first-order multi-agent systems with undirected and fixed topologies to achieve the rigid formation were proposed in

[13] using the tool of complex Laplacian. Circular formation control problems for first-order multi-agent systems in one-dimensional and three-dimensional space were addressed in [14] and [15], respectively. A distributed controller observer strategy for tracking control of the centroid and of the relative formation of a multi-robot system with first-order dynamics was presented in [16]. Xie and Wang [17] proposed sufficient conditions for second-order multi-agent systems with undirected topologies to achieve time-invariant formations. Formation control problems for second-order multi-agent systems with heterogeneous communication delays were investigated in [18]. In [19], formation stabilization problems for second-order multi-agent systems with undirected and connected topologies were dealt with by incorporating the relative motion constraints into the navigation functions. Decentralized robust formation controllers for multi-robot systems with uncertainties and time delays were proposed in [20]. Necessary and sufficient conditions for second-order multi-agent systems with fixed directed topologies and switching undirected topologies to achieve time-varying formations were derived, respectively, in [21, 22], where the results were applied to solve the time-varying formation control problems of multiple unmanned aerial vehicles.

In the aforementioned work on formation control of multi-agent systems, only formation stabilization or maintenance problems were considered. In some practical applications, forming a predefined formation is only the first step for a multi-agent system, and there exist higher level tasks for the multi-agent system, such as tracking the trajectory generated by an actual/virtual leader or enclosing a moving target. In these scenarios, formation tracking problems arise. Consensus based formation control and trajectory tracing problems for multi-agent robot systems with first-order dynamics were addressed in [23]. Ren and Sorensen [24] studied formation tracking problems for first-order multi-agent systems with one virtual leader which provides the desired position trajectory for a group of followers. For first-order multi-agent systems with switching topologies, a distributed strategy was proposed in [25] to solve the target enclosing problem which can be regarded as a formation tracking problem with one leader. A differential game approach was applied to deal with the formation tracking problem for first-order multi-agent systems with collision avoidance in [26]. It should be pointed out that in [23–26], the dynamics of each agent is limited to be first-order. However, the dynamics of a broad class of vehicles can only be described by second-order models as the motion of a practical vehicle is often controlled by using engines or motors which provide forces and torques. Although formation tracking problems for nonholonomic multi-agent systems were addressed in [27, 28], the formation tracking strategy is centralized due to that each agent should track the reference trajectory to achieve the formation maintenance and tracking control.

Motivated by the aforementioned analysis, distributed time-varying formation tracking problems for second-order multi-agent systems with one leader is studied in this paper. Firstly, a distributed distributed formation tracking protocol is constructed using local neighboring information, where both the desired formation and the tracking trajectories of the leader can be time-varying. Secondly, by constructing a nonsingular transformation matrix utilizing the properties of the Laplacain matrix, necessary and sufficient conditions for second-order multi-agent systems to achieve time-varying formation tracking are proposed, which include the time-varying formation tracking feasibility constraint. Then an approach to determine the gain matrix of the formation tracking protocol is given by solving an algebraic Riccati equation, the solvability of which can be guaranteed. It is shown that the proposed results can be applied to deal with the moving target enclosing problems and consensus tracking problems of second-order multi-agent systems with directed topologies. Finally, a numerical example for a group of vehicles to enclose a moving target is given.

Compared with the previous works, the contributions of the current paper are twofold. Firstly, the states of the followers are not only required to achieve the predefined time-varying formation but also need to track the time-varying state of the leader. In [11–22], only formation control problems were considered, where there exists no explicit leader and the formation in [12], [17–19] and [26] is time-invariant. In [29–32], only consensus control problems or consensus tracking control problems were addressed. Note that in many practical applications, there exists a leader to provide the global reference trajectory, and the time-varying formation will bring the derivative of the formation information to both the analysis and design. The results in [11–22] and [29–32] cannot be applied to deal with the time-varying formation tracking control problems in the current paper directly. Secondly, the criteria for second-order multi-agent systems with one leader to achieve time-varying formation tracking are both necessary and sufficient. In [27, 28], only sufficient conditions were obtained and the results are centralized. The obtained results can be applied to solve the target enclosing problems and consensus tracking problems for multi-agent systems with one target/leader. However, in [24–26], the criteria for first-order multi-agent systems to achieve formation tracking are only sufficient. In addition, the dynamics of each agent in [12–16] and [23–26] is limited to be first-order.

The remainder of this paper is organized as follows. Preliminaries on graph theory and the problem formulation are given in Section 2. Time-varying formation tracking analysis and protocol design problems are dealt with in Section 3. A numerical simulation example is presented for illustration in Section 4. Conclusions are drawn in Section 5.

Throughout this paper, for simplicity of notation, let 0 be the zero matrix of appropriate size with zero vector and zero number as special cases. Denote by $\mathbf{1}_N$ a column vector of size N with 1 as its elements. Let I_N represent an identity matrix with dimension N and $\otimes$ denote the Kronecker product.

2 Preliminaries and Problem Description

In this section, basic concepts on graph theory are introduced and the problem description is presented.

2.1 Basic Concepts on Graph Theory

A weighted directed graph $G = \{Q, E, W\}$ consists of a set of vertices $Q = \{q_1, q_2, \cdots, q_N\}$, a set of edges $E \subseteq \{(q_i, q_j) : q_i, q_j \in Q, i \neq j\}$ and a weighted adjacency matrix $W = [w_{ij}] \in \mathbb{R}^{N\times N}$ with nonnegative elements w_{ij}. Denoted by $q_{ij} = (q_i, q_j)$ the edge from node q_i to node q_j, where q_{ij} is also named the incoming edge of node q_j and node q_i is called a neighbor of node q_j. For any $i, j \in \{1, 2, \cdots, N\}$, $w_{ij} > 0$ if and only if $q_{ji} \in E$ and $w_{ij} = 0$ otherwise. $N_i = \{q_j \in Q : q_{ji} \in E\}$ denotes the neighbor set of node q_i. The in-degree of node q_i is defined as $\deg_{in}(q_i) = \sum_{j=1}^{N} w_{ij}$. Denote by $D = \operatorname{diag}\{\deg_{in}(q_i), i = 1, 2, \cdots, N\}$ the degree matrix of G. The Laplacian matrix of G is defined as $L = D - W$. A directed path from node q_{i_1} to node q_{i_r} is a sequence of ordered edges with the form of $(q_{i_k}, q_{i_{k+1}})$ where $q_{i_k} \in Q$ $(k = 1, 2, \cdots, r-1)$. More details on graph theory can be found in [33].

2.2 Problem Description

Consider a multi-agent system with N agents. The interaction topology of the multi-agent system can be described by a directed graph G with agent i being the node q_i in G. For $i, j \in \{1, 2, \cdots, N\}$, the interaction channel from agent i to agent j is denoted by the edge q_{ij}, and the corresponding interaction strength is denoted by w_{ji}.

Definition 1 An agent is called a *leader* if its corresponding node in the directed graph does not have the incoming edge. An agent is called a *follower* if its corresponding node in the directed graph has at least one incoming edge.

Suppose that there are $N - 1$ followers and one leader. Let $F = \{1, 2, \cdots, N-1\}$ be the follower subscript set. The dynamics of the follower i $(i \in F)$ is described by

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \alpha_x x_i(t) + \alpha_v v_i(t) + u_i(t), \end{cases} \tag{1–1}$$

where $x_i(t) \in \mathbb{R}^n$, $v_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ are the position, velocity and control input vectors of follower i, respectively, and $\alpha_x \in \mathbb{R}$ and $\alpha_v \in \mathbb{R}$ are known damping constants. The dynamics of leader N is described by

$$\begin{cases} \dot{x}_N(t) = v_N(t), \\ \dot{v}_N(t) = \alpha_x x_N(t) + \alpha_v v_N(t), \end{cases} \tag{1–2}$$

where $x_N(t) \in \mathbb{R}^n$ and $v_N(t) \in \mathbb{R}^n$ are the position and velocity vectors of the leader N. For simplicity of presentation, let $n = 1$ if not otherwise specified.

However, all the results hereafter are still valid for the n-dimensional ($n > 1$) case by introduction of the Kronecker product.

Remark 1 From Eq. 1, one sees that α_x and α_v are the damping gains corresponding to the position and velocity, respectively. The dynamics described by Eq. 1 can be treated as a generalized second-order model. In the case where $\alpha_x = 0$ and $\alpha_v = 0$, the dynamics of each agent becomes the double-integrator ones studied in [11, 17–19, 21, 22, 34].

The time-varying formation for the followers is specified by $h_F(t) = [h_1^T(t), h_2^T(t), \cdots, h_M^T(t)]^T$, where $h_i(t) = [h_{ix}(t), h_{iv}(t)]^T$ $(i \in F)$ is the piecewise continuously differentiable formation vector for follower i. Let $\phi_k(t) = [x_k(t), v_k(t)]^T$ $(k = 1, 2, \cdots, N)$.

Definition 2 Multi-agent system (1) with one leader is said to achieve time-varying formation tracking if for any given bounded initial states,

$$\lim_{t\to\infty} (\phi_i(t) - h_i(t) - \phi_N(t)) = 0 \ (i \in F). \tag{2}$$

Remark 2 The formation vector $h_F(t)$ is predefined and each follower knows its formation information $h_i(t)$ $(i \in F)$. It is reasonable for the followers to know the formation information $h_i(t)$ $(i \in F)$ since their control object is to realize the predefined formation specified by $h_i(t)$. For any follower i $(i \in F)$, the neighboring formation information $h_j(t)$ $(j \in N_i)$ can be transferred to it via the communication network (refer to [21] and [22] for more details on the implementation in practical application). It should be pointed out that $h_i(t)$ $(i \in F)$ is not the trajectory for each follower to follow. From Definition 2, one sees that $h_i(t)$ $(i \in F)$ represents the relative offset vector of $\phi_i(t)$ with respect to $\phi_N(t)$ and is only used to specify the desired time-varying formation.

Remark 3 When the formation tracking is achieved, the state of the leader may lie inside or outside the time-varying formation $h_F(t)$. In the case where $\lim_{t\to\infty} \sum_{i=1}^{N-1} h_i(t) = 0$, it follows from Eq. 2 that $\lim_{t\to\infty}\left(\sum_{i=1}^{M} \phi_i(t)\big/(N-1) - \phi_N(t)\right) = 0$, which means that $\phi_N(t)$ lies in the center of the time-varying formation $h_F(t)$. Therefore, by choosing $\lim_{t\to\infty} \sum_{i=1}^{N-1} h_i(t) = 0$. Definition 2 become the definitions for target enclosing or target pursuing problems with one target studied in [3] and [16]. It should be pointed out that choosing $h_F(t)$ satisfying $\lim_{t\to\infty} \sum_{i=1}^{M} h_i(t) = 0$ will not bring additional conservatism as for a given time-varying formation shape, the time-varying vector $h_F(t)$ to characterize the formation shape is not unique. Moreover, from Definition 2, one sees that in the case where $h_F(t) \equiv 0$, the formation tracking problem becomes the well-known consensus tracking or leader-follower consensus problem. Therefore, target enclosing problem and consensus tracking problem can be regarded as special cases of the time-varying formation tracking problems discussed in the current paper.

Consider the following distributed time-varying formation tracking protocol

$$\begin{aligned} u_i(t) = K &\sum_{j\in N_i, j\neq N} w_{ij}(t)\left((\phi_i(t)-h_i(t))-(\phi_j(t)-h_j(t))\right) \\ &+K w_{iN}(t)\left((\phi_i(t) - h_i(t)) - \phi_N(t)\right) \\ &-\alpha h_i(t) + \dot{h}_{iv}(t), \end{aligned} \tag{3}$$

where $i \in F$, N_i represents the neighbor set of agent i, $\alpha = [\alpha_x, \alpha_v]$ and $K = [k_{11}, k_{12}]$ is a constant gain matrix.

From Definition 1, one gets that the Laplacian matrix L has the following form

$$L = \begin{bmatrix} L_1 & L_2 \\ 0 & 0 \end{bmatrix}, \tag{4}$$

where $L_1 \in \mathbb{R}^{(N-1)\times(N-1)}$ and $L_2 \in \mathbb{R}^{(N-1)\times 1}$.

Let $\phi_F(t) = [\phi_1^T(t), \phi_2^T(t), \cdots, \phi_{N-1}^T(t)]^T$, $B_1 = [1, 0]^T$, and $B_2 = [0, 1]^T$. Under protocol (3), multi-agent system (1) can be written in a compact form as follows

$$\begin{cases} \dot{\phi}_F(t) = \left(I_{N-1}\otimes\left(B_1B_2^T+B_2\alpha\right)+L_1\otimes B_2K\right)\phi_F(t)+(L_2\otimes B_2K)\,\phi_N(t)+\left(I_{N-1}\otimes B_2B_2^T\right)\dot{h}_F(t) \\ \qquad -(L_1\otimes B_2K - I_{N-1}\otimes B_2\alpha)\,h_F(t), \\ \dot{\phi}_N(t) = \left(B_1B_2^T + B_2\alpha\right)\phi_N(t). \end{cases} \tag{5}$$

Remark 4 From protocol (3), one sees that only the neighboring relative state and formation information is required since $(\phi_i(t)-h_i(t))-(\phi_j(t)-h_j(t))$ can be rewritten as $(\phi_i(t)-\phi_j(t))-(h_i(t)-h_j(t))$. Protocol (3) is an extension to the traditional consensus tracking protocols (see, e.g., [29]) and can also be named as the consensus based formation tracking protocol. In the case where $h_F(t) \equiv 0$, protocol (3) becomes the consensus tracking protocol in [29]. The traditional consensus tracking protocols cannot be applied to deal with the time-varying formation tracking problems in the current paper due to that all the states of the followers in the former reach an agreement with the state of the leader while all the states of the followers in the latter should keep time-varying formation with respect to the state of the leader. From Eq. 5, one sees that both the formation information $h_F(t)$ and its derivative $\dot{h}_F(t)$ have effects on $\phi_F(t)$, which means that the analysis and design for time-varying formation control are much complicated than those for consensus tracking control. In some practical applications, only partial neighboring information is available. In such cases, protocol (3) may not be applied directly and one may construct an observer to estimate the required information (see, e.g., [16] and [30]).

The current paper mainly focuses on the following two problems for multi-agent system (1) under protocol (3): (i) under what conditions the time-varying formation tracking can be achieved with one leader; and (ii) how to design protocol (3) to achieve time-varying formation tracking with one leader.

3 Time-Varying Formation Tracking Analysis and Protocol Design

In this section, time-varying formation tracking analysis and design problems for multi-agent system (5) with one leader are studied. Necessary and sufficient conditions for multi-agent system (5) with one leader to achieve time-varying formation tracking are proposed. An approach to design protocol (3) is presented.

Assumption 1 *For each follower, there exists at least one directed path from the leader to it.*

If Assumption 1 holds, the following lemma can be obtained.

Lemma 1 *([35]): If the directed interaction topology G satisfies Assumption 1, then all the eigenvalues of L_1 have positive real parts; each entry of $-L_1^{-1}L_2$ is nonnegative and each row of $-L_1^{-1}L_2$ has a sum equal to one.*

Lemma 2 *([36]): The system $\dot{\varphi}(t) = M\varphi(t)$, where M is a 2×2 complex matrix with characteristic polynomial $f(s) = s^2 + a_1 s + a_2$, is asymptotically stable if and only if $Re(a_1) > 0$ and $Re(a_1)Re(a_1\bar{a}_2) - Im(a_2)^2 > 0$.*

Theorem 1 *Multi-agent system (1) with one leader achieves time-varying formation tracking under protocol (3) if and only if for any $i \in F$, the formation tracking feasibility condition $\lim_{t\to\infty}(h_{iv}(t) - \dot{h}_{ix}(t)) = 0$ is satisfied and*

$$\begin{cases} \alpha_v + \mathrm{Re}(\lambda_i)k_{12} < 0, \\ (\alpha_v + \mathrm{Re}(\lambda_i)k_{12})\,\psi_i + \mathrm{Im}(\lambda_i)^2 k_{11}^2 < 0, \end{cases} \tag{6}$$

where

$$\psi_i = \alpha_v\alpha_x + \mathrm{Re}(\lambda_i)\,(\alpha_v k_{11} + \alpha_x k_{12}) + \left(\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2\right) k_{11}k_{12}.$$

Proof Let $\phi(t) = [\phi_F^T, \phi_N^T]^T$. Multi-agent system (5) can be rewritten as

$$\begin{aligned} \dot{\phi}(t) = & \left(\begin{bmatrix} I_{N-1}\otimes(B_1B_2^T + B_2\alpha) & 0 \\ 0 & B_1B_2^T + B_2\alpha \end{bmatrix}\right. \\ & \left. + L\otimes B_2K\right)\phi(t) \\ & - \begin{bmatrix} L_1\otimes B_2K + I_{N-1}\otimes B_2\alpha \\ 0 \end{bmatrix} h_F(t) \\ & + \begin{bmatrix} I_{N-1}\otimes B_2B_2^T \\ 0 \end{bmatrix}\dot{h}_F(t). \end{aligned} \tag{7}$$

Define $\theta_i(t) = \phi_i(t) - h_i(t)$ $(i \in F)$, $\theta_F(t) = [\theta_1^T(t), \theta_2^T(t), \cdots, \theta_{N-1}^T(t)]^T$ and $\xi(t) = [\theta_F^T, \phi_N^T]^T$. Then $\xi(t) = \phi(t) - [I, 0]^T h_F(t)$ and multi-agent system (7) can be transformed into

$$\begin{aligned} \dot{\xi}(t) = & \left(\begin{bmatrix} I_{N-1}\otimes\left(B_1B_2^T + B_2\alpha\right) & 0 \\ 0 & B_1B_2^T + B_2\alpha \end{bmatrix}\right. \\ & \left. + L\otimes B_2K\right)\xi(t) \\ & + \begin{bmatrix} I_{N-1}\otimes B_1B_2^T \\ 0 \end{bmatrix} h_F(t) \\ & - \begin{bmatrix} I_{N-1}\otimes B_1B_1^T \\ 0 \end{bmatrix}\dot{h}_F(t). \end{aligned} \tag{8}$$

From Lemma 1, all the eigenvalues of L_1 have positive real parts. Denote by λ_i $(i \in F)$ the eigenvalue of with $0 < \mathrm{Re}(\lambda_1) \le \mathrm{Re}(\lambda_2) \le \cdots \le \mathrm{Re}(\lambda_{N-1})$. There exists a nonsingular matrix $U_F \in \mathbb{R}^{(N-1)\times(N-1)}$ such that $U_F^{-1} L_1 U_F = J_F$ with J_F being the Jordan canonical form of L_1. Let

$$T = \begin{bmatrix} U_F & \mathbf{1}_{N-1} \\ 0 & 1 \end{bmatrix}.$$

Then one has

$$T^{-1} = \begin{bmatrix} U_F^{-1} & -U_F^{-1}\mathbf{1}_{N-1} \\ 0 & 1 \end{bmatrix}.$$

From Lemma 1, one has $-L_1^{-1} L_2 = \mathbf{1}_{N-1}$; that is,

$$L_1 \mathbf{1}_{N-1} + L_2 = 0. \tag{9}$$

It follows from Eq. 9 that

$$T^{-1} L T = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Define $\varsigma(t) = (U_F^{-1} \otimes I_2)\theta_F(t) - (U_F^{-1}\mathbf{1}_{N-1} \otimes I_2)\phi_N(t)$ and $\bar{\xi}(t) = [\varsigma^H(t), \phi_N^H(t)]^H$. Then one gets that

$$\left(T^{-1} \otimes I_2\right)\xi(t) = \bar{\xi}(t), \tag{11}$$

and system (8) can be converted into

$$\begin{aligned} \dot{\varsigma}(t) = &\left(I_{N-1} \otimes \left(B_1 B_2^T + B_2 \alpha\right) + J_F \otimes B_2 K\right)\varsigma(t) \\ &+ \left(U_F^{-1} \otimes B_1 B_2^T\right) h_F(t) \\ &- \left(U_F^{-1} \otimes B_1 B_1^T\right)\dot{h}_F(t), \end{aligned} \tag{12}$$

$$\dot{\phi}_N(t) = \left(B_1 B_2^T + B_2 \alpha\right)\phi_N(t). \tag{13}$$

Let

$$\xi_f(t) = (T \otimes I_2)\,[0, \phi_N^T]^T, \tag{14}$$

$$\xi_{\bar{f}}(t) = (T \otimes I_2)\,[\varsigma^H(t), 0]^H. \tag{15}$$

Since $\left[0, \phi_N^T\right]^T = e_N \otimes \phi_N(t)$, where $e_N \in \mathbb{R}^N$ with 1 as its N th entry and 0 elsewhere, it follows from Eq. 14 that

$$\xi_f(t) = T e_N \otimes \phi_N(t) = \mathbf{1}_N \otimes \phi_N(t). \tag{16}$$

From Eqs. 11, 14 and 15, one has

$$\xi(t) = \xi_f(t) + \xi_{\bar{f}}(t). \tag{17}$$

It holds from Eqs. 14 and 15 that $\xi_f(t)$ and $\xi_{\bar{f}}(t)$ are linear independent as $T \otimes I_2$ is nonsingular. From Eqs. 16 and 17, it can be obtained that

$$\xi_{\bar{f}}(t) = \phi(t) - \begin{bmatrix} I \\ 0 \end{bmatrix} h_F(t) - \mathbf{1}_N \otimes \phi_N(t), \tag{18}$$

that is,

$$\xi_{\bar{f}}(t) = \begin{bmatrix} \phi_F(t) - h_F(t) - \mathbf{1}_{N-1} \otimes \phi_N(t) \\ 0 \end{bmatrix}. \tag{19}$$

It follows from Eq. 19 that multi-agent system (1) with one leader achieves time-varying formation tracking under protocol (3) if and only if

$$\lim_{t\to\infty} \xi_{\bar{f}}(t) = 0. \tag{20}$$

Due to Eq. 15 and the fact that $T \otimes I_2$ is nonsingular, it holds that Eq. 20 is equivalent to

$$\lim_{t\to\infty} \varsigma(t) = 0, \tag{21}$$

which means that $\varsigma(t)$ represents the time-varying formation tracking error. From Eq. 12, the time-varying formation tracking error $\varsigma(t)$ converge to zero if and only if $I_{N-1} \otimes (B_1 B_2^T + B_2\alpha) + J_F \otimes B_2 K$ is Hurwitz and

$$\lim_{t\to\infty} \left((U_F^{-1} \otimes B_1 B_2^T) h_F(t) - (U_F^{-1} \otimes B_1 B_1^T)\dot{h}_F(t)\right) = 0. \tag{22}$$

From the structure of $\bar{J}$, one gets that $I_{N-1} \otimes (B_1 B_2^T + B_2\alpha) + J_F \otimes B_2 K$ is Hurwitz if and only if $B_1 B_2^T + B_2\alpha + \lambda_i B_2 K$ is Hurwitz. It can be obtained that

$$B_1 B_2^T + B_2\alpha + \lambda_i B_2 K = \begin{bmatrix} 0 & 1 \\ \alpha_x + \lambda_i k_{11} & \alpha_v + \lambda_i k_{12} \end{bmatrix}. \tag{23}$$

The characteristic polynomial of Eq. 23 is

$$p(s) = s^2 - (\alpha_v + \lambda_i k_{12})\, s - (\alpha_x + \lambda_i k_{11}), \tag{24}$$

where s is a complex variable. Using Lemma 2, one gets that $I_{N-1} \otimes (B_1 B_2^T + B_2\alpha) + J_F \otimes B_2 K$ is Hurwitz if and only if condition (6) holds. Because $U_F^{-1} \otimes I_2$ is nonsingular, pre-multiplying the both sides of Eq. 22 by $U_F \otimes I_2$ gives

$$\lim_{t\to\infty} \left((I_{N-1} \otimes B_1 B_2^T) h_F(t) - (I_{N-1} \otimes B_1 B_1^T)\dot{h}_F(t)\right) = 0. \tag{25}$$

that is, $\lim_{t\to\infty}(h_{iv}(t) - \dot{h}_{ix}(t)) = 0$ $(i \in F)$, which means the formation tracking feasibility condition is both necessary and sufficient. Therefore, the conclusion of Theorem 1 can be obtained. □

Remark 5 The formation tracking feasibility condition reveals that for multi-agent system (1) with one leader under protocol (3), the derivative of the position component of the desired formation vector should be equal to the velocity component of the desired formation vector eventually, which is determined by the second-order dynamics of each agent. In the case where $\lim_{t\to\infty}\sum_{i=1}^{N-1} h_i(t) = 0$, necessary and sufficient conditions for linear systems to achieve target enclosing can be obtained from Theorem 1. In [16], the dynamics of each agent is first-order and the criteria are only sufficient. Moreover, in the case where the time-varying formation $h_F(t) \equiv 0$, Theorem 1 presents necessary and sufficient conditions for multi-agent system (1) under protocol (3) to achieve consensus tracking with one leader.

Based on Theorem 1, an approach to determine the gain matrix in the protocol (3) is proposed in the following theorem.

Theorem 2 *If the time-varying formation tracking feasibility condition in Theorem 1 holds, multi-agent system (1) achieves time-varying formation tracking by protocol (3) with* $K = -\delta[Re(\lambda_1)]^{-1}R^{-1}B_2^T P$, *where* $\delta > 0.5$ *is a given constant and P is the positive solution to the following algebraic Riccati equation*

$$P(B_1B_2^T + B_2\alpha) + \left(B_1B_2^T + B_2\alpha\right)^T P - PB_2R^{-1}B_2^T P + I = 0, \quad (26)$$

where $R = R^T > 0$ *is any given constant matrix.*

Proof Consider the stability of the following subsystem

$$\dot{\bar{\varsigma}}_i(t) = \left(B_1B_2^T + B_2\alpha + \lambda_i B_2K\right)\bar{\varsigma}_i(t) \ (i \in F). \quad (27)$$

Construct the following Lyapunov candidate function

$$V_i(t) = \bar{\varsigma}_i^H(t)P\bar{\varsigma}_i(t).$$

Taking the derivative of $V_i(t)$ along the trajectory of subsystem (27) gives

$$\begin{aligned}\dot{V}_i(t) = \bar{\varsigma}_i^H(t)\Big(P\left(B_1B_2^T + B_2\alpha\right) &+ \left(B_1B_2^T + B_2\alpha\right)^T P \\ &+ \lambda_i^H(B_2K)^T P + \lambda_i PB_2K\Big)\bar{\varsigma}_i(t). \end{aligned} \quad (28)$$

Substituting $K = -\delta[\mathrm{Re}(\lambda_1)]^{-1}R^{-1}B_2^T P$ and $P(B_1B_2^T + B_2\alpha) + (B_1B_2^T + B_2\alpha)^T P = PB_2R^{-1}B_2^T P - I$ into Eq. 28 one has

$$\begin{aligned}\dot{V}_i(t) = &-\bar{\varsigma}_i^H(t)\bar{\varsigma}_i(t) + (1 - 2\delta[\mathrm{Re}(\lambda_1)]^{-1}\mathrm{Re}(\lambda_i))\bar{\varsigma}_i^H(t) \\ &\times (PB_2R^{-1}B_2^T P)\bar{\varsigma}_i(t).\end{aligned} \quad (29)$$

Due to that $1 - 2\delta[\mathrm{Re}(\lambda_1)]^{-1}\mathrm{Re}(\lambda_i) < 0$, it follows from Eq. 29 that $\lim_{t\to\infty}\bar{\varsigma}_i(t) = 0$; which means that $B_1B_2^T + B_2\alpha + \lambda_i B_2K$ $(i \in F)$ is Hurwitz. Since the time-varying formation tracking feasibility condition in Theorem 1 is satisfied, it follows from the proof of Theorem 1 that multi-agent system (1) achieves time-varying formation tracking by the designed protocol (3). The proof for Theorem 2 is completed. □

Remark 6 From Theorem 2, one gets that the gain matrix K of protocol (3) can be determined by solving the algebraic Riccati equation (26), which is simple

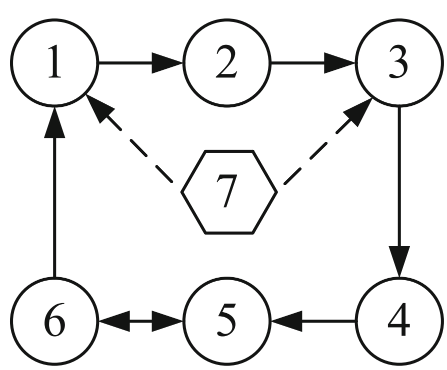
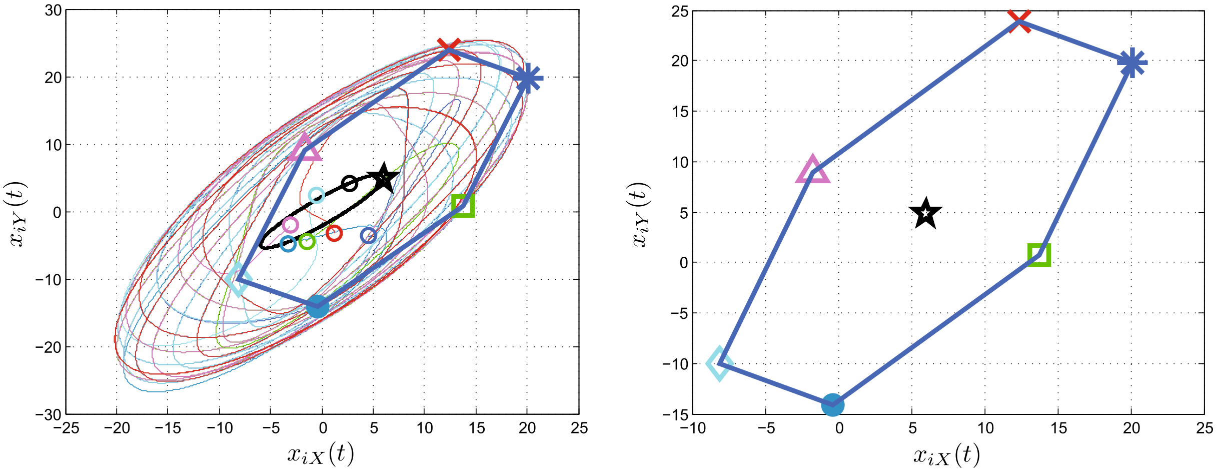


Fig. 1 Directed interaction topology G



(a) Position trajectories within t = 80s

(b) Position snapshots at t = 80s

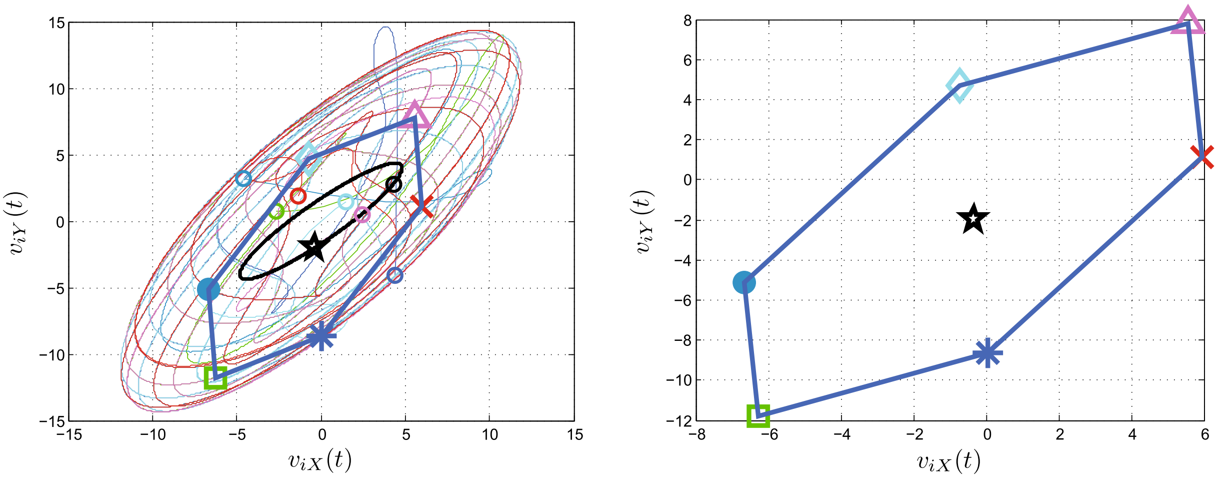
Fig. 2 Position trajectories and snapshots of the seven vehicles

and easy to implement. Due to that $(B_1 B_2^T, I)$ is stabilizable, the existence of K can be guaranteed.

4 Applications in Target Enclosing of Multiple Vehilces

In this section, the obtained results are applied to deal with the moving target enclosing problems of a group of vehicles. The robustness of the proposed results under the influence of the stochastic noises in the relative state and relative desired formation is also shown.

Example 1 Suppose that there are seven vehicles, where six of them are followers and one is the target (leader). The six followers are required to maintain a time-varying circular formation while surrounding the moving target in the horizontal XY plane ($n = 2$). The directed interaction topology among the seven vehicles is shown in Fig. 1. For simplicity, it is assumed that G is 0-1 weighted. Suppose that the dynamics of each vehicle is described by Eq. 1 with $\alpha_x = -0.64$, $\alpha_v = 0$, $\phi_k(t) = [x_{kX}(t), v_{kX}(t), x_{kY}(t), v_{kY}(t)]^T$ $(k = 1, 2, \cdots, 7)$ and $u_i(t) = [u_{iX}(t), u_{iY}(t)]^T$ $(i \in F)$. The



(a) Velocity trajectories within t = 80s

(b) Velocity snapshots at t = 80s

Fig. 3 Velocity trajectories and snapshots of the seven vehicles

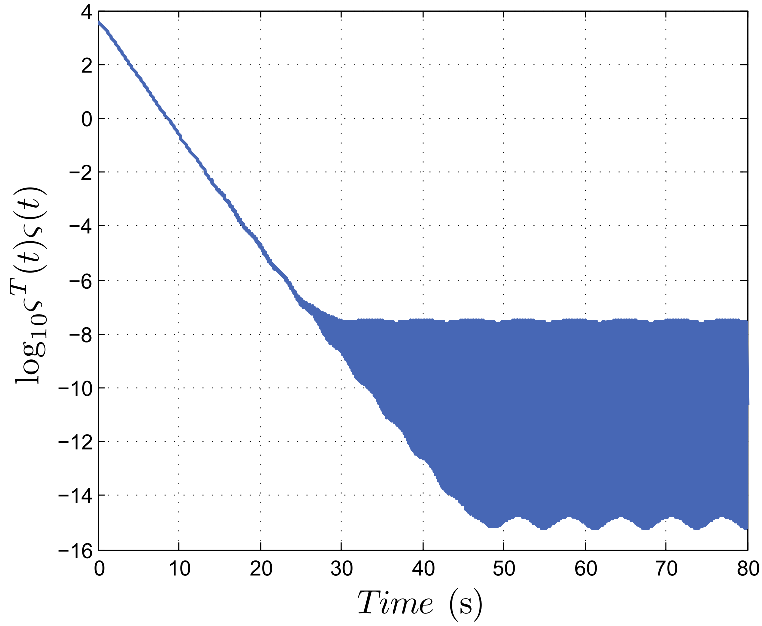


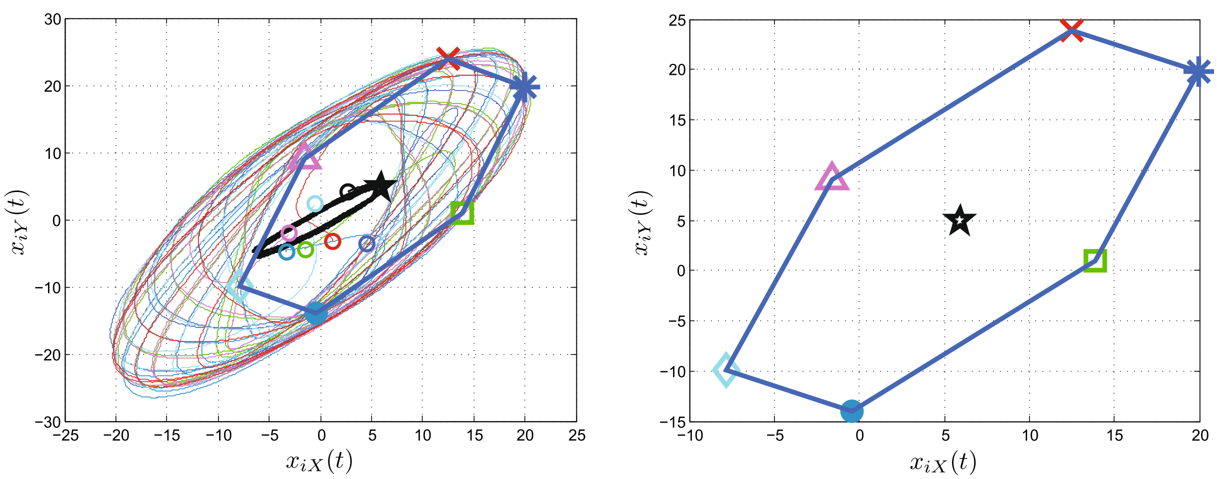
Fig. 4 Curve of the formation tracking error in the logarithmic scale

time-varying circular formation for the followers is specified by

$$h_i(t)=\begin{bmatrix} r\sin(\omega t+(i-1)\pi/3)-r\cos(\omega t+(i-1)\pi/3) \\ r\omega\sin(\omega t+(i-1)\pi/3)+r\omega\cos(\omega t+(i-1)\pi/3) \\ 2r\sin(\omega t+(i-1)\pi/3) \\ 2r\omega\cos(\omega t+(i-1)\pi/3) \end{bmatrix}(i\in F),$$

where $r = 10$ and $\omega = 0.5$. From $h_F(t)$, one gets that $\lim_{t\to\infty}\sum_{i=1}^{6} h_i(t) = 0$, which means that when the desired formation tracking is achieved, the states of the six followers will keep a time-varying circular parallel hexagon around the target.

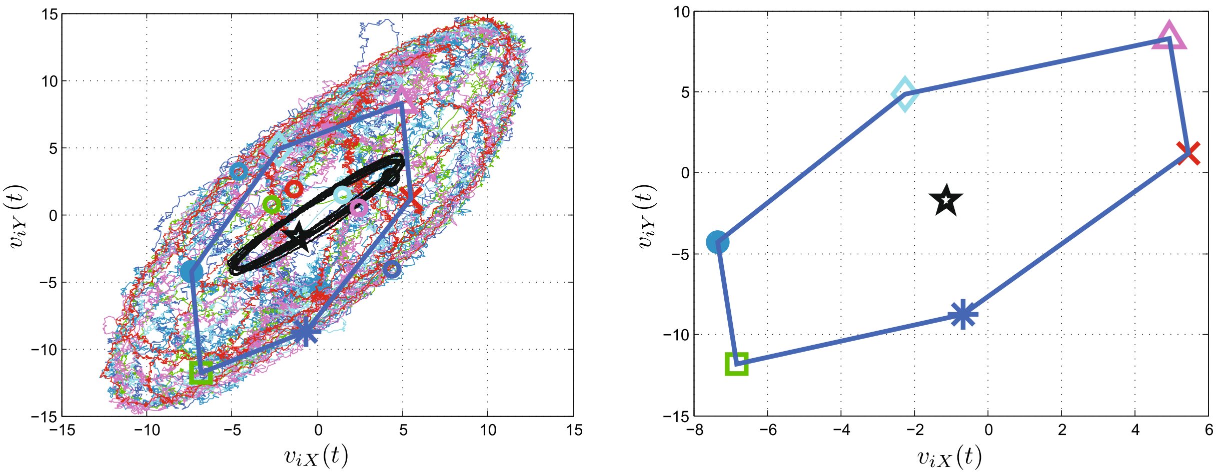
It can be verified that the formation tracking feasibility condition in Theorem 1 is satisfied. Choose $\delta = 0.6$ and $R = I$. Using the approach in Theorem 2, one gets the gain matrix K as $K = I_2 \otimes [-1.7264, -4.5655]$. Choose the initial states of the seven agents as $\phi_{kj}(t) = 10(\Theta - 0.5)$ $(k = 1, 2, \cdots, 7;\ j = 1, 2, 3, 4)$. Figures 2 and 3 show the state trajectories and snapshots of the seven vehicles



(a) Position trajectories within $t = 80$s

(b) Position snapshots at $t = 80$s

Fig. 5 Position trajectories and snapshots of the seven vehicles under the influence of the stochastic noises in the states and formation vectors



(a) Velocity trajectories within $t = 80$s

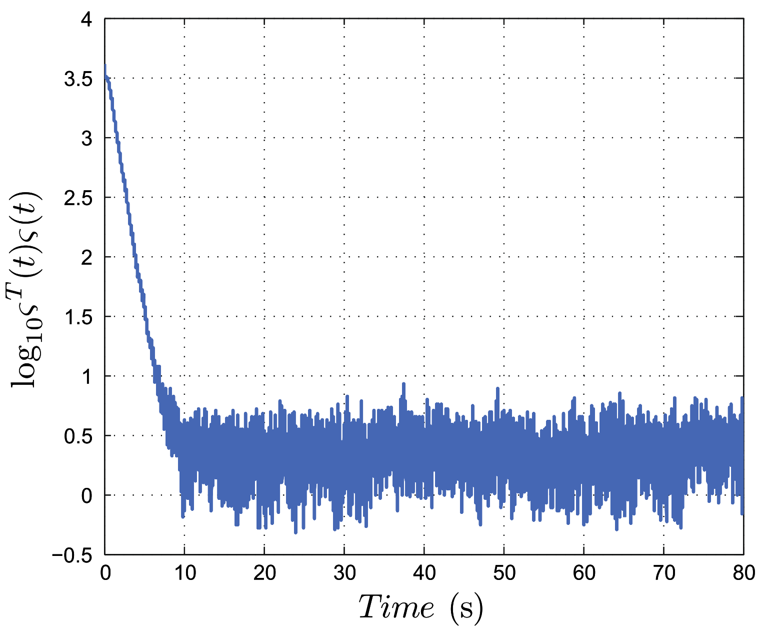
(b) Velocity snapshots at $t = 80$s

Fig. 6 Velocity trajectories and snapshots of the seven vehicles under the influence of the stochastic noises in in the states and formation vectors

within $t = 80$s, where the initial states are marked by circles, and the final states of the followers and the target are denoted by the star, point, x-mark, square, diamond, triangle and pentagram, respectively. Figure 4 shows the curve of the time-varying formation tracking error $\varsigma(t)$ in the logarithmic scale within $t = 80$s. From Figs. 2, 3 and 4, one sees that (i) the states of the six followers form a time-varying circular parallel hexagon, (ii) the state of the moving target lies in the center of the hexagon, and (iii) the moving direction of the leader is time-varying. Therefore, the desired time-varying formation tracking with one leader is achieved.

Example 2 To show the robustness of the proposed results under the influence of the stochastic measurement noises in the relative state and relative desired formation, consider the time-varying formation tracking problem in Example 1. Note that the state of each vehicle is $\phi_k(t) \in \mathbb{R}^{4\times 1}$ $(k \in \{1, 2, \cdots, 7\})$ and the

Fig. 7 Curve of the formation tracking error in the logarithmic scale under the influence of the stochastic noises in the states and formation vectors



desired formation for each follower is $h_i(t) \in \mathbb{R}^{4\times 1}$ $(i \in F)$. To demonstrate the influence of the stochastic noises in the relative state and relative desired formation to the formation tracking control, inject the stochastic noise terms $\Phi^{\phi_k}_{noise}(t) \in \mathbb{R}^{4\times 1}$ and $\Phi^{h_i}_{noise}(t) \in \mathbb{R}^{4\times 1}$ to the states of the seven vehicles and the desired formation vector, respectively, where each component of $\Phi^{\phi_k}_{noise}(t)$ and $\Phi^{h_i}_{noise}(t)$ is randomly drawn from the standard uniform distribution on the open interval $(-3, 3)$. All the parameters are the same as those in Example 1. Figures 5 and 6 show the state trajectories and snapshots of the seven vehicles within $t = 80$s under the influence of the stochastic noises. Figure 7 shows the curve of the time-varying formation tracking error $\varsigma(t)$ in the logarithmic scale within $t = 80$s under the influence of the stochastic noises. From Figs. 5, 6 and 7, one sees that under the influence of the stochastic noises in the relative state and relative desired formation, the desired time-varying formation tracking can still be realized with bounded errors. Moreover, by further comparing Figs. 2 and 3 with Figs. 5 and 6, respectively, one can see that both the position and velocity trajectories in Figs. 5 and 6 exhibit bounded stochastic drifts caused by the noises, and the stochastic noises have a stronger effects on the velocity trajectories than on the position ones due to the differential relationship between velocity and position. Therefore, the obtained results can work in the presence of bounded stochastic noises.

5 Conclusions

Distributed time-varying formation tracking control problems for second-order multi-agent systems with one leader were studied, where the states of followers form a predefined time-varying formation while tracking the state of the leader. A distributed formation tracking protocol was constructed using only neighboring relative information. Necessary and sufficient conditions for second-order multi-agent systems with one leader to achieve time-varying formation tracking were presented. An approach to design the formation tracking protocol was proposed by solving an algebraic Riccati equation. The obtained results can be applied to deal with the target enclosing problems and consensus tracking problems for second-order multi-agent systems with one target/leader. Based on this result, it is of interest to further study robust time-varying formation tracking control problems for multi-agent systems with stochastic dynamics. Another meaningful future research topic is to study time-varying formation tracking control problems using observer based protocols in which only partial neighboring information is available. Moreover, time-varying formation tracking control problem, where the desired formation can be changed online to adapt to the different external environments, is also a practical and interesting topic for the future study.

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