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Distributed Cooperative Control of Multiple Nonholonomic Mobile Robots

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Abstract In this paper, the distributed cooperative control problem is considered for multiple type $(1, 2)$ nonholonomic mobile robots. Firstly, a local change of coordinates and feedback is proposed to transform the original nonholonomic system to a new transformed system. Secondly, a distributed controller for the transformed system is designed by using information of the intrinsic system and its neighbors to make the state converge to the same value asymptotically. Furthermore, it shows that the same value can be confined to the origin, which means that the problem of cooperatively converging to a stationary point of a group of nonholonomic systems can be practically solved. Finally, due to the communication delays are inevitable in practice, new distributed controllers for the transformed system are also proposed making the state converge to the same value or zero asymptotically with considering communication delays. The proposed methods are then extended to the case where the nonholonomic mobile robot needs to form a prescribed formation other than agreeing on a same value. The stability of the proposed methods is proved rigorously. Simulation results confirm the effectiveness of the proposed methods.

Keywords Distributed control · Nonholonomic mobile robots · Formation control · Cooperative control

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1 Introduction

In recent years, there has been an increasing research interest in the distributed synchronization control of multi-agent systems due to its potential applications in many areas, such as formation control [1, 2], design of distributed sensor networks [3], flocking control [4, 5], etc. Some seminal works are [6, 7], just to name a few.

A large number of effective control approaches have focused on two control problems of networked systems, i.e., leaderless consensus problems and leader-following consensus problems. For leaderless consensus problems, controllers are designed to drive all the agents to a common value, which depends on initial conditions (see [8, 9]). As for leader-following consensus problems, controllers are designed to make all the follower nodes track the trajectory of the leader node (see [10, 11]). Besides, there are also many works investigated for different types of agent dynamics including first-order integrator systems [12, 13], second-order integrator systems [14, 15] and higher-order integrator systems [16, 17]. However, many practical cooperative control applications involve agents that are nonlinear and nonholonomic. The stabilization problem of nonholonomic system cannot be solved by many methods of classical linear system for the fact nonholonomic system fails to meet the three necessary conditions of the theorem of Brockett [18]. Thus the above mentioned methods cannot the solve the cooperative control of multiple nonholonomic agents. To solve the single nonholonomic system control problem, many scholars have done a lot of relevant research in this area (see [19–23], etc.). But most of the methods focused on the single nonholonomic system cannot solve the cooperative control of multiple nonholonomic systems directly, because we consider multiple nonholonomic mobile robots and the associated controller is distributed in nature-for each robot has access to the state of its neighbors only. Motivated by those observations, the authors in [2, 24–28] have focused on the cooperative control of multiple nonholonomic agents. In [2], Lin, Francis, and Maggiore have studied the feasibility problem of achieving a specified formation among a group of nonholonomic unicycles by local distributed control. In [24], Dong and Farrell presented two controllers for cooperative control problems of nonholonomic systems. One distributed controller was proposed to make a group of nonholonomic mobile agents cooperatively converge to some stationary point; The other controller was proposed to make a group of mobile agents converge to and track a target point which moves along a desired trajectory under various communication scenarios. And they also extended the methods to solve the problem of cooperative control of multiple nonholonomic dynamic systems with uncertainty in [25]. In [26], Liu and Jiang proposed a new class of distributed nonlinear controller for leader-following formation control of unicycle robots by using nonlinear small-gain design methods. In [27], Dong studied the distributed tracking control of multiple nonholonomic chained systems. Different from their works in [24, 25], the assumption that all follower robots have access to the information of the leader robot is not needed. In other words, for each robot, the available information for feedback is its own information and its neighbours' information. In [28], Cao, Jiang, and Yue have also investigated the consensus problems of multiple nonholonomic systems. Distributed controller was constructed by using the theory of cascaded systems. Different to previous assumptions on the group reference such as persistent excitation or converging to nonzero constant in [24], the condition on the group reference signal has been further relaxed.

Campion, Basin, and D'Andréa-Novel claimed that the interesting nonholonomic wheeled mobile robots are type $(2, 0)$, $(2, 1)$, $(1, 1)$, $(1, 2)$ robots in [29]. In this paper, we study distributed cooperative control problem of multiple type $(1, 2)$ nonholonomic mobile robots. This kind of systems is more complicated, compared with type $(2, 0)$, type $(2, 1)$ and type $(1, 1)$. The idea exploited in this paper can be used to investigate the same problem of the other three nonholonomic wheeled mobile robots. The main contributions of this paper are threefold. First, a local change of coordinates and feedback is proposed to transform the original nonholonomic system to a new transformed system. Second, distributed controllers for the new transformed system are designed by using its own information and its neighbours' information to make the state converge to the same value or zero asymptotically with and without considering communication delays. Third, extension is provided to extend the proposed schemes to the case, where the nonholonomic mobile robot needs to form a stable formation other than agreeing on a same value.

The remainder of this paper is organized as follows. In Section 2, some notions and preliminaries about the algebraic graph theory are briefly introduced, and the kinematic of type $(1, 2)$ and the distributed cooperative control problem of type $(1, 2)$ are presented. In Section 3, under two different communication scenarios, distributed controllers are designed to ensure that the state of each transformed system converges to the same value or zero asymptotically. Extensions are provided in Section 4. In Section 5, the simulation results are shown to illustrate the performance of the proposed methods. Some conclusions are given in the last Section.

2 Problem Statement

2.1 Basic Graph Theory and Notations

In this subsection, some notions and preliminaries about the algebraic graph theory are briefly introduced.

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ denote a directed graph, where $\mathcal{V} = \{1, \ldots, N\}$ is the set of nodes corresponding to each robot, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. $(i, j) \in \mathcal{E}$ means that robot j can obtain information from robot i, but not necessarily vice versa for a directed graph. In this paper, self-loop is not allowed in the graph, that is, $(i, i) \notin \mathcal{E}$. $N_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ denotes the neighbors of robot i. A matrix $A = [a_{ij}] \in R^{N\times N}$ denotes the adjacency matrix of $\mathcal{G}$, where $a_{ij} > 0$ iff $(j, i) \in \mathcal{E}$, else $a_{ij} = 0$. It is assumed that the topology is fixed which means A is time-invariant. A matrix $L = D - A$ is called the Laplacian matrix of $\mathcal{G}$, where $D = diag(d_1, \ldots, d_N)$ is the in-degree matrix with $d_i = \sum_{j=1}^{N} a_{ij}$. A direct path from robot i to robot j is a sequence of successive edges in the form $\{(i, l), (l, m), \ldots, (k, j)\}$. Graph $\mathcal{G}$ is strongly connected if any two robots (i, j) with $i \neq j$, there is a direct path from robot i to robot j. A directed graph $\mathcal{G}$ has a spanning tree, if there exists a robot i such that there is a direct path from robot i to every other robot in the graph, where the robot i is called the root of graph $\mathcal{G}$. A directed graph $\mathcal{G}$ is balanced if $\underline{1}^T L = 0$, where $\underline{1}$ is a vector with element one. Bidirectional graph is a special case of a directed graph, if $(i, j) \in \mathcal{E}$, then $(j, i) \in \mathcal{E}$. Meanwhile, it is stipulated that $a_{ij} = a_{ji}$ in bidirectional graph.

2.2 Kinematic of the Mobile Robots

Consider a group of $N(N \geq 2)$ type $(1, 2)$ nonholonomic mobile robots as shown in Fig. 1 Each robot has two steering wheels (conventional centered orientable wheels) and one castor wheel (conventional off-centered orientable wheel). (x_i, y_i) denotes the position P_i of the center of the ith $(i = 1, 2, \ldots, N)$ robot's mass, θ_i denotes the angle between $x_{i1}-$axis and $X-$axis, and β_{i1} and β_{i2} denote angles between

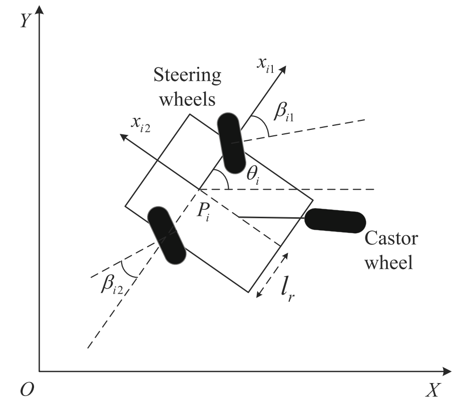


Fig. 1 Type (1,2) nonholonomic mobile robot

the orientation of the plane of steering wheels and $x_{i1}-$axis, $l_r(> 0)$ is half of the width of the ith robot. The nonholonomic constraint of the ith robot is defined by [29]

$$\begin{cases} (\cos\beta_{i1}, \sin\beta_{i1}, l_r \sin\beta_{i1}) H(\theta_i)\dot{\xi}_i = 0, \\ (-\cos\beta_{i2}, -\sin\beta_{i2}, l_r \sin\beta_{i2}) H(\theta_i)\dot{\xi}_i = 0, \end{cases} \tag{1}$$

where $\xi_i = (x_i, y_i, \theta_i)$, and

$$H(\theta_i) = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In addition, Eq. 1 can be specifically written as

$$\begin{cases} \dot{x}_i = -l_r \nu_{i1}[\sin\beta_{i1}\sin(\theta_i+\beta_{i2}) + \sin\beta_{i2}\sin(\theta_i+\beta_{i1})], \\ \dot{y}_i = l_r \nu_{i1}[\sin\beta_{i1}\cos(\theta_i+\beta_{i2}) + \sin\beta_{i2}\cos(\theta_i+\beta_{i1})], \\ \dot{\theta}_i = \nu_{i1}\sin(\beta_{i2}-\beta_{i1}), \quad \dot{\beta}_{i1} = \nu_{i2}, \quad \dot{\beta}_{i2} = \nu_{i3}, \end{cases} \tag{2}$$

where $q_i = [x_i, y_i, \theta_i, \beta_{i1}, \beta_{i2}]^T$ is the state of the ith robot, and $\nu_{i1}, \nu_{i2}, \nu_{i3}$ are the velocity of castor wheel and two angular velocities of steering wheels of the ith robot, respectively.

2.3 Cooperative Control Problem

The chained form systems were first introduced in [30] as a class of systems to which one could convert a number of interesting examples, and for which it was easy to derive steering control laws. However, only the systems that have two input and one chain were focused on. In our manuscript, the type $(1, 2)$ nonholonomic mobile robot has three inputs and two chains. Thus, the state feedback and coordinate transformation proposed in [30] cannot be utilized directly. The sufficient conditions for converting a multiple-input and multiple-chain system with nonholonomic constraints into a chained form via state feedback and a coordinate transformation were presented in [31, 32]. Here, we invoke the coordinate and state transformation which is similar to that in [32]. Then, to simplify the distributed cooperative controller design, a novel change of states by adding $\int_0^t \omega(s)ds$ based on chained form is proposed as follows.

$$
\begin{aligned}
z_{i1} &= \theta_i - \int_0^t \omega(s)ds,\\
z_{i2} &= x_i \cos\theta_i + y_i \sin\theta_i,\\
z_{i3} &= x_i \sin\theta_i - y_i \cos\theta_i,\\
z_{i4} &= -x_i \sin\theta_i + y_i \cos\theta_i - 2l_r \tfrac{\sin\beta_{i1}\sin\beta_{i2}}{\sin(\beta_{i2}-\beta_{i1})}\\
&\quad +\gamma_1\omega(x_i\cos\theta_i + y_i\sin\theta_i),\\
z_{i5} &= x_i \cos\theta_i + y_i \sin\theta_i - l_r \tfrac{\sin(\beta_{i1}+\beta_{i2})}{\sin(\beta_{i2}-\beta_{i1})}\\
&\quad +\gamma_2\omega(x_i\sin\theta_i - y_i\cos\theta_i),\\
u_{i1} &= \nu_{i1}\sin(\beta_{i2}-\beta_{i1}),\\
u_{i2} &= -\nu_{i1}\sin(\beta_{i2}-\beta_{i1})(x_i\cos\theta_i + y_i\sin\theta_i)\\
&\quad +2l_r\nu_{i3}\tfrac{\sin^2\beta_{i1}}{\sin^2(\beta_{i2}-\beta_{i1})}\\
&\quad -2l_r\nu_{i2}\tfrac{\sin^2\beta_{i2}}{\sin^2(\beta_{i2}-\beta_{i1})} + l_r\nu_{i1}\sin(\beta_{i1}+\beta_{i2}),\\
u_{i3} &= \nu_{i1}\sin(\beta_{i2}-\beta_{i1})(x_i\sin\theta_i - y_i\cos\theta_i)\\
&\quad +l_r\nu_{i3}\tfrac{\sin(2\beta_{i1})}{\sin^2(\beta_{i2}-\beta_{i1})}\\
&\quad -l_r\nu_{i2}\tfrac{\sin(2\beta_{i2})}{\sin^2(\beta_{i2}-\beta_{i1})} + 2l_r\nu_{i1}\sin\beta_{i1}\sin\beta_{i2},
\end{aligned} \tag{3}
$$

where $\omega = \rho\sin t$, and ρ, γ_1, γ_2 are positive constants.

Taking derivative of Eq. 3, we have

$$
\begin{aligned}
\dot z_{i1} &= u_{i1} - \omega,\\
\dot z_{i2} &= -\gamma_1 z_{i2}\omega^2 + \omega z_{i4} + (u_{i1}-\omega)(z_{i4} - \gamma_1\omega z_{i2}),\\
\dot z_{i3} &= -\gamma_2 z_{i3}\omega^2 + \omega z_{i5} + (u_{i1}-\omega)(z_{i5} - \gamma_2\omega z_{i3}),\\
\dot z_{i4} &= u_{i2} + \gamma_1\dot\omega z_{i2} + \gamma_1\omega u_{i1} z_{i4} - \gamma_1^2\omega^2 u_{i1} z_{i2},\\
\dot z_{i5} &= u_{i3} + \gamma_2\dot\omega z_{i3} + \gamma_2\omega u_{i1} z_{i5} - \gamma_2^2\omega^2 u_{i1} z_{i3}.
\end{aligned} \tag{4}
$$

Remark 1 It should be noted that because of the local nature of the state and feedback transformations (3), the laws designed for the transformed system (4) do not guarantee global stability properties for the original model (2) of the ith type (1,2) nonholonomic mobile robot. Indeed, since the coordinate transformation and state feedback are well defined over the subset $\Omega_i = \{(x_i, y_i, \theta_i, \beta_{i1}, \beta_{i2}) \in R^5 | \beta_{i1} \neq \beta_{i2}\ mod\ \pi\}$. We have that only within such a domain can we obtain "global" stability.

Definition 1 The distributed cooperative control problem of multiple type (1,2) nonholonomic mobile robots (2) discussed in this paper is to design the distributed control input $u_i = [u_{i1}, u_{i2}, u_{i3}]^T$ for the ith system (4) using $z_i = [z_{i1}, z_{i2}, z_{i3}, z_{i4}, z_{i5}]^T$ and the relative state z_l of its neighbors for $l \in N_i$ such that z_i is bounded and $\lim_{t\to\infty}(z_i(t) - z_j(t)) = 0$ for $1 \le i \neq j \le N$.

Remark 2 The control laws are required to make the state z_i of each transformed system converge to the same value $c(t)$ with $c(t) = [c_1, c_2(t), c_3(t), c_4, c_5]^T$, where c_1, c_4, and c_5 are constants which are unknown and depend on robots' initial conditions and communication between robots, and $c_2(t), c_3(t)$ are bounded functions. Furthermore, if $\lim_{t\to\infty}(u_{i1}(t) - \omega(t)) = 0$, $c_1 = 0$, $c_4 = 0$, and $c_5 = 0$, then $c_2 = 0$, $c_3 = 0$ (see Lemma 2). Since the system (2) discussed in this paper is nonholonomic, by the theorem of Brockett [18], the state q_i of each original system (2) cannot be stabilized at a stationary point by a smooth pure state feedback controller which is a smooth function of its own state q_i and the states q_l of its neighbors for $l \in N_i$. To overcome this difficulty, we design cooperative control laws such that the state z_i of each transformed system (4) converges to a moving vector $c(t)$. Then, we will state that $c(t)$ can also be confined to the origin, which means that cooperatively converging to a stationary point of a group of nonholonomic systems (2) can be practically solved. For details, please refer to the remarks after Theorem 2.

An additional assumption on the communication topology is given below.

Assumption 1 The communication digraph $\mathcal{G}$ has a spanning tree and $\mathcal{G}$ with weight matrix A is balanced.

Remark 3 Note that this assumption is very common which has appeared in relevant literature such as Dong [33]. And it is much more relaxed than undirected connected graph as has been made in Hou, Cheng, and Tan [8], Ou, Du, and Li [34], Feng and Wen [35].

The following lemmas are useful in our design and analysis of distributed controllers.

Lemma 1 *(Dong and Farrell [24]) If the digraph $\mathcal{G}$ has a spanning tree and the Laplacian matrix L of the digraph $\mathcal{G}$ with weight matrix $A = [a_{ij}](a_{ij} \geq 0)$, then*

$$\lim_{t\to\infty} e^{\mu t}(e^{-Lt} - \underline{1}w^T) = 0$$

for any $\mu \in [0, Re(\lambda_2(L)))$, where λ_2 is the nonzero eigenvalue of L with the smallest real part, w satisfies $w^T L = 0$ and $w^T \underline{1} = 1$.

Lemma 2 *(Dong [33]) If the digraph $\mathcal{G}$ has a spanning tree and the Laplacian matrix L of the digraph $\mathcal{G}$ with weight matrix A is balanced, the matrix $L^T + L$ is semidefinite. Furthermore, if $\lim_{t\to\infty} x^T(L^T + L)x = 0$ for a vector $x = [x_1, x_2, \ldots, x_N]^T$, then*

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0, 1 \leq i \neq j \leq N. \tag{5}$$

Before proceeding further, the following additional lemma is required.

Lemma 3 *For the ith transformed system (4), if $u_{i1} - \omega, z_{i4}, z_{i5}$ are bounded and converge to zero asymptotically, then z_{i2}, z_{i3} are bounded and converge to zero asymptotically.*

Proof Consider the Lyapunov function candidate

$$V_1 = \frac{1}{2}\left(z_{i2}^2 + z_{i3}^2\right). \tag{6}$$

Differentiating V_1 along with solutions of system (4), we get

$$\begin{aligned}\dot{V}_1 &= -\omega^2\left(\gamma_1 z_{i2}^2 + \gamma_2 z_{i3}^2\right) + \omega z_{i2} z_{i4} + \omega z_{i3} z_{i5} \\ &\quad + z_{i2}(u_{i1} - \omega)(z_{i4} - \gamma_1 \omega z_{i2}) \\ &\quad + z_{i3}(u_{i1} - \omega)(z_{i5} - \gamma_2 \omega z_{i3}) \\ &\leq -2\underline{\gamma}\omega^2 V_1 + 2\varphi_1 V_1 + 2\varphi_2\sqrt{V_1},\end{aligned} \tag{7}$$

where $\underline{\gamma} = \min\{\gamma_1, \gamma_2\}$, and

$$\varphi_1 = \bar{\gamma}|\omega||u_{i1} - \omega|, \varphi_2 = \tfrac{1}{\sqrt{2}}(|z_{i4}| + |z_{i5}|)|u_{i1}|, \tag{8}$$

with $\bar{\gamma} = \max\{\gamma_1, \gamma_2\}$.

Due to boundedness of ω, and $\lim_{t\to\infty}(u_{i1} - \omega) = 0$, $\lim_{t\to\infty} z_{i4}(t), z_{i5}(t) = 0$, we have $\lim_{t\to\infty} \varphi_1(t), \varphi_2(t)=0$. In order to facilitate the following analysis, we take $\sigma = \sqrt{V_1}$, then $D^+\sigma \leq -\underline{\gamma}\omega^2\sigma + \varphi_1\sigma + \varphi_2$, where D^+ is the upper Dini derivative. Thus, we get

$$\begin{aligned}\sigma(t) &\leq e^{\int_0^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}\sigma(0) \\ &\quad + \int_0^t e^{\int_\tau^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}\varphi_2(\tau)d\tau.\end{aligned} \tag{9}$$

Note that

$$\begin{aligned}\int_0^t -\underline{\gamma}\omega^2(s)ds &= \int_0^t -\underline{\gamma}\rho^2\sin^2 s ds \\ &= -\underline{\gamma}\rho^2\left(\frac{t}{2} - \frac{\sin 2t}{4}\right).\end{aligned}$$

With this observation in mind, since $\lim_{t\to\infty}\varphi_1(t)=0$, there always exists $T_1 > 0$ such that $\varphi_1(t) \leq \frac{\underline{\gamma}\rho^2}{4}$ for all $t \geq T_1$. Define function $\bar{\varphi}_1(t) = \sup_{0\leq\tau\leq t}\varphi_1(\tau)$, the following equation can be achieved

$$\begin{aligned}\int_0^t\varphi_1(s)ds &= \int_0^{T_1}\varphi_1(s)ds + \int_{T_1}^t\varphi_1(s)ds \\ &\leq \bar{\varphi}_1(T_1)T_1 + \tfrac{\underline{\gamma}\rho^2}{4}(t - T_1).\end{aligned} \tag{10}$$

Thus

$$\begin{aligned}&\int_0^t(-\underline{\gamma}\omega^2(s) + \varphi_1(s))ds \\ &\leq -\underline{\gamma}\rho^2\left(\tfrac{t}{2} - \tfrac{\sin 2t}{4}\right) + \bar{\varphi}_1(T_1)T_1 + \tfrac{\underline{\gamma}\rho^2}{4}(t - T_1) \\ &\leq -\tfrac{\underline{\gamma}\rho^2}{4}t + \tfrac{\underline{\gamma}\rho^2}{4} + \bar{\varphi}_1(T_1)T_1 - \tfrac{\underline{\gamma}\rho^2}{4}T_1.\end{aligned} \tag{11}$$

Hence

$$\begin{aligned}&\lim_{t\to\infty} e^{\int_0^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}\sigma(0) \\ &= e^{\lim_{t\to\infty}\int_0^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}\sigma(0) = 0.\end{aligned} \tag{12}$$

Next, we will show that

$$\lim_{t\to\infty}\int_0^t e^{\int_\tau^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}\varphi_2(\tau)d\tau = 0. \tag{13}$$

Due to $\lim_{t\to\infty}\varphi_2(t)$=0, $\forall\eta > 0$, we can always find that $T_2 \geq T_1$ such that $\varphi_2(t) \leq \eta$ for all $t \geq T_2$. Define function $\bar{\varphi}_2(t) = \sup_{0\leq\tau\leq t}\varphi_2(\tau)$, we get

$$\begin{aligned}
&\int_0^t e^{\int_\tau^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}\varphi_2(\tau)d\tau \\
&= \int_0^{T_2} e^{\int_\tau^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}\varphi_2(\tau)d\tau \\
&\quad + \int_{T_2}^t e^{\int_\tau^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}\varphi_2(\tau)d\tau \\
&\leq \bar{\varphi}_2(T_2)\int_0^{T_2} e^{\int_0^\tau(\underline{\gamma}\omega^2(s)-\varphi_1(s))ds}d\tau\, e^{\int_0^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds} \\
&\quad +\eta\int_{T_2}^t e^{\int_\tau^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}d\tau \\
&\leq \zeta + \eta\int_{T_2}^t e^{\int_\tau^t\left(-\underline{\gamma}\omega^2(s)+\frac{\underline{\gamma}\rho^2}{4}\right)ds}d\tau \\
&\leq \zeta + \eta\int_{T_2}^t e^{-\underline{\gamma}\rho^2\left(\frac{t-\tau}{2}-\frac{1}{4}\sin 2t+\frac{1}{4}\sin 2\tau\right)+\frac{\underline{\gamma}\rho^2}{4}(t-\tau)}d\tau \\
&\leq \zeta + \eta\int_{T_2}^t e^{\frac{\underline{\gamma}\rho^2}{2}}e^{-\frac{\underline{\gamma}\rho^2}{4}(t-\tau)}d\tau \\
&\leq \zeta + \eta\frac{4}{\underline{\gamma}\rho^2}e^{\frac{\underline{\gamma}\rho^2}{2}},
\end{aligned}$$

where $\zeta = \psi e^{\int_0^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}$ with $\psi = \bar{\varphi}_2(T_2)\int_0^{T_2} e^{\int_0^\tau(\underline{\gamma}\omega^2(s)-\varphi_1(s))ds}d\tau$. Due to boundedness of ψ, thus $\lim_{t\to\infty}\zeta(t) = 0$. Furthermore $\forall\varepsilon > 0$, there exists $T_3 > 0$ such that $\zeta(t) \leq \frac{\varepsilon}{2}$ for all $t \geq T_3$. Choose $\eta = \frac{\underline{\gamma}\rho^2}{8}e^{-\frac{\underline{\gamma}\rho^2}{2}}\varepsilon$, we have

$$\int_0^t e^{\int_\tau^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}\varphi_2(\tau)d\tau \leq \varepsilon, \forall t \geq \max\{T_2, T_3\}$$

which implies $\lim_{t\to\infty}\int_0^t e^{\int_\tau^t(-\underline{\gamma}\omega^2(s)+\varphi_1(s))ds}\varphi_2(\tau)d\tau = 0$.

Therefore, it can be concluded that the right-side of inequality (9) will converge to zero as $t\to\infty$. Consequently, $\sigma(t)$ is bounded and tends to zero asymptotically, which also suggests that V_1, z_{i2}, z_{i3} are bounded and converge to zero asymptotically. This completes the proof. □

Remark 4 It should be noted that the proof of Lemma 3 is different from that in Lemma 6 of [24] and Lemma 2 of [25]. The requirements for the convergence of φ_1, φ_2 must be exponential in [24] and [25], which are relaxed to be asymptotical here, and the proof here is much more rigorous.

Lemma 4 *If $u_{i1}-\omega$, $u_{j1}-\omega$, $z_{i4}, z_{j4}, z_{i5}, z_{j5}$ are bounded and $u_{i1}-\omega$, $u_{j1}-\omega$, $z_{i4}-z_{j4}$, $z_{i5}-z_{j5}$ asymptotically converge to zero for $1 \leq i \neq j \leq N$, then $z_{i2}, z_{i3}, z_{j2}, z_{j3}$ are bounded and $z_{i2}-z_{j2}$ and $z_{i3}-z_{j3}$ converge to zero asymptotically.*

Proof First, we will prove that z_{i2} and z_{i3} are bounded for $i = 1,\ldots,N$. By Eq. (4) and using means of variation of constants and initial integral methods, we have

$$\begin{aligned}
z_{i2}(t) &= e^{-\int_0^t g_{i11}(s)ds}(z_{i2}(0) \\
&\quad +\int_0^t g_{i12}(\sigma)e^{\int_0^\sigma g_{i11}(s)ds}d\sigma), \\
z_{i3}(t) &= e^{-\int_0^t g_{i21}(s)ds}(z_{i2}(0) \\
&\quad +\int_0^t g_{i22}(\sigma)e^{\int_0^\sigma g_{i21}(s)ds}d\sigma),
\end{aligned} \tag{14}$$

where

$$g_{i11} = \gamma_1\omega^2 + (u_{i1}-\omega)\gamma_1\omega,\ g_{i12} = u_{i1}z_{i4},$$
$$g_{i21} = \gamma_2\omega^2 + (u_{i1}-\omega)\gamma_2\omega,\ g_{i22} = u_{i1}z_{i5}.$$

Since $u_{i1}-\omega$, ω, z_{i4}, z_{i5} are all bounded, thus $g_{i11}, g_{i12}, g_{i21}, g_{i22}$ are bounded. Furthermore, it can be proved that z_{i2} and z_{i3} are bounded by Eq. 14.

Let $e_{ij2} = z_{i2} - z_{j2}$ for $1 \leq i \neq j \leq N$, we have

$$\begin{aligned}
\dot{e}_{ij2} &= -\gamma_1 z_{i2}\omega^2 + \omega z_{i4} + (u_{i1}-\omega)(z_{i4}-\gamma_1\omega z_{i2}) \\
&\quad +\gamma_1 z_{j2}\omega^2 - \omega z_{j4} - (u_{j1}-\omega)(z_{j4}-\gamma_1\omega z_{j2}) \\
&= -\gamma_1\omega^2 e_{ij2} + \varphi_{ij1}(t),
\end{aligned} \tag{15}$$

where $\varphi_{ij1}(t) = \omega(z_{i4}-z_{j4}) + (u_{i1}-\omega)(z_{i4}-\gamma_1\omega z_{i2}) - (u_{j1}-\omega)(z_{j4}-\gamma_1\omega z_{j2})$. Since $z_{i4}-z_{j4}$ and $u_{i1}-\omega$ asymptotically converge to zero, and ω, $z_{i2}, z_{i4}, z_{j2}, z_{j4}$ are bounded, thus $\varphi_{ij1}(t)$ converge to zero asymptotically. Choose the following Lyapunov function

$$V_2 = \frac{1}{2}e_{ij}^2. \tag{16}$$

Using the mimicking argument as the proof of Lemma 3, it can be easily proved that $\lim_{t\to\infty}e_{ij2}(t) = 0$, namely, $z_{i2}-z_{j2}$ converges to zero asymptotically for $1 \leq i \neq j \leq N$. Also, with the similar technique, the conclusion that $z_{i3}-z_{j3}$ asymptotically converges to zero can be given. □

3 Controller Design and Stability Analysis

3.1 Closed-loop System Stability

In this subsection, we will design the distributed control input u_i for the ith system (4) using z_i and the relative state z_l of its neighbors for $l \in N_i$ such that

z_i is bounded and $\lim_{t\to\infty}(z_i(t) - z_j(t)) = 0$ for $1 \le i \ne j \le N$. The structure of system (4) suggests z_{i1}, z_{i4}, z_{i5} can be directly controlled via u_{i1}, u_{i2}, u_{i3}. Now we are ready to choose the distributed controller u_i as

$$\begin{aligned} u_{i1} &= -\sum_{j=1}^{N} a_{ij}(z_{i1} - z_{j1}) + \omega, \\ u_{i2} &= -\sum_{j=1}^{N} a_{ij}(z_{i4} - z_{j4}) - \gamma_1\dot{\omega}z_{i2} - \gamma_1\omega u_{i1}z_{i4} \\ &\quad +\gamma_1^2\omega^2 u_{i1}z_{i2}, \\ u_{i3} &= -\sum_{j=1}^{N} a_{ij}(z_{i5} - z_{j5}) - \gamma_2\dot{\omega}z_{i3} - \gamma_2\omega u_{i1}z_{i5} \\ &\quad +\gamma_2^2\omega^2 u_{i1}z_{i3}. \end{aligned} \tag{17}$$

Remark 5 The first term of Eq. 17 is a weighted sum of the relative state information between system i and its neighbors. And the terms $\omega, -\gamma_1\dot{\omega}z_{i2} - \gamma_1\omega u_{i1}z_{i4} + \gamma_1^2\omega^2 u_{i1}z_{i2}, -\gamma_2\dot{\omega}z_{i3} - \gamma_2\omega u_{i1}z_{i5} + \gamma_2^2\omega^2 u_{i1}z_{i3}$ are the canceling terms, which are designed to cancel the extra parts.

Substituting control input (17) into system (4), we can get the following closed-loop error system for z_{i1}, z_{i4}, z_{i5}

$$\begin{aligned} \dot{z}_{i1} &= -\sum_{j=1}^{N} a_{ij}(z_{i1} - z_{j1}), \\ \dot{z}_{i4} &= -\sum_{j=1}^{N} a_{ij}(z_{i4} - z_{j4}), \\ \dot{z}_{i5} &= -\sum_{j=1}^{N} a_{ij}(z_{i5} - z_{j5}). \end{aligned} \tag{18}$$

Theorem 1 *Consider the closed-loop system consisting of N transformed systems (4) satisfying Assumption 1, the proposed distributed controller (17). Then the state z_i of the ith transformed system (4) in the closed-loop system is bounded and $\lim_{t\to\infty}(z_i(t) - z_j(t)) = 0$ for $1 \le i \ne j \le N$.*

Proof By Eq. 18, we have

$$\dot{Z}_1 = -LZ_1, \dot{Z}_4 = -LZ_4, \dot{Z}_5 = -LZ_5, \tag{19}$$

where $Z_q = [z_{1q}, z_{2q}, \ldots, z_{Nq}]$ for $q = 1, 4, 5$, and L is the Laplacian matrix of $\mathcal{G}$. Therefore

$$Z_1 = e^{-Lt}Z_1(0), Z_4 = e^{-Lt}Z_4(0), Z_5 = e^{-Lt}Z_5(0).$$

By Lemma 1, we have

$$\begin{aligned} \lim_{t\to\infty} Z_1(t) &= \underline{1}w^T Z_1(0) =: c_1\underline{1}, \\ \lim_{t\to\infty} Z_4(t) &= \underline{1}w^T Z_4(0) =: c_4\underline{1}, \\ \lim_{t\to\infty} Z_5(t) &= \underline{1}w^T Z_5(0) =: c_5\underline{1}. \end{aligned} \tag{20}$$

It is apparent that $\lim_{t\to\infty}(z_{1i}(t) - z_{1j}(t)) = 0, \lim_{t\to\infty}(z_{4i}(t) - z_{4j}(t)) = 0, \lim_{t\to\infty}(z_{5i}(t) - z_{5j}(t)) = 0$ for $1 \le i \ne j \le N$.

By utilizing Eqs. 17 and 20, we can prove that $u_{l1} - \omega$ is bounded and converges to zero asymptotically for $l = 1, \ldots, N$. Then according to Lemma 4, we have that z_{l2} and z_{l3} are bounded. In addition, the conclusion that $z_{i2} - z_{j2}$ and $z_{i3} - z_{j3}$ converge to zero asymptotically for $1 \le i \ne j \le N$ can also be given by Lemma 4, namely, $\lim_{t\to\infty}(z_{l2}(t) - c_2(t)) = 0$, $\lim_{t\to\infty}(z_{l3}(t) - c_3(t)) = 0$ for $l = 1, \ldots, N$, where $c_2(t)$ and $c_3(t)$ are unknown but bounded functions. □

Remark 6 A distributed control law for system (4) is given by Eq. 17. Control law (17) can make z_l for $l = 1, \ldots, N$ converge to $c(t)$ asymptotically with $c(t) = [c_1, c_2, c_3, c_4, c_5]^T$. By Eq. 3, it is easy to prove that

$$\lim_{t\to\infty} \bar{q}_i(t) - \bar{q}_j(t) = 0, 1 \le i \ne j \le N \tag{21}$$

where $\bar{q}_l = [x_l, y_l, \theta_l]^T$ for $l = 1, \ldots, N$.

The following theorem shows that we can make z_i converge to zero. We redesign the distributed controller u_i as

$$\begin{aligned} u_{i1} &= -\sum_{j=1}^{N} a_{ij}(z_{i1} - z_{j1}) - p_i z_{i1} + \omega, \\ u_{i2} &= -\sum_{j=1}^{N} a_{ij}(z_{i4} - z_{j4}) - q_i z_{i4} - \gamma_1\dot{\omega}z_{i2} \\ &\quad -\gamma_1\omega u_{i1}z_{i4} + \gamma_1^2\omega^2 u_{i1}z_{i2}, \\ u_{i3} &= -\sum_{j=1}^{N} a_{ij}(z_{i5} - z_{j5}) - k_i z_{i5} - \gamma_2\dot{\omega}z_{i3} \\ &\quad -\gamma_2\omega u_{i1}z_{i5} + \gamma_2^2\omega^2 u_{i1}z_{i3}, \end{aligned} \tag{22}$$

where $p_i \ge 0, q_i \ge 0, k_i \ge 0$, and $\sum_{i=1}^{N} p_i > 0$, $\sum_{i=1}^{N} q_i > 0, \sum_{i=1}^{N} k_i > 0$.

Remark 7 These terms $p_i z_{i1}, q_i z_{i4}, k_i z_{i5}$ in Eq. 22 can also be considered as relative information between robot i and a virtual robot with its state being zero.

Theorem 2 *Consider the closed-loop system consisting of N transformed systems (4) satisfying Assumption 1, the proposed distributed controller (22) with the parameters satisfying $p_i \ge 0, q_i \ge 0, k_i \ge 0$, and $\sum_{i=1}^{N} p_i > 0$, $\sum_{i=1}^{N} q_i > 0, \sum_{i=1}^{N} k_i > 0$. Then the state z_i of the ith transformed system (4) in the closed-loop system is bounded and converges to zero asymptotically, i.e., $\lim_{t\to\infty} z_i(t) = 0$ for $i = 1, \ldots, N$.*

Proof With the distributed controller u_i defined in Eq. 22, we have

$$\begin{aligned}\dot{z}_{i1} &= -\sum_{j=1}^{N} a_{ij}(z_{i1}-z_{j1}) - p_i z_{i1},\\ \dot{z}_{i4} &= -\sum_{j=1}^{N} a_{ij}(z_{i4}-z_{j4}) - q_i z_{i4},\\ \dot{z}_{i5} &= -\sum_{j=1}^{N} a_{ij}(z_{i5}-z_{j5}) - k_i z_{i5}.\end{aligned} \tag{23}$$

Choose the Lyapunov function

$$V = \frac{1}{2}\sum_{i=1}^{N}\left(z_{i1}^2 + z_{i4}^2 + z_{i5}^2\right). \tag{24}$$

Differentiating V along the solutions of Eq. 23 yields

$$\begin{aligned}\dot{V} &= -\tfrac{1}{2}Z_1^T(L^T+L)Z_1 - \tfrac{1}{2}Z_4^T(L^T+L)Z_4\\ &\quad -\tfrac{1}{2}Z_5^T(L^T+L)Z_5\\ &\quad -\sum_{i=1}^{N}\left(p_i z_{i1}^2 + q_i z_{i4}^2 + k_i z_{i5}^2\right),\end{aligned} \tag{25}$$

where $Z_q = [z_{1q}, z_{2q}, \ldots, z_{Nq}]$ for $q = 1, 4, 5$, and L is the Laplacian matrix of $\mathcal{G}$. Since $L^T + L$ is positive semidefinite, $\dot{V} \leq 0$, hence that $V(t)$ is bounded and z_{i1}, z_{i4}, z_{i5} are bounded. According to the definition (24), Barbalat's Lemma [36] can be employed to prove that that $\lim_{t\to\infty}\dot{V}(t) = 0$. Thus we obtain,

$$\begin{aligned}&\lim_{t\to\infty} p_l z_{l1}^2, q_l z_{l4}^2, k_l z_{l5}^2 = 0, \quad l = 1, \ldots, N\\ &\lim_{t\to\infty} Z_1^T(L^T+L)Z_1 = 0,\\ &\lim_{t\to\infty} Z_4^T(L^T+L)Z_4 = 0,\\ &\lim_{t\to\infty} Z_5^T(L^T+L)Z_5 = 0.\end{aligned} \tag{26}$$

Since there is at least one integer m such that $p_m > 0$, $\lim_{t\to\infty} z_{m1}(t) = 0$. By applying Lemma 2, $\lim_{t\to\infty}(z_{i1}(t) - z_{j1}(t)) = 0$ for $1 \leq i \neq j \leq N$. Hence, $\lim_{t\to\infty} z_{l1}(t) = 0$ for $l = 1, \ldots, N$. And $\lim_{t\to\infty} z_{l4}(t) = 0, \lim_{t\to\infty} z_{l5}(t) = 0$ can also be proved in the similar argument.

By utilizing Eqs. 22 and 26, we can prove that $u_{l1} - \omega$ is bounded and converges to zero asymptotically for $l = 1, \ldots, N$. Then according to Lemma 3, we have z_{l2}, z_{l3} are bounded and converge to zero asymptotically. In summary, the state z_l of the lth transformed system (4) in the closed-loop system is bounded and converges to zero asymptotically, i.e., $\lim_{t\to\infty} z_l(t) = 0$ for $l = 1, \ldots, N$. □

Remark 8 By Eq. 3 and $\lim_{t\to\infty} z_l(t) = 0$, we have $\lim_{t\to\infty}[\theta_l(t) - \rho(1-\cos t)] = 0$, which means that θ_l converges to a neighborhood Bd of the origin with radius ρ. And from the second equation and third equation of Eq. 3, we can also get x_l, y_l are bounded and asymptotically converge to zero provided z_{l2}, z_{l3} are bounded and converge to zero asymptotically. From the fourth equation and fifth equation of Eq. 3, it can also be proved that if $\lim_{t\to\infty} z_l(t) = 0$, then $\lim_{t\to\infty}\beta_{l1}(t) = k_{l1}\pi$ and $\lim_{t\to\infty}\beta_{l2}(t) = k_{l2}\pi$, $k_{l1}, k_{l2} \in Z$. Thus, the problem of cooperatively converging to a stationary point of a group of nonholonomic systems (2) is practically solved. In addition, if ρ decreases, then the θ_l becomes small. However, the performance of x_l, y_l becomes bad, i.e., the convergence rate of z_{l2}, z_{l3} to zero decreases. Therefore, there is a tradeoff between small θ_l and a large convergence rate of x_l, y_l when one chooses ρ.

3.2 Closed-loop System Stability with Communication Delays

In practice, there are always time delays due to communication and other factors. In this subsection, we will consider communication delays in the control design and analysis. For simplicity, in this paper we assume that all communication delays are constant.

Assumption 2 The communication digraph $\mathcal{G}$ is bidirectional and strongly connected.

Under Assumption 2, the distributed controller is designed as

$$\begin{aligned}u_{i1}(t) &= -\sum_{j=1}^{N} a_{ij}(z_{i1}(t) - z_{j1}(t-\tau_i)) + \omega(t),\\ u_{i2}(t) &= -\sum_{j=1}^{N} a_{ij}(z_{i4}(t) - z_{j4}(t-\tau_i)) - \gamma_1\dot{\omega}(t)z_{i2}(t)\\ &\quad -\gamma_1\omega(t)u_{i1}(t)z_{i4}(t) + \gamma_1^2\omega^2(t)u_{i1}(t)z_{i2}(t),\\ u_{i3}(t) &= -\sum_{j=1}^{N} a_{ij}(z_{i5}(t) - z_{j5}(t-\tau_i)) - \gamma_2\dot{\omega}(t)z_{i3}(t)\\ &\quad -\gamma_2\omega(t)u_{i1}(t)z_{i5}(t) + \gamma_2^2\omega^2(t)u_{i1}(t)z_{i3}(t),\end{aligned} \tag{27}$$

where communication delay $\tau_i(\geq 0)$ is a positive constant.

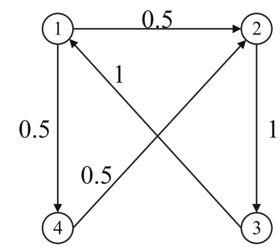


Fig. 2 The communication graph $\mathcal{G}_1$

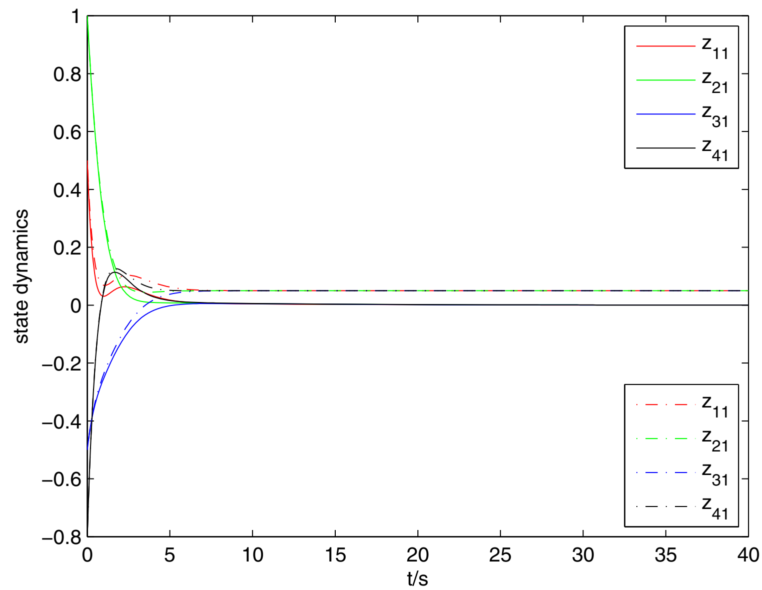


Fig. 3 Profiles of the states z_{i1} with controller (17) and controller (22)

Substituting the distributed controller (27) into system (4), we can get the following closed-loop error system for z_{i1}, z_{i4}, z_{i5}

$$\begin{aligned} \dot{z}_{i1}(t) &= -\sum_{j=1}^{N} a_{ij}(z_{i1}(t) - z_{j1}(t-\tau_i)), \\ \dot{z}_{i4}(t) &= -\sum_{j=1}^{N} a_{ij}(z_{i4}(t) - z_{j4}(t-\tau_i)), \\ \dot{z}_{i5}(t) &= -\sum_{j=1}^{N} a_{ij}(z_{i5}(t) - z_{j5}(t-\tau_i)). \end{aligned} \tag{28}$$

Theorem 3 *Consider the closed-loop system consisting of N transformed systems (4) satisfying Assumption 2, and the proposed distributed controller (27). Then the state z_i of the ith transformed system (4) in the closed-loop system is bounded and* $\lim_{t\to\infty}(z_i(t) - z_j(t)) = 0$ *for* $1 \le i \ne j \le N$.

Proof Let

$$\begin{aligned} V(t) = &\tfrac{1}{2}\sum_{i=1}^{N}\Big(z_{i1}^2(t) + \sum_{j=1}^{N} a_{ij}\int_{t-\tau_i}^{t} z_{j1}^2(s)ds\Big) \\ &+\tfrac{1}{2}\sum_{i=1}^{N}\Big(z_{i4}^2(t)+\sum_{j=1}^{N} a_{ij}\int_{t-\tau_i}^{t} z_{j4}^2(s)ds\Big) \\ &+\tfrac{1}{2}\sum_{i=1}^{N}\Big(z_{i5}^2(t)+\sum_{j=1}^{N} a_{ij}\int_{t-\tau_i}^{t} z_{j5}^2(s)ds\Big). \end{aligned} \tag{29}$$

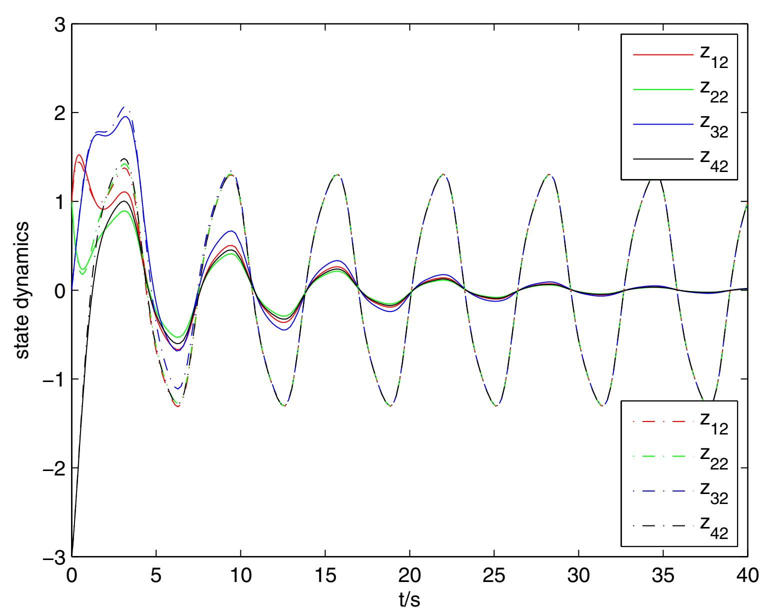


Fig. 4 Profiles of the states z_{i2} with controller (17) and controller (22)

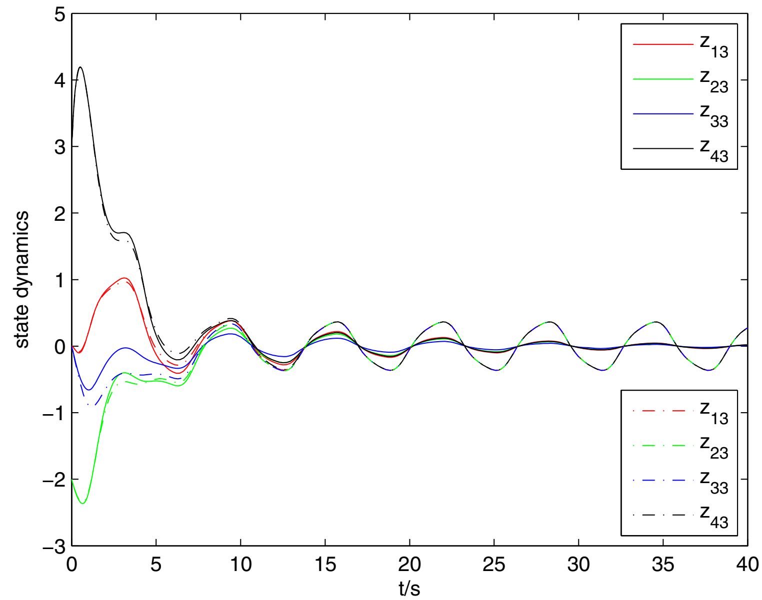


Fig. 5 Profiles of the states z_{i3} with controller (17) and controller (22)

Differentiate V along the solutions of Eq. 28 yields

$$\begin{aligned}\dot{V}(t) = & -\tfrac{1}{2}\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(z_{i1}(t) - z_{j1}(t-\tau_i))^2 \\ & -\tfrac{1}{2}\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(z_{i4}(t) - z_{j4}(t-\tau_i))^2 \\ & -\tfrac{1}{2}\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(z_{i5}(t)-z_{j5}(t-\tau_i))^2 \le 0,\end{aligned} \tag{30}$$

where the fact that the communication graph $\mathcal{G}$ is bidirectional has been used. By the invariance principle [37], z_{l1}, z_{l4}, and z_{l5} will converge to constants for $l = 1, \ldots, N$. The following proof is the same as the proof in Theorem 1, but omitted here. □

Remark 9 In practice, there are always time delays due to communication and other factors. In our manuscript, we take time delays into account in our design of distributed protocol and we allow the delays to be arbitrarily large. In the theorem, communication delays only appear in the neighbors states. This

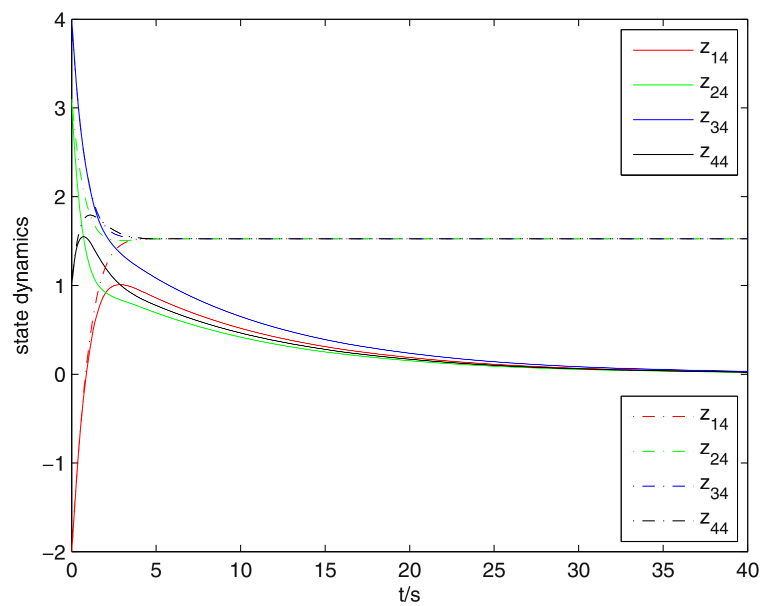


Fig. 6 Profiles of the states z_{i4} with controller (17) and controller (22)

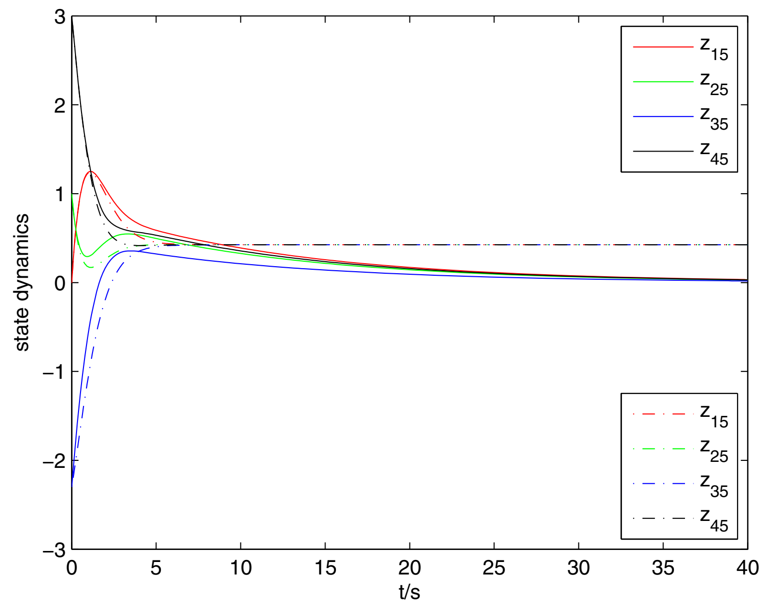


Fig. 7 Profiles of the states z_{i5} with controller (17) and controller (22)

assumption is reasonable because the communication delay is the dominated delay among all other time delays. The first term of Eq. 22 can be treated as a weighted sum of the relative state information between the current states of system i and the delayed state information of its neighboring. By applying the invariance principle, it is proved that our proposed cooperative control laws are still effective even existing communication delay. Assumption 2 is stronger than Assumption 1, since the existence of delays is in the communication.

Corresponding to Theorem 2, we have the following delayed version result.

Theorem 4 *Consider the system consisting of N transformed systems (4) satisfying Assumption 2, and use distributed controller given by*

$$\begin{aligned} u_{i1}(t) &= -\sum_{j=1}^{N} a_{ij}(z_{i1}(t) - z_{j1}(t-\tau_i)) - p_i z_{i1}(t) + \omega(t), \\ u_{i2}(t) &= -\sum_{j=1}^{N} a_{ij}(z_{i4}(t) - z_{j4}(t-\tau_i)) - q_i z_{i4}(t) - \gamma_1 \dot{\omega}(t) z_{i2}(t) \\ &\quad -\gamma_1 \omega(t) u_{i1}(t) z_{i4}(t) + \gamma_1^2 \omega^2(t) u_{i1}(t) z_{i2}(t), \\ u_{i3}(t) &= -\sum_{j=1}^{N} a_{ij}(z_{i5}(t) - z_{j5}(t-\tau_i)) - k_i z_{i5}(t) - \gamma_2 \dot{\omega}(t) z_{i3}(t) \\ &\quad -\gamma_2 \omega(t) u_{i1}(t) z_{i5}(t) + \gamma_2^2 \omega^2(t) u_{i1}(t) z_{i3}(t), \end{aligned} \tag{31}$$

with the parameters satisfying $p_i \geq 0, q_i \geq 0, k_i \geq 0$, and $\sum_{i=1}^{N} p_i > 0$, $\sum_{i=1}^{N} q_i > 0$, $\sum_{i=1}^{N} k_i > 0$, where communication delay $\tau_i (\geq 0)$ is a positive constant. Then the state z_i of the ith transformed system (4) in the closed-loop system is bounded and converges to zero asymptotically, i.e., $\lim_{t\to\infty} z_i(t) = 0$ for $i = 1, \ldots, N$.

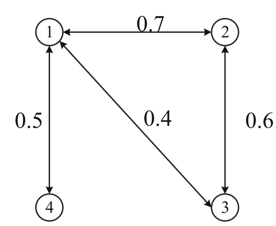
Proof The proof is analogous as that of Theorem 2 and Theorem 3 and is omitted here. □

4 Extensions

In practical applications, multiple type (1, 2) nonholonomic mobile robots may need to achieve a prescribed formation other than rendezvousing at a common value. It is shown that, if convergence to a common value is feasible, then other formations can also be obtained by the simple transformation.

Definition 2 The formation control problem discussed in this paper is to design a distributed controller for the ith system (2), based on its state information

Fig. 8 The communication graph $\mathcal{G}_2$ with time-delays



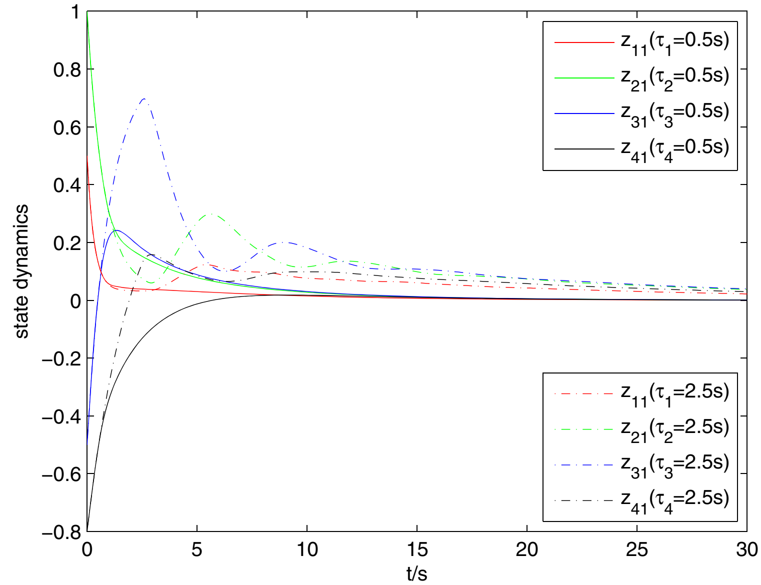


Fig. 9 Profiles of the states z_{i1} with communication delays $\tau = 0.5$s and $\tau = 2.5$s

q_i and the relative state q_l of its neighbors for $l \in N_i$ such that

$$\lim_{t\to\infty}\left(\begin{bmatrix} x_i - x_j \\ y_i - y_j \end{bmatrix} - \begin{bmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{bmatrix}\begin{bmatrix} p_{ix} - p_{jx} \\ p_{iy} - p_{jy} \end{bmatrix}\right) = 0,$$
$$\lim_{t\to\infty}(\theta_i(t) - \theta_j(t)) = 0, 1 \le i \ne j \le N \tag{32}$$

$$\lim_{t\to\infty}\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} p_{ix}, \lim_{t\to\infty}\sum_{i=1}^{N} y_i = \sum_{i=1}^{N} p_{iy},$$
$$\lim_{t\to\infty}\sum_{i=1}^{N}\beta_{i1} = k_1\pi, \lim_{t\to\infty}\sum_{i=1}^{N}\beta_{i1} = k_2\pi, \quad k_1, k_2 \in Z \tag{33}$$

where χ is a free variable, and p_{ix}, p_{iy} are the prescribed displacements between the state value x_i, y_i of robot i and the system consensus value, which is

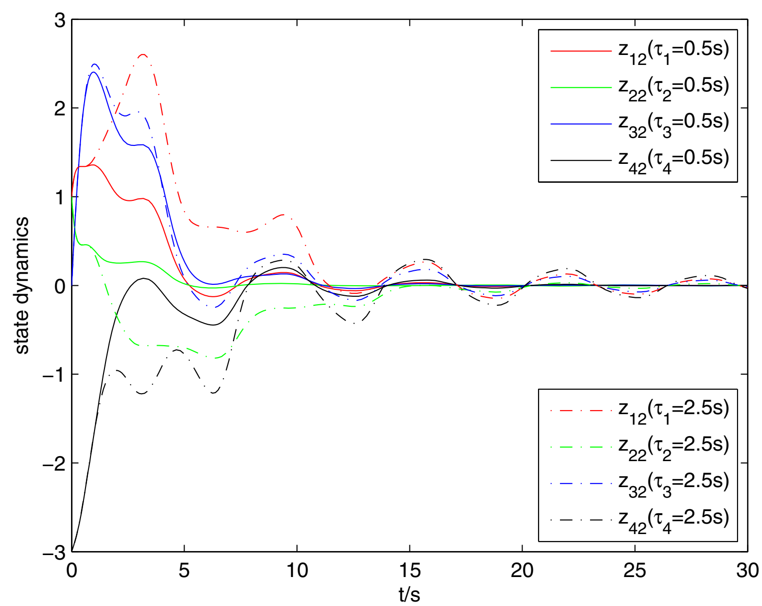


Fig. 10 Profiles of the states z_{i2} with communication delays $\tau = 0.5$s and $\tau = 2.5$s

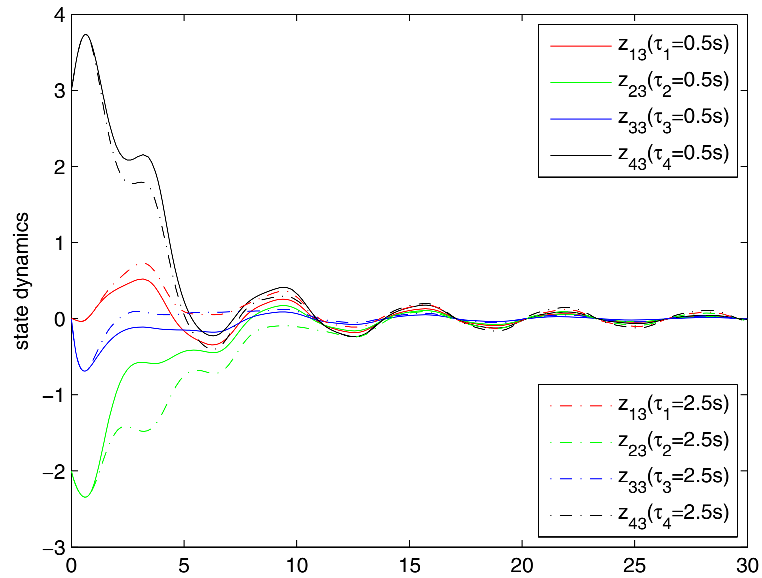


Fig. 11 Profiles of the states z_{i3} with communication delays $\tau = 0.5$s and $\tau = 2.5$s

unknown and depends on robots' initial conditions and communication between robots.

Let

$$\begin{aligned}
\bar{z}_{i1} &= \theta_i - \int_0^t \omega(s)ds, \\
\bar{z}_{i2} &= (x_i - p_{ix})\cos\theta_i + (y_i - p_{iy})\sin\theta_i, \\
\bar{z}_{i3} &= (x_i - p_{ix})\sin\theta_i - (y_i - p_{iy})\cos\theta_i, \\
\bar{z}_{i4} &= -\bar{z}_{i3} - 2l_r\frac{\sin\beta_{i1}\sin\beta_{i2}}{\sin(\beta_{i2}-\beta_{i1})} + \gamma_1\omega\bar{z}_{i2}, \\
\bar{z}_{i5} &= \bar{z}_{i2} - l_r\frac{\sin(\beta_{i1}+\beta_{i2})}{\sin(\beta_{i2}-\beta_{i1})} + \gamma_2\omega\bar{z}_{i3}, \\
\bar{u}_{i1} &= v_{i1}\sin(\beta_{i2}-\beta_{i1}), \\
\bar{u}_{i2} &= -v_{i1}\sin(\beta_{i2}-\beta_{i1})\bar{z}_{i2} + 2l_r v_{i3}\frac{\sin^2\beta_{i1}}{\sin^2(\beta_{i2}-\beta_{i1})} \\
&\quad -2l_r v_{i2}\frac{\sin^2\beta_{i2}}{\sin^2(\beta_{i2}-\beta_{i1})} + l_r v_{i1}\sin(\beta_{i1}+\beta_{i2}), \\
\bar{u}_{i3} &= v_{i1}\sin(\beta_{i2}-\beta_{i1})\bar{z}_{i3} + l_r v_{i3}\frac{\sin(2\beta_{i1})}{\sin^2(\beta_{i2}-\beta_{i1})} \\
&\quad -l_r v_{i2}\frac{\sin(2\beta_{i2})}{\sin^2(\beta_{i2}-\beta_{i1})} + 2l_r v_{i1}\sin\beta_{i1}\sin\beta_{i2}, \quad (34)
\end{aligned}$$

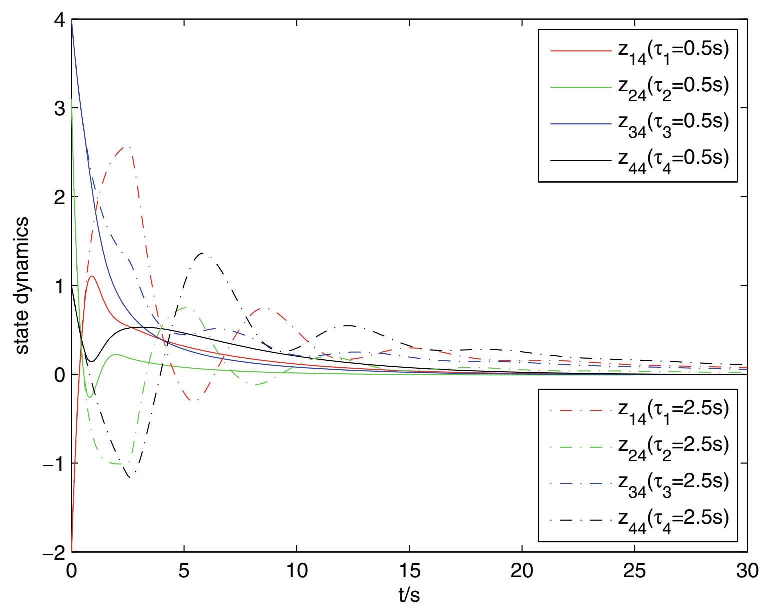


Fig. 12 Profiles of the states z_{i4} with communication delays $\tau = 0.5$s and $\tau = 2.5$s

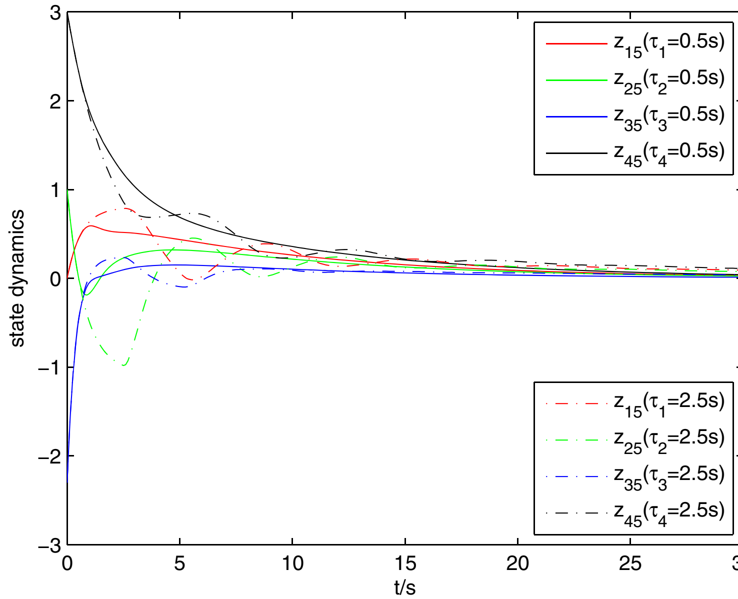


Fig. 13 Profiles of the states z_{i5} with communication delays $\tau = 0.5$s and $\tau = 2.5$s

where $\omega = \rho \sin t$, and ρ, γ_1, γ_2 are positive constants. Taking derivative of Eq. 34, we have

$$\begin{aligned}
\dot{\bar{z}}_{i1} &= \bar{u}_{i1} - \omega, \\
\dot{\bar{z}}_{i2} &= -\gamma_1 \bar{z}_{i2}\omega^2 + \omega \bar{z}_{i4} + (\bar{u}_{i1} - \omega)(\bar{z}_{i4} - \gamma_1 \omega \bar{z}_{i2}), \\
\dot{\bar{z}}_{i3} &= -\gamma_2 \bar{z}_{i3}\omega^2 + \omega \bar{z}_{i5} + (\bar{u}_{i1} - \omega)(\bar{z}_{i5} - \gamma_2 \omega \bar{z}_{i3}), \\
\dot{\bar{z}}_{i4} &= \bar{u}_{i2} + \gamma_1 \dot{\omega} \bar{z}_{i2} + \gamma_1 \omega \bar{u}_{i1} \bar{z}_{i4} - \gamma_1^2 \omega^2 \bar{u}_{i1} \bar{z}_{i2}, \\
\dot{\bar{z}}_{i5} &= \bar{u}_{i3} + \gamma_2 \dot{\omega} \bar{z}_{i3} + \gamma_2 \omega \bar{u}_{i1} \bar{z}_{i5} - \gamma_2^2 \omega^2 \bar{u}_{i1} \bar{z}_{i3}.
\end{aligned} \tag{35}$$

Lemma 5 *If* $\lim_{t\to\infty}(\bar{Z}_i(t) - \bar{Z}_j(t)) = 0$ *for* $1 \le i \ne j \le N$, *then Eq. 32 holds, where* $\bar{Z}_i(t) = [\bar{z}_{i1}, \bar{z}_{i2}, \bar{z}_{i3}, \bar{z}_{i4}, \bar{z}_{i5}]^T$. *Furthermore, if* $\lim_{t\to\infty} \bar{Z}_l(t) = 0$ *for* $l = 1, \ldots, N$, *then Eqs. 32 and 33 hold.*

By replacing z_{ij} in Eqs. 17, 22, 27, and 31 with $\bar{z}_{ij}$ for $j = 1, \ldots, 5$, similar control algorithms can be obtained. By Lemma 5, the formation control problem is also solved.

5 Simulations

We consider some examples to illustrate the proposed design schemes and verify the established theoretical results. Consider the system (4) discussed in Section 2.3. Let $N = 4$ and the initial values of each system be

$$z_1 = [0.5, 1, 0, -2, 0]^T, z_2 = [1, 1, -2, 3.1, 1]^T,$$
$$z_3 = [-0.5, 0, 0, 4, -2.3]^T, z_4 = [-0.8, -3, 3, 1, 3]^T.$$

Case 1 The communication graph $\mathcal{G}_1$ without communication delays is described in Fig. 2. Note that this communication graph $\mathcal{G}_1$ satisfies Assumption 1. The corresponding adjacency matrix A_1 is given by

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \end{bmatrix}. \tag{36}$$

Two simulations are respectively implemented for the distributed control law (17) and the distributed control law (22) with $p_1 = 0.5, q_2 = 0.5, k_3 = 0.5$ and other control parameters are all zero. We choose the parameter $\rho = 1$ in local change of coordinates and feedback (3). The simulations are conducted by the Matlab "ode45" method. The trajectories of states versus time plotted using solid line and dash-dot line shown in Figs. 3, 4, 5, 6 and 7 are corresponding to the distributed controller (17) and the distributed controller (22), respectively. Note that the states do not converge to zero directly, but are the same as its neighbors'. It demonstrates that if z_{i1}, z_{i4}, z_{i5} converge to nonzero constants, then z_{i2} and z_{i3} are bounded. Furthermore, if z_{i1}, z_{i4}, and z_{i5} converge to zero asymptotically, then z_{i2} and z_{i3} also converge to zero asymptotically.

Case 2 The communication graph $\mathcal{G}_2$ with communication delays is described in Fig. 8. Note that this communication graph $\mathcal{G}_2$ satisfies Assumption 2.

The corresponding adjacency matrix A_2 is given by

$$A_2 = \begin{bmatrix} 0 & 0.7 & 0.4 & 0.5 \\ 0.7 & 0 & 0.6 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

To simplify the simulation, we assume all the communication delays are common to each system, namely $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau$. The simulation is implemented for the distributed control law (27). We choose the parameter $\rho = 1$ in local change of coordinates and feedback (3), $p_1 = 1.5, q_2 = 1.5, k_3 = 1.5$ and other control parameters are all zero. In order to better analyze the influence of communication delays for the system, τ is set to be 0.5s, 2.5s in the two simulations, respectively. The simulations are performed by the Matlab "dde23" method. The trajectories of states versus time plotted using solid line and dash-dot line shown in Figs. 9, 10, 11, 12 and 13 are corresponding to the time delay $\tau = 0.5$s and $\tau = 2.5$s, respectively. Figs. 9–13 verify the fact that the states of every system (4) converge to zero asymptotically even with communication delays. It also indicates that the asymptotical convergence of the states can also be achieved for large constant delays. However, the cooperative performance is bad if communication delays are large.

6 Conclusion

In this paper, the distributed cooperative control problem has been investigated for type $(1, 2)$ nonholonomic mobile robots. Four distributed controllers are designed to ensure that the state of the transformed system converges to the common value or zero asymptotically with and without considering communication delays. Extension is also provided to extend the proposed schemes to the case, where the nonholonomic mobile robot needs to form a stable formation other than rendezvousing at a common value. The stability of the proposed methods is proved rigorously. Simulation results confirm the effectiveness of the proposed methods. It is our future work to solve the consensus problem for multiple nonholonomic mobile robots based on visual servoing.

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