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Time-Optimal Control of a Hovering Quad-Rotor Helicopter

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Abstract. The time-optimal control problem of a hovering quad-rotor helicopter is addressed in this paper. Instead of utilizing the Pontryagin's Minimum Principle (PMP), in which one needs to solve a set of highly nonlinear differential equations, a nonlinear programming (NLP) method is proposed. In this novel method, the count of control steps is fixed initially and the sampling period is treated as a variable in the optimization process. The optimization object is to minimize the sampling period such that it will be below a specific minimum value, which is set in advance considering the accuracy of discretization. To generate initial feasible solutions of the formulated NLP problem, genetic algorithms (GAs) are adopted. With the proposed method, one can find a time-optimal movement of the helicopter between two configurations. To show the feasibility of the proposed method, simulation results are included for illustration.

Key words: nonlinear programming, quad-rotor helicopters, time-optimal control.

1. Introduction

A quad-rotor helicopter is an under-actuated dynamic rotorcraft with four input forces and six output coordinates. This system is an autonomous vehicle capable of quasi-stationary (hover and near hover) flight. Unlike conventional helicopters that have variable pitch angle rotors, a quad-rotor helicopter has four fixed pitch angle rotors. In this manner, it has higher payload capacity and better maneuverability in comparison with a conventional one.

Varying the rotor speeds of all four rotors, thereby changing the lift forces, generates the motion of a quad-rotor helicopter. The helicopter tilts towards the direction of a slow spinning rotor, which enables acceleration along that direction [3]. Spinning directions of the rotors are set to balance the moments,

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therefore eliminating the need for a tail rotor. This is also used to produce the desired yaw motion. A good controller should properly arrange the rotor speeds such that only the desired states will be changed [18].

In recent years, many researchers been addressed the control problem associated with quad-rotor helicopters [1–4, 8, 9, 12, 14–16, 18, 20]. Typical missions for a quad-rotor helicopter include attitude stabilization and the movement from one configuration (position and attitude) to another.

Dynamic modeling and configuration of a commercial four-rotor helicopter were proposed by McKerrow [14] and Hamel et al. [9]. Nonlinear control problems for hovering quad-rotor helicopters such as feedback linearization control and back-stepping control laws were studied by Altug et al. [2] and Mistler et al. [15]. When the roll and yaw angles are set to zero, a hovering four-rotor helicopter can be viewed as a planar vertical take off and landing (PVTOL) aircraft. Therefore, based on the dynamic model of a PVTOL aircraft, Castillo et al. [4], Hamel et al. [8] designed controllers for yaw angular displacement, and pitch and roll movements of a hovering quad-rotor helicopter. In experiments, the vision-based stabilization and output tracking control for hovering four-rotor helicopters have been researched by Altug et al. [1], Castillo et al. [4], Lozano et al. [12], and Suter [20]. A dynamic feedback controller has also been developed and its performance and robustness are tested in simulation by Mokhtari et al. [16].

Time-optimal problems of control systems have attracted the attention of many researchers, especially in aerospace [5, 10] and robotics [19, 21, 22] in the past few years. Jan and Chiou [10] developed a sliding mode control method that can be used to perform a spacecraft large angle maneuver in minimum-time. Chern et al. [5] investigated the time-optimal aerobraking maneuver of the shuttle-type space vehicle at a constant altitude. However, to the best knowledge of the authors, previous researchers have not addressed the time-optimal control problem of a hovering quad-rotor helicopter yet. This motivates the research in this paper and a NLP method will be proposed to carry out a hovering quad-rotor helicopter motion maneuver in minimum-time.

The time-optimal motion-planning (TOMP) problem for a hovering quad-rotor helicopter is to find the time-optimal motion in an unobstructed atmosphere between two configurations, where the initial and final velocities must be zero. Usually, this TOMP problem leads to the utilization of the PMP [17], in which one needs to solve a set of differential equations. Since these equations are usually nonlinear and highly coupled, one will have two-point boundary value problems, which are intractable in numerical computation.

Recently, a NLP method that does not utilize the PMP was developed by one of the authors to solve the time-optimal control problem of linear systems [6]. The basic idea of this method is that instead of considering a fixed sampling period as is usually the case, the count of control steps is fixed initially and the sampling period is treated as a variable in the optimization process. The optimization object is to minimize the sampling period below a specific minimum value, which is set in advance considering the accuracy of discretization. With this approach, the optimization procedure requires only two iterations in most linear cases, thereby reducing the computation time dramatically.

Extending the concept in [6] to nonlinear systems, this paper will show how to generate time-optimal motion between two configurations for a hovering quad-rotor helicopter with four independently driven rotors. However, since the quad-rotor helicopter system is nonlinear, it will be a difficult task to find a feasible solution for the formulated NLP problem. Therefore, a GA-based approach will also be proposed to generate feasible solutions for the TOMP problem. Simulation examples will be given to verify the feasibility of the proposed method.

The rest of this paper is as follows. In Section 2, dynamical equations of a quad-rotor helicopter will be derived. Then the TOMP problem between two configurations of the helicopter will be formulated as a NLP one in Section 3. In Section 4, GAs will be used to generate initial feasible solutions of the NLP problem. Problem solution and simulation results are shown in Sections 5 and 6, respectively. Finally, conclusions and discussion are given in Section 7.

2. Dynamical Equations of a Quad-Rotor Helicopter

To illustrate the motion of a helicopter, a schematic diagram is given as shown in Figure 1. In the working space of the helicopter, an earth-frame and a body-frame will be defined. The earth-frame denotes a frame that everything discussed can be referenced and the body-frame is a frame attached to the helicopter. In order to maintain hovering flight of the quad-rotor helicopter, the magnitudes of four driven forces, F_i, $i = 1, 2, 3, 4$, are equally tuned to one fourth of the weight of the helicopter initially. The vertical motion along z-axis of the body-frame can be obtained by changing the speeds of all the four rotors simultaneously. The forward motion along x-axis of the body-frame can be achieved by changing the speeds of rotors 1 and 3 reversely and retaining the speeds of rotors 2 and 4. The lateral motion along y-axis of the body-frame can be reached by changing the speeds of rotor 2 and 4 reversely and retaining the speeds of rotors 1 and 3. The yaw motion is related to the difference between the moments created by the rotors. To turn in a clockwise direction, rotors 2 and 4 should increase speeds to overcome the speeds of rotors 1 and 3. On the other hand, to turn in a counter-clockwise direction, rotors 3 and 1 should increase speeds to overcome the speeds of rotors 2 and 4. Table I summarizes the nomenclature used in this paper.

The rotational transformation matrix between the earth-frame and the body-frame can be obtained based on Euler angles.

$$R_{\mathrm{EB}} = R_\psi R_\theta R_\phi = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & -\cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \quad (1)$$

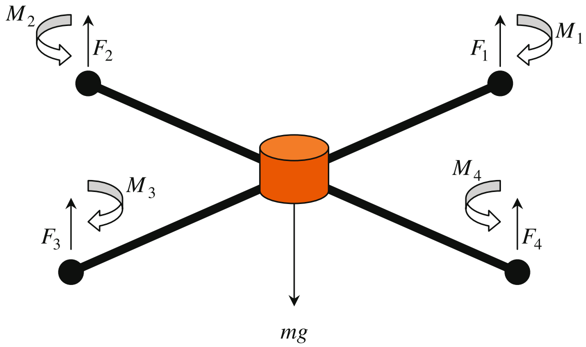


Figure 1. A schematic diagram of the quad-rotor helicopter.

The transformation of velocities between the body-frame and the earth-frame coordinates is then

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_{\mathrm{EB}} \begin{bmatrix} u_{\mathrm{B}} \\ v_{\mathrm{B}} \\ w_{\mathrm{B}} \end{bmatrix} \tag{2}$$

Similarly, the accelerations, rotational velocities, positions, forces and moments can be transformed based on R_{EB} between the coordinate systems.

In the body-frame, the forces are defined as

$$F_{\mathrm{B}} = \begin{bmatrix} F_{x\mathrm{B}} \\ F_{y\mathrm{B}} \\ F_{z\mathrm{B}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sum_{k=1}^{4} F_k \end{bmatrix} \tag{3}$$

Table I. The nomenclature used in this paper.

Nomenclature	Definition
I_x, I_y, I_z	Moments of inertia along x, y, z directions
ϕ, θ, ψ	Roll angle, pitch angle, and yaw angle
u_{B}, v_{B}, w_{B}	Velocities along x, y, z directions of the body-frame
u, v, w	Velocities along x, y, z directions of the earth-frame
x, y, z	Coordinates of center of gravity in the earth-frame
$F_{x\mathrm{B}}$, $F_{y\mathrm{B}}$, $F_{z\mathrm{B}}$	Body forces along x, y, z directions of the body-frame
F_x, F_y, F_z	Body forces along x, y, z directions of the earth-frame
u_1	$u_1 = F_1 + F_2 + F_3 + F_4$
u_2	$u_2 = F_4 - F_2$
u_3	$u_3 = F_3 - F_1$
u_4	$u_4 = F_1 - F_2 + F_3 - F_4$
$\mathbf{u}_1$	$\mathbf{u}_1 = [u_1(0), u_1(1), \ldots, u_1(N-1)]$
$\mathbf{u}_2$	$\mathbf{u}_2 = [u_2(0), u_2(1), \ldots, u_2(N-1)]$
$\mathbf{u}_3$	$\mathbf{u}_3 = [u_3(0), u_3(1), \ldots, u_3(N-1)]$
$\mathbf{u}_4$	$\mathbf{u}_4 = [u_4(0), u_4(1), \ldots, u_4(N-1)]$

In the earth-frame, the forces can be described as

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = R_{\mathrm{EB}}\, F_{\mathrm{B}} = \left(\sum_{k=1}^{4} F_k\right) \begin{bmatrix} \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ \cos\phi\cos\theta \end{bmatrix} \tag{4}$$

Therefore, equations of motion in the earth-frame are represented as

$$m\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} F_x - K_1\dot{x} \\ F_y - K_2\dot{y} \\ F_z - mg - K_3\dot{z} \end{bmatrix} \tag{5}$$

where K_i, $i = 1, 2, 3$, are the drag coefficients. Note that these coefficients are negligible at low speeds. As a result, equations of motion can be derived based on the balance of forces and moments.

$$\ddot{\phi} = \frac{l(F_3 - F_1 - K_4\dot{\phi})}{I_x} \tag{6}$$

$$\ddot{\theta} = \frac{l(F_4 - F_2 - K_5\dot{\theta})}{I_y} \tag{7}$$

$$\ddot{\psi} = \frac{(M_1 - M_2 + M_3 - M_4 - K_6\dot{\psi})}{I_z} \tag{8}$$

where l is the length from the center of gravity of the helicopter to each rotor, M_i, $i = 1, 2, \ldots, 4$, are the moments of rotors, and I_x, I_y, and I_z represent the moments of inertia along x, y, z directions. In terms of F_1, F_2, F_3, and F_4, equation (8) can be also written as

$$\ddot{\psi} = \frac{(F_1 - F_2 + F_3 - F_4 - K_6'\dot{\psi})}{I_z'} \tag{9}$$

where $I_z' = I_z/\mathrm{C}$, $K_6' = K_6/C$, and C is the scaling factor.

For convenience of computing the TOMP problem, the inputs are defined as

$$\begin{aligned} u_1 &= F_1 + F_2 + F_3 + F_4 \\ u_2 &= F_4 - F_2 \\ u_3 &= F_3 - F_1 \\ u_4 &= F_1 - F_2 + F_3 - F_4 \end{aligned} \tag{10}$$

The inputs then can be represented in matrix form as

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \tag{11}$$

Then the individual forces will be

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 1 & -2 & 0 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{12}$$

Therefore, the dynamical equations of the four-rotor helicopter become

$$\ddot{x} = \frac{(\sin\psi \sin\phi + \cos\psi \sin\theta \cos\phi)u_1 - K_1\dot{x}}{m} \tag{13}$$

$$\ddot{y} = \frac{(\sin\psi \sin\theta \cos\phi - \cos\psi \sin\phi)\, u_1 - K_2\dot{y}}{m} \tag{14}$$

$$\ddot{z} = \frac{(\cos\phi \cos\theta)\, u_1 - K_3\dot{z}}{m} - g \tag{15}$$

$$\ddot{\phi} = \frac{(u_3 - K_4\dot{\phi})\, l}{I_x} \tag{16}$$

$$\ddot{\theta} = \frac{(u_2 - K_5\dot{\theta})\, l}{I_y} \tag{17}$$

$$\ddot{\psi} = \frac{(u_4 - K_6'\dot{\psi})}{I_z'} \tag{18}$$

For a hovering flight, velocities and angles of roll, pitch, and yaw must be zeros, and the four driven forces are $F_1(0) = F_2(0) = F_3(0) = F_4(0) = \frac{mg}{4}$. Substituting these conditions into equations (13) through (18), one will find that zero accelerations will be induced.

3. TOMP between Two Configurations

3.1. PROBLEM FORMULATION

The TOMP problem of the hovering helicopter between two configurations is to find the control inputs that will move the system from an initial configuration to a desired final configuration while minimizing the traveling time. With the dynamics in equations (13) through (18), the TOMP problem can be formulated as follows:

PROBLEM 1: For the quad-rotor helicopter described in equations (13) through (18), assuming that the initial configuration is given as

$$(x(0),\ y(0),\ z(0),\ \phi(0),\ \theta(0),\ \psi(0)) = (x_0,\ y_0,\ z_0,\ \phi_0,\ \theta_0,\ \psi_0) \tag{19}$$

$$(\dot{x}(0),\ \dot{y}(0),\ \dot{z}(0),\ \dot{\phi}(0),\ \dot{\theta}(0),\ \dot{\psi}(0)) = (0,\ 0,\ 0,\ 0,\ 0,\ 0) \tag{20}$$

determine the control inputs $u_1(t)$, $u_2(t)$, $u_3(t)$, and $u_4(t)$ for $t \in [0, t_f]$ to minimize

$$J = t_f \tag{21}$$

subject to

$$\left(x\left(t_f\right),\ y\left(t_f\right),\ z\left(t_f\right),\ \phi\left(t_f\right),\ \theta\left(t_f\right),\ \psi\left(t_f\right)\right) = (x_f,\ y_f,\ z_f,\ \phi_f,\ \theta_f,\ \psi_f) \tag{22}$$

$$\left(\dot{x}\left(t_f\right),\ \dot{y}\left(t_f\right),\ \dot{z}\left(t_f\right),\ \dot{\phi}\left(t_f\right), \dot{\theta}\left(t_f\right),\ \dot{\psi}\left(t_f\right)\right) = (0,\ 0,\ 0,\ 0,\ 0,\ 0) \tag{23}$$

and

$$u_{i,\min} \le u_i(t) \le u_{i,\max} \text{ for } t \in \left[0,\ t_f\right];\ i = 1, 2, \ldots, 4 \tag{24}$$

where $(x_f, y_f, z_f, \phi_f, \theta_f, \psi_f)$ is the desired final configuration.

It is obvious that Problem 1 is a very difficult problem due to the nature of the nonlinear and coupled relation of the quad-rotor helicopter system. To cope with the difficulty, Problem 1 will be formulated and solved in the discrete-time domain by numerical methods. By extending the concept in [6], it will be shown how to determine the time-optimal movement of a hovering quad-rotor helicopter between configurations. The first step is to divide the interval $t \in [0, t_f]$ into N equal time intervals, where N is the number of control steps [6]. That is

$$t_i - t_{i-1} = \Delta t = t_f / N \qquad \text{for } i = 1,\ 2, \ldots, N \tag{25}$$

If the acceleration is assumed to be constant for each sub-interval, and the drag coefficient in equation (16) is assumed to be neglected, one will obtain

$$
\begin{aligned}
\dot{\phi}(i) &= \dot{\phi}(i-1) + \ddot{\phi}(i-1)\cdot\Delta t \\
&= \dot{\phi}(0) + \sum_{k=0}^{i-1} \ddot{\phi}(k)\cdot\Delta t \\
&= \dot{\phi}(0) + \frac{l}{I_x}\sum_{k=0}^{i-1} u_3(k)\cdot\Delta t \quad \text{for } i = 1,\ 2,\ldots,\ N
\end{aligned} \tag{26}
$$

where $\dot{\phi}(i)$ and $u_3(i)$ denote $\dot{\phi}(i\cdot\Delta t)$ and $u_3(i\cdot\Delta t)$, respectively.

From the above equation, one can find that $\dot{\phi}(N)$ is a function of the initial angular velocity $\dot{\phi}(0)$, the input variables $u_3(0)$, $u_3(1),\ldots,$ $u_3(N-1)$, and the sampling period Δt. This means that

$$\dot{\phi}(N) = f_1(\dot{\phi}(0),\ \mathbf{u}_3,\ \Delta t) \tag{27}$$

where $\mathbf{u}_3$ is defined in Table I.

In a similar way, one also can obtain

$$\dot{\theta}(N) = f_2(\dot{\theta}(0),\ \mathbf{u}_2,\ \Delta t) \tag{28}$$

$$\dot{\psi}(N) = f_3(\dot{\psi}(0),\ \mathbf{u}_4,\ \Delta t) \tag{29}$$

$$\phi(N) = f_4(\phi(0),\ \dot{\phi}(0),\ \mathbf{u}_3,\ \Delta t) \tag{30}$$

$$\theta(N) = f_5(\theta(0),\ \dot{\theta}(0),\ \mathbf{u}_2,\ \Delta t) \tag{31}$$

$$\psi(N) = f_6(\psi(0),\ \dot{\psi}(0),\ \mathbf{u}_4,\ \Delta t) \tag{32}$$

where $\mathbf{u}_2$ through $\mathbf{u}_4$ are defined in Table I.

The linear velocities and displacements can also be represented as

$$\dot{x}(N) = f_7(\dot{x}(0),\ \theta(0),\ \phi(0),\ \psi(0),\ \dot{\theta}(0),\ \dot{\phi}(0), \dot{\psi}(0),\ \mathbf{u}_1,\ \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4,\ \Delta t) \tag{33}$$

$$\dot{y}(N) = f_8(\dot{y}(0),\ \theta(0),\ \phi(0),\ \psi(0),\ \dot{\theta}(0),\ \dot{\phi}(0),\ \dot{\psi}(0),\ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4,\ \Delta t) \tag{34}$$

$$\dot{z}(N) = f_9(\dot{z}(0),\ \theta(0),\ \phi(0),\ \psi(0), \dot{\theta}(0), \dot{\phi}(0), \dot{\psi}(0),\ \mathbf{u}_1,\ \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4,\ \Delta t) \tag{35}$$

$$x(N) = f_{10}(x(0), \dot{x}(0), \theta(0), \phi(0), \psi(0),\ \dot{\theta}(0), \dot{\phi}(0), \dot{\psi}(0), \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \Delta t) \tag{36}$$

$$y(N) = f_{11}(y(0), \dot{y}(0), \theta(0), \phi(0), \psi(0), \dot{\theta}(0), \dot{\phi}(0), \dot{\psi}(0), \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \Delta t) \tag{37}$$

$$z(N) = f_{12}(z(0), \dot{z}(0), \theta(0), \phi(0), \psi(0), \dot{\theta}(0), \dot{\phi}(0), \dot{\psi}(0), \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \Delta t) \tag{38}$$

where $\mathbf{u}_1$ through $\mathbf{u}_4$ are defined in Table I.

With equations (27) through (38), Problem 1 is now turned into a standard constrained NLP problem as follows:

PROBLEM 2: Given the initial configuration in equations (22) and (23), determine the values of $u_1(0)$, $u_1(1)$,…, $u_1(N-1)$, $u_2(0)$, $u_2(1)$,…, $u_2(N-1)$, $u_3(0)$, $u_3(1)$,…, $u_3(N-1)$, $u_4(0)$, $u_4(1)$,…, $u_4(N-1)$, and Δt to minimize

$$J = N \cdot \Delta t \tag{39}$$

subject to

$$\Delta t > 0 \tag{40}$$

$$(x(N), y(N), z(N), \phi(N), \theta(N), \psi(N)) = (x_f, y_f, z_f, \phi_f, \theta_f, \psi_f) \tag{41}$$

$$(\dot{x}(N), \dot{y}(N), \dot{z}(N), \dot{\phi}(N), \dot{\theta}(N), \dot{\psi}(N)) = (0, 0, 0, 0, 0, 0) \tag{42}$$

$$u_{i,\min} \le u_i(j) \le u_{i,\max} \text{ for } i = 1, 2, \ldots, 4; j = 0, 1, \ldots, N-1 \tag{43}$$

where $(x(N),\ y(N),\ z(N),\ \phi(N),\ \theta(N),\ \psi(N))$ and $(\dot{x}(N), \dot{y}(N), \dot{z}(N), \dot{\phi}(N), \dot{\theta}(N), \dot{\psi}(N))$ are defined in equations (27) through (38).

3.2. CHOICE OF CONTROL STEPS AND SAMPLING PERIOD

Although the TOMP problem of a hovering quad-rotor helicopter can be formulated as shown in Problem 2, there still exist several difficulties to be solved. One difficulty is the choice of the value of control steps N It is obvious that a larger value of N will give more freedom for the input variables. However, this also means more computation burden for Problem 2. For linear system without constraints on the input variables, it has been shown that the initial choice of N must be greater than the dimension of state variables [6]. Though no similar rules can be followed for nonlinear systems, an integer that is large than the dimension of state variables will be chosen as an initial value of N in this paper.

Another difficulty is the choice of the sampling period. From the viewpoint of discretization accuracy, it is obvious that smaller sampling period value will result in a more accurate model. Therefore, a limitation of the sampling period, say Δt_{limit}, should be chosen. If the value of Δt obtained in Problem 2 is greater than Δt_{limit}, then a new value of control steps will be chosen according to

$$N_{\text{new}} > \frac{N \cdot \Delta t}{\Delta t_{\text{limit}}} \tag{44}$$

4. Initial Feasible Solutions

Most NLP algorithms usually need an initial feasible solution to start the optimization process. In Problem 2, an initial feasible solution means a set of $u_1(0)$, $u_1(1),\ldots,$ $u_1(N-1)$, $u_2(0)$, $u_2(1),\ldots,$ $u_2(N-1)$, $u_3(0)$, $u_3(1),\ldots,$ $u_3(N-1)$, $u_4(0)$, $u_4(1),\ldots,$ $u_4(N-1)$, and Δt satisfying the constraints in equations (40) through (43). It is obvious that these solutions are not easy to be found since the constraints are highly nonlinear and coupled. Therefore, an approach based on GAs is developed to generate initial feasible solutions.

The theoretical basis of GAs is that chromosomes (solutions) better suited to the environment (evaluation) will have greater chance of survival and better chance of producing offspring. The evolutionary process is based primary on the mutation and crossover operators. The crossover operator combines the features of two parents to form two offspring. The mutation operator arbitrarily alters one or more genes of a selected chromosome, which increases the variability of the population. These two operators can further be divided into static and dynamic, where static ones do not change over the life of the population while dynamic ones are functions of time.

In the evolutionary process to generate initial feasible solutions of Problem 2, genetic operators such as real number encoding, arithmetical crossover and non-uniform mutation will be implemented. Moreover, dynamic mutation and cross-over, enlarged sampling space and ranking mechanism will also be used to expedite the convergence of the evolutionary process.

4.1. CHROMOSOME REPRESENTATIONS

How to encode a solution of the problem into a chromosome is a key issue for GAs. In this paper, since the parameters to be determined are all real, real number encoding technique will be used. Once the real-code chromosomes are used, the next step is to determine the number of genes in a chromosome. If the number of control steps is N, then the chromosomes will contains $(4N+1)$ genes, which denote $u_1(0)$, $u_1(1),\ldots,$ $u_1(N-1)$, $u_2(0)$, $u_2(1),\ldots,$ $u_2(N-1)$, $u_3(0)$, $u_3(1),\ldots,$ $u_3(N-1)$, $u_4(0)$, $u_4(1),\ldots,$ $u_4(N-1)$, and Δt, respectively. For a chromosome $\mathbf{x}=[x_1, x_2,\ldots, x_{4N+1}]$, one can find that the first $4N$ genes are within the ranges $[u_{i,\min}, u_{i,\max}]$ for $i=1, 2,\ldots, 4$, and the lower bound of the last gene is greater than zero.

4.2. CROSSOVER AND MUTATION OPERATIONS [7]

Arithmetical crossover and non-uniform mutation will be introduced in this section. For two real-coded chromosomes $\mathbf{x}_1$ and $\mathbf{x}_2$, the operation of arithmetical crossover is defined as follows:

$$\mathbf{x}_1' = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \tag{45}$$

$$\mathbf{x}_2' = \lambda\mathbf{x}_2 + (1-\lambda)\mathbf{x}_1 \tag{46}$$

where $\lambda \in (0, 1)$.

For a given parent $\mathbf{x}$, if a gene x_k of it is selected for mutation, then the resulting offspring will be randomly selected from one of the following two choices.

$$x_k' = x_k + \left(x_k^U - x_k\right)\cdot r \cdot \left(1-\frac{gen}{G}\right)^b \tag{47}$$

$$x_k' = x_k - \left(x_k - x_k^L\right)\cdot r \cdot \left(1-\frac{gen}{G}\right)^b \tag{48}$$

where x_k^U and x_k^L are the upper and lower bounds of x_k; r is a random number from [0,1]; gen is the generation number; G is the maximal generation number, and b is a parameter determining the degree of non-uniformity.

In addition to arithmetical crossover and non-uniform mutation, dynamic crossover and mutation probability rates will also be used for fast convergence. The crossover and mutation rates are defined as follows:

$$\text{crossover rate} = \exp\left(-\frac{gen}{G}\right) \tag{49}$$

$$\text{mutation rate} = \exp\left(-\frac{gen}{4G}\right) - 1 \tag{50}$$

4.3. ENLARGE SAMPLING SPACE

To generate good offspring, a method for selection of parents will be necessary. For selection methods that are developed based on regular sampling space, parents are replaced by their offspring soon after they give birth. In this manner, some fitter chromosomes will be worse than their parents. To cope with this problem, the selection method in this paper will be performed in enlarged sampling space, in which both parents and offspring have the same chance of competition for survival. Moreover, since more random perturbation is allowed in enlarged sampling space, high crossover and mutation will be allowed in the evolutionary process.

4.4. RANKING MECHANISM

In proportional selection procedure, the selection probability of a chromosomes is proportional to its fitness. This scheme exhibits some undesirable properties such as a few super chromosomes will dominate the process of selection in early generations. Moreover, competition among chromosomes will be less strong and a random search behavior will emerge in later generations. Therefore, the ranking mechanism is used in this paper to mitigate these problems, in which the chromosomes are selected proportionally to their ranks rather than actual evaluation values. This means that the fitness will be an integer number from 1 to P, where P is the population size. The best chromosomes will have a fitness value equal to P and the worst one will have a fitness value equal to 1.

5. Problem Solution

The details of the proposed method can be summarized as follows:

Algorithm A (Generating an initial feasible solution)

Step 1: Define the fitness function.

Step 2: Determine the population size, the crossover rate according to equation (49), and the mutation rate according equation (50).

Step 3: Produce an initial generation in a random way.

Step 4: Evaluate the fitness for each member of generation.

Step 5: With the crossover rate in Step 2, generate offspring according to equations (45) and (46), in which the ranking mechanism is used for selection of chromosomes.

Step 6: With mutation rate in Step 2, generate offspring according to Equations (47) and (48).

Step 7: Select the members of the new generation from the parents in the old generation and the offspring in Step 5 and Step 6 according to their fitness values.

Step 8: Repeat the procedure in Step 5 through Step 7 until the number of generations reaches a prescribed value.

Algorithm B (Solution of Problem 2)
Step 1: Choose a value of Δt_{limit} and an integer N.
Step 2: Formulate the TOMP problem as a NLP problem as shown in Problem 2 with the chosen value N.
Step 3: Use Algorithm A to find an initial feasible solution of Problem 2.
Step 4: Use any NLP algorithm to determine the minimum value of Δt in Problem 2 based on the initial feasible solution obtained in Step 3.
Step 5: If $\Delta t > \Delta t_{\text{limit}}$, then choose a new value of N according to e+quation (44) and go to Step 2. Otherwise, continue.
Step 6: $N \cdot \Delta t$ is the minimal traveling time.

6. Simulation Results

In this simulation example, the hovering quad-rotor helicopter will be moved from the initial configuration

$$(x(0),\ y(0),\ z(0),\ \phi(0),\ \theta(0),\ \psi(0)) = (0.3\text{m}, 0.4\text{m}, 1.5\text{m}, 0, 0, 0) \tag{51}$$

$$(\dot{x}(0),\ \dot{y}(0),\ \dot{z}(0),\ \dot{\phi}(0),\ \dot{\theta}(0),\ \dot{\psi}(0)) = (0, 0, 0, 0, 0, 0) \tag{52}$$

to the desired final configuration

$$(x(N), y(N),\ z(N),\ \phi(N),\ \theta(N),\ \psi(N)) = (0, 0, 0, 0, 0, 0) \tag{53}$$

$$(\dot{x}(N),\ \dot{y}(N),\ \dot{z}(N),\ \dot{\phi}(N),\ \dot{\theta}(N),\ \dot{\psi}(N)) = (0, 0, 0, 0, 0, 0) \tag{54}$$

in a time-optimal manner.

For convenience, the dynamical equations used in this example will be the same as those in [1, 2]. This means that the force to moment ratio C is chosen to be 1.3. The length between rotors and center of gravity l is chosen to be 21 cm The inertia matrix elements are calculated with a point mass analysis as I_x = 0.0142 kg m^2, I_y = 0.0142 kg m^2, and I_z = 0.0071 kg m^2. Mass of the helicopter is chosen to be 0.56 kg Gravity is selected as g = 9.81 m/s^2 and maximum thrust is taken as $F_{\max}$ = 10 N. Therefore, the inputs are limited to

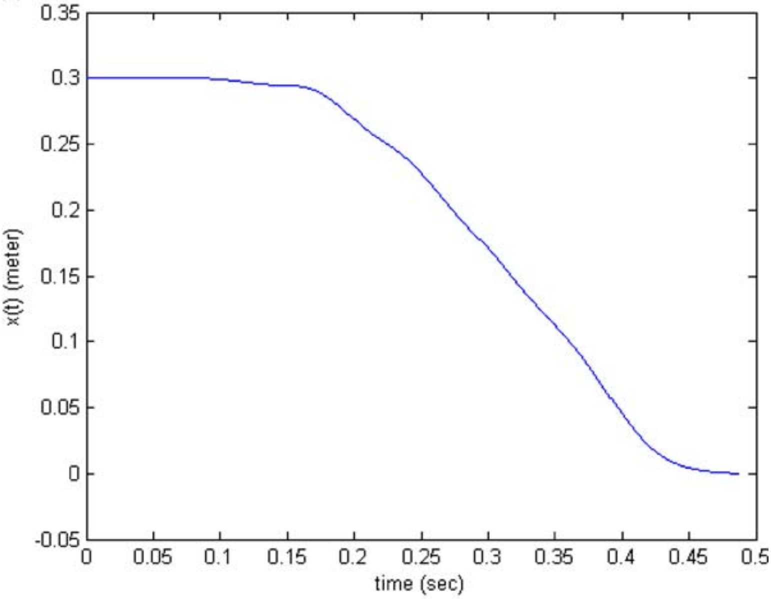
$$-10\,\text{N} \le u_1 \le 40\,\text{N} \tag{55}$$

$$-20\,\text{N} \le u_2 \le 20\,\text{N} \tag{56}$$

$$-20\,\text{N} \le u_3 \le 20\,\text{N} \tag{57}$$

$$-20\,\text{N} \le u_4 \le 20\,\text{N} \tag{58}$$

(a)



(b)

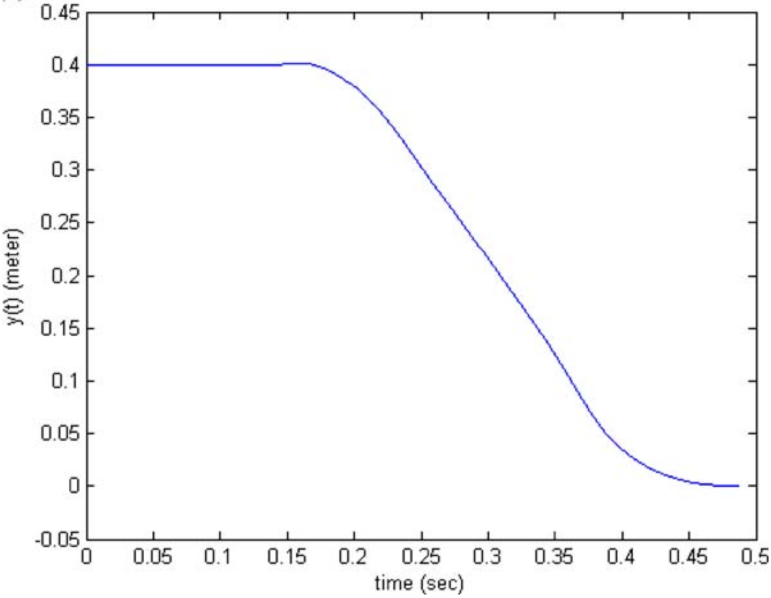


Figure 2. (a) Plot of $x(t)$ for $N = 21$. (b) Plot of $y(t)$ for $N = 21$. (c) Plot of $z(t)$ for $N = 21$. (d) Plot of $\phi(t)$ for $N = 21$. (e) Plot of $\theta(t)$ for $N = 21$. (f) Plot of $\psi(t)$ for $N = 21$. (g) The time-optimal path of the quad-rotor helicopter.

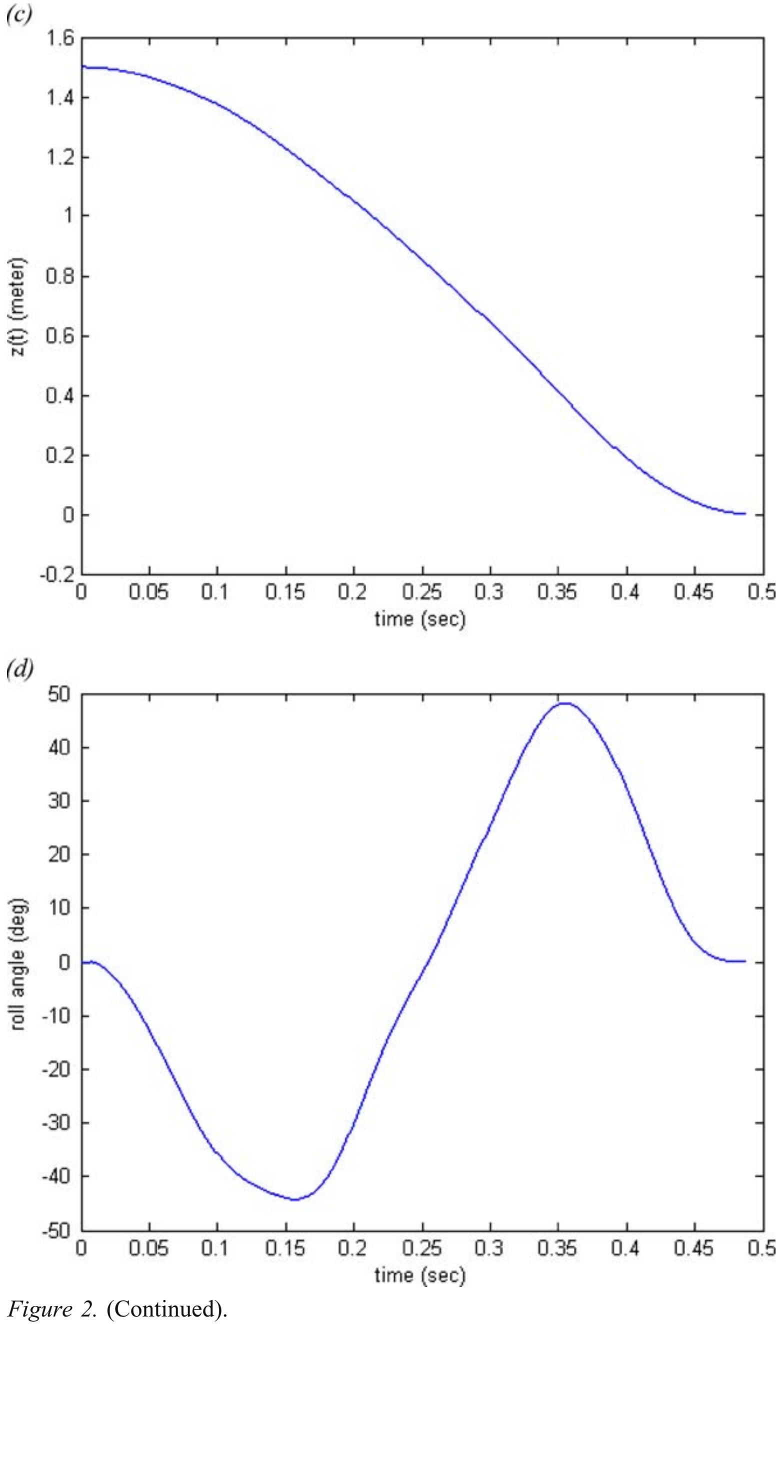
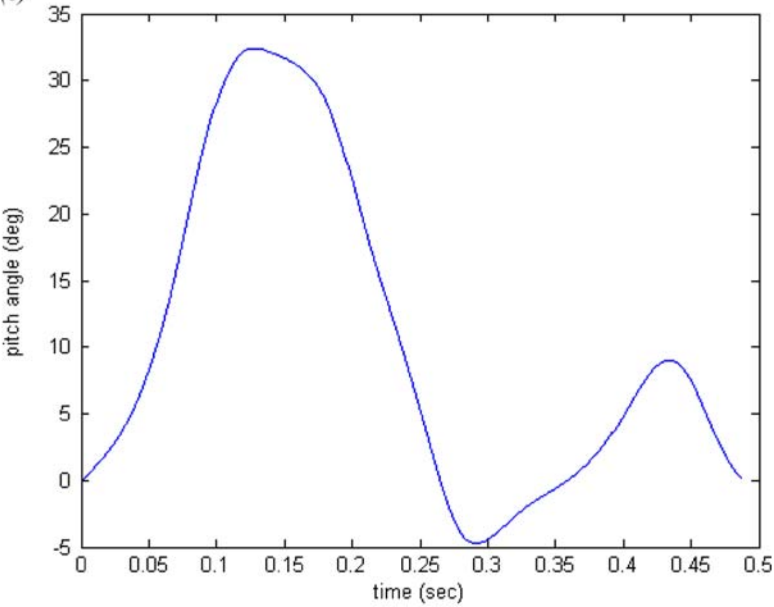


Figure 2. (Continued).

(e)



(f)

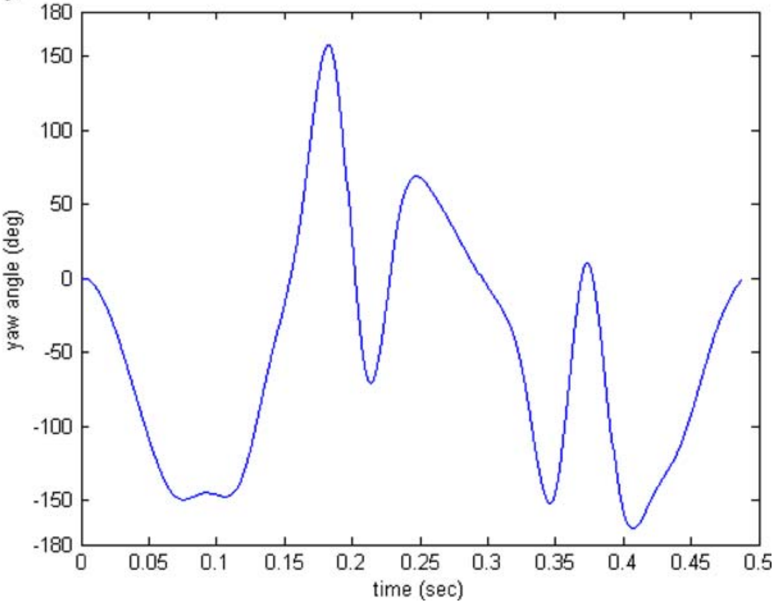


Figure 2. (Continued).

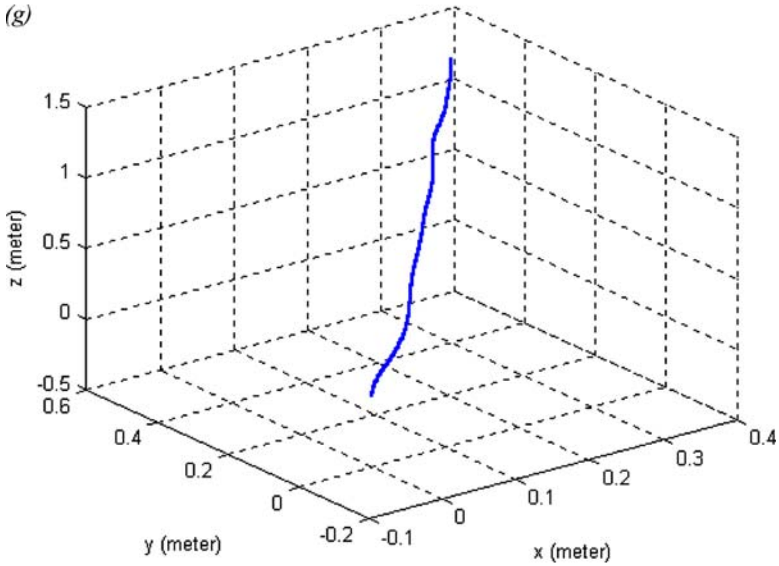


Figure 2. (Continued).

In applying Algorithm A to generate an initial feasible solution, the fitness function will be defined as

$$fitness = \frac{1}{1 + e^2 + \dot{e}^2} \tag{59}$$

where

$$e^2 = \Big[\big(x_f - x(N)\big)^2 + \big(y_f - y(N)\big)^2 + \big(z_f - z(N)\big)^2 + \big(\phi_f - \phi(N)\big)^2 + \big(\theta_f - \theta(N)\big)^2 + \Big(\psi_f - \psi(N)\Big)^2\Big] \tag{60}$$

and

$$\dot{e}^2 = \Big[\Big(\dot{x}(N)\Big)^2 + \big(\dot{y}(N)\big)^2 + \big(\dot{z}(N)\big)^2 + \big(\dot{\phi}(N)\big)^2 + \big(\dot{\theta}(N)\big)^2 + \big(\dot{\psi}(N)\big)^2\Big] \tag{61}$$

In applying GAs, the population size and the maximal generation number are chosen to be 50 and 100, respectively. During the simulation, the MATLAB Optimization Toolbox will be used, and the value of Δt_{limit} and the initial value of N are chosen to be 0.025 (s) and 11, respectively.

Applying Algorithm B with $N = 11$, the values of Δt and $N \cdot \Delta t$ are found to be 0.0459 (s) and 0.5029 (s), respectively. Since $\Delta t > \Delta t_{\text{limit}}$, the value of N will be updated according to equation (44), and the new value of N is chosen to be 21.

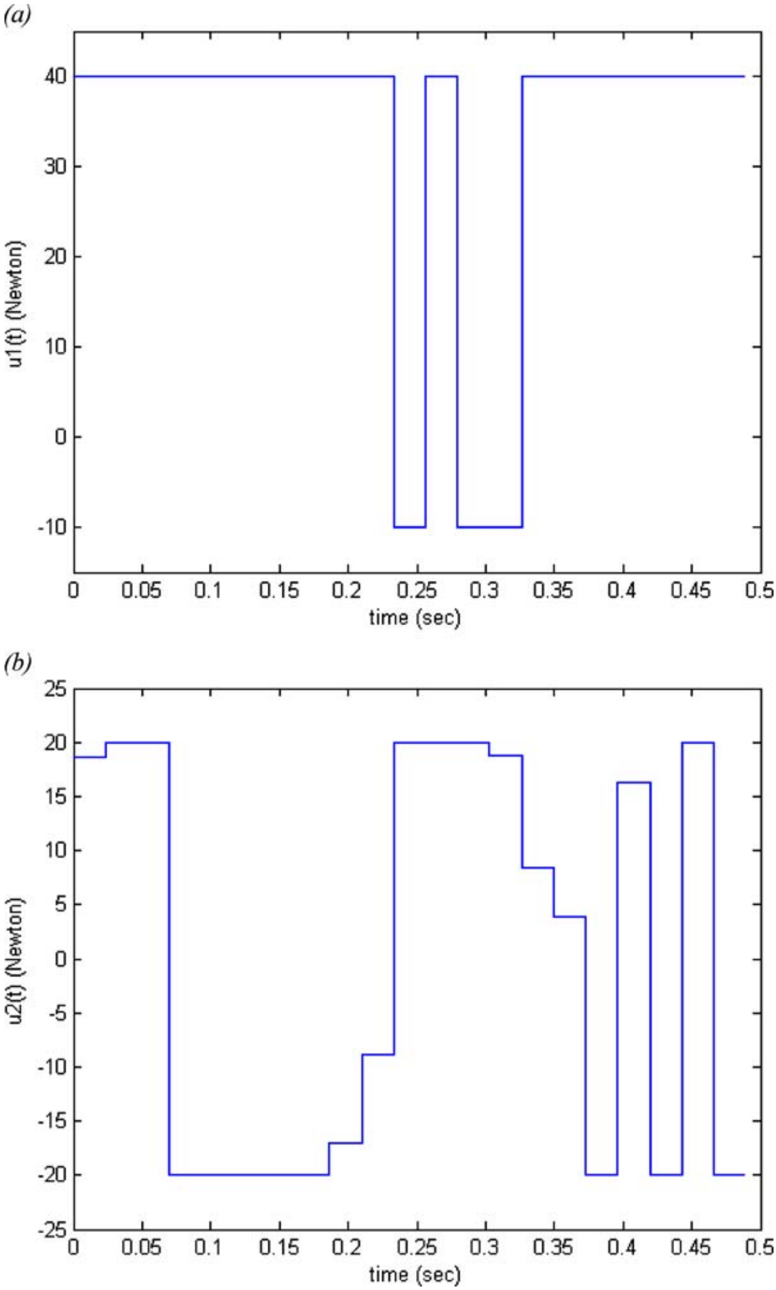
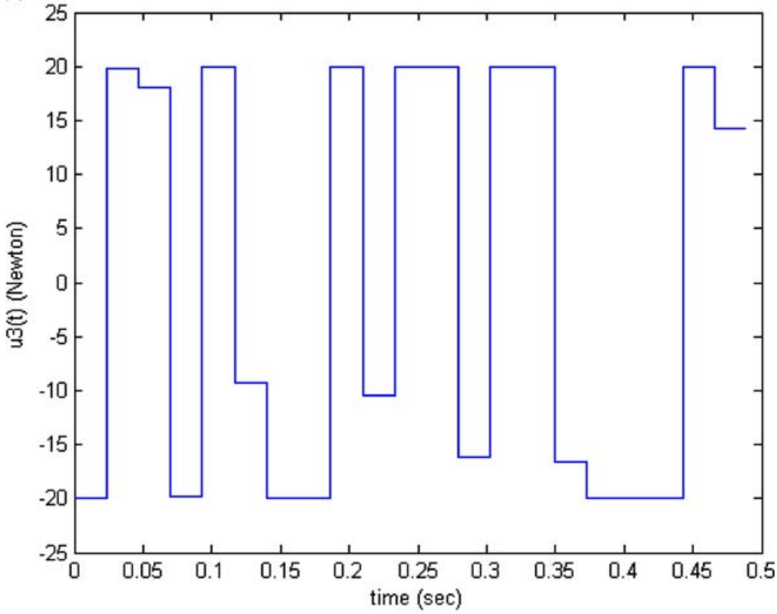


Figure 3. (a) Plot of $u_1(t)$ for $N = 21$. (b) Plot of $u_2(t)$ for $N = 21$. (c) Plot of $u_3(t)$ for $N = 21$. (d) Plot of $u_4(t)$ for $N = 21$.

(c)



(d)

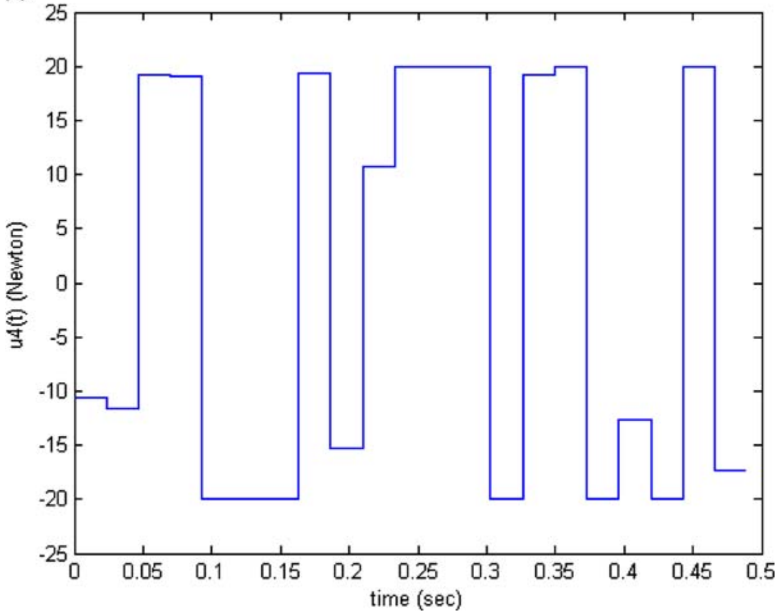


Figure 3. (Continued).

Applying Algorithm B with $N = 21$, the values of Δt and $N \cdot \Delta t$ are found to be 0.0233 (s) and 0.4893 (s), respectively, and the simulation results are shown in Figures 2 and 3.

7. Conclusions and Discussion

This paper presented a novel method to solve the TOMP problem of a hovering quad-rotor helicopter with four variant rotor speeds to change the lift forces. The first step is to transform the problem into a NLP problem by an iterative procedure. Then the optimization process is started to find the minimal traveling time. Different from the methods that utilizing the PMP, the major advantage of the proposed method is that one does not need to solve a set of highly nonlinear differential equations.

In solving a NLP problem, an initial feasible solution is usually needed. Therefore, a GA-based method is also proposed for generation of initial feasible solutions, in which dynamic mutation and crossover, enlarged sampling space, and ranking mechanism are used.

In the proposed method, one may ask why the optimal solution cannot be obtained by applying the GAs directly? From theoretical point of view, this task is possible to be done. However, in practice, the major difficulty is that the feasibility of the solution is very easy to be violated during the evolutionary process. This explains why the time-optimal solution cannot be obtained by applying the GAs directly.

It can be proved that the solution obtained satisfying the Kuhn–Tucker condition [13], which is a criterion used to check a local minimum. In addition, from the simulation results in Figure 3, one also can find that at least one of the four control inputs saturated at any time instant. This means that the solution is in the form of bang–bang control [11]. If a solution does not satisfy the Kuhn–Tucker condition or not in the form of bang–bang control, then one can conclude that the solution is not a global minimum. However, since the solution obtained meets both criterions simultaneously, it will be hard to determine whether the solution is globally optimal or not. More effort will be needed if one is interested in this issue.

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References

1. Altug, E., Ostrowski, J. P., and Taylor, C. J.: Quadrotor control using dual camera visual feedback, in: *Proc. of IEEE International Conference on Robotics and Automation*, 2003, pp. 4294–4299.

2. Altug, E., Ostrowski, J. P., and Mahony, R.: Control of a quadrotor helicopter using visual feedback, in: *Proc. of IEEE International Conference on Robotics and Automation*, 2002, pp. 72–77.
3. Bouabdallah, S., Murrieri, P., and Siegwart, R.: Design and control of an indoor micro quadrotor, in: *Proc. of IEEE International Conference on Robotics and Automation*, 2004, pp. 4393–4398.
4. Castillo, P., Dzul, A., and Lozano, R.: Real-time stabilization and tracking of a four-rotor mini rotorcraft, *IEEE Trans. Control Syst. Technol.* **12** (2004), 510–516.
5. Chern, J. S., Hong, Z. C., and Chen Y. T.: Approximate chattering arc for minimum time flight, *Acta Astronaut.* **35** (1995), 623–631.
6. Chung, T. S. and Wu, C. J.: A computationally efficient numerical algorithm for the minimum-time control problem of continuous systems, *Automatica* **28** (1992), 841–847.
7. Gen, M. and Cheng, R.: *Genetic Algorithm and Engineering Design*, Wiley, New York, 1997.
8. Hamel, T., Mahony, R., and Chriette, A.: Visual servo trajectory tracking for a four rotor VTOL aerial vehicle, in: *Proc. of IEEE International Conference on Robotics and Automation*, 2002, pp. 2781–2786.
9. Hamel, T., Mahony, R., Lozano, R., and Ostrowski, J.: Dynamic modeling and configuration stabilization for an X4-Flyer, in: *Proc. of International Federation of Automatic Control Symposium*, 2002.
10. Jan, Y. W. and Chiou, J. C.: Minimum-time spacecraft maneuver using sliding-mode control, *Acta Astronaut.* **54** (2003), 69–75.
11. Lewis, F. L.: *Optimal Control*, Wiley, New York, 1986.
12. Lozano, R., Castillo, P., Garcia, P., and Dzul, A.: Robust prediction-based control for unstable delay systems, in: *Proc. of IEEE International Conference on Decision and Control*, 2003, pp. 4014–4019.
13. Luenberger, D. G.: *Linear and Nonlinear Programming*, Addison-Wesley, Reading, 1973.
14. McKerrow, P.: Modelling the Draganflyer four-rotor helicopter, in: *Proc. of IEEE International Conference on Robotics and Automation*, 2004, pp. 3596–3601.
15. Mistler, V., Benallegue, A., and M'Sirdi, N. K.: Exact linearization and noninteracting control of a 4 rotors helicopter via dynamic feedback, in: *Proc. of IEEE International Workshop on Robot and Human Interactive Communication*, 2001, pp. 586–593.
16. Mokhtari, A. and Benallegue, A.: Dynamic feedback controller of Euler angles and wind parameters estimation for a quadrotor unmanned aerial vehicle, in: *Proc. of IEEE International Conference on Robotics and Automation*, 2004, pp. 2359–2366.
17. Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F.: *The Mathematical Theory of Optimal Processes*, Gordon and Beach, New York, 1986.
18. Pounds, P., Mahony, R., Hynes, P., and Roberts, J.: Design of a four-rotor aerial robot, in: *Proc. of Australasian Conference on Robotics and Automation*, 2002, pp. 145–150.
19. Soueres, P. and Laumond, J. P.: Shortest paths synthesis for a car-like robot, *IEEE Trans. Automat. Contr.* **41** (1996), 672–688.
20. Suter, D., Hamel, T., and Mahony, R.: Visual servo control using homography estimation for the stabilization of an X4-flyer, in: *Proc. of IEEE International Conference on Decision and Control*, 2002, pp. 2872–2877.
21. Wu, W., Chen, H., and Woo, P. Y.: Optimal motion planning for a wheeled mobile robot, in: *Proc. IEEE International Conference on Robotics and Automation*, 1999, pp. 41–46.
22. Zheng, Y. and Moore, P.: The design of time-optimal control for two-wheel driven carts tracking a moving target, in: *Proc. of IEEE International Conference on Decision and Control*, 1995, pp. 3831–3836.