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Maintenance optimization in failure-prone systems under imperfect preventive maintenance

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Abstract

In the majority of the existing preventive optimization models only costs related to maintenance actions are accounted for, while breakdown and operational costs are usually ignored. Liao et al. (J Intell Manuf 21(6):875–884, 2010) proposed a preventive maintenance model to deal with this shortcoming. In the present paper, we revisit and discuss the results provided in Liao et al. (2010) and point out some inconsistencies in the maintenance optimization model proposed therein. Accordingly, we develop a new maintenance optimization model and discuss some of its main cost components. Furthermore, optimality conditions are also formally investigated and a solution method is provided. Numerical experiments are conducted to illustrate the validity of the proposed approach and results are compared with those provided in the original paper by Liao et al. (2010).

Keywords Stochastic process · Reliability · Preventive maintenance · Imperfect repair · Optimization

Introduction

The growing importance and necessity of the maintenance activity in many industrial fields has led to an increasing interest in the development and implementation of maintenance optimization models for stochastic deteriorating systems. The original known results of replacements and maintenance policies were summarized and extensively discussed in Barlow and Proschan (1996). Since that, many papers and books appeared in the literature (Jardine et al. 2006; Wang and Pham 2006; Gertsbakh 2005; Nakagawa 2008). Many maintenance models have also been developed to include production scheduling constraints (Haoues et al. 2016; Aghezzaf et al. 2016; Zied et al. 2014; Liao 2013; Ben-Daya and Rahim 2001).

Traditional preventive maintenance (PM) models assume that the system after PM is either *as good as new* (i.e., perfect PM or replacement) or *as bad as old* (i.e., minimal repair). The more realistic and generalized approach is to assume that the system after PM lies somewhere between as good as new and as bad as old, which is called imperfect PM. Multiple maintenance optimization models in systems under imperfect PM have been reported in the literature. Malik (1979) introduced the age reduction PM model according to which a system becomes *younger* whenever it undergoes a PM. Nakagawa and his co-authors (Nakagawa 1986, 1988) [see also (Nakagawa and Mizutani 2009; Nakagawa 2008)] introduced the concept of hazard rate increased factor according to which the failure rate increases with PM. Lin et al. (2000) the hybrid failure rate approach which combines failure rate adjustment and age reduction approaches. This hybrid model is used as a modeling approach in Lin et al. (2001) for repairable systems with two categories of failure modes, namely the maintainable and non-maintainable failure modes. Only the system failure rate corresponding to the maintainable failure mode is altered whenever PM actions are performed. In line with the work of Lin et al. (2001), El-Ferik and Ben-Daya (2006) developed an age-based hybrid model for systems with two categories of failure modes and discussed the existence and uniqueness of optimal imperfect PM policy. The authors in Khatab et al. (2017), Khatab (2015) proposed a PM optimization model for a system subject to stochastic degradations that undergoes PM whenever its reliability reach a given threshold. Zhang et al. (2015) proposed an imperfect maintenance model that is applicable to systems whose sensor information can be modeled by stochastic processes. Liu et al. (2012) proposed a three-step approach selecting the best imperfect maintenance model for a given situation: a goodness-of-fit

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test, a Bayesian approach for selecting the most adequate model among several competitive candidates, and a framework that incorporates the model selection results into the PM decision making. Recently, Zhang and Xie (2017) developed an ameliorated improvement factor model for imperfect maintenance and its goodness of fit to give practical grounds to the imperfect maintenance model. One common feature of most applications of the imperfect maintenance models is that they deal only with new systems.

As pointed out by Liao et al. (2010), the majority of the existing PM optimization models only consider costs related to maintenance actions. However, the breakdown due to a system' failure may induce economic losses. Furthermore, operational conditions of the system becomes worse with time and may then impact system reliability, product quality and delivery time. Consequently, both breakdown and operational costs must be accounted for in maintenance decision making. To deal with this shortcoming, Liao et al. (2010) integrated a sequential imperfect maintenance policy into a maintenance model for a continuously monitored system subject to degradation. The hybrid hazard rate approach of Lin et al. (2000) is used to model imperfect PM. A maintenance optimization model is then proposed where breakdown and operational cots are explicitly accounted for. The system undergoes PM whenever its reliability reaches a given threshold level R_0. In the case where the system fails before reaching the threshold R_0, a minimal repair is then carried out. After a number N of PM cycles, the system is replaced by a new one. The objective of the maintenance optimization problem consists then on finding the joint optimal reliability threshold R_0 together with the optimal number N of PM cycles to minimize the total expected total cost rate in the infinite time span.

The present paper revisits the maintenance model developed in Liao et al. (2010). It aims to remedy some inconsistencies reported in Liao et al. 2010. A new maintenance optimization model is then developed and fully discussed. Furthermore, optimality conditions are also discussed through two propositions and their proofs to show the existence and uniqueness of an optimal solution. These results are then used to build up a fix-and-optimize numerical procedure to compute the optimal solution of the optimization problem. The present work assume cost structures adopted in Liao et al. (2010) but is still general to deal with other cost structures. Using the data set in Liao et al. 2010, numerical experiments are then conducted and fully discussed. The results of these experiments are compared to those obtained in Liao et al. (2010). The overall results obtained demonstrate the accuracy and the validity of the proposed maintenance optimization model.

The remainder of the paper is organized as follows. Acronym, notations and the main working assumptions are listed in section "Acronym, notation and assumptions". Section "Maintenance optimization model" is devoted to the problem definition and modeling. The fundamental principles and the relevant cost components of this decision-making problem are developed and thoroughly discussed. The resulting optimization problem is proposed and analytically discussed in section "The mathematical optimization model". This section presents the main theoretical contributions. Section "Solution method" presents a numerical procedure to solve the proposed optimization model. The validity and accuracy of the proposed approach is demonstrated on a comprehensive test case in section "Test case". Conclusions and future works are drawn in section "Conclusion". Proofs of the two propositions and the lemma are reported in appendices A, B and C.

Acronym, notation and assumptions

Acronym:

PM: Preventive maintenance

Notation:

N	Number of PM cycles, a decision variable
$\lambda_k(t)$	Failure rate of the system during the kth PM cycle
X_k	Calendar age of the system right before the kth PM
Y_k	Effective age of the system right before the kth PM
T_k	Duration of the the kth PM cycle
R_0	Reliability threshold, a decision variable
a_k, b_k	The respective age reduction and adjustment coefficients of the kth PM
C_r	Replacement cost
C_p	Imperfect PM cost
C_m	Minimal repair cost
C_i	Cost component of the operational cost ($i = 0, 1, 2$)
C_b	Breakdown cost
δ_k	Total operational cost incurred during the kth PM cycle
$\mathbb{E}[MC]$	Expected total maintenance cost
$\mathbb{E}[OC]$	Expected total operational cost
$\mathbb{E}[BC]$	Expected total breakdown cost
$\mathcal{J}(R_0, N)$	Expected total cost rate, the objective function
$\mathbb{E}[\mathcal{C}]$	Expected total cost in a replacement cycle
$\mathbb{E}[\mathcal{T}]$	Expected duration of a replacement cycle

Assumptions

1. The system is new at the beginning of each replacement cycle.

2. the planning horizon is infinite.
3. The system is failure-prone and its lifetimes are characterized by a continuous and increasing failure rate.
4. The times spent in PM, minimal repair and replacement are negligible.

Maintenance optimization model

Let us consider a system with stochastic deterioration. As in Liao et al. (2010), the system is designed to operate for N PM cycles. At the end of the Nth cycle, the system is replaced by a new and identical one. The system undergoes PM whenever its reliability reaches a threshold level R_0. In the case where the system fails before reaching the threshold R_0, a minimal repair is then carried out. A minimal repair only restores the system to a working condition but does not improve the system's reliability (Wang and Pham 2006). PM actions are imperfect and modeled according to the hybrid hazard rate model proposed by Lin et al. (2000). The objective is to find the optimal values of two decision variables, namely the reliability threshold R_0 and the number N of PM including the replacement, which minimize the expected total cost rate $\mathcal{J}(R_0, N)$ over an infinite time horizon.

According to the above maintenance strategy, the system is subjected to a regenerative process starting and ending at the instants of the replacement operations. It follows from the theory of renewal reward processes that the long-run expected total cost per unit of time $\mathcal{J}(R_0, N)$ is the average total cost $\mathbb{E}(\mathcal{C})$ in a replacement cycle divided by the average length $\mathbb{E}(\mathcal{T})$ of that cycle:

$$\mathcal{J}(R_0, N) = \frac{\mathbb{E}[\mathcal{C}]}{\mathbb{E}[\mathcal{T}]}. \tag{1}$$

The expected total cost $\mathbb{E}[\mathcal{C}]$ in a cycle is defined as:

$$\mathbb{E}[\mathcal{C}] = \mathbb{E}[MC] + \mathbb{E}[OC] + \mathbb{E}[BC], \tag{2}$$

where $\mathbb{E}[MC]$, $\mathbb{E}[OC]$ and $\mathbb{E}[BC]$ refer, respectively, the expected total maintenance/replacement cost, the expected total operational cost and the expected total breakdown cost. In what follows, these costs are fully defined and discussed. Their respective expressions are revisited, especially those corresponding to maintenance operations and breakdown costs.

Maintenance cost

Cost of a replacement is denoted as C_r, and costs of PM and minimal repair are, respectively, denoted by C_p and C_m. PM actions are imperfect and modeled according to the hybrid hazard rate model (Lin et al. 2000). The principle of this imperfect PM model is shown in Fig. 1 where X_k $(k = 1, 2, \ldots, N)$ are the instants where the PM actions (including the final replacement) are performed on the system. In this figure, $T_k = X_k - X_{k-1}$ represents the time interval between the $(k-1)$th and the kth PM actions, with $X_0 = 0$ meaning that the first PM instant is measured from time 0 when the system was new. Accordingly, one may observe that $X_k = \sum_{i=1}^{k} T_i$ represents the time to perform the kth PM.

To evaluate the failure rate of the system subjected to PM, let us first evaluate the effective age, hereafter denoted by Y_k, of the system right before it undergoes the kth PM. The effective age Y_1 corresponds to the time $T_1 = X_1$ to reach the instant of the first PM. The effective age Y_2 is obtained as $Y_2 = a_1 Y_1 + T_2$ where a_1 is the age reduction coefficient of the first PM, and so on. We then have the following generalized recursive equation:

$$Y_k = a_{k-1} Y_{k-1} + T_k, \tag{3}$$

where a_k $(k \geq 1)$ stands for the age reduction coefficients such that $0 \leq a_1 < a_2 < \cdots < 1$ with $a_0 = 0$ and $Y_0 = 0$.

Now let us evaluate the failure rate of the system in each PM cycle, i.e. within the time interval $[X_{k-1}, X_k)$. In the first PM cycle, the failure rate is equal to $\lambda_1(t)$. After the first PM $(k = 1)$ the failure rate becomes $b_1\lambda_1(a_1 Y_1)$ when it was $\lambda_1(Y_1)$ right before the first PM. Thus in the PM cycle $[X_1, X_2)$, the failure rate is expressed as $b_1\lambda_1(a_1 Y_1 + t)$ for all $t \in [0, X_2 - X_1)$ (i.e. $t \in [0, T_2)$) where b_1 is the adjustment coefficient of the PM. Following the previous reasoning, the failure rate after the second PM $(k = 2)$ becomes $b_2 b_1 \lambda_1(a_2 Y_2)$ when it was $b_1\lambda_1(Y_2)$ right before the second PM. It follows that the failure rate of the system during the third PM cycle, i.e during the time interval $[0, T_3)$ is $b_2 b_1 \lambda_1(t + a_2 Y_2)$ for all $t \in [0, T_3)$. More generally, the failure rate $\lambda_k(t)$ of the system during the kth PM cycle (i.e. before performing the kth PM) is:

$$\lambda_k(t) = B_{k-1}\lambda_1(t + a_{k-1}Y_{k-1}) \text{ if } k > 1 \text{ and } t \in [0, T_k), \tag{4}$$

where $B_{k-1} = \prod_{i=1}^{k-1} b_i$ with $B_0 = 1$ and $1 \leq b_1 \leq b_2 \leq \cdots$.

Since PM times T_k $(k = 1, 2, \ldots)$ correspond to instants where the system's reliability reaches the threshold level R_0 it follows that:

$$R_0 = \exp\left(-B_{k-1}\int_{a_{k-1}Y_{k-1}}^{Y_k} \lambda_1(t)\mathrm{d}t\right), \tag{5}$$

which implies that:

$$\ln(R_0) = -B_{k-1}\int_{a_{k-1}Y_{k-1}}^{Y_k} \lambda_1(t)\mathrm{d}t. \tag{6}$$

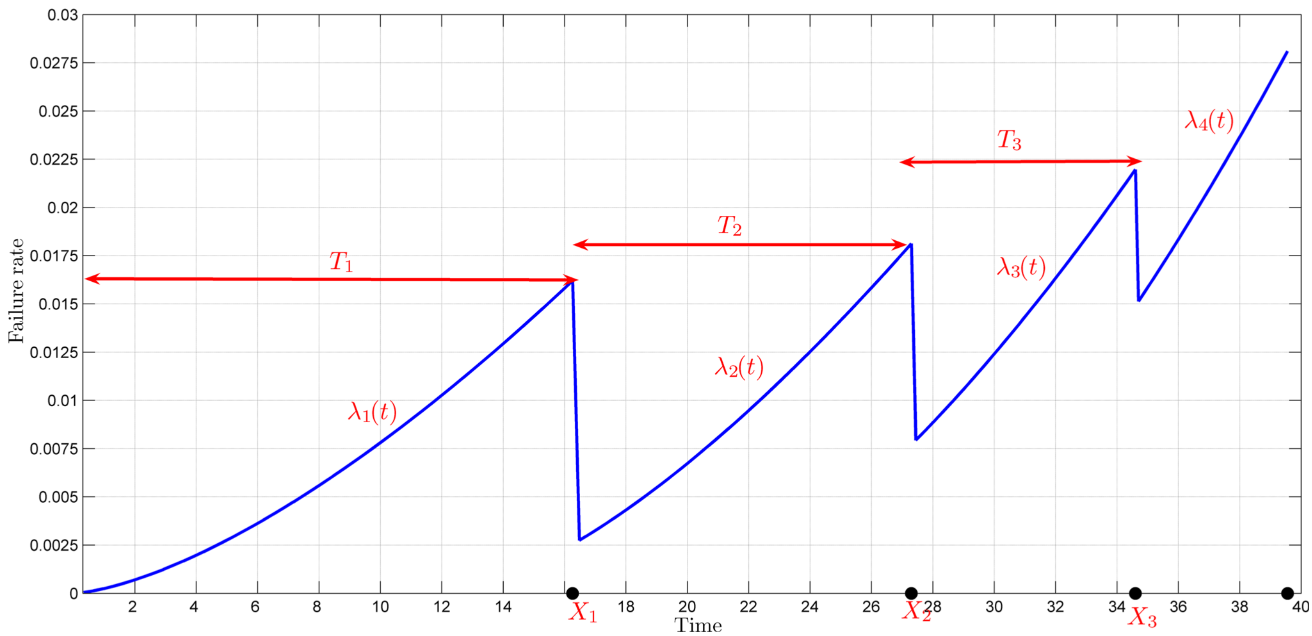

Fig. 1 Illustration of the hybrid hazard rate approach of imperfect PM

The expected total maintenance cost $\mathbb{E}[MC]$ incurred for minimal repair cots and PM including replacement costs during a renewal cycle is:

$$\mathbb{E}[MC] = C_r + (N-1)C_p + C_m \sum_{k=1}^{N} \left(B_{k-1} \int_{a_{k-1}Y_{k-1}}^{Y_k} \lambda_1(t)\mathrm{d}t \right), \quad (7)$$

where $B_{k-1} \int_{a_{k-1}Y_{k-1}}^{Y_k} \lambda_1(t)\mathrm{d}t = \int_0^{T_k} \lambda_k(t)\mathrm{d}t$ represents the expected number of failures occurring during the kth PM cycle. From the result of Eq. (6), the expected total maintenance cost becomes:

$$\mathbb{E}[MC] = C_r + (N-1)C_p - NC_m \ln(R_0). \quad (8)$$

Operational cost

In line with the work in Liao et al. (2010), the operational cost structure used to describe the cost incurred during the operating process is denoted as $C_o(k,t)$ and composed of three cost components such that:

$$C_o(k,t) = C_0 + k \cdot C_1 + t \cdot C_2, \quad (9)$$

where C_0 is a fixed cost rate, $k \cdot C_1$ is a cost rate related to the number of PM performed on the system, and $t \cdot C_2$ is a cost rate related to the cumulative operating time of the system.

According to the cost structure defined in Eq. (9), the expected total operational cost $\mathbb{E}[OC]$ incurred during the overall replacement cycle is computed as:

$$\mathbb{E}[OC] = \sum_{k=1}^{N} \delta_k, \quad (10)$$

where $\delta_k = \int_0^{T_k} C_o(k,t)\mathrm{d}t$ is the expected total operational cost incurred during the kth PM cycle.

Following the cost function in Eq. (9) together with the fact that $T_k = Y_k - a_{k-1}Y_{k-1}$ [see Eq. (3)], the expected total operational cost $\mathbb{E}[OC]$ can also equivalently be computed as:

$$\mathbb{E}[OC] = \sum_{k=1}^{N} \left(C_0 + kC_1 + \frac{C_2}{2}(Y_k - a_{k-1}Y_{k-1}) \right)(Y_k - a_{k-1}Y_{k-1}). \quad (11)$$

Breakdown cost

In contrast to the computation approach used in Liao et al. (2010), the expected total breakdown cost $\mathbb{E}[BC]$ is charged not only once per PM cycle but rather every time the system undergoes a maintenance action being a minimal repair or a PM including a replacement. Therefore, if one consider the kth PM cycle, the corresponding breakdown cost is computed as the product of breakdown cost by the sum of the expected number of failures occurring in that PM cycle and a PM performed at the end of the kth PM cycle . Accordingly, we suggest the following total expected breakdown cost $\mathbb{E}[BC]$ incurred during a renewal cycle such that:

$$\mathbb{E}[BC] = C_b \left(N + \sum_{k=1}^{N} \int_0^{T_k} \lambda_k(t)\mathrm{d}t \right), \tag{12}$$

where the term NC_b refers to the breakdown cost du to the sum of $(N-1)$ imperfect repair and a perfect repair (replacement activity at the end of the Nth PM cycle), and the term $C_b \sum_{k=1}^{N} \int_0^{T_k} \lambda_k(t)\mathrm{d}t$ refers to the breakdown cost as a consequence of minimal repair carried out during N PM cycles.

Combining the results of Eqs. (4)–(6), the total expected breakdown cost is computed as:

$$\mathbb{E}[BC] = NC_b(1 - \ln(R_0)). \tag{13}$$

According to Eqs. (8), (11) and (13), the expected total cost $\mathbb{E}[\mathcal{C}]$ in a replacement cycle is then computed as:

$$\begin{aligned}\mathbb{E}[\mathcal{C}] &= C_r + (N-1)C_p - NC_m \ln(R_0) \\ &\quad + NC_b(1-\ln(R_0)) + \sum_{k=1}^{N} \Bigg(C_0 + kC_1 \\ &\quad + \frac{C_2}{2}(Y_k - a_{k-1}Y_{k-1}) \Bigg) (Y_k - a_{k-1}Y_{k-1}).\end{aligned} \tag{14}$$

Length of a replacement cycle $\mathbb{E}(\mathcal{T})$

At the beginning of a replacement cycle, a new system is put into operation and will undergo a series of N PM cycles. At the instant of the Nth PM, the system is renewed. The length of this replacement cycle is therefore the sum of the times between the PM actions. We then have:

$$\mathbb{E}(\mathcal{T}) = \sum_{k=1}^{N} T_k, \tag{15}$$

From Eq. (3) we have that $T_k = Y_k - a_{k-1}Y_{k-1}$. Therefore, the expected length $\mathbb{E}(\mathcal{T})$ of a replacement cycle becomes:

$$\mathbb{E}(\mathcal{T}) = \sum_{k=1}^{N-1} (1-a_k)Y_k + Y_N. \tag{16}$$

The mathematical optimization model

The optimization problem considered is to find the decision variables defining the optimal joint maintenance strategy (R_0^*, N^*) which minimize the expected total cost rate $\mathcal{J}(R_0, N) = \frac{\mathbb{E}[\mathcal{C}]}{\mathbb{E}[\mathcal{T}]}$. According to Eqs. (14) and (16), the cost rate function $\mathcal{J}(R_0, N)$ proposed is then computed as:

$$\begin{aligned}\mathcal{J}(R_0, N) &= \frac{C_r + (N-1)C_p - NC_m \ln(R_0) + NC_b(1-\ln(R_0))}{\sum_{k=1}^{N-1}(1-a_k)Y_k + Y_N} \\ &\quad + \frac{\sum_{k=1}^{N} \left(C_0 + kC_1 + \frac{C_2}{2}(Y_k - a_{k-1}Y_{k-1}) \right)(Y_k - a_{k-1}Y_{k-1})}{\sum_{k=1}^{N-1}(1-a_k)Y_k + Y_N}.\end{aligned} \tag{17}$$

The optimization problem in Eq. (17) contains a continuous decision variable R_0 in addition to a discrete decision variable N. Unfortunately, the optimal solutions that minimize Eq. (17) are in general difficult to obtain analytically even for simple lifetime distributions. Nevertheless, in what follows, some useful properties are derived to build a fix-and-optimize numerical method to minimize Eq. (17). Two propositions will be developed to help in building the proposed solution method. The first proposition examines conditions under which the optimal N exists when the reliability threshold R_0 is fixed. The second proposition focuses on the evaluation of the optimal value of R_0 in the case where te number N of PM cycles is fixed.

Proposition 1 (N^* when R_0 is fixed) *For fixed values of R_0, if $\frac{\delta_{k+1}}{T_{k+1}} > \frac{\delta_k}{T_k}$ $(k \geq 1)$, the optimal number N^* of PM is unique, finite and it is solution of:*

$$\frac{1}{T_{N^*}} < \frac{\mathcal{J}(R_0, N^*)}{C_p + C_b - (C_m + C_b)\ln(R_0)} < \frac{1}{T_{N^*+1}}. \tag{18}$$

Proof See Appendix A. □

In the above proposition, the ratio $\frac{\delta_k}{T_k}$ represents the total expected operational cost rate incurred during the kth PM cycle. The assumption according to which $\frac{\delta_{k+1}}{T_{k+1}} > \frac{\delta_k}{T_k}$ $(k \geq 1)$ means that the total expected operational cost rate increases by the increasing of the number of PM. This assumption is reasonable since the operational cost increases by the increasing of both operational time and the number of imperfect PM performed during the replacement cycle.

The result of the following Lemma 1 will be used later in Proposition 2. The lemma computes the partial derivative $\frac{\partial Y_k}{\partial R_0}$.

Lemma 1 *The partial derivative $\frac{\partial Y_k}{\partial R_0}$ is given by:*

$$\frac{\partial Y_k}{\partial R_0} = \frac{-1}{B_{k-1}\lambda_1(Y_k)R_0}, \tag{19}$$

where we recall that $B_k = \prod_{i=1}^{k} b_i$ with $B_0 = 1$.

Proof See Appendix B □

Proposition 2 (R_0^* *when N is fixed) For fixed values of N, the optimal value of the reliability threshold R_0^* is solution of:*

$$
\begin{aligned}
\ln(R_0) &= \frac{(C_r - C_p) + N(C_p + C_b}{N(C_b + Cm)} \\
&+ \frac{\sum_{k=1}^{N} (kC_1 + C_2(Y_k - a_{k-1}Y_{k-1}))(Y_k - a_{k-1}Y_{k-1})}{N(C_b + Cm)} \\
&- \left(1 - \frac{\sum_{k=1}^{N} \frac{(kC_1 + C_2(Y_k - a_{k-1}Y_{k-1}))(a_{k-1}b_{k-1}\lambda_1(Y_k) - \lambda_1(Y_{k-1}))}{B_{k-1}\lambda_1(Y_{k-1})\lambda_1(Y_k)}}{N(C_b + Cm)}\right) \\
&\times \left(\frac{\sum_{k=1}^{N-1} (1 - a_k)Y_k + Y_N}{\sum_{k=1}^{N-1} \frac{1-a_k}{B_{k-1}\lambda_1(Y_k)} + \frac{1}{B_{N-1}\lambda_1(Y_N)}}\right) \qquad (20)
\end{aligned}
$$

Proof See Appendix C □

In the case where N is fixed, Proposition 1 provides a necessary condition to be satisfied for any optimal values of N. Proposition 2 also provides a necessary condition that R_0 must satisfy when N is fixed. In the following section, the results of these propositions will be used to build up a fix-and-optimize numerical solution method.

Solution method

A general approach for minimizing Eq. 17 based on the above propositions is as follows. For each value of R_0 solution of Eq. 20 from Proposition 2 is calculated as a function of the remaining decision variable N. This solution is unique because the logarithm function is monotonous. Finally, using Eq. (18) from Proposition 1, a unique value of N can be computed. An optimal solution is the solution (R_0, N) that has the least expected cost rate value. The pseudo-code of the proposed algorithm is given below.

Algorithm 1 Pseudo-code for the minimization of Eq. (17)

1: Input data: $\lambda_1(t)$, C_r, C_p, C_m, C_b, C_0, C_1, C_2, and coefficients a_k and b_k.
2: Set ΔR: the step size used to loop through values of R_0.
3: Set $R_0 = 1$.
4: **while** $R_0 > 0$ **do**
5: Using Eq. (20) of Proposition 2, compute R_0 as a function of N.
6: Using Eq. (18) of Proposition 1, solve for N and compute the value of the objective function $\mathcal{J}^*(R_0, N)$.
7: Store the current values of $\mathcal{J}^*(R_0, N)$, R_0, and N
8: $R_0 = R_0 - \Delta R$
9: **end while**
10: Select the lowest value of $\mathcal{J}^*(R_0, N)$ and its corresponding decision variables R_0 and N from all the data stored in Step (7).
11: Set $R_0^* = R_0$, $N^* = N$, and $\mathcal{J}^*(R_0, N) = \mathcal{J}(R_0^*, N^*)$

In contrast to the initial work by Liao et al. (2010), in our solution approach no restrictions are made on the search intervals of both decision variables R_0 and N. Indeed, in the above algorithm, no upper limit had to be specified for the search for the optimal number of PM cycles, nor a restricted range of reliability threshold had to be imposed for the search of the optimal reliability threshold. The computation of the optimal solution is more general and formally performed with respect to the results of Propositions (1) and (2).

The following section investigates a test case to illustrate our proposed approach. The above algorithm is implemented and the obtained mathematical model is solved in order to derive maintenance decisions. The results are compared with those obtained in the original paper (Liao et al. 2010).

Test case

In this section, numerical experiments are conducted. To compare our results to those obtained by Liao et al. (2010), all the input data are the same as those used in Liao et al. (2010). Accordingly, the proposed approach is applied in a system whose lifetimes are Weibull distributed with the shape parameter β and the scale parameter η set, respectively, to $\beta = 5$ and $\eta = 200$. Its corresponding failure rate $\lambda_1(t)$ is then given as:

$$\lambda_1(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}.$$

The other input data are set as follows. Costs related to maintenance, namely replacement, imperfect PM, minimal repair costs and breakdown cost are set to $C_r = 5000$, $C_p = 1200$, $C_m = 2000$ and $C_b = 500$, respectively. The operational cost components are set to $C_0 = 4$, $C_1 = 1.2$ and $C_2 = 0.05$. Finally, the values of the adjustment factors b_k and the age reduction factors, k_k $(k = 1, 2, \cdots)$ of the imperfect PM model are obtained from the following formula:

$$a_k = \frac{2k}{5k+9}, \quad b_k = \frac{13k+3}{11k+4}.$$

Experiment #1

In this experiment, we assume that the system is required to operate with a level of reliability not less than $R_0 = 90\%$ (i.e., the system must undergo PM whenever its reliability reaches the threshold value $R_0 = 90\%$). In this case, the system reliability level is an input parameter set by the decision-maker rather a decision variable. The objective of the maintenance decision maker is then to find the optimal value of the number N.

Figure 2 depicts the expected total cost rate $\mathcal{J}(R_0, N)$ versus the number N of PM cycles. The optimal value is found to be $N^* = 5$ meaning that the system undergoes 4 imperfect PM actions and then is replaced at the end of the 5th PM cycle. The optimal solution suggests to operate the system under the following maintenance plan. This maintenance plan is different from that found in Liao et al. (2010)

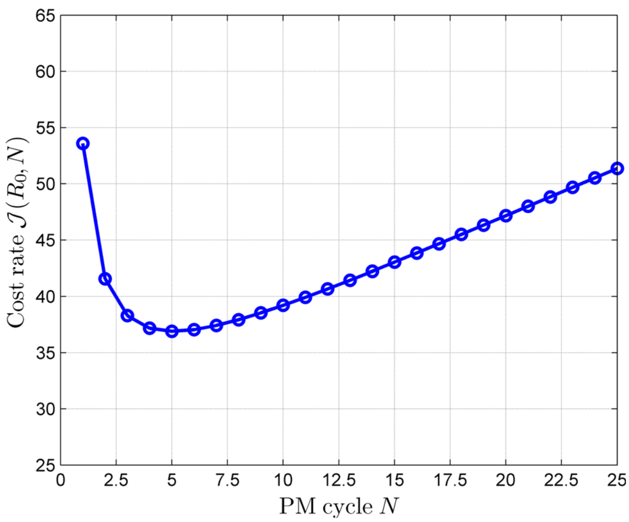

Fig. 2 Total expected cost rate $\mathcal{J}(R_0, N)$ versus the number of PM cycles: case of Experiment #1

Table 1 Optimal PM intervals: case of Experiment #1

k	1	2	3	4	5
T_k	127.52	107.67	96.67	89.31	83.76

which suggested an optimal maintenance of $N = 8$ PM cycles, resulting then in a difference of three additional PM cycles.

The optimal PM intervals are given in Table 1. The first PM is performed at time $T_1 = 127.52$, i.e. 127.52 time units after the system is put into operation. The second PM is performed $T_2 = 107.67$ time units after the first one and so on. The replacement of the system is done $T_5 = 83.76$ time units after the 4th PM. From Table 1, one may observe that the time intervals between PM and replacement actions decrease as the number of PM performed increases. This result is due to the fact that a PM action not only reduces the effective age of the system but also changes the slope of the failure rate. Thus, the more the system ages the more frequent the PM actions are. This results in an non-periodic maintenance strategy which is indeed practically reasonable and more suitable for deteriorating industrial systems. The optimal expected total cost rate induced by the suggested maintenance plan is $\mathcal{C}(R_0, N^*) = 36.90$ (see also Fig. 2).

Experiment #2

In this experiment, the maintenance optimization problem is solved with the same input data used in Experiment #1 except that the reliability level R_0 is now considered as a decision variable rather than a predetermined value. Applying the algorithm in section "Solution method" with $\Delta R0 = 1\%$,

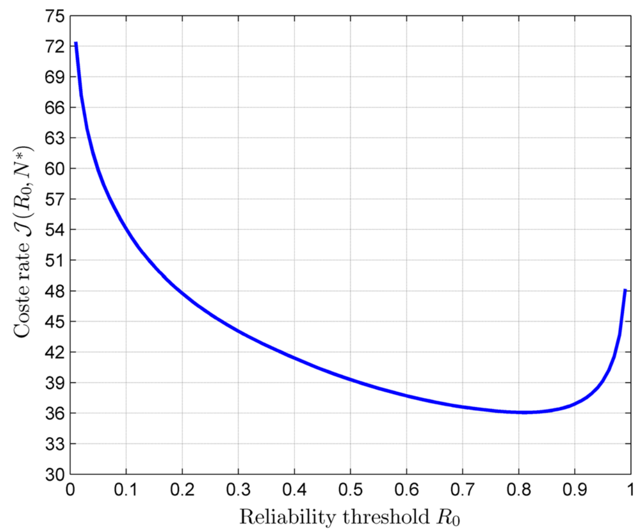

Fig. 3 Total expected cost rate $\mathcal{J}(R_0, N^*)$ versus the reliability threshold R_0: case of Experiment #2

Fig. 3 shows the variation of the minimum expected total cost rate $\mathcal{J}(R_0, N^*)$ obtained for different values of the reliability threshold R_0. One may observe that Fig. 3 is convex and so that it is different from Fig. 2 in Liao et al. (2010). In our case, one may observe the explicit trade-off between the total expected cost rate and the reliability threshold. Indeed, in the case where the reliability threshold is set to a high value, this implies that more frequent PM actions need to be carried out. The resulting total expected cost rate becomes then more important. Similarly, a low value of the reliability threshold allows a long and an uninterrupted use of the system but with, however, an increased risk of failures which increase the number of minimal repair, and by the way increases the expected total cost rate. An appropriate reliability threshold is then required to ensure a balance between the total expected costs of maintenance, breakdown in addition to the total expected operational cost.

The optimal values found for the decision variables R_0 and N are, respectively, 81% and 5. The resulting optimal maintenance plan consists to perform 4 PM actions and a replacement at the instant of the 5th PM. Time intervals between consecutive maintenance actions are shown in Table 2. Once again, from Table 2, one may also observe that the time intervals to PM actions are decreasing as the PM number increases, due to the hybrid hazard rate model. The resulting total expected cost rate is $\mathcal{J}(R_0^*, N^*) = 36.07$.

Table 2 Optimal PM intervals: case of Experiment #2

k	1	2	3	4	5
T_k	146.48	123.68	111.05	102.59	96.22

Conclusion

This paper aimed in revisiting the maintenance optimization model developed in Liao et al. (2010). The system studied is stochastic deteriorating and operate under imperfect maintenance. The work contributes to the current state of knowledge on systems' maintenance by incorporating operational and breakdown costs into the development of an optimal imperfect maintenance strategy for a system subject to stochastic degradation. Imperfect PM action are carried out whenever the system's reliability reaches a given threshold. After a number of PM actions, the system is then replaced with an identical one. A mathematical model was developed to determine the optimal maintenance policy to minimize the long-run expected total cost. Two decision variables were considered: the reliability threshold and the number of PM intervals.

The present paper highlights some inconsistencies in the initial work (Liao et al. 2010). A new optimization model is then developed and fully discussed. Optimality conditions were also discussed and resulted in two propositions. These results were then used in a fix-and-optimize numerical procedure to determine the optimal solution of the problem. Using the data set in Liao et al. (2010), numerical experiments were then conducted and fully discussed. The results of these experiments are compared to those obtained in Liao et al. (2010). The overall results obtained demonstrate the accuracy and the validity of our approach.

The present work can be extended to deal with more general repair process instead of minimal repair carried out between instants of PM. This work can also be extended to include warranty and inspection aspects. An interesting issue would then be to consider a bi-variate (age and usage) model to assess the actual reliability or condition of the system.

Appendix A: Proof of Proposition 1

When the values of R_0 is fixed, the expected total cost rate function becomes a discrete uni variate function of N and is denoted here simply as $\mathcal{J}(N)$. To alleviate the notations, we further write the expected total cost rate function as:

$$\mathcal{J}(N) = \frac{\alpha + N\gamma + \varphi(N)}{\psi(N)},$$

where:

$$\alpha = C_r - C_p,$$
$$\gamma = C_p + C_b - (C_m + C_b)\ln(R_0),$$
$$\varphi(N) = \sum_{k=1}^{N} \delta_k, \text{ and } \psi(N) = \sum_{k=1}^{N} T_k.$$

A number N of PM is optimal if it satisfies the following two conditions:

1. $\mathcal{J}(N) < \mathcal{J}(N-1)$, and
2. $\mathcal{J}(N) < \mathcal{J}(N+1)$.

The first condition implies that:

$$\mathcal{J}(N) < \frac{\alpha + (N-1)\gamma + \varphi(N-1)}{\psi(N-1)},$$

Noting that $\delta_N = \varphi(N) - \varphi(N-1)$ and $\psi(N) = \psi(N-1) + T_N$, then the above equation can equivalently be written as:

$$\mathcal{J}(N) < \frac{a + N\gamma + \varphi(N)}{\psi(N) - T_N} - \frac{\gamma + \delta_N}{\psi(N) - T_N},$$

and then we have:

$$\mathcal{J}(N) - \left(\frac{a+N\gamma+\varphi(N)}{\psi(N)}\right)\frac{\psi(N)}{\psi(N)-T_N} < -\frac{\gamma + \delta_N}{\psi(N) - T_N},$$

which in turn leads to:

$$\mathcal{J}(N)\left(\frac{-T_N}{\psi(N) - T_N}\right) < -\frac{\gamma + \delta_N}{\psi(N) - T_N},$$

From the fact that $\psi(N) > T_N$ (i.e., any PM interval is always smaller than the sum of all PM intervals), the above equation finally gives:

$$\mathcal{J}(N) > \frac{\gamma + \delta_N}{T_N}.$$

Since $\delta_N \geq 0$, then we get:

$$\mathcal{J}(N) > \frac{\gamma}{T_N}. \tag{21}$$

A similar procedure can be followed with condition (2) above to get:

$$\mathcal{J}(N) < \frac{\gamma}{T_{N+1}}. \tag{22}$$

Combining inequalities (21) and (22) gives the result of Proposition (1):

$$\frac{1}{T_N} < \frac{\mathcal{J}(N)}{\gamma} < \frac{1}{T_{N+1}}.$$

Now let us show that if the condition $\frac{\delta_{k+1}}{T_{k+1}} > \frac{\delta_{k+1}}{T_{k+1}}$, then a number N which is solution of the above inequality is unique and finite. To do so, let us return back to conditions (1) and (2) above. From condition (1), it follows that:

$$\mathcal{J}(N) - \mathcal{J}(N-1) < 0.$$

Substituting $\mathcal{J}(N)$ and $\mathcal{J}(N-1)$ by their respective values and computing the difference $\mathcal{J}(N) - \mathcal{J}(N-1)$, we have:

$$\mathcal{J}(N) - \mathcal{J}(N-1) = \frac{\alpha + N\gamma + \varphi(N)}{\psi(N)} - \frac{\alpha + (N-1)\gamma + \varphi(N-1)}{\psi(N-1)},$$

From the fact that $\psi(N-1) = \psi(N) - T_N$ and $\varphi(N-1) = \varphi(N) - \delta_N$ the above equation can be simplified as:

$$\mathcal{J}(N) - \mathcal{J}(N-1) = \frac{\gamma\left(\psi(N) - NT_N\right) + \left(\psi(N)\delta_N - T_N\varphi(N)\right) - \alpha T_N}{\psi(N-1)\psi(N)}. \tag{23}$$

Since $\mathcal{J}(N) - \mathcal{J}(N-1) < 0$ together with the fact that $\psi(N)$ is positive valued, it follows that the numerator of the above equation is negative. This implies that:

$$\xi(N) < \frac{\alpha}{\gamma},$$

where the discrete function $\xi(N)$ is defined as:

$$\xi(N) = \frac{\gamma\left(\psi(N) - NT_N\right) + \left(\psi(N)\delta_N - T_N\varphi(N)\right)}{\gamma T_N}.$$

By considering condition (2) and following the same reasoning as done from condition (1), condition (2) is written as:

$$\xi(N+1) > \frac{\alpha}{\gamma}.$$

The function $\xi(N)$ as defined above is increasing in N. Indeed, computing the difference $\xi(N+1) - \xi(N)$, we get:

$$\xi(N+1) - \xi(N) = \frac{\psi(N)\left(T_N - T_{N+1}\right)}{T_N T_{N+1}} + \psi(N)\left(\frac{\delta_{N+1}}{T_{N+1}} - \frac{\delta_N}{T_N}\right).$$

All T_k terms are non-negative. Furthermore, the term $T_N - T_{N+1}$ is positive because the imperfect PM model used guarantees that $T_N > T_{N+1}$ as, by definition of the hybrid failure rate, each PM interval is shorter than its predecessor. This together with the assumption stating that $\frac{\delta_{k+1}}{T_{k+1}} > \frac{\delta_k}{T_k}$, it follows that $\xi(N+1) - \xi(N)$ is strictly positive for all values of N. Hence the discrete function $\xi(N)$ is strictly increasing.

Now, it will be shown that the solution is finite as well. On one hand, by definition of the hybrid failure rate, the term $\psi(N) - NT_N$ is positive. On the other hand, the term $\psi(N)\delta_N - T_N\varphi(N)$ is equivalently computed as:

$$\psi(N)\delta_N - T_N\varphi(N) = \sum_{k=1}^{N}(T_k\delta_N - T_N\delta_k) = \sum_{k=1}^{N} T_k T_N\left(\frac{\delta_N}{T_N} - \frac{\delta_k}{T_k}\right)$$

From the assumption stating that $\frac{\delta_{k+1}}{T_{k+1}} > \frac{\delta_k}{T_k}$ for all $k \geq 1$, it follows that the term $\psi(N)\delta_N - T_N\varphi(N)$ is also positive. Therefore, the discrete function $\xi(N)$ is such that:

$$\gamma\xi(N) > \frac{\psi(N) - NT_N}{T_N}.$$

From the fact that $\psi(N) > T_1 + (N-1)T_N$, the above inequality implies that:

$$\gamma\xi(N) > \frac{T_1 + (N-1)T_N - NT_N}{T_N},$$

which can also be written as:

$$\gamma\xi(N) > \left(\frac{T_1}{T_N} - 1\right).$$

We have that $\xi(N)$ is increasing in N, this together with the fact that:

$$\lim_{N\to+\infty}\left(\frac{T_1}{T_N}\right) = \infty,$$

it follows that,

$$\lim_{N\to+\infty}\xi(N) = +\infty.$$

In summary, we have that the function $\xi(N)$ is strictly increasing and tends to $+\infty$ as N tends to $+\infty$. Thus, one may conclude that there exist a finite and unique N^* for which conditions (1) and (2) are satisfied. Therefore, the first integer satisfying these two conditions is the global minimum of the expected total cost rate $\mathcal{J}(N)$.

Appendix B: Proof of Lemma 1

Let us denote by $H_1(t) = \int_0^t \lambda_1(x)\mathrm{d}x$, the cumulative failure rate of the system at the start of a replacement cycle where a system is new. From Eq. (6), we have that:

$$H_1(Y_k) = H_1(b_{k-1}Y_{k-1}) - \frac{\ln(R_0)}{B_{k-1}}.$$

It follows that:

$$\frac{\partial H_1(Y_k)}{\partial Y_k} = -\frac{\partial}{\partial Y_k}\left(\frac{\ln(R_0)}{B_{k-1}}\right) = -\left(\frac{1}{B_{k-1}}\right)\left(\frac{\partial \ln(R_0)}{\partial R_0}\right)\left(\frac{\partial R_0}{\partial Y_k}\right) = -\left(\frac{1}{B_{k-1}}\right)\left(\frac{1}{R_0}\right)\left(\frac{\partial R_0}{\partial Y_k}\right). \quad (24)$$

Since $\frac{\partial H_1(Y_k)}{\partial Y_k} = \lambda_1(Y_k)$, it follows that:

$$\lambda_1(Y_k) = -\left(\frac{1}{B_{k-1}}\right)\left(\frac{1}{R_0}\right)\left(\frac{\partial R_0}{\partial Y_k}\right),$$

then we get:

$$\frac{\partial Y_k}{\partial R_0} = \frac{-1}{B_{k-1}\lambda_1(Y_k)R_0}$$

Appendix C: Proof of Proposition 2

For fixed values of the number N of PM, the optimal value of the reliability threshold R_0^* is obtained by solving the following partial derivative:

$$\frac{\partial \mathcal{J}(R_0, N)}{\partial R_0} = 0.$$

Before starting the computation of the above partial derivative, one may observe that due to the operational cost structure the total expected operational cost rate $\frac{\mathbb{E}[OC]}{\mathbb{E}[\mathcal{T}]}$ is such that:

$$\frac{\mathbb{E}[OC]}{\mathbb{E}[\mathcal{T}]} = \frac{\sum_{k=1}^{N}\left(C_0T_k + kC_1T_K + \frac{C_2}{2}T_k^2\right)}{\sum_{k=1}^{N}T_k} = C_0 + \frac{\sum_{k=1}^{N}\left(kC_1T_K + \frac{C_2}{2}T_k^2\right)}{\sum_{k=1}^{N}T_k}.$$

From the above equation, one may conclude that the term induced by the fixed operational cost rate C_0 has no impact on the computation of the partial derivative $\frac{\partial\mathcal{J}(R_0,N)}{\partial R_0}$. Accordingly, writing the partial derivative $\frac{\partial\mathcal{J}(R_0,N)}{\partial R_0}$ and setting it equal to 0 leads to the following equality:

$$\frac{\sum_{k=1}^{N-1}(1-a_k)Y_k + Y_N}{\sum_{k=1}^{N-1}(1-a_k)\frac{\partial Y_k}{\partial R_0} + \frac{\partial Y_N}{\partial R_0}} = \frac{C_r + (N-1)C_p + NC_b - N(C_m+C_b)\ln(R_0) + \sum_{k=1}^{N}\left(kC_1 + \frac{C_2}{2}(Y_k - a_{k-1}Y_{k-1})\right)(Y_k - a_{k-1}Y_{k-1})}{\frac{N(C_m+C_b)}{R_0} + \frac{\partial\left(\sum_{k=1}^{N}\left(kC_1+\frac{C_2}{2}(Y_k-a_{k-1}Y_{k-1})\right)(Y_k-a_{k-1}Y_{k-1})\right)}{\partial R_0}}.$$

In the above equation, substituting the expression of the partial derivative $\frac{\partial Y_k}{\partial R_0} : \frac{\partial Y_k}{\partial R_0} = \frac{-1}{B_{k-1}\lambda_1(Y_k)R_0}$, obtained from Lemma 1, yields the following equality:

$$\frac{\sum_{k=1}^{N-1}(1-a_k)Y_k + Y_N}{\sum_{k=1}^{N-1}\frac{1-a_k}{B_{k-1}\lambda_1(Y_k)} + \frac{1}{B_{N-1}\lambda_1(Y_N)}} = \frac{C_r + (N-1)C_p + NC_b - N(C_m+C_b)\ln(R_0) + \sum_{k=1}^{N}\left(kC_1 + \frac{C_2}{2}(Y_k - a_{k-1}Y_{k-1})\right)(Y_k - a_{k-1}Y_{k-1})}{N(C_m+C_b) - \sum_{k=1}^{N}\frac{(kC_1+C_2(Y_k-a_{k-1}Y_{k-1}))(a_{k-1}b_{k-1}\lambda_1(Y_k)-\lambda_1(Y_{k-1}))}{B_{k-1}\lambda_1(Y_{k-1})\lambda_1(Y_k)}},$$

which is equivalently written as:

$$\left(\frac{\sum_{k=1}^{N-1}(1-a_k)Y_k + Y_N}{\sum_{k=1}^{N-1}\frac{1-a_k}{B_{k-1}\lambda_1(Y_k)} + \frac{1}{B_{N-1}\lambda_1(Y_N)}}\right)\left(N(C_m+C_b) - \sum_{k=1}^{N}\frac{(kC_1 + C_2(Y_k - a_{k-1}Y_{k-1}))(a_{k-1}b_{k-1}\lambda_1(Y_k) - \lambda_1(Y_{k-1}))}{B_{k-1}\lambda_1(Y_{k-1})\lambda_1(Y_k)}\right)$$
$$= (C_r + (N-1)C_p + NC_b) + \sum_{k=1}^{N}\left(kC_1 + \frac{C_2}{2}(Y_k - a_{k-1}Y_{k-1})\right)(Y_k - a_{k-1}Y_{k-1}) - N(C_m+C_b)\ln(R_0).$$

Dividing each side of the above equation by the quantity $N(C_m + C_b)$, and performing some basic algebraic operations, we then obtain the result of the proposition.

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