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Option pricing formulas for uncertain financial market based on the exponential Ornstein–Uhlenbeck model

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Abstract Uncertain finance is an application of uncertainty theory in the field of finance. This paper investigates the uncertain financial market based on the exponential Ornstein–Uhlenbeck model. European option pricing formulas and American option pricing formulas are derived via the α-path method. Finally, some mathematical properties of the uncertain option pricing formulas are discussed.

Keywords Uncertain variable · Uncertain process · Exponential Ornstein–Uhlenbeck model · Option pricing

Introduction

Black and Shocles (1973) and Merton (1973) used the geometric Brownian motion to construct a theory for pricing the options. From then on, Black–Scholes formula has been pivotal to the growth and success of financial engineering.

As we all know, when using probability theory, a fundamental premise is that the estimated probability distribution is close enough to the long-run cumulative frequency. Otherwise, the law of large numbers is no longer valid and probability theory is no longer applicable. However, in many situations, there are not enough (or even no) historical data. Then we have to invite some domain experts to evaluate their belief degree that each event will occur. In order to model human belief degrees, uncertainty theory was established by Liu (2007) and refined by Liu (2010a). Nowadays, uncertainty theory has become a branch of axiomatic mathematics with diverse applications such as uncertain logic (Li and Liu 2009), uncertain risk analysis (Liu 2010c), uncertain set (Liu 2010b, 2013a) and uncertain game (Yang and Gao 2013, 2014). In order to describe dynamic uncertain systems, (Liu 2008) introduced uncertain process. Besides, Liu (2009) designed canonical Liu process which could be seen as a counterpart of Brownian motion. Based on canonical Liu process, Liu introduced uncertain calculus (Liu 2009) and uncertain differential equations (Liu 2008). After that, Chen and Liu (2010) proved the existence and uniqueness theorem of solution of uncertain differential equation. The concept of stability of uncertain differential equation was presented by Liu (2009). Later on Yao et al. (2013) proved some stability theorems of uncertain differential equation. Chen and Liu (2010), Liu (2012) and Yao (2013a) developed many methods to solve the uncertain differential equations. Furthermore, Yao and Chen (2013) designed the α-path method, which produced an inverse uncertainty distribution of the solution. Based on α-path, Yao (2013b) presented some formulas to calculate the extreme value, first hitting time, and time integral of solution of uncertain differential equation.

As a different doctrine, based on the assumption that stock price follows a geometric canonical process, uncertainty theory was first introduced into finance by Liu (2009) in 2009. And Liu (2009) also proposed an uncertain stock model and derived its European option price formulas. Later on, Chen (2011) studied American option pricing formulas for uncertain stock market, Sun and Chen (2013) derived Asian option price formulas, and Yao (2015) proved a no-arbitrage theorem for this type of uncertain stock model. In addition,

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Peng and Yao (2010) proposed a different uncertain stock model and derived some option price formulas. Liu et al. (2012) proposed an uncertain currency model and explored some mathematical properties of it. Under the assumption of short interest rate following uncertain processes, Chen and Gao (2013) derived the term-structure equation to value the zero-coupon bond. Besides, Jiao and Yao (2015) investigated another type of uncertain interest rate model. For exploring the recent developments of uncertain finance, the readers may consult (Liu 2013b) and the book by Liu (2015).

Peng–Yao stock model (Peng and Yao 2010) incorporated a general economic behavior: mean reversion. That is, when the stock price is too high, the price tends to fall more likely; when the stock price is too low, the price tends to rise more likely. However, this model is linear mean reversion. In this paper, we shall use a nonlinear mean reversion, assuming that the stock price follows uncertain counterpart of the exponential Ornstein–Uhlenbeck model. European option pricing formulas and American option pricing formulas are derived, respectively. Furthermore, some mathematical properties are discussed.

The rest of the paper is organized as follows. Some preliminary concepts of uncertain process are recalled in "Preliminaries" section. The exponential Ornstein–Uhlenbeck model for uncertain markets is formulated in "Exponential Ornstein–Uhlenbeck model" section. "European option price" section gives European option pricing formulas and discusses some properties of the formulas. "American opition price" section gives American option pricing formulas and discusses some properties of the formulas. Finally, a brief summary is given in "Conclusion " section.

Preliminaries

Uncertainty theory was established by Liu (2007) and refined by Liu (2010a). Nowadays, uncertainty theory has become a branch of axiomatic mathematics for modelling human belief degrees. In this section, we will introduce some basic results in uncertainty theory. For more detailed expositions of uncertainty theory with applications (Gao 2013; Liu 2013a, b, 2014), the readers may consult Liu's recent book (Liu 2015).

Definition 1 (Liu 2007) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Besides, in order to provide the operational law, (Liu 2009) defined the product uncertain measure on the product σ-algebre $\mathcal{L}$ as follows.

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2 (Liu 2007) An uncertain variable is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, such that, for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

In order to describe uncertain variables in practice, the concept of uncertainty distribution was introduced.

Definition 3 (Liu 2007) The uncertainty distribution of an uncertain variable ξ is defined as

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any real number x.

An uncertainty distribution $\Phi(x)$ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$. And $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ. In this paper, we assume that all the payoffs are characterized by regular uncertain variables.

Definition 4 (Liu 2009) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^{m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

The operational law of uncertain variables was proposed by Liu (2010c) to calculate the inverse uncertainty distribution of strictly monotonous function.

Theorem 1 (Liu 2010c) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with $x_{m+1}, x_{m+2}, \ldots, x_n$, then*

$$\xi = f(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Definition 5 (Liu 2007) Let ξ be an uncertain variable. Then the expected value of ξ is defined as

$$\mathrm{E}[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}\mathrm{d}x$$

provided that at least one of the two integral is finite.

If ξ is a regular uncertain variable with uncertainty distribution Φ, then its expected value can be briefed as

$$\mathrm{E}[\xi] = \int_0^{+\infty} (1-\Phi(x))\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\mathrm{d}x = \int_0^{1} \Phi^{-1}(x)\mathrm{d}x.$$

Definition 6 (Liu 2008) Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B,

$$\{X_t \in B\} = \{\gamma \in \Gamma \mid X_t(\gamma) \in B\}$$

is an event.

Definition 7 (Liu 2008) An uncertain process X_t is said to have independent increments if

$$X_{t_0},\ X_{t_1} - X_{t_0},\ X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where t_0 is the initial time and $t_1, t_2, \ldots, t_k$ are any time with $t_0 < t_1 < \cdots < t_k$.

Definition 8 (Liu 2008) An uncertain process X_t is said to have stationary increments if, for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$.

Definition 9 (Liu 2009) An uncertain process C_t is said to be a canonical process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1},\ x \in \Re.$$

Definition 10 (Liu 2009) Let X_t be an uncertain process and let C_t be a canonical Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$

Then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_t \cdot (C_{t_i+1} - C_{t_i})$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

Definition 11 (Liu 2008) Suppose C_t is a canonical Liu process, and f and g are two functions. Then

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is called an uncertain differential equation. A solution is a Liu process X_t that satisfies the equation identically in t.

Definition 12 (Yao and Chen 2013) Let α be a number with $0 < \alpha < 1$. An uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t,$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of standard normal uncertain variable, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Theorem 2 (Yao and Chen 2013) *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t,$$

respectively. Then the solution X_t has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = X_t^\alpha.$$

Theorem 3 (Yao and Chen 2013) *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t,$$

respectively. Then for any monotone function J, we have

$$E[J(X_t)] = \int_0^1 J(X_t^\alpha)d\alpha.$$

Theorem 4 (Yao 2013b) *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t,$$

respectively. Then for any time $s > 0$ and strictly increasing function $J(x)$, the supremum

$$\sup_{0\le t\le s} J(X_t)$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} J(X_t^\alpha).$$

Theorem 5 (Yao 2013b) *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t,$$

respectively. Then for any time $s > 0$ and strictly decreasing function $J(x)$, the supremum

$$\sup_{0\le t\le s} J(X_t)$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} J(X_t^{1-\alpha}).$$

Exponential Ornstein–Uhlenbeck model

In this section, we give the exponential Ornstein–Uhlenbeck model for uncertain markets.

Let X_t be the stock price and Y_t be the bond price. Suppose that the stock price X_t follows a geometric canonical process. Then Liu's stock model (Liu 2008) is written as follows,

$$\begin{cases} dX_t = \mu X_t dt + \sigma X_t dC_t \\ dY_t = rY_t dt \end{cases} \tag{1}$$

where r is the riskless interest rate, μ is the stock drift, σ is the stock diffusion, and C_t is a canonical process. This model represents that the stocks have constant expected rate of return.

Besides, Peng–Yao's stock model (Peng and Yao 2010) has the following form

$$\begin{cases} dX_t = (m - \alpha X_t)dt + \sigma X_t dC_t \\ dY_t = rY_t dt \end{cases} \tag{2}$$

where $r > 0, m > 0, \alpha > 0$ and $\sigma > 0$ are constants. This model incorporates a general economic behavior: mean reversion. That is, when the stock price is too high, the price tends to fall more likely; when the stock price is too low, the price tends to rise more likely. However, this model is linear mean reversion. In the sequel, we shall give the following nolinear model

$$\begin{cases} dX_t = \mu(1 - c\ln X_t)X_t dt + \sigma X_t dC_t \\ dY_t = rY_t dt \end{cases} \tag{3}$$

where $r > 0, c > 0, \sigma > 0$ and μ are constants. For $c = 0$, it is clearly Liu's stock model (1). This model is the exponential Ornstein–Uhlenbeck model.

Now, we use the α-path's method to discuss the exponent OU model under the uncertain environment.

Theorem 6 *Suppose that the stock price follows the model*

$$dX_t = \mu(1 - c\ln X_t)X_t dt + \sigma X_t dC_t,$$

where X_t represents the stock price at the moment t. Then the α-path of X_t is

$$X_t^\alpha = \exp\left(\exp(-\mu ct)\ln X_0 + (1 - \exp(-\mu ct))\left(\frac{1}{c} + \frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right).$$

Proof By Definition 12, we have

$$dX_t^\alpha = \mu(1 - c\ln X_t^\alpha)X_t^\alpha dt + \sigma X_t^\alpha \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}dt.$$

So

$$d\ln X_t^\alpha = \left(\mu + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)dt - \mu c\ln X_t^\alpha dt.$$

The above ordinary differential equation has a solution

$$\ln X_t^\alpha = \frac{\mu + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}}{\mu c}(1 - \exp(-\mu ct)) + \exp(-\mu ct)\ln X_0.$$

Then

$$X_t^\alpha = X_0^{\exp(-\mu ct)} \exp\left((1-\exp(-\mu ct))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)$$
$$= \exp\left(\exp(-\mu ct)\ln X_0 + (1-\exp(-\mu ct))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right).$$

The α-path's formula of X_t is verified.

European option price

A European option gives one the right, but not the obligation, to buy or sell a stock at a specified time for a specified price. Suppose that a European call option has a strike price K and an expiration time T. If X_T is the final price of the underlying stock, then the payoff from buying a European call option is $(X_T-K)^+$. Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rT)(X_T-K)^+$. Hence the European call option should be the expected present value of the payoff. Then this option has a price

$$f_c = \exp(-rT)E[(X_T-K)^+]. \tag{4}$$

Theorem 7 (European call option pricing formula) *Suppose a European call option for the stock model (3) has a strike price K and an expiration time T. Then the European call option pricing formula is*

$$f_c = \exp(-rT)\int_m^1\left(\exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)-K\right)\mathrm{d}\alpha,$$

where

$$m=\frac{e^A}{1+e^A},\quad A=\frac{\mu\pi}{\sigma\sqrt{3}}\left(\frac{c(\ln K-\exp(-\mu cT)\ln X_0)}{1-\exp(-\mu cT)}-1\right).$$

Proof From Theorems 2 and 6, we get

$$\Phi_T^{-1}(\alpha) = X_T^\alpha = \exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right).$$

By Theorem 3, we have

$$f_c = \exp(-rT)E[(X_T-K)^+]$$
$$= \exp(-rT)\int_0^1\left(\exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)-K\right)^+\mathrm{d}\alpha,$$

Letting

$$\exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right) > K,$$

we have

$$\alpha > m,$$

where

$$m = \frac{e^A}{1+e^A},$$
$$A = \frac{\mu\pi}{\sigma\sqrt{3}}\left(\frac{c(\ln K-\exp(-\mu cT)\ln X_0)}{1-\exp(-\mu cT)}-1\right).$$

Then

$$f_c = \exp(-rT)\int_m^1\left(\exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)-K\right)\mathrm{d}\alpha,$$

The European call option pricing formula is verified.

Theorem 8 (Monotonicity of European call option pricing model) *Suppose the stock price follows the stock model (3), which has a strike price K and an expiration time T. Then f_c has the following properties:*

1. *f_c is a decreasing function of K;*
2. *f_c is a decreasing function of r;*
3. *f_c is an increasing function of X_0.*

Proof 1. By equation (4) we get assertion clearly.
2. Since $\exp(-rt)$ is a decreasing function of r, the result is obvious.
3. Let

$$G = \exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right).$$

Then

$$f_c = \exp(-rT)\int_0^1 (G-K)^+ \mathrm{d}\alpha.$$

Since

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}X_0} &= \exp\left((1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\\ &\quad \frac{\mathrm{d}\exp(\exp(-\mu cT)\ln X_0)}{\mathrm{d}X_0}\\ &= \exp\left((1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right.\\ &\quad \left. -\mu cT + (\exp(-\mu cT)-1)\ln X_0\right)\\ &> 0.\end{aligned}$$

It is obvious that f_c is an increasing function of X_0 since G is an increasing function of X_0. The Monotonicity of European call option pricing formula is verified.

Suppose that a European put option has a strike price K and an expiration time T. If X_T is the final price of the underlying stock, then the payoff from buying a European put option is $(K-X_T)^+$. Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rT)(K-X_T)^+$. Hence the European call option should be the expected present value of the payoff. Then this option has a price

$$f_p = \exp(-rT)E[(K-X_T)^+]. \tag{5}$$

Theorem 9 (European put option pricing formula) *Suppose a European call option for the stock model (3) has a strike price K and an expiration time T. Then the European put option pricing formula is*

$$f_p = \exp(-rT)\int_0^m \left(K-\exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\right)\mathrm{d}\alpha,$$

where

$$m=\frac{e^A}{1+e^A},\ A=\frac{\mu\pi}{\sigma\sqrt{3}}\left(\frac{c(\ln K-\exp(-\mu cT)\ln X_0)}{1-\exp(-\mu cT)}-1\right).$$

Proof From Theorems 2 and 6, we get

$$\Phi_T^{-1}(\alpha) = X_T^\alpha = \exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right).$$

By Theorem 3, we have

$$\begin{aligned} f_p &= \exp(-rT)E[(K-X_T)^+]\\ &= \exp(-rT)\int_0^1 \left(K-\exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\right)^+\mathrm{d}\alpha.\end{aligned}$$

Let

$$\exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right) < K,$$

we have

$$\alpha < m = \frac{e^A}{1+e^A},$$

where

$$A = \frac{\mu\pi}{\sigma\sqrt{3}}\left(\frac{c(\ln K-\exp(-\mu cT)\ln X_0)}{1-\exp(-\mu cT)}-1\right).$$

So

$$f_p = \exp(-rT)\int_0^m \left(K-\exp\left(\exp(-\mu cT)\ln X_0 + (1-\exp(-\mu cT))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\right)\mathrm{d}\alpha,$$

The European put option pricing formula is verified.

Theorem 10 (Monotonicity of European put option pricing model) *Suppose the stock price follows the stock model (3), which has a strike price K and an expiration time T. Then f_p has the following properties:*

1. *f_p is a decreasing function of K;*
2. *f_p is a decreasing function of r;*
3. *f_p is a decreasing function of X_0.*

Proof The proof is omitted here for it is similar to that of Theorem 8.

American opition price

An American option gives one the right, but not the obligation, to buy or sell a stock before a specified time for a specified price. Suppose that an American call option has a strike price K and an expiration time T. If X_T is the final price of the underlying stock, then the payoff from buying an American call option is the supremum of $(X_t - K)^+$ over the time interval $[0, T]$. Considering the time value of money resulted from the bond, the present value of this payoff is the supremum of $\exp(-rt)(X_t - K)^+$. Hence an American call option should be the expected present value of the payoff. Then this option has a price

$$F_c = E\left[\sup_{0\le t\le T} \exp(-rt)(X_t - K)^+\right]. \tag{6}$$

Theorem 11 (American call option pricing formula) *Suppose an American call option for the stock model (3) has a strike price K and an expiration time T. Then the American call option pricing formula is*

$$F_c = \int_0^1 \sup_{0\le t\le T} \exp(-rt)\left(\exp\left(\exp(-\mu ct)\ln X_0 + (1-\exp(-\mu ct))\left(\frac{1}{c} + \frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right) - K\right)^+ \mathrm{d}\alpha.$$

Proof We note that $\exp(-rt)(X_t - K)^+$ is an increasing function of X_t. According to Theorem 4, we get that the inverse distribution function of $\sup_{0\le t\le T}\exp(-rt)(X_t - K)^+$ is

$$\Psi_t^{-1}(\alpha) = \sup_{0\le t\le T} \exp(-rt)(X_t^\alpha - K)^+.$$

From Theorem 6, we have

$$X_t^\alpha = \exp\left(\exp(-\mu ct)\ln X_0 + (1-\exp(-\mu ct))\left(\frac{1}{c} + \frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right).$$

Thus we get

$$\Psi_t^{-1}(\alpha) = \sup_{0\le t\le T} \exp(-rt)\left(\exp\left(\exp(-\mu ct)\ln X_0 + (1-\exp(-\mu ct))\left(\frac{1}{c} + \frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right) - K\right)^+.$$

So

$$\begin{aligned} F_c &= E\left[\sup_{0\le t\le T} \exp(-rt)(X_t - K)^+\right] \\ &= \int_0^1 \Psi_t^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 \sup_{0\le t\le T} \\ &\quad \times \exp(-rt)\left(\exp\left(\exp(-\mu ct)\ln X_0 + (1-\exp(-\mu ct))\left(\frac{1}{c} + \frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\right)\right) - K\right)^+ \mathrm{d}\alpha. \end{aligned}$$

The American call option pricing formula is verified.

Theorem 12 (Monotonicity of American call option pricing model) *Suppose the stock price follows the stock model (3), which has a strike price K and an expiration time T. Then F_c has the following properties:*

1. *F_c is a decreasing function of K;*
2. *F_c is a decreasing function of r;*
3. *F_c is an increasing function of T;*
4. *F_c is an increasing function of X_0.*

Proof The proof is omitted here for it is similar to that of Theorem 8.

Suppose that an American put option has a strike price K and an expiration time T. If X_T is the final price of the underlying stock, then the payoff from buying an American put option is the supremum of $(K - X_t)^+$ over the time interval $[0, T]$. Considering the time value of money resulted from the bond, the present value of this payoff is the supremum of $\exp(-rt)(K - X_t)^+$. Hence an American call option should be the expected present value of the payoff. Then this option has a price

$$F_p = E\left[\sup_{0\le t\le T} \exp(-rt)(K - X_t)^+\right]. \tag{7}$$

Theorem 13 (American put option pricing formula) *Suppose an American put option for the stock model (3) has a strike price K and an expiration time T. Then the American put option pricing formula is*

$$F_p = \int_0^1 \sup_{0\le t\le T} \exp(-rt)\left(K - \exp\left(\exp(-\mu ct)\ln X_0 + (1-\exp(-\mu ct))\left(\frac{1}{c} + \frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{1-\alpha}{\alpha}\right)\right)\right)^+ \mathrm{d}\alpha.$$

Proof We note that $\exp(-rt)(K - X_t)^+$ is an decreasing function of X_t. According to Theorem 5, we get that the inverse distribution function of $\sup_{0\le t\le T}\exp(-rt)(X_t - K)^+$ is

$$\Psi_t^{-1}(\alpha) = \sup_{0\le t\le T} \exp(-rt)(X_t^{1-\alpha} - K)^+.$$

From Theorem 6, we have

$$X_t^{1-\alpha} = \exp\left(\exp(-\mu ct)\ln X_0 + (1-\exp(-\mu ct))\left(\frac{1}{c} + \frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{1-\alpha}{\alpha}\right)\right).$$

Thus we get

$$\Psi_t^{-1}(\alpha) = \sup_{0\le t\le T} \exp(-rt)\Bigg(K - \exp\Bigg(\exp(-\mu ct)\ln X_0 + (1-\exp(-\mu ct))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{1-\alpha}{\alpha}\right)\Bigg)\Bigg)^+.$$

So

$$F_P = E\left[\sup_{0\le t\le T}\exp(-rt)(K-X_t)^+\right] = \int_0^1 \Psi_t^{-1}(\alpha)\mathrm{d}\alpha. = \int_0^1 \sup_{0\le t\le T}\exp(-rt)\Bigg(K-\exp\Bigg(\exp(-\mu ct)\ln X_0 + (1-\exp(-\mu ct))\left(\frac{1}{c}+\frac{\sigma\sqrt{3}}{\mu c\pi}\ln\frac{1-\alpha}{\alpha}\right)\Bigg)\Bigg)^+ \mathrm{d}\alpha.$$

The American put option pricing formula is verified.

Theorem 14 (Monotonicity of American put option pricing model) *Suppose the stock price follows the stock model (3), which has a strike price K and an expiration time T. Then F_p has the following properties:*

1. *F_p is an increasing function of K;*
2. *F_p is a decreasing function of r;*
3. *F_p is an increasing function of T;*
4. *F_p is a decreasing function of X_0.*

Proof The proof is omitted here for it is similar to that of Theorem 8.

Conclusion

In this paper, we investigated the option pricing problems for uncertain financial market. European option price formulas and American option price formulas are calculated by the α-path method for the exponential Ornstein–Uhlenbeck model. At the same time, some properties of these formulas are discussed.

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