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Joint optimisation of maintenance and production policies with subcontracting and product returns

Hajej Zied · Dellagi Sofiene · Rezg Nidhal

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Abstract This paper, an extension of our previous research, deals with the problem of jointly optimizing maintenance, production and inventory costs considering subcontracting and product returns. The manufacturing system, which fails randomly, has to satisfy a random product demand during a finite planning horizon under a required service level. The portion of products returned by the customers that are still in saleable condition are collected in the principle store from which customer demand is filled, while the portion that are non-conformal are collected in a second store and then remanufactured by a subcontractor. This study is validated by a real industrial case presented in this paper.

Keywords Optimization · Integrated production-maintenance strategies · Variable production rates · Degradation · Functional age approach · Subcontracting · Product returns

Introduction

The consideration of integrated maintenance policies and production plans to achieve a global optimum has recently become an important research area. There is still a lack of tools to evaluate the production systems in the presence of maintenance activities. Due to factors like random demand, random failure, and inventory constraints, this problem is not easy to model and solve.

H. Zied (✉) · D. Sofiene · R. Nidhal
LGIPM-INRIA, Université Paul Verlaine, Metz, France
e-mail: hajej@univ-metz.fr

D. Sofiene
e-mail: dellagi@univ-metz.fr

R. Nidhal
e-mail: rezg@univ-metz.fr

The linear decision rule developed by Holt et al. (1960) can be considered as an important contribution for strategic production planning decision. This analytical rule is determined from the minimization of quadratic cost functions subject to inventory and workforce balance equations. As a result, it provides an optimal smoothing solution for aggregate inventory, production and workforce levels. Silva Filho and Cezarino (2004) showed that it is possible to extend the unconstrained HMMS model in order to deal with a chance-constrained stochastic production planning problem under the hypothesis of imperfect inventory information variables. The HMMS model is usually applied as a benchmarking tool for comparing different aggregate production planning approaches as well as to provide managers with insights about the use of the company's material resources. However, there are doubts about its applicability for practical industrial purposes (APICS 1994), one being that quadratic cost functions are a questionable assumption, and another being that it fails to provide a reliable production plan since it does not take into account constraints on the decision variables. The first criticism is overcome by the argument that quadratic costs are an interesting way of evaluating production processes (Hax and Candea 1984); for example, quadratic inventory costs (i.e. holding cost) are incurred for both negative (backorder) and positive inventory (Parlar 1985). By contrast, the second criticism is more valid since not including physical constraints into the problem formulation can lead to dire consequences in practice.

The simultaneous study of maintenance policies and production planning and control has been the subject of several studies. The effect of maintenance policies on just-in-time production systems has been studied by Abdul-nour et al. (1995). Some studies have examined the conditions of building buffer stocks to guarantee the continuous supply of the subsequent production unit during the interruptions of service due to repair or preventive maintenance. Rezg et al. (2004) presented a joint optimization of preventive maintenance and stock control in a production line made up of N machines. In the same context of integrating maintenance and production, Rezg et al. (2008) developed a mathematical model and a numerical procedure which allow determining a joint optimal inventory control and age-based preventive maintenance policy for a randomly failing production system. Chelbi and Ait-Kadi (2004) developed an analytical model to determine both the buffer stock size and the preventive maintenance period for an unreliable production unit which is subject to regular preventive maintenance of random duration.

Other related works appearing in the literature include Cheung and Hausman (1997), who simultaneously optimize strategic stock and an age-based maintenance policy. In the same context, Gharbi and Kenne (2000) and Kenne and Gharbi (2001) studied the optimal flow control for a manufacturing system subject to random failures, repair and preventive maintenance. Recently, Chelbi and Rezg (2006) developed an integrated model of production and inventory for a randomly failing system subject to a minimum required availability level.

Furthermore, in the manufacturing system field, the relationships between enterprises are leaning towards more cooperation and collaboration. In this context, many companies employ subcontracting in order to compensate for inadequate technology, to manufacture a product competitively or to meet delivery deadlines. New maintenance/production strategies which take into account subcontracting are studied by Dellagi et al. (2007). Dellagi et al. (2007) developed and optimized a new maintenance policy incorporating subcontractor constraints. Its deals with a case study which demonstrates the influence of the subcontractor constraints on the optimal maintenance strategy adopted. Also dealing with this frame work, Dahane et al. (2010) studied analytically the problem of the integration of subcontracting activity and the number of subcontracting tasks to be performed during a maintenance cycle.

Many maintenance models assume that the system is maintained under fixed operational and environmental conditions. In this context, Özekici (1995) proposes to use an intrinsic age of the system instead of the actual age to account for variable environmental conditions, while Martorell et al. (1999) use accelerated life models. Schutz et al. (2009) proposed a model incorporating periodic and sequential preventive maintenance policies for a system that performs various missions over a finite planning horizon. Each mission can have different characteristics that depend on operational and environmental conditions. Moreover, the failure rate increases with time and according to the use of the equipment, a situation seldom studied in the literature. Among these works, we can cite Hu et al. (1994) who discussed the optimality conditions of the hedging point policy for production systems for which the failure rate of machines depends on the production rate. Others like Liberopoulos, G. and Caramanis (1994) studied the optimal flow control of single-part-type production systems with homogeneous Markovian machine failure rates dependent on the production rate.

Motivated by the lack of consideration of the system's failure rate variation as a function of the production rate, Hajej et al. (2009) dealt with combined production and maintenance plans for a randomly failing manufacturing system satisfying a random demand over a finite horizon. In the case of linear machine degradation, the failure rate depends on time and on the production rate which is variable over the planning horizon. Their approach is to establish an optimal production plan combined with a preventive maintenance policy which minimizes the total cost subject to a required service level constraint. Hajej et al. (2011a), using an analytical approach based on a stochastic optimization model and using the operational age concept, reveal the significant influence of the production rate on the deterioration of the manufacturing system and consequently on the integrated production/maintenance policy. For their part, Hajej et al. (2011b) dealt with the problem of a joint maintenance and production policy under a subcontracting constraint. They first establish an optimal production plan which minimizes the total inventory and production cost taking into consideration the subcontractor constraint. Secondly, using this optimal production plan, they derive an optimal maintenance schedule which minimizes the total maintenance cost.

This paper stems from an actual case study of a low-cost textile company whose randomly failing manufacturing system has to satisfy a random product demand. We develop a stochastic dynamic model in order to establish simultaneously optimal production and maintenance plans taking into account the influence of products returned by the customer, some of which are still new and therefore saleable, and others that are non-conformal which are sent to the subcontractor for recycling and remanufacturing.

This remainder of this paper is organized as follows. Section "Modelling the production/maintenance problem" formulates a general linear quadratic stochastic control model that represents the production/maintenance policy of interest. The decision variables are the production rate in each period along with the preventive maintenance period. We deal with this problem in the context, rarely considered in the literature, where the system's failure rate depends on both time and the variable production rate. In section "Analytical study", considering the influence of the variable production rates on the degradation of the manufacturing system and based on the functional age approach, we further develop the analytical study of the total costs. An industrial example is discussed in section “Numerical example” and a general conclusion is provided in section “Conclusion”.

Modelling the production/maintenance problem

In this section, a stochastic optimal control problem with constraints is formulated. It represents a production/maintenance planning problem with constraints on inventory, production and preventive maintenance variables. This optimization model is developed from the classical HMMS model (Holt et al. 1960).

Description of the industrial problem

The industrial case study upon which this research is based, illustrated in Fig. 1, will now be described. It concerns a low-cost textile company located in Morocco specialized in manufacturing clothing for hunting and fishing, fashion products, uniforms, bedding, transport bags and fabric or leather upholstery. The company achieved its success through a high quality outsourcing network comprised of some of the best clothing companies.

The company's production system consists of a chain stitching and assembly machine (machine M in Fig. 1) which can achieve a volume of up to 20,000 pieces per day. The company also includes a central purchasing department located in France (storage S_1 in Fig. 1) which receives and strives to meet random customer demands. In the event that, upon the receipt of the product, the customer decides he is not satisfied with its quality, he has the right to return the product within a specific deadline. The returned products that are in saleable condition are then stored in the central purchasing department (S_1) in order to be relisted for sale. Meanwhile, the returned product that is defective will be repaired, recycled or remanufactured by a subcontractor located in France (storage S_2 and machine M_s), following which the product will be returned to the central purchasing department.

In terms of reliability, the production system is subject to failures whose frequency increases with the production rate variation and the natural degradation of the equipment. The ensuing multi-criteria integrated maintenance/production planning problem seeks to minimize the costs of maintenance actions and the loss of customer demand while improving the availability of the production equipment. On the one hand, production planning is subject to the constraints of time (delay), cost and quality product. On the other hand, the maintenance plan, subject to the constraints of equipment reliability, attempts to maximize the availability of the production equipment taking into account the planning of production tasks.

In this context, the research work proposed in this manuscript is to determine joint maintenance and production planning policies taking into account not only the influence of product returned by the customer but also that of the production rate variation.

Mathematical formulation of the problem

Machine M is subject to random failures. The random single-product demand, characterized by a normal probability distribution with a known average and standard deviation, is satisfied from S_1. The finite horizon is divided into H production periods of equal lengths Δt.

The costs of this system consist of the holding costs at the two storages, the manufacturing production cost, the

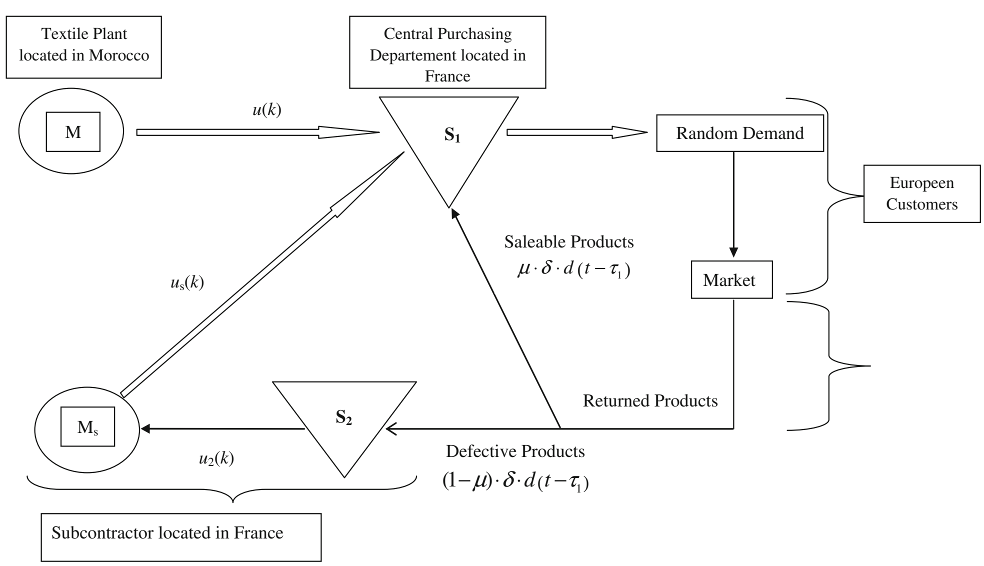


Fig. 1 Problem description

subcontractor's remanufacturing cost and the maintenance cost. The model is represented as an optimal control problem with two state variables, namely the inventory levels in both storages, together with the three control variables, which are: the manufacturing and remanufacturing rates as well as the preventive maintenance periods over the finite horizon.

The probability degradation law of machine M is described by the probability density function associated with its time to failure, $f(t)$. Its failure rate, $\lambda(t)$, increases with both time and the production rate, $u(t)$. Failures of machine M can be reduced through preventive maintenance activities scheduled periodically at certain time intervals. The subcontractor machine M_s is characterized by its production rates $u_s(k)$ at each period k, its unit production cost C_{prs} and its availability rate β_s.

The objective is to minimize the sum of the inventory costs at the two stores, the manufacturing and remanufacturing costs along with the costs associated with the maintenance policy. Our approach is based on the estimation of the new age of the machine called functional age. The model is in Fig. 1.

Notation

The following parameters are used in the mathematical formulation of the model:

μ	Percentage of returned product that is saleable and sent back to store S_1
δ	Percentage of backordered product
τ_1	Backorder delay
Δt	Length of a production period
H	Number of production periods in the planning horizon
$H \cdot \Delta t$	Length of the finite planning horizon
$u(k)$	Production rate of machine M during period k $(k = 0, 1, \ldots, H)$ $U = \{u(0), u(2), \ldots, u(H-1)\}$
$u_s(k)$	Production rate of the subcontractor machine M_s during period k (k=0, 1,…,H)
$u_2(k)$	Rate at which product is withdrawn from store S_2 by the subcontractor M_2during period k (k=0, 1,…,H)
$d(k)$	Average demand during period k (k=0, 1,…,H)
$V_{d(k)}$	Variance of demand during period k (k=0,1,…,H)
$S_1(k)$	Inventory level of first store at the end of period k (k=0, 1,…,H)
$\hat{S}_1(k)$	Average inventory level of first store during period k (k=0, 1,…,H)
$S_2(k)$	Inventory level of second store at the end of period k (k=0, 1,…,H)
$\hat{S}_2(k)$	Average inventory level of second store during period k (k=0, 1,…,H)
C_{pr}	Unit production cost of machine M
C_{prs}	Unit production cost of subcontractor machine M_s
Cs	Inventory holding cost of one product unit during one period at the two stores
C_M	Total maintenance cost
C_{pm}	Preventive maintenance action cost
C_{cm}	Corrective maintenance action cost
mu	Monetary unit
U_{max}	Maximal production rate of machine M_1
θ	Probability index related to customer satisfaction and expressing the service level
$f(t)$	Probability density function associated with the time to failure of M_1
$F(t)$	Probability distribution function associated with the time to failure of M_1, $F(t) = \int_0^t f(x)dx$
$R(t)$	Reliability function, equal to 1-$F(t)$
$\lambda_n(t)$	Nominal failure rate corresponding to the maximal production rate
$\lambda_k(t)$	Machine failure rate function during period k $(k = 0, 1,\ldots,H)$
β_s	Machine M_s availability rate
S_1^0	Initial inventory level of first store
S_2^0	Initial inventory level of second store

The problem formulation

The main objective of this subsection is to describe the mathematical model of the problem. The idea is to minimize the expected production and holding costs and the maintenance cost over a finite time horizon. The demand is satisfied at the end of each period.

This kind of problem can be formulated as a stochastic quadratic optimization problem under a stock threshold level constraint, with the production rates corresponding to each period as the decision variables. Let $f_k(S_1(k), S_2(k), u(k))$ represent holding and production costs and $C_M(U, N)$ represent maintenance costs according to the production plan defined by the vector U and the number of preventive maintenance actions N. We formulate the problem as follows:

$$\underset{(U,N)}{Min}\left(\left\{\sum_{k=0}^{H-1} f_k\left(S_1(k), S_2(k), u(k)\right) + f_H\big(S_1(H), S_2(H)\big)\right\} + \{C_M(U,N)\}\right) \quad (1)$$

Subject to:

$$S_1(k+1) = S_1(k) + u_s(k) + u(k) + \mu \cdot \delta \cdot d(k-\tau_1) - d(k) \quad k = 0, 1, \ldots, H-1 \quad (2)$$

$$\text{Prob}\left[S_1(k+1) \geq 0\right] \geq \theta \quad k = 0, 1, \ldots, H-1 \quad (3)$$

$$S_2(k+1) = S_2(k) + (1-\mu) \cdot \delta \cdot d(k-\tau_1)$$

$$-u_2(k) \quad k = 0, 1, \ldots, H-1 \tag{4}$$

$$u_s(k) = \beta_s \cdot u_2(k) \quad 0 < \beta_s < 1 \tag{5}$$

$$0 \le u(k) \le U_{\max} \quad k = 0, 1, \ldots, H-1 \tag{6}$$

Observe that the expected total cost incurred during the last period H does not depend on the production rate $u(k)$. The criterion $f(\cdot)$ is generic but whenever convexity is assumed, the uncertainties can be handled directly by computing the expected value of the cost. The maintenance cost $C_M(\cdot)$ is characterized by the preventive and corrective maintenance costs and the expected number of failures. Constraint (2) denotes the inventory balance equation for the principle store. The service level requirement constraint for each period is expressed by constraint (3). The index θ is a measure of the probability chosen by the manager in the range [0,1] which can be interpreted as the tradeoff between backlogged sales (by choosing $\theta \epsilon [0, 1/2]$) and customer satisfaction ($\theta \epsilon [1/2, 1]$). For example $\theta = 0.95$ implies demand is expected to be satisfied at least 95 % of time (i.e. high customer service level), whereas $\theta = 0.2$ means that backlogging occurs at least 80 % of the time (i.e., low customer service level). The inventory balance equation for the second store is described by constraint (4). The relation (5) defines the subcontractor production rate according to their availability β_s. Finally, the last constraint defines an upper bound on the production level during each period k.

The production policy

The randomness of demand is handled by using the certainty-equivalence principle while the use of a quadratic cost function allows penalizing both excess and shortage in the inventory level.

The expected cost including production and holding costs for the period k is given by:

$$f_k(S_1(k), S_2(k), u(k)) = f_{u(k)}(u(k), u_s(k)) + f_{s(k)}(S_1(k), S_2(k))$$

where the expected production costs for period k $= f_{u(k)}$, $(u(k), u_s(k)) = C_{p_r} \times E\{u(k)^2\} + C_{ps} \times E\{u_s(k)^2\}$ and the expected holding costs of period k = $f_{s(k)}(S_1(k), S_2(k)) = C_s \times \left(E\{S_1(k)^2\} + E\{S_2(k)^2\}\right)$

Note that *E{}* denotes the mathematical expectation operator.

The quadratic total expected cost of production and inventory over the finite horizon $H \cdot \Delta$ t can then be expressed as follows:

$$f(u) = C_s \times \left(E\left\{S_1(H)^2\right\} + E\left\{S_2(H)^2\right\}\right) + \sum_{k=1}^{H-1}\left[C_{p_r} \times u(k)^2 + C_{ps} \times E\left\{u_s(k)^2\right\} + C_s \times \left(E\left\{S_1(k)^2\right\} + E\left\{S_2(k)^2\right\}\right)\right]$$
$$\text{with } k \in \{0, 1, \ldots, H-1\} \tag{7}$$

The maintenance policy

The maintenance strategy under consideration is the well known preventive maintenance policy with minimal repair at failure.

The interval $[0, H]$ is partitioned equally into N parts each of length T. Perfect preventive maintenance or replacement is performed periodically at times $i \cdot T$, $i = 0, 1, \ldots, N$, following which the unit is as good as new. When a unit fails between preventive maintenance actions, only minimal repair is made (returning the unit to the state “as bad as old”), and hence the failure rate remains undisturbed by any repair at failure. It is assumed that the repair and replacement times are negligible.

It is assumed that the total time required to perform both types of maintenance activities (preventive and corrective) does not exceed the horizon H. Using the Cox model (Cox 1972), which established a parametric relationship between risk factors (related to the operational and environmental conditions of each period) and the hazard rate, we define a failure law that establishes a parametric relation between the production rate of each period and the nominal distribution.

We assume that:

$$\lambda_k(t, u(k)) = \lambda_n(t) \cdot g(u(k)) \tag{8}$$

where $\lambda_k(t, u(k))$ represents the instantaneous failure rate function at period k as a function of the production rate $u(k)$, $\lambda_n(t)$is the failure rate for nominal conditions which is equivalent to the failure rate with maximal production, and $g(u(k)) = \frac{u(k)}{U_{\max}}$.

For the maintenance strategy, the idea is to minimize the total cost of maintenance taking into account the costs incurred by the preventive and corrective maintenance actions and the expected number of failures that can occur during the production horizon $H \cdot \Delta t$. This strategy, based on the functional age concept, reveals the significant influence of the production rate on the deterioration of the manufacturing system. Formally, the maintenance cost is expressed as follows:

$$C_M(U(u(1), u(1), \ldots u(H-1)), N) = C_{pm} \times (N-1) + C_{cm} \times \varphi_M(U, N) \tag{9}$$

where $\varphi(U, N)$is the average number of failures as a function of the production plan defined by the vector $U$and the number of preventive maintenance actions N.

Analytical study

Production policy

The purpose of this subsection is to develop and optimize the expected production and holding costs over the finite time horizon H such that the demand is satisfied at the end of each period. The problem can be formulated as a stochastic optimization problem under a threshold inventory level constraint. Due to the stochastic nature of our problem, the constraints and the dimensionality, to try to obtain an optimal solution can become a hard task. An approach that transforms the stochastic problem into a deterministic equivalent is necessary. This deterministic problem maintains the main properties of the original problem, that is, the linearity of the inventory balance equation (2), the convexity of the HMMS's functional costs described by equation (7) and the random demand described by Gaussian processes.

Before proceeding, the following notation is introduced:

Mean variables: $E\{S_1(k)\} = \hat{S}_1(k)$, $E\{S_2(k)\} = \hat{S}_2(k)$, $E\{u(k)\} = \hat{u}(k)$, $E\{u_s(k)\} = \hat{u}_s(k)$

Variance variables: $V_{u(k)} = V_{u_s(k)} = 0$. (Note that this reflects the fact that the control variables $u(k)$ and $u_s(k)$ are deterministic).

Production and inventory costs

Let us now convert the objective function (1) to a deterministic equivalent.

Lemma 1

$$f(u) = \sum_{k=0}^{H} C_s \times \Big[\hat{S}_1(k)^2 + \hat{S}_2(k)^2 + \Big(k\cdot\delta^2\cdot(1-\mu)^2 + \Big(k + (k-1)\cdot(\mu\cdot\delta)^2\Big)\Big)\cdot\sigma_d^2\Big] + \sum_{k=1}^{H-1}\Big(C_{p_r}\times\hat{u}(k)^2 + C_{ps}\times\hat{u}_s(k)^2\Big) \qquad (10)$$

Proof It is assumed that the demand variable has its first and second statistic moments perfectly known for each period k, that is, $E\{d(k)\} = \hat{d}(k)$ and $V_{d(k)} = \sigma_d^2$ for each k.

The inventory variables $S_1(k)$ and $S_2(k)$ are statistically described respectively by their means $E\{S_1(k)\} = \hat{S}_1(k)$ and $E\{S_2(k)\} = \hat{S}_2(k)$ as well as their variances $V_{S_1(k)} = E\{(S_1(k) - \hat{S}_1(k))^2\}$ and $V_{S_2(k)} = E\{(S_2(k) - \hat{S}_2(k))^2\}$

The inventory balance equation (2) can be reformulated as:

$$\hat{S}_1(k+1) = \hat{S}_1(k) + \hat{u}_s(k) + \hat{u}(k) + \mu\cdot\delta\cdot\hat{d}(k-\tau_1) - \hat{d}(k) \quad k = 0,1,\ldots,H-1 \qquad (11)$$

Likewise, the inventory balance equation (4) can be reformulated as:

$$\hat{S}_2(k+1) = \hat{S}_2(k) + (1-\mu)\cdot\delta\cdot\hat{d}(k-\tau_1) - \hat{u}_2(k) \qquad (12)$$

- If we take the difference between Eqs. (2) and (11) we obtain:

$$\begin{aligned}
S_1(k+1) - \hat{S}_1(k+1) &= \Big(S_1(k) - \hat{S}_1(k)\Big) + \Big(\underbrace{u_s(k) - \hat{u}_s(k)}_{0}\Big) + \Big(\underbrace{u(k) - \hat{u}(k)}_{0}\Big) \\
&\quad + \mu\cdot\delta\cdot\Big(d(k-\tau_1) - \hat{d}(k-\tau_1)\Big) - \Big(d(k) - \hat{d}(k)\Big) \\
\Rightarrow \Big(S_1(k+1) - \hat{S}_1(k+1)\Big)^2 &= \Big(\Big(S_1(k) - \hat{S}_1(k)\Big) + \mu\cdot\delta\cdot\Big(d(k-\tau_1) - \hat{d}(k-\tau_1)\Big) - \Big(d(k) - \hat{d}(k)\Big)\Big)^2 \\
\Rightarrow E\Big\{\Big(S_1(k+1) - \hat{S}_1(k+1)\Big)^2\Big\} &= E\Big\{\Big(S_1(k) - \hat{S}_1(k)\Big)^2\Big\} \\
&\quad + E\Big\{(\mu\cdot\delta)^2\Big(d(k-\tau_1) - \hat{d}(k-\tau_1)\Big)^2\Big\} \\
&\quad + E\Big\{\Big(d(k) - \hat{d}(k)\Big)^2\Big\} \\
&\quad + 2\cdot E\Big\{\Big(S_1(k) - \hat{S}_1(k)\Big)\cdot\mu\cdot\delta\cdot\Big(d(k-\tau_1) - \hat{d}(k-\tau_1)\Big)\Big\} \\
&\quad - 2\cdot E\Big\{\Big(S_1(k) - \hat{S}_1(k)\Big)\cdot\Big(d(k) - \hat{d}(k)\Big)\Big\} \\
&\quad - 2\cdot E\Big\{\Big(d(k) - \hat{d}(k)\Big)\cdot\mu\cdot\delta\cdot\Big(d(k-\tau_1) - \hat{d}(k-\tau_1)\Big)\Big\}
\end{aligned}$$

Since $S_1(k)$ and $d(k)$ are independent random variables, we can deduce that:

$$\begin{aligned}
&E\Big\{\Big(S_1(k) - \hat{S}_1(k)\Big)\cdot\Big(d(k) - \hat{d}(k)\Big)\Big\} \\
&\quad = E\Big\{S_1(k) - \hat{S}_1(k)\Big\}\, E\Big\{d(k) - \hat{d}(k)\Big\} \\
&E\Big\{\Big(S_1(k) - \hat{S}_1(k)\Big)\cdot\mu\cdot\delta\cdot\Big(d(k-\tau_1) - \hat{d}(k-\tau_1)\Big)\Big\} \\
&\quad = \mu\cdot\delta\cdot E\Big\{S_1(k) - \hat{S}_1(k)\Big\}\cdot E\Big\{d(k-\tau_1) - \hat{d}(k-\tau_1)\Big\} \\
&E\Big\{\Big(d(k) - \hat{d}(k)\Big)\cdot\mu\cdot\delta\cdot\Big(d(k-\tau_1) - \hat{d}(k-\tau_1)\Big)\Big\} \\
&\quad = \mu\cdot\delta\cdot E\Big\{d(k-\tau_1) - \hat{d}(k-\tau_1)\Big\}\cdot E\Big\{d(k) - \hat{d}(k)\Big\}
\end{aligned}$$

Hence

$$\begin{aligned}
&E\Big\{S_1(k) - \hat{S}_1(k)\Big\} = E\{S_1(k)\} - E\Big\{\hat{S}_1(k)\Big\} = 0 \\
&E\Big\{d(k) - \hat{d}(k)\Big\} = E\{d(k)\} - E\Big\{\hat{d}(k)\Big\} = 0 \\
&E\Big\{d(k-\tau_1) - \hat{d}(k-\tau_1)\Big\} = E\{d(k-\tau_1)\} \\
&\quad - E\Big\{\hat{d}(k-\tau_1)\Big\} = 0
\end{aligned}$$

Therefore

$$\Rightarrow E\left\{\left(S_1(k+1)-\hat{S}_1(k+1)\right)^2\right\}$$
$$=E\left\{\left(S_1(k)-\hat{S}_1(k)\right)^2\right\}+(\mu\cdot\delta)^2$$
$$\cdot E\left\{\left(d(k-\tau_1)-\hat{d}(k-\tau_1)\right)^2\right\}+E\left\{\left(d(k)-\hat{d}(k)\right)^2\right\}$$

If we assume that $V_{S_1}(k=0)=0$ and that σ_{d_k} is constant and equal to σ_d for all periods, we can deduce that

$$V_{s_1}(k+1)=V_{s_1}(k)+\left((\mu\cdot\delta)^2+1\right)\cdot\sigma_d^2 \quad k\geq 1,\tau_1\geq 1$$

For $k=0$, $V_{s_1}(1)=\sigma_d^2$,

$$V_{s_1}(k)=\sigma_d^2\left(k+(k-1)\cdot(\mu\cdot\delta)^2\right)$$

Since $V_{S_1(k)}=E\left\{(S_1(k)-\hat{S}_1(k))^2\right\}=E\left\{S_1(k)^2\right\}-\hat{S}_1(k)^2$, we can write

$$E\left\{S_1(k)^2\right\}-\hat{S}_1(k)^2=\sigma_d^2\left(k+(k-1)\cdot(\mu\cdot\delta)^2\right).$$

Hence,

$$E\left\{S_1(k)^2\right\}=\sigma_d^2\left(k+(k-1)\cdot(\mu\cdot\delta)^2\right)+\hat{S}_1(k)^2 \tag{13}$$

- If we take the difference between Eqs. (4) and (12) we obtain:

$$S_2(k+1)-\hat{S}_2(k+1)$$
$$=\left(S_2(k)-\hat{S}_2(k)\right)+(1-\mu)\cdot\delta\cdot\left(d(k-\tau_1)-\hat{d}(k-\tau_1)\right)$$
$$\Rightarrow E\left\{\left(S_2(k+1)-\hat{S}_2(K+1)\right)^2\right\}$$
$$=E\left\{\left(S_2(k)-\hat{S}_2(k)\right)^2\right\}$$
$$+(1-\mu)^2\cdot\delta^2\cdot E\left\{\left(d(k-\tau_1)-\hat{d}(k-\tau_1)\right)^2\right\}$$
$$+2\cdot(1-\mu)\cdot\delta\cdot E\left\{\left(d(k-\tau_1)-\hat{d}(k-\tau_1)\right)\right.$$
$$\left.\cdot\left(S_2(k)-\hat{S}_2(k)\right)\right\}$$

Since $S_2(k)$ and $d(k-\tau_1)$ are independent random variables, we can deduce that:

$$(1-\mu)\cdot\delta\cdot E\left\{\left(d(k-\tau_1)-\hat{d}(k-\tau_1)\right)\cdot\left(S_2(k)-\hat{S}_2(k)\right)\right\}$$
$$=(1-\mu)\cdot\delta\cdot E\left\{\left(d(k-\tau_1)-\hat{d}(k-\tau_1)\right)\right\}$$
$$\cdot E\left\{\left(S_2(k)-\hat{S}_2(k)\right)\right\}$$

with

$$E\left\{\left(d(k-\tau_1)-\hat{d}(k-\tau_1)\right)\right\}$$
$$=E\{d(k-\tau_1)\}-E\{\hat{d}(k-\tau_1)\}=0$$
$$E\left\{\left(S_2(k)-\hat{S}_2(k)\right)\right\}=E\{S_2(k)\}-E\{\hat{S}_2(k)\}=0$$

Therefore,

$$\Rightarrow E\left\{S_2(k+1)-\hat{S}_2(K+1)\right\}$$
$$=E\left\{\left(S_2(k)-\hat{S}_2(k)\right)^2\right\}$$
$$+(1-\mu)^2\cdot\delta^2\cdot E\left\{\left(d(k-\tau_1)-\hat{d}(k-\tau_1)\right)^2\right\}$$
$$\Rightarrow V_{s_2}(k+1)=V_{s_2}(k)+\delta^2\cdot(1-\mu)^2\cdot\sigma_d^2$$

If we assume that $V_{S_2}(k=0)=0$ and that σ_{d_k} is constant and equal to σ_d for all the periods, we can deduce that:

$$V_{s_2}(k)=k\cdot\delta^2\cdot(1-\mu)^2\cdot\sigma_d^2$$

Since $Var_{S_2(k)}=E\left\{\left(S_2(k)-\hat{S}_2(k)\right)^2\right\}=E\left\{S_2(k)^2\right\}-\hat{S}_2(k)^2$,

We can write $E\left\{S_2(k)^2\right\}-\hat{S}_2(k)^2=k\cdot\delta^2\cdot(1-\mu)^2\cdot\sigma_d^2$. Hence,

$$E\left\{S_2(k)^2\right\}=k\cdot\delta^2\cdot(1-\mu)^2\cdot\sigma_d^2+\hat{S}_2(k)^2 \tag{14}$$

Substituting (13) and (14) in (7) we obtain:

$$f(u)=C_s\times\left(E\left\{S_1(H)^2\right\}+E\left\{S_2(H)^2\right\}\right)$$
$$+\sum_{k=1}^{H-1}\left(C_{p_r}\times u(k)^2+C_{ps}\times E\left\{u_s(k)^2\right\}\right)$$
$$+\sum_{k=1}^{H-1}C_s\times\left(E\left\{S_1(k)^2\right\}+E\left\{S_2(k)^2\right\}\right)$$

with $k\in\{0,1,\ldots,H-1\}$

$$=>\ f(u)=\sum_{k=0}^{H}C_s\times\left[\hat{S}_1(k)^2+\hat{S}_2(k)^2\right.$$
$$\left.+k\cdot\delta^2\cdot(1-\mu)^2\cdot\sigma_d^2+\left(k+(k-1)\cdot(\mu\cdot\delta)^2\right)\cdot\sigma_d^2\right]$$
$$+\sum_{k=1}^{H-1}\left(C_{p_r}\times\hat{u}(k)^2+C_{ps}\times\hat{u}_s(k)^2\right)$$

Service level constraint

Constraints on state and decision variables significantly increase the complexity of solving an optimization problem. Generally in stochastic cases it is complicated just to guarantee feasibility, though one possibility of overcoming such difficulty is to consider probabilistic constraints. Another important transformation changes the service level constraint into equivalent, but deterministic, inequalities by specifying through the following lemma a minimum cumulative production quantity depending on the service level requirements.

Lemma 2 *We recall that* θ *defines the targeted service level as expressed by constraint (3), repeated below:*

$$\text{Prob}\left[S_1(k+1) \geq 0\right] \geq \theta \quad \text{with} \quad 0 \leq u(k) \leq U_{\max}.$$

Then, for $k = 0, 1, \ldots, H-1$ *we have:*

$$\text{Prob}\big(S_1(k+1) \geq 0\big) \geq \theta \Rightarrow \Big(u(k) \geq \big(V_{d(k)} \times V_{d(k-\tau)}\big) \times \varphi^{-1}(\theta) - S_1(k) - u_s(k) + \hat{d}(k) - \mu \cdot \delta \cdot \hat{d}(k-\tau_1)\Big) \quad k = 0, 1, \ldots, H-1 \tag{15}$$

φ : *Cumulative Gaussian distribution function with mean* $\left(\frac{1}{V_{dk-\tau}} \times \hat{d}(k) - \frac{\delta}{V_d} \times \hat{d}(k-\tau)\right)$. *and finite variance* $\left(\left(\frac{1}{V_{d,k-\tau}}\right)^2 \times V_{d,k} + \left(-\frac{\delta}{V_d}\right)^2 \times V_{d,k-\tau} \geq 0\right)$

φ^{-1} : *Inverse distribution function*

Proof

$$S_1(k+1) = S_1(k) + u_s(k) + u(k) + \mu \cdot \delta \cdot d(k-\tau_1) - d(k)$$

$$\text{Prob}\left(S_1(k+1) \geq 0\right) \geq \theta$$

$$\text{Prob}\big(S_1(k) + u_s(k) + u(k) + \mu \cdot \delta \cdot d(k-\tau_1) - d(k) \geq 0\big) \geq \theta$$

$$\text{Prob}\big(S_1(k) + u_s(k) + u(k) + \mu \cdot \delta \cdot d(k-\tau_1) \geq d(k)\big) \geq \theta$$

$$\text{Prob}\left(S_1(k) + u_s(k) + u(k) + \mu \cdot \delta \cdot d(k-\tau_1) - \hat{d}(k) \geq d(k) - \hat{d}(k)\right) \geq \theta$$

$$\text{Prob}\left(d(k) - \hat{d}(k) \leq S_1(k) + u_s(k) + u(k) + \mu \cdot \delta \cdot d(k-\tau_1) - \hat{d}(k)\right) \geq \theta$$

$$\text{Prob}\left(d(k) - \hat{d}(k) - \mu \cdot \delta \cdot d(k-\tau_1) + \mu \cdot \delta \cdot \hat{d}(k-\tau_1) \leq S_1(k) + u_s(k) + u(k) - \hat{d}(k) + \mu \cdot \delta \cdot \hat{d}(k-\tau_1)\right) \geq \theta$$

$$\text{Prob}\left(\frac{d(k) - \hat{d}(k) - \mu \cdot \delta \times \left(d(k-\tau) - \hat{d}(k-\tau)\right)}{V_{d,k} \times V_{d,k-\tau}} \leq \frac{S_1(k) + u_s(k) + u(k) - \hat{d}(k) + \mu \cdot \delta \cdot \hat{d}(k-\tau_1)}{V_{d,k} \times V_{d,k-\tau}}\right) \geq \theta$$

□

$$\text{Prob}\left(\frac{1}{V_{d,k-\tau}} \times \frac{d(k) - \hat{d}(k)}{V_{d,k}} - \frac{\mu \cdot \delta}{V_{d,k}} \times \frac{d(k-\tau) - \hat{d}(k-\tau)}{V_{d,k-\tau}} \leq \frac{S_1(k) + u_s(k) + u(k) - \hat{d}(k) + \mu \cdot \delta \cdot \hat{d}(k-\tau_1)}{V_{d,k} \times V_{d,k-\tau}}\right) \geq \theta \tag{16}$$

Note that $X = \left(\frac{d(k)-\hat{d}(k)}{V_{d,k}}\right)$ is a Gaussian random variable with an identical distribution as $d(k)$.

and that $Y = \left(\frac{d(k-\tau_1)-\hat{d}(k-\tau_1)}{V_{d,k-\tau_1}}\right)$ is a Gaussian random variable with an identical distribution as $d(k-\tau)$.

This formulation is equivalent to $\text{Prob}(A \times X + B \times Y \leq C) \geq \theta$ with $A = \frac{1}{V_{d,k-\tau}}$ and $B = -\frac{\mu \cdot \delta}{V_{d,k}}$.

$X' = A \times X$ is a Gaussian random variable with an identical distribution as $f_{X'} = \frac{1}{A} \times f\left(\frac{y}{A}\right)$, with mean $A \times \hat{d}(k) = \frac{1}{V_{d,k-\tau}} \times \hat{d}(k)$ and variance $A^2 \times V_{d,k} = \left(\frac{1}{V_{d,k-\tau}}\right)^2 \times V_{d,k} \geq 0$ while $Y' = B \times Y$ is a Gaussian random variable with an identical distribution as $f_{Y'} = -\frac{1}{B} \times f\left(\frac{y}{B}\right)$, with mean $B \times \hat{d}(k-\tau_1) = -\frac{\mu \cdot \delta}{V_{d,k}} \times \hat{d}(k-\tau_1)$ and variance $B^2 \times V_{d,k-\tau_1} = \left(\frac{\mu \cdot \delta}{V_{d,k}}\right)^2 \times V_{d,k-\tau_1} \geq 0$.

Thus $T' = X' + Y'$ is a Gaussian random variable with an identical distribution as $h = f_{X'} * f_{Y'}$, with mean $A \times \hat{d}(k) + B \times \hat{d}(k-\tau_1)$ and variance $A^2 \times V_{d,k} + B^2 \times V_{d,k-\tau} \geq 0$ with $A = \frac{1}{V_{d,k-\tau}}$ and $B = -\frac{\mu \cdot \delta}{V_{d,k}}$.

φ is a cumulative Gaussian distribution function of T'.

$$(16) \Rightarrow \varphi\left(\frac{S_1(k) + u_s(k) + u(k) - \hat{d}(k) + \mu \cdot \delta \cdot \hat{d}(k-\tau_1)}{V_{d,k} \times V_{d,k-\tau}}\right) \geq \theta \tag{17}$$

Since $\lim_{d_k \to -\infty} \varphi_{d_k} = 0$ and $\lim_{d_k \to +\infty} \varphi_{d_k} = 1$, the function ϕ_{d_k} is strictly increasing, and we note that it is indefinitely differentiable. That's why we conclude that φ_{d_k} is invertible.

Thus,

$$\frac{S_1(k) + u_s(k) + u(k) - \hat{d}(k) + \mu \cdot \delta \cdot \hat{d}(k-\tau_1)}{V_{d,k} \times V_{d,k-\tau}} \geq \varphi^{-1}(\theta) \tag{18}$$

$$\Rightarrow u(k) \geq \left(V_{d,k} \times V_{d,k-\tau}\right) \times \varphi^{-1}(\theta) - S_1(k) - u_s(k) + \hat{d}(k) - \mu \cdot \delta \cdot \hat{d}(k-\tau_1)$$

It can consequently be concluded that

$$\text{Prob}\big(S_1(k+1) \geq 0\big) \geq \theta \;\Rightarrow\; \big(u(k) \geq \left(V_{d,k} \times V_{d,k-\tau}\right) \times \varphi^{-1}(\theta) - S_1(k) - u_s(k) + \hat{d}(k) - \mu \cdot \delta \cdot \hat{d}(k-\tau_1)\big) \quad k = 0, 1, \ldots, H-1$$

Maintenance policy

For the maintenance policy, we seek to determine the optimal maintenance strategy characterized by the optimal number N^* of preventive maintenance actions and the time between them T^*, as given by Eq. (19).

$$T^* = \frac{H}{N^*} \tag{19}$$

The analytic expression of the total maintenance cost is as follows, with $N \in \{1, 2, 3 \ldots\}$.

$$C_M(U, N) = (N-1) \cdot C_{pm} + C_{cm} \cdot \varphi_M(U, N) \tag{20}$$

where $\varphi_M(U, N)$corresponds to the expected number of failures that occur during the horizon H, considering the production rate in each production period Δt. We recall that the manufacturing system considered in this study is composed of a machine M characterized by the reliability function $R_k(t, u(k))$ $(k = 0, 1 \ldots N-1)$.

The average number of failures is defined as follows:

$$\varphi_M(U, N) = \sum_{i=0}^{\dim(U)} \int_{t_{p_i}}^{t_{p_{i+1}}} \lambda_{p_{i+1}}(t, u(k))\, dt \tag{21}$$

where $t_{p_i} = t_{p_{i-1}} + \Delta t$, $(t_{p_0} = 0)$, the system age at the end of period p_i, and $\lambda_{p_i}(t, u(k))$= the failure law after p_i periods.

However, the relationship defined by Eq. (21) is not accurate because it does not ensure the continuity of the reliability function. This situation is rectified via the use of the functional age concept. In essence, if two successive production periods $p_i[(i-1) \cdot \Delta t, i \cdot \Delta t]$ and $p_{i+1}[i \cdot \Delta t, (i+1) \cdot \Delta t]$ have different production rates u_i and u_{i+1}, the continuity of the reliability function requires that the following relation holds.

$$R_{u(i)}(t_{u(i)}) = R_{u(i+1)}(t_{u(i)}) \tag{22}$$

where

$t_{u(i)}$: system age at the end of the period where the production rate is equal to $u(i)$.

$R_{u(i)}$: reliability function corresponding to the production rate $u(i)$.

From this relationship, we note that the reliability function at the end of period p_i is equal to the reliability function of the beginning of period p_{i+1}. Therefore, this relation to hold, the production rate $u(i)$ during $i \cdot \Delta t$ is equivalent to the production rate $u(i+1)$ during a different duration denoted d_{ui+1}. This new duration characterizes the operational age of the system and Eq. (22) becomes:

$$R_{u(i)}(t_{u(i)}) = R_{u(i+1)}(\Gamma_{u(i+1)}) \tag{23}$$

where $\Gamma_{u(j+1)} = \sum_{i=1}^{j} d_{u(i)}$ $\quad j = \{1, 2, \ldots, H\}$ represents the functional age of the system at the beginning of period $i+1$.

The average number of failures occurring during the production plan is given by:

$$\varphi_M(U, N) = \sum_{n=1}^{N} \underbrace{\left(\sum_{i=d(n)}^{f(n)} \int_{\Gamma_i}^{\Gamma_i+\sigma_i} \lambda_i(t, u(i))dt \right)}_{\substack{average\ number\ of\ failure \\ for\ each\ PM\ intervals}} \tag{24}$$

We assume that notation $\Gamma_i = \Gamma_{u(i)}$

We can observe from Fig. 2 that the production periods can be linked to several maintenance intervals. For example, the production period p_2 is divided between maintenance intervals A and B. So, the preventive maintenance intervals are constituted of several production periods. The maintenance interval B is composed of parts of production periods p_2 and p_4 and all of period p_3.

Calculating the average number of failures that occur during the production plan is determined for each maintenance interval, which is itself divided into several production periods. Consequently, it's necessary to know the values of d(n) and f(n) that respectively represent the beginning and the end periods of the interval $n = \{1, 2, \ldots, N\}$.

The duration of these production periods for each interval is denoted σ_i, with $\sigma_i \leq \Delta t$. Note from Fig. 2 that the production periods p_2 and p_4 are represented respectively by $d(B)$ and $f(B)$ for the maintenance interval B. The duration of production period p_2 in this interval is given by $(2 \cdot \Delta t - T)$, and the duration of period p_4 is equal to $(2 \cdot T - (3 \cdot \Delta t))$.

From Eq. (8):

$$\lambda_i(t, u(i)) = \frac{u(i)}{U_{\max}} \cdot \lambda_n(t) \tag{25}$$

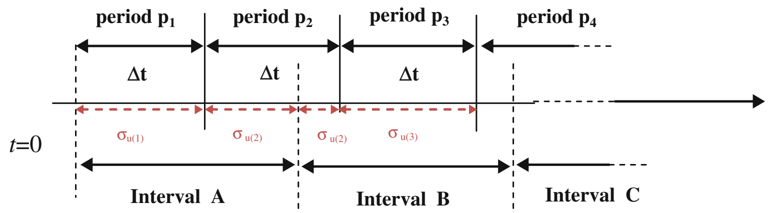
Hence

$$\varphi_M(U, N) = \sum_{n=1}^{N} \left(\sum_{i=d(n)}^{f(n)} \int_{\Gamma_i}^{\Gamma_i+\sigma_i} \frac{u(i)}{U_{\max}} \cdot \lambda_n(t)dt \right) \tag{26}$$

Although the failure rate varies according to the production rates, the system reliability must be continuous over time. Thus, the role of the functional age is to guarantee the continuity of the reliability function by defining a representative duration of the system age. Consequently, the determination of the functional age requires knowledge of the reliability at the end of the previous period as well as the production rates for the previous and next periods.

With the functional age-based model, the virtual age is given by:

Fig. 2 Maintenance policy example



Lemma 3

$$\Gamma_i = R_i^{-1}\left(R_{i-1}\left(t_{u(i-1)}\right)\right) i \geq 2 \quad \text{with} \quad t_{u(i)} = \Gamma_{i-1} + \sigma_{u(i)} \tag{27}$$

$\sigma_{u(i)}$: *duration of period corresponding to the beginning and the end of each maintenance interval*

As the case, the duration of a period is expressed by:

$$\sigma_{u(i)} = \begin{cases} \Delta t & \text{if } d(n) \neq i \neq f(n) \\ \sum_{l=1}^{d(n)} \Delta t - (n-1) \cdot T & \text{if } d(n) = i \neq f(n) \\ n \cdot T - \sum_{l=1}^{f(n)-1} \Delta t & \text{if } d(n) \neq i = f(n) \\ T & \text{if } d(n) = i = f(n) \end{cases}$$

For the more detailing of the functional age concept, see the Appendix.

Numerical example

The effectiveness of the method discussed in the previous section is illustrated here via a numerical example. A company whose sales are subject to the effects of seasonality seeks to develop a production plan which minimizes total cost over a finite planning horizon $H = 72$ periods each of $\Delta t = 1$ month duration. For the maintenance policy, the machine M has a degradation law characterized by a Weibull distribution with scale and shape parameters respectively equal to $\beta = 16.79$ and $\alpha = 3$, while $C_{cm} = 3000$ mu and $C_{pm} = 500$ mu. The only information known about M_s is its availability rate $\beta_s = 0.9$. In the case of not satisfaction of customer with the product's quality, he has the right to return it within a specific deadline. The returned products defined by a rate $\delta \epsilon \{0.1, 0.3, 0.5\}$. The products witch are in saleable condition will be stored in the central purchasing department in order to be relisted for sale and is defined by a rate $\mu \epsilon \{0.5, 0.8\}$. Meanwhile, the returned products that are defective will be repaired, recycled or remanufactured by a subcontractor located in France, following which the product will be returned to the central purchasing department is given by the following rate $(1 - \mu)$.

The values of the other parameters are as follows:

$C_{pr} = 7$ mu, $C_{prs} = 25$ *mu*, $U_{max} = 500$, $\theta = 0.9$, $C_s = 0.65$ *mu*, $S_1^0 = 0$, $S_2^0 = 0$, $V_{d_k} = 4.52$ and $\tau_1 = 1$.

The average demand is presented in Table 1 below

The economically production plans $\left(U^* = \{u^*(0), u^*(2), \ldots, u^*(H-1)\}\right)$ of principle machine are presented in Figs. 3, 4, and 5.

Case1: $\mu = 0.5$: Figs. 3, 4, 5, 6, 7, and 8.

Case2: $\mu = 0.8$: Figs. 9, 10, and 11.

Figures 6, 7, 8, 9, 10, and 11 show the following results, where ζ^* is the optimal total cost: The above figures illustrates the minimum total cost for different values of the number, N, of PM actions to be performed. For each value of δ and μ, we calculate the economical production plan and the optimal number of preventive maintenance. The obtained minimal total cost for each value of δ and μ as following:

For $\mu = 0.5$

For $\delta = 0.1$: $N^* = 4$, $\zeta^* = 761.88$ *mu* and $T^* = H/N^* = 18\Delta t$.

For $\delta = 0.3$: $N^* = 3$, $\zeta^* = 731.46$ *mu* and $T* = H/N* = 24\Delta t$.

For $\delta = 0.5$: $N^* = 2$, $\zeta^* = 548.05$ *mu* and $T^* = H/N^* = 36\Delta t$.

For $\mu = 0.8$

Table 1 The monthly mean demand

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
350	420	340	392	431	444	442	340	392	375	392	400
350	370	395	415	431	444	442	340	392	375	400	420
350	340	392	370	431	392	500	350	320	420	365	480
300	290	360	370	460	460	358	400	442	340	392	375
400	431	400	390	350	290	358	360	460	442	392	450
400	450	360	350	300	392	375	400	431	400	390	450

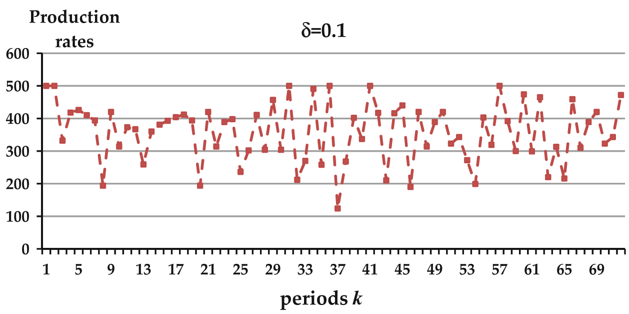


Fig. 3 Optimal production plan for $\delta = 0.1$

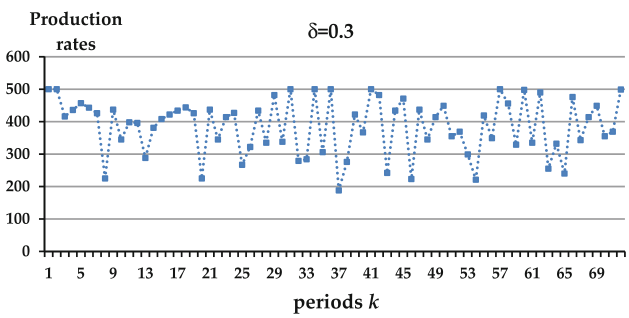


Fig. 4 Optimal production plan for $\delta = 0.3$

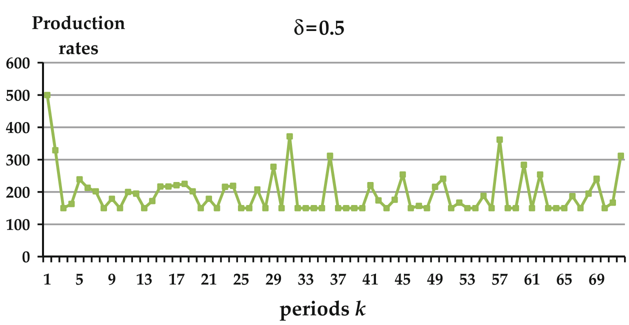


Fig. 5 Optimal production plan for $\delta = 0.5$

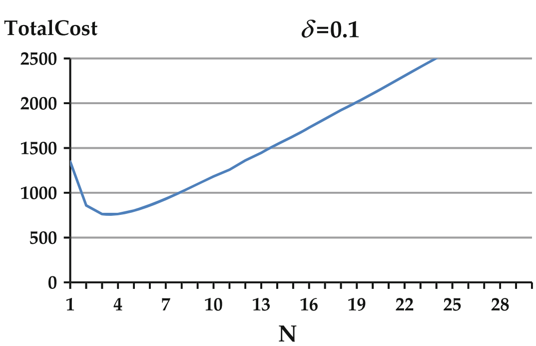


Fig. 6 Optimal total cost for$\delta = 0.1 (N^* = 4, \zeta^* = 761.88\ mu)$

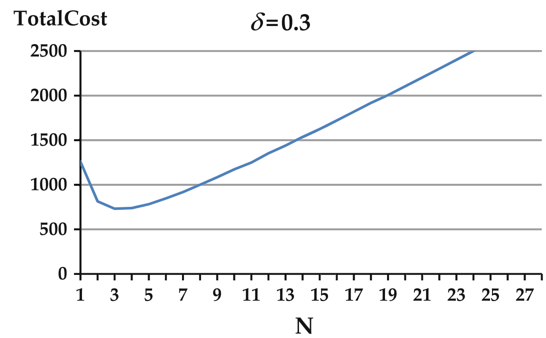


Fig. 7 Optimal total cost for $\delta = 0.3 (N^* = 3, \zeta^* = 731.46\ mu)$

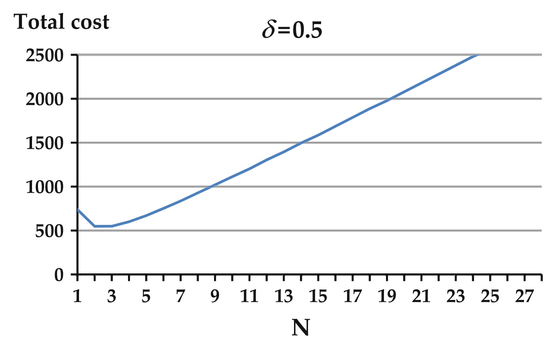


Fig. 8 Optimal total cost for $\delta = 0.5 (N^* = 2, \zeta^* = 548.05\ mu)$

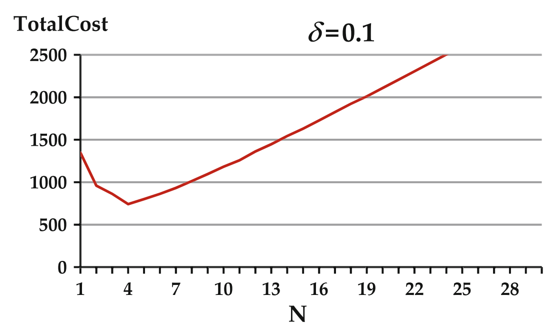


Fig. 9 Optimal total cost for $\delta = 0.1 (N^* = 4, \zeta^* = 741.89\ mu)$

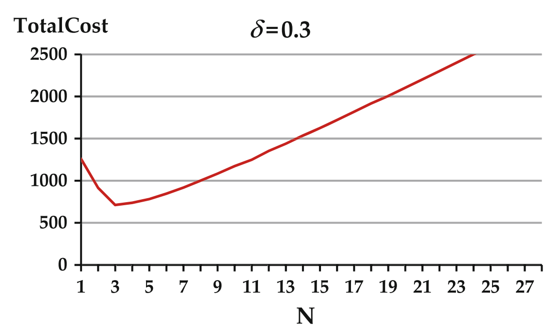


Fig. 10 Optimal total cost for $\delta = 0.3 (N^* = 3, \zeta^* = 711.56\ mu\ mu)$

For $\delta = 0.1 : N^* = 4, \zeta^* = 741.89\ mu$ and $T^* = H/N^* = 18\Delta t$.

For $\delta = 0.3 : N^* = 3, \zeta^* = 711.56\ mu$ and $T^* = H/N^* = 24\Delta t$.

For $\delta = 0.5 : N^* = 2, \zeta^* = 458.52\ mu$ and $T^* = H/N^* = 36\Delta t$.

Hence, the higher the value of δ (percentage of backordered product) and μ (percentage of returned product that

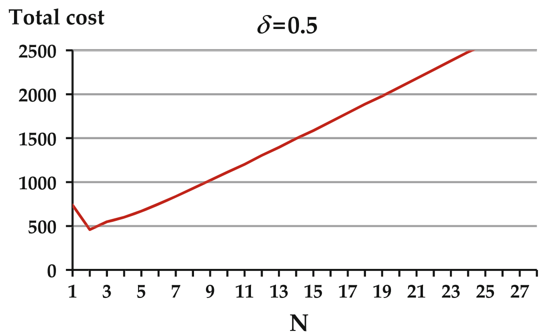


Fig. 11 Optimal total cost for $\delta = 0.5(N^* = 2, \zeta^* = 458.52)$

is saleable and sent back to store S_1), the lower of the optimal total cost and the number of preventive maintenance actions. The best integrated policy consists in performing two preventive maintenance actions over the time horizon $H.\Delta t$ at $\mu = 0.8$ and $\delta = 0.5$, yielding $\zeta^* = 458.52$ *mu*. This cost is significantly lower than that obtained for $\mu = 0.8$ ($\delta = 0.1, \delta = 0.3$) and $\mu = 0.5(\delta = 0.1, \delta = 0.3, \delta = 0.5)$. This can be explained by the fact that, as δ increases, the production rates decrease. Similarly if μ increases, the production rates decrease, resulting in fewer preventive maintenance actions.

Conclusion

Based on an industrial case study, this paper considered a manufacturing system that is subject to random failures. A minimal repair is performed each time a failure occurs. In order to reduce the failure frequency, preventive maintenance actions are scheduled as a function of the production rate. A random product demand must be satisfied over a finite planning horizon with a given required service level. Meantime, products returned by the customers which are in saleable condition are re-stocked, while those returned products that are non-conformal are sent to a subcontractor for recycling and remanufacturing. A jointly optimal production plan and preventive maintenance program is obtained via a linear quadratic stochastic optimization problem. A key aspect of this study is that the production system's failure rate increases with both time and the production rate. We used the *HMMS* model with a maintenance policy based on the functional age concept taking into consideration the influence of the production rates on the system deterioration.

For future research, we propose to consider many practical situations in which preventive maintenance actions are imperfect and have non-negligible durations. It would be interesting to assess the impact of such a situation on the optimal production plan.

Appendix

The following figure illustrated the concept of functional age. Two production periods (period1 and period 2), with different production rates ($u(1)$ and $u(2)$), are represented by the curves of reliability R_1(t) and R_2(t). The reliability of system at the end of first production period (period 1) amounted to $R_1(t_1)$. In order to guarantee this reliability in the beginning of the second period (period 2), we assumed that the system worked during period d_1 under the production rate $u(2)$ of second period (See Fig. a1).

The various steps for determining the functional age are as following:

- Determination of date t_1 using the production period
- Determination of the reliability $R_1(t_1)$.
- determination of date Γ_1 in function to reliability $R_1(t_1)$ $= R_2(\Gamma_1)$

The length of preventive maintenance intervals is fixed. The maintenance plan time is divided into $(N + 1)$ regular

Fig. a1 Functional age

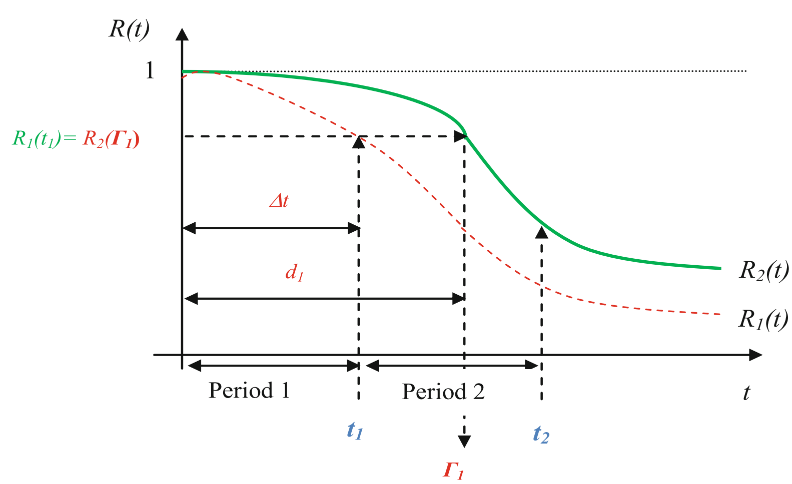


Fig. a2 Case 1

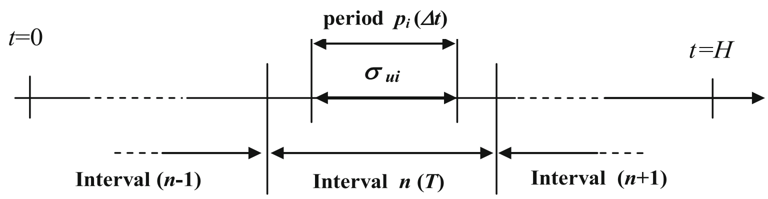


Fig. a3 Case 2

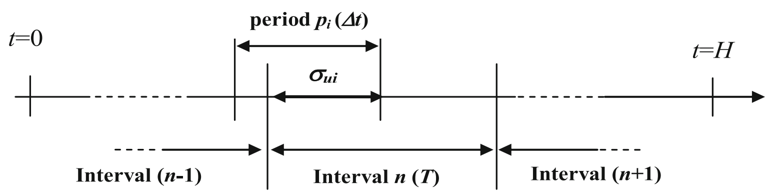


Fig. a4 Case 3

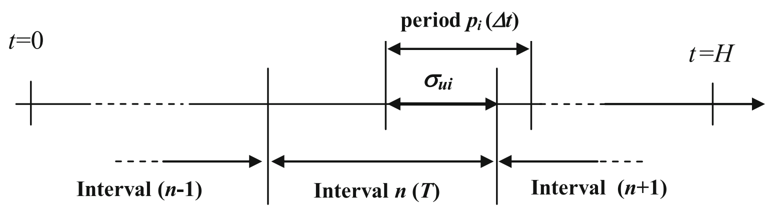
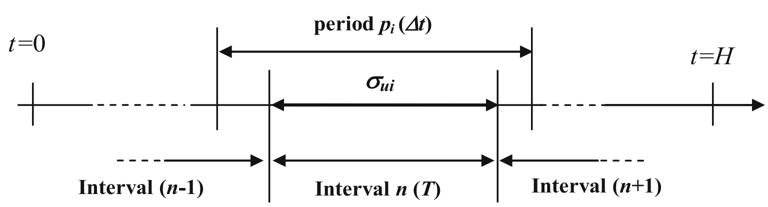


Fig. a5 Case 4



intervals. The preventive maintenances activities can be carried out during the production period.

Hence, it is necessary to determine the periods corresponding to the beginning and the end of each interval. (See Figs. a2, a3, a4, a5).

- Case 1 if period $p_i \subset T$, the production period is completely performed $(d(n) \neq i \neq f(n))$.

The duration of period p_i in the interval n is given by:

$$\sigma_{u_i} = \Delta t$$

- Case 2 if period p_i begins the preventive maintenance interval and it does not finish $(d(\text{n}) = i \neq f(\text{n}))$

The duration of period p_i in the interval n is given by:

$$\sigma_{u_i} = \sum_{l=1}^{d(n)} \Delta t - (n-1) \cdot T$$

- Case 3 if period p_i does not begins the preventive maintenance interval and it finishes $(d(\text{n}) \neq i = f(\text{n}))$

The duration of period p_i in the interval n is given by:

$$\sigma_{u_i} = n \cdot T - \sum_{l=1}^{f(n)-1} \Delta t \quad \text{or} \quad \sigma_{u_i} = \Delta t - \left(\sum_{l=1}^{f(n)} \Delta t - n \cdot T \right)$$

- Case 4 if period p_i begins and finishes the preventive maintenance interval $(d(\text{n}) = i = f(\text{n}))$

The duration of period p_i in the interval n is given by:

$$\sigma_{u_i} = T$$

Finally, as the case, the duration of a period is expressed by:

$$\sigma_{u_i} = \begin{cases} \Delta t & si\ d(n) \neq i \neq f(n) \\ \sum_{l=1}^{d(n)} \Delta t - (n-1) \cdot T & si\ d(n) = i \neq f(n) \\ n \cdot T - \sum_{l=1}^{f(n)-1} \Delta t & si\ d(n) \neq i = f(n) \\ T & si\ d(n) = i = f(n) \end{cases}$$

The terms correspond to the periods at the ends of preventive maintenance intervals, $P_{d(n)}$ and $P_{f(n)}$ are given by

$$d\left(n\right)=\begin{cases}1 & if\ \text{n}=1\\ i\ \ \text{solution}\ \ of\begin{cases}\sum\limits_{l=1}^{i}\Delta t>\left(n-1\right)\cdot T\\ \sum\limits_{l=1}^{i-1}\Delta t\le\left(n-1\right)\cdot T\end{cases} & if\ \ no\end{cases}$$

$$f\left(n\right)=\begin{cases}H & if\ \ \text{n}=\text{H}\\ i\ \ \text{solution}\ \ of\begin{cases}\sum_{l=1}^{i}\Delta t\ge n\cdot T\\ \sum_{l=1}^{i-1}\Delta t<n\cdot T\end{cases} & if\ \ no\end{cases}$$

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