Journal of Educational Change (2024) 25:173–196
https://doi.org/10.1007/s10833-023-09487-5

# Primary teachers' early and retrospective instructional vision of mathematical inquiry

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Accepted: 9 June 2023 / Published online: 17 July 2023


## Abstract

A key challenge in implementing inquiry-based learning in mathematics has been raising teachers' confidence and skills with unfamiliar pedagogical practices. The nature of inquiry in particular challenges traditional notions of teaching mathematics that dominate the field. Few studies have explored how teachers' perceptions of the nature of inquiry evolve as they adopt and gain experience over time teaching mathematics with inquiry. This article draws on interviews from ten primary teachers about their anticipated and initial experiences, then again after five years of experience. Using instructional vision as a lens, analysis of their perspectives of inquiry at each juncture provided insights into how teachers were confronted by and then persisted through early challenges to make mathematical inquiry a regular part of their pedagogy. This paper provides new insights of teachers' vision of their role into adopting ambitious pedagogies over time.

**Keywords** Mathematical inquiry · Inquiry-based learning · Teacher learning · Qualitative longitudinal research · Ambitious pedagogy

## Introduction

Mathematics education reform has been a focus of educational change for decades. Reform grounded in the work of John Dewey (1910) was later advanced by the National Council of Teachers of Mathematics (NCTM, 1987). "Many labels have been used to characterize the 'sides' of these debates, with the umbrella terms reform and traditional often applied" (Munter et al., 2015, p. 2). Reformers envision classrooms that engage students in practices that allow them to apply

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their knowledge to authentic problems (Spillane et al., 2018). Being able to adapt knowledge to a novel task has been positively associated with students' engagement, enjoyment. and achievement (Collie & Martin, 2017). Inquiry-based learning has been advocated as a way forward, improving students' twenty-first century skills (creative and critical thinking, communication, collaboration), participation, facilitation of transfer, and even addressing declining enrolment in advanced mathematics (Dorier & García, 2013; Lazonder & Harmsen, 2016).

Inquiry-based learning in mathematics, or mathematical inquiry, is broadly described as "student-centred ways of teaching in which students raise questions, explore situations and develop their own ways towards solutions" (Maaß & Artigue, 2013, p. 780). As a pedagogy, it is generally described in contrast with direct instruction that uses short, closed tasks that focus on practicing specific skills taught by the teacher (Munter et al., 2015). Researchers have emphasised how inquiry requires a shift in classroom tasks and discourses situated in well-designed activities that scaffold students' ideas and develop their mathematical authority (Munter, 2014; Munter & Wilhelm, 2021; Uiterwijk-Luijk et al., 2019). However this shift has been slow in mathematics.

> Teachers not only need to learn how to create new forms of student work, how to foster an environment that focusses on the questions students find interesting, and how to put students in new roles, they also need to value and believe in supporting an inquiry-based culture in the classroom. (Uiterwijk-Luijk et al., 2019, pp. 1-2)

Rather than consider the low uptake as resistance, it is important to consider the demands placed on teachers as well as potential mismatch between teachers' and researchers' priorities (Kazakbaeva, in press; Kennedy, 2009). Mathematics teachers value inquiry but find its practices confronting (Hallman-Thrasher, 2016; Marshall et al., 2009; van Es et al., 2017). The nature of inquiry challenges teachers' roles, beliefs, and identities in the classroom (Dobber et al., 2017; Munter, 2014; Stipek et al., 2001; Voet & De Wever, 2019; Webb et al., 2014). Even with professional development, early challenges that teachers face with ambitious or innovative pedagogies can lead to their premature rejection in future attempts (Krajcik et al., 1998). Teachers' personal perspective on how they experience ambitious pedagogies is therefore critical to understanding what for them is changing and how they perceive that change. "Change then, most often begins with a transformation of people's personal perceptions and projects and flows outwards into the social and institutional domain" (Goodson, 2001, pp. 56–57). For teachers, generating a personal vision of their "ideal classroom practice" (Hammerness, 2001, p. 143) for themselves raises tensions between maintaining and disrupting continuity, that is "images of what is and … images of *what could be*" (Zittoun & Cerchia, 2013, p. 306, emphasis in original). The nature of inquiry in particular can sharply contrast with typical pedagogies in mathematics which neglect the kinds of complex, ambiguous tasks that one experiences outside of school.

Most studies about teaching inquiry have highlighted its complexities and often examine teachers' experiences at a point in time. However, teachers' experiences are unlikely static, so situating their experience in their own context and

over time is critical to gaining insight into teachers' experience and finding ways to support them in the long term (Silver et al., 2019). The aim of this paper is to investigate teachers' perspectives of the nature of mathematical inquiry through the theoretical lens of instructional vision (Munter & Wilhelm, 2021) as they gained experience teaching inquiry over a number of years. The research question addressed in this paper is, *How do teachers shift their perception of the nature of inquiry as they experience teaching mathematical inquiry in their classroom over time?* In particular, the article draws on ten teachers' interpretations of their anticipated and initial experiences, then again after substantial experience (five or more years). Understanding teachers' experiences from their own perspectives is significant as it can help move the field beyond identification of challenges towards practical solutions.

## Literature

This section outlines the potential and pedagogical challenges in teaching mathematical inquiry, specifically in relation to the nature of inquiry and subsequent pedagogical demands. It also introduces instructional vision as a theoretical lens to understanding their experiences and perceptions about teaching mathematical inquiry.

### Mathematical inquiry

Mathematical inquiry encompasses a broad range of pedagogical approaches described by authors, for example reform-based, dialogic, ambitious, or student-centred teaching. Most studies about mathematical inquiry focus on closed-ended problems with multiple possible solution pathways (e.g., Spillane et al., 2018; Yackel & Cobb, 1996). However, problems in life are typically complex, open-ended, and contain ambiguities that require negotiation, referred to as ill-structured problems (Reitman, 1965). New areas related to mathematics, like computational thinking and data science, demand that students can manage ambiguity and work collaboratively and with persistence (Pérez, 2018).

This study positions mathematical inquiry as an ambitious pedagogy in which students address ill-structured problems that rely on mathematical evidence. For example in the mathematical inquiry, *Do younger students have healthier lunches than older students?* (Allmond et al., 2010), children (aged 8–10) debate ambiguities such as what is meant by "healthy" and how a lunch containing both healthy and unhealthy items can be evaluated overall. Problems such as this ask students to grapple with making sense of mathematical ideas using the context and scaffolds to make decisions about how to approach the problem. In addition to analysing and reporting on their data, students design what kinds of mathematical evidence would convince others, clarify the ambiguities in the problem statement, and weigh the practical constraints of collecting the data they need. Understanding the *nature of inquiry* is critical for teachers to be able to facilitate learning. Teachers' perception of the nature of inquiry must be aligned, or calibrated, with both their practice and

their vision of where they see their practice in the future. Given that direct instruction is the most common approach to teaching mathematics, how can teachers perceive of their role when they themselves have not experienced teaching and learning mathematics through inquiry? If they approach teaching mathematical inquiry in the same way that they would teach students using direct instruction, it turns into a cookbook style of lesson which removes students' opportunities for decision-making (see Schoenfeld, 2013). In contrast, if teachers approach mathematical inquiry in opposition to direct instruction, perceiving the nature of inquiry as unstructured, then students are left without sufficient support (see Swain & Swan, 2007). Bråten et al. (2017) "recommend that more research, including in-depth qualitative studies, be conducted to identify both external and individual factors that either support or preclude teachers from effectively calibrating their epistemic cognition with advocated classroom practices" (p. 264).

The perception of mathematical inquiry as unstructured is unfortunate as "discovery learning" is generally regarded as ineffective (Bakker, 2018; Hmelo-Silver et al., 2007). Indeed, if inquiry is unstructured, then outcomes can be quite poor. Lazonder and Harmsen's (2016) meta-analysis of 72 studies examined the impact of teacher guidance on inquiry-based learning in mathematics and science. Their investigation suggested that in comparison to explicit instruction, little to no guidance was less effective, while adequate guidance provided the highest levels of learning. Finding a reasonable balance of control has been raised as both highly challenging and essential in the literature (Furtak et al., 2012; Hmelo-Silver et al., 2007). Finding such a balance requires teachers to coordinate both how they perceive of the nature of inquiry and their capacity to operationalise their perception, for example skills in pressing students to elaborate and clarify their ideas (Webb et al., 2019).

Ambitious pedagogies like mathematical inquiry that use open-ended problems demand significant changes in teachers' practice (Dobber et al., 2017; Lee et al., 2019; Munter, 2014). How does this change occur? Anticipating and envisioning details of enactment are central to teachers' capacity to adopt ambitious pedagogies like inquiry (Heyd-Metzuyanim, 2019; Inoue et al., 2019; Silver et al., 2019). "One's vision may be sharply defined, with distinct images and interactions, or the vision may be blurry, with vague images and indistinct activity" (Hammerness, 2001, p. 145). Progressing complex open-ended tasks may take a number of directions (productive and less so) and encounter unanticipated obstacles along the way (Cowie et al., 2018; Krajcik et al., 1998; Stein et al., 2008; van Uum et al., 2016). How teachers respond when the lesson deviates from what was planned is critical. Choppin (2011) observed that in response to seeing their students struggle, teachers who focused on students' mathematical reasoning and strategies tended to adapt tasks towards greater complexity and opportunity to engage in powerful mathematics, and perceived discussions as essential to learning (p. 193). In contrast, teachers who focused on whether students' answers were correct or incorrect tended to reduce task complexity and rely more on direct instruction. Choppin's study suggests that teachers' roles in the adaptation of tasks signal different orientations of the teacher towards mathematics, with teachers focused on correct answers seeing mathematics as acquisition and those focused on discussion and reasoning seeing mathematics as participation (see Sfard & Cobb, 2014, for more about mathematics as acquisition

and participation). Singer-Gabella et al. (2016) argued that more needs to be studied about teachers' developmental learning and approaches to managing diverse student ideas within their own classroom contexts.

Teachers struggle with their own confidence, balancing control between themselves and students, managing unexpected issues and addressing students' unscripted ideas (Dobber et al., 2017; Spillane et al., 2018; Webb et al., 2014). They need to be supported to "develop their capacity to teach in ways that approximate the ambitious content and pedagogy advanced by reformers" (Spillane et al., 2018, p. 534) and to feel part of a learning community where they and their teaching peers have or can develop sufficient skills, capacity, and opportunity to be successful (Horn, 2010; Uiterwijk-Luijk et al., 2017; Valentine & Bolyard, 2019). Bandura (2001) argued that one's vague intentionality and idealised anticipation of actions are challenged by the specifics that arise when one reacts to and reflects on their actions in practice. How they perceive and manage the gap can be critical in whether they proceed with challenge.

### Teachers' instructional vision

Thompson et al. (2013) contend that teachers' "personal theories about 'what counts' as productive teaching and learning" (p. 579), i.e., their critical pedagogical discourses, are what influence teachers' actions and it is this narrative that must change in order for teachers to change their practice.

> Critical pedagogical discourses organize and influence one's perceptions of the past and the future in terms of perceiving problems and opportunities, approaches to problem solving, and other choices made in instructional contexts … Critical pedagogical discourses are not necessarily consistent with teaching choices, but reflect what individuals believe 'should have been done' even if they cannot or will not translate these discourses into action. (ibid)

Critical pedagogical discourses interact with teachers' perception and engagement with discourses in the social and institutional environments that teachers work. Alignment of these discourses can more strongly influence teachers' pedagogical choices in the classroom. Teachers' vision must also be embedded in their own context, that is, the one in which teachers imagine themselves working (Hammerness, 2001). Munter and Wilhelm (2021) used the term instructional vision to capture "the discourse that teachers or others currently employ to characterize the kind of 'ideal classroom practice' to which they aspire but have not yet necessarily mastered" (p. 343). Jansen et al. (2020) defined instructional vision as "an individual teacher's idealized image of how she intends to teach in the future" (p. 184). They reflect a teachers' intended actions, what is valued as well as teachers' vision of how they see their current practice, typically in relation to reform-based practices (i.e., "what and how someone currently sees", p. 184). Vision is a subset of identity, described as a

> set of self-understandings, or narratives created to explain relationships between the self and the work of mathematics teaching, that are constantly negotiated between how we see ourselves in the past, how we see ourselves

> right now and who we want to become. … Instructional visions are more concrete and specific than beliefs. (p. 185)

Vaughn et al. (2021) argued that teachers should not be expected to adopt a singular vision because one's vision evolves over time and needs to resonate with their own personal values and commitments. Furthermore, seeing instructional vision as a dynamic *set* rather than fixed singular entity emphasises its fluid and multi-faceted quality, just as teachers have multiple identities and not just a single identity (Akkerman & Meijer, 2011). Using teachers' instructional vision (specific or set) of their current and future classrooms has become a powerful lens through which to understand what influences teachers' change in practice. Munter and his colleagues have been at the forefront of this work (Munter, 2014; Munter & Correnti, 2017; Munter & Wilhelm, 2021), building on Hammerness' (2006) research of teachers' embodied vision. Others have contributed to research on how teachers use vision to improve their practice, for example in how preservice teachers progress their vision of ambitious teaching practices through readings, discussion, rehearsals, and decomposition of practice (Arbaugh et al., 2021; Jansen et al., 2020).

What and how instructional vision influences teachers to improve their practice is key to enabling this change. One of the most persuasive scaled studies comes from Munter and Correnti (2017). Working with practicing teachers, they studied 200 teachers over four years to investigate what influenced teachers to change their practice towards one that is more reform-oriented (see Munter, 2014, for elaboration of reform practices in relation to the role of the teacher, classroom discourse, tasks and student engagement). The authors highlighted the key contribution that mathematical knowledge for teaching (MKT) plays, suggesting that with low levels of MKT, "teachers may be less likely to envision classrooms centered on students' responses to rich mathematical tasks and teachers playing a facilitating role and sharing authority with students" (p. 350). However, previous research had suggested that MKT on its own does not predict improvement in teachers' instructional practice. That is, although teachers' mathematical knowledge for teaching influenced the *quality of their practice*, Munter's research argued that it was their instructional vision of high-quality instruction that provided impetus for growth towards *improved practice*. In a related study, Munter and Wilhelm (2021) followed 117 middle school teachers across 30 schools for at least three years. In addition to individual factors (e.g., MKT, instructional quality), they found that the social context in which teachers worked (e.g., their peers' instructional vision, collegial networks) also played a key role in the growth of teachers' own instructional vision.

In research with preservice teachers, Arbaugh et al. (2021) used Munter's (2014) rubric of instructional vision to understand how preservice teachers included reform-based practices as part of their instructional vision of the role of the teacher. They noted that teachers' instructional vision often emerges *from practice* rather than being set prior to practice. In a related study, Jansen et al. (2020) examined 81 primary teacher education graduates in relation to their instructional vision of reform-oriented teaching (as taught in their teacher education program) 2–3 years post-graduation, particularly one that valued conceptual understanding, student discourse, and engaging students in productive struggle. Their study acknowledged that

how teachers talk about their idealised practice is only probabilistically related to their instructional vision. They noted that the teachers in their study emphasised different pedagogical commitments, with those embracing a reform-based vision tending to emphasise "a stronger commitment to honouring students' ideas and treating the learning process as a collective endeavour, while still holding a commitment to treating mathematics with integrity" (p. 202). The teachers in their study struggled to resolve conflicts between their values from their teacher education program and their enacted practice. However, Jansen et al. argued that if teachers experience reform-based content and pedagogies as learners, then they were in a better position to weigh the benefits and challenges of these approaches as they gained experience as teachers. They recognised that their study solely focused on self-report, however it allowed them to "gain insight into how teachers construct and activate aspects of vision in context" (p. 203).

In this study, instructional vision for teaching mathematical inquiry is considered as what teachers aim to achieve as a teacher by teaching mathematics through inquiry. Their vision represents a deeply personal perspective of teaching that negotiates their beliefs, sense of agency, knowledge of what and how children learn, personal and professional histories and experiences and contextual factors and constraints (Vaughn et al., 2021). This study operationalised teachers' instructional vision as how they talk about their current and idealised practices. This allowed the study to encompass the potentialities, current ideas (including challenges) and reflections teachers had about their practice.

## Method

### Qualitative longitudinal research

Qualitative longitudinal research is rare in classroom research because of tensions and ethical complexities incurred (Derrington, 2019; McCoy, 2017). For example, funding is uncertain and short-term, conventional definitions of "participant" are often too simple, data collection and analysis can become unwieldy, attrition and school pressures can interrupt participation, informed consent and continuing involvement must be re-negotiated when school leadership shifts, research staff change and take time to enculturate, and publication can take many years (Thomson & Holland, 2003). Therefore, although much of the research literature seeks causal explanation to produce predictive mechanisms, longitudinal research is often too complex to be able to point to and isolate causality. Bruner (1990) emphasised a similar notion about the stories people tell about their experiences:

> We obviously cannot track people through life and observe or interrogate them each step of the way. Even if we could, doing so would transform the meaning of what they were up to. And, in any case, we would not know how to put the bits and pieces together at the end of the inquiry. One viable alternative is obvious—to do the inquiry retrospectively, through *autobiography*. … An account of what one thinks one did in what settings in what ways for what felt

> purposes. … It does not matter whether the account conforms to what others might say … [nor] whether the account is 'self-deceptive' or 'true'. Our interest, rather, is only in what the person thought [s/]he did, what he thought he was doing it for, what kind of plights he thought he was in, and so on. (p. 119-120)

In a longitudinal study, shifts are experienced by both the participants and the researcher (McCoy, 2017). Rather than see this shift as disruptive, this article sought to exploit its dynamic nature by analysing how teachers interpreted their experience early in their journey (before and after their first experience teaching mathematical inquiry) and then in retrospect. Although a primary interest for longitudinal researchers, *change* is blurred by an unfolding narrative of participants, how they interpret and re-interpret events at different times, learning by the researcher over time and its effect on interpretations of the data. McCoy argued that qualitative longitudinal research is unique in its orientation to time, change and iteration. The passage of *time*—for participants, contexts and the researcher—gives this approach to research access to perspectives on experience not available in point-in-time studies. That is, longitudinal studies shift between participants' experiences, including future anticipations, extant understandings and reflections on the past, with each in different subjective temporal contexts. Therefore, this study explored change, not directly, but "how individuals interpret and respond to such change" (p. 443). Finally, *iterative* analysis over time suggests that perspectives are contextually shaped and situated, creating opportunities for fresh insights and revisiting potential transitions that may not have been noticed before.

## Context of the study

The context of the study is relevant to understanding the interviews, however this paper is not seeking to make causal claims about the effectiveness of the professional development or support that teachers received. The data reported here come from a project seeking to understand how teachers' initial experiences and practices evolved as they learned to adopt mathematical inquiry over time. Although not originally intended as a longitudinal study, funding enabled it to continue over seven years. This meant that initial data collection that was intensive in the early phase (e.g., regular teacher interviews) was unmanageable with a small team as the number of teachers increased and data collection shifted to focus primarily on classroom observations. The benefit of this was that conversations turned less formal and were more regular as we spent time in the classroom. This likely strengthened the rapport and trust between teachers and the research team. The aspect of the study reported here focuses on the interviews conducted when teachers initially joined the study and at the end of the study.

Throughout the project, teachers participated in three full professional learning days per year, mostly in smaller cohorts (4–10 teachers) to maintain relevance and a relaxed atmosphere. In most cases, the teachers in rural schools met together and the teachers in suburban schools met together. These workshops were typically conducted at the university and consisted of three 2-hour sessions, separated by tea and

lunch breaks. In the first session, teachers shared challenges, strategies and benefits that they anticipated or experienced. This created a sense of community among the teachers over time as they validated their experiences, compared ideas, sought each other's input and offered advice. The middle session typically engaged the teachers as learners in a content-rich mathematical inquiry or workshopped specific skills (e.g., adapting a closed question into a more open one). Finally, teachers shared their proposed lesson sequence ideas with peers at the end of the day and had time for collaborative planning. Teacher participants only interacted with one another within their own project schools, except in the professional development days, sharing their experiences in the first session.

Teachers were asked to teach one mathematical inquiry sequence (two or more lessons) per term.[1] They designed their own sequence of lessons or adapted resources that they found, shared, or were offered by the project. A member of the research team observed (and often videotaped) these inquiry lessons and supported the teachers through informal conversation. Observations provided insight into the teachers' classroom contexts and the presence of the research team added some accountability. Informal conversations also took place in teachers' classrooms when the research team observed lessons, and the observing researcher would sometimes wander while children were working to ask questions or probe thinking. Some teachers took account of this and listened or asked for assistance in the lesson, but this was never expected.

## Participants

The data retrospectively analysed in this article come from a series of studies that sought to understand primary teachers' experiences and practice as they adopted mathematical inquiry over time (Makar, 2007; Makar & O'Brien, 2013). In each of the three funding rounds (1.5–3 years each, involving 1–5 primary schools), teacher cohorts were invited to join the research and those already involved stayed on as funding continued. Because the teachers who started in early phases remained in later stages, they helped enculturate new teachers who joined later. At the heart of the research was an ethos ensuring a positive, long-term relationship between the researcher and teacher participants. Ethically and practically, it was critical that teachers and schools benefited from the relationship and that their participation was continually invited rather than assumed (Warin, 2011). Trust and rapport were prioritised to ensure participants benefited from the interaction, but it also enabled the research team to attend to aspects of teaching that the teachers cared about (Cai et al., 2017).

In all, fifty-eight teachers were involved in at least one funding round (most in the final phase). The pool of participants included ten teachers (Table 1) who engaged with the research for five or more years, creating a longitudinal study embedded within the research. These ten teachers came from two schools who

[1] The Australian school year consists of four 10-week terms.

**Table 1** Participants in this paper

| Starting | Participants (pseudonyms) | School | Grade levels |
|---|---|---|---|
| Cohort 1 | Cate, Julia, Natasha | Suburban | 4-7 |
| Cohort 2a | Bronte, Rebecca, Ruby | Rural | P-4 |
| Cohort 2b | Carla*, James, Jade, Mojo | Rural | 4-7 |

*Carla was from the suburban school but joined later, so participated with Cohort 2

participated in the initial two funding phases of the research (Cohort 1 started in the first funding round; Cohort 2 in the second funding round); four teachers came from a school in a middle-class suburb of a major Australian city, while six of the teachers came from a rural school in a low socio-economic region over 50 km away.

## Data collection and analysis

Teachers were interviewed by the author to understand their experience learning to teach mathematical inquiry. In the semi-structured, audio-recorded interviews, teachers were asked about benefits and challenges they anticipated or experienced for themselves and their students. The data in this article come from those sections of the interviews. The interviews were intentionally conducted in a supportive and conversational style and the teachers were encouraged to elaborate on issues that they raised. The data used in this article consists of interview data from three periods: (1) Pre-interview prior to their first sequence of lessons teaching mathematical inquiry; (2) post-interview following their first sequence of lessons teaching mathematical inquiry; and (3) the final interview at the end of the overall project, seven years from the start of the first funding cycle.

The interview transcripts were analysed by the author to seek insights into how the teachers discussed and explained their experiences in relation to the nature of inquiry. The analysis was adapted from Powell et al. (2003) with interview transcripts first read carefully and annotated to obtain an overall picture of teachers' perspectives. A theoretical lens of teachers' instructional vision of mathematical inquiry was used as a way to seek an understanding of teachers' evolving perspective of the nature of inquiry by reviewing critical episodes that suggested emerging insights about a teacher's instructional vision in relation to the nature of inquiry. Overall, the analysis was a highly intensive process needed to reflect, refine, distinguish and clarify nuances that arose. Text was drafted around key ideas that emerged from this process, with excerpts selected that concisely expressed an idea, and provided useful insights into the teachers' inferred instructional vision of the nature of inquiry. The overall writing was refined and literature updated to improve the focus and remove repetition.

## Results

The results below use the theoretical lens of teachers' instructional vision to illustrate how teachers articulated their perspective of the nature of inquiry at the beginning of their involvement (before and after their first lesson sequence) and again as they reflected back on their perspective at the end of the study.

### Anticipated experiences

In anticipating what inquiry would look like in their classroom, the teachers' words suggested a fragmented instructional vision of their inquiry classrooms. Without experience, their initial visions were quite diverse—a mixture of optimism, uncertainty and apprehension (see Dobber et al., 2017 and Heyd-Metzuyanim, 2019 for their similar observations). In most cases, the teachers had a hazy, idealised vision of what a classroom that taught mathematics through inquiry might look like ("images of *what could be*", Zittoun & Cerchia, 2013, p. 306), but not how to reach or enact their vision.

Before the first experience, most of the teachers anticipated that teaching mathematics through inquiry would improve student learning but also expected it to challenge both their students and themselves. When asked to discuss what they thought mathematical inquiry was, many had a vision of inquiry as similar to discovery learning, where teachers leave students to determine their own pathways.

Julia: [Inquiry is] where they can choose which way, which direction they'd like to take.
Carla: It is certainly not teacher-directed. [It's] just letting them see what they can come up with.
Mojo: Discovery learning.
Ruby: Well, setting a problem and then maybe giving a little bit of guidelines to the kids and then basically letting them go for it.

These responses were not surprising as teachers may initially have a romantic view of constructivism (Ernest, 1994), that is, a vision of inquiry as being a shift from "telling" to "not telling" (Swain & Swan, 2007, p. 33). Munter (2014) outlined this perspective as teachers seeing themselves in a purely facilitative role (little input from the teacher). As an instructional vision of an inquiry classroom, this perspective was quite vague. The teachers didn't yet express clarity about specific skills, practices, or supports that will be needed. Some could, however, imagine challenges that both they and the students would encounter. The balance of student and teacher control was one area that was raised by the teachers.

AUTHOR[2]: What's going to be the most challenging? …
Natasha: Letting go!
Cate: Not having the investigation go the way you think it should.
Mojo: Thinking on your feet, a little bit. You sort of think, "Okay, this is what I want them to do. This is where they are going to go." You sort of have in your mind that this is the steps that they are going to go through, but then after step two they go that way (off plan).

Although most teachers were upbeat about students *having* more control, they worried about what they saw as *relinquishing* control (Dobber et al., 2017; Uiterwijk-Luijk et al., 2019; van Uum et al., 2016). The teachers also raised a number of logistical issues they expected based on their current experiences, for example they anticipated challenges in managing noise, resources, curricular demands, and keeping a lesson on track.

Rebecca: [Our students will] make their own board game and that's how we are going to sort of get the maths investigation happening. But *how*, I don't know.
James: I tried to do something when I first came here. … Went well for about 5 minutes and then rulers were flying. … Where do you draw the line, [between] scaffolding and leading them in a direction?

Rebecca and James were both optimistic about trying to teach with mathematical inquiry but struggled with a specific instructional vision of how to enact inquiry practices. That is, the *idea* of mathematical inquiry resonated with the teachers, but details of implementation remained hazy (Munter, 2014; van Es et al., 2017). As expressed by these teachers, prior to teaching mathematical inquiry they anticipated benefits and challenges for themselves and their students. Their instructional vision of teaching mathematical inquiry encapsulated a potential for developing students' confidence, but also revealed their own lack of confidence in "letting go" or in leaving students to pursue their own pathways. Apparent contradictions arose between valuing benefits of student direction and having concerns about their own and their students' capacity to achieve these benefits (Stipek et al., 2001).

### Initial experiences with inquiry

The teachers were interviewed again after their first sequence of lessons teaching mathematical inquiry in order to understand their initial experience. As typically happens in inquiry, teachers' lessons did not go as planned (see also similar findings in Hallman-Thrasher, 2016 and Uiterwijk-Luijk et al., 2019). Because few of the teachers had this 'unexpected' aspect of inquiry as part of their early instructional vision (contrast with Mojo and James above), they often interpreted the unexpected outcomes as poor planning on their part.

[2] AUTHOR denotes the author.

Natasha: I didn't completely anticipate how the children were going to go and what that would mean for … the underlying skills that they needed.

Mojo: We didn't really give it as much thought as we possibly should have … It was too hard, I think, for them.

Like Natasha and Mojo, most of the teachers explicitly criticised themselves in their post-interviews for unexpected issues that arose. Disappointment can derail a teacher's sense of being efficacious in their teaching and see them abandon further attempts (Krajcik et al., 1998). This suggests that the initial experience could be a unique and vital period for teachers as they re-calibrate their instructional vision with their experience.

Several positive outcomes also came out of their initial experiences teaching mathematical inquiry. For example, during the first sequence of inquiry lessons, the teachers also became more aware of students' own ideas (van Uum et al., 2016). Although they had expected students to take control of their ideas, they hadn't envisioned what those ideas might be (Stein et al., 2008). In their first sequence of lessons, Ruby's Grade 2 class had investigated kite design, Bronte's Grade 1 students compared how matchbox cars travelled down ramps of varying incline, and Cate's Grade 4 class collected data to explore whether particular traits like a 'hitchhiker's thumb' or attached earlobe might be inherited.[3] All three teachers expressed surprise at how students' own ideas enhanced the learning experience (Collie & Martin, 2017; Land et al., 2019; van Es et al., 2017).

Ruby: They really enjoyed it (investigating kites). They picked up on it a lot and (untranslated) kites that they actually made, you know – size, materials – they actually looked into a lot of that design. … You just never know where the kids are going to take it, and the types of things they're going to come up with!

Bronte: Well, I know the kids really enjoyed it. They're the ones that came up with the ideas of changing the ramps and (untrans) and speeds and other stuff – changing the – from the flat ground to a ramp that had a lip to a ramp that had no lip.

Cate: Some of mine picked up on, it was the *interpretation* of what actually constituted it (inherited traits). One child who was doing the index finger said it was cut-and-dried for somebody like mine, who was short and somebody like his, who was long, but there were some [index fingers] that were sort of, not short and not long. And another child said yes with [both] the hitchhiker's thumb *and* straight thumb. … I think some of the kids had switched-on that there's data collected that might be dicey or might be different.

Ruby and Bronte both were surprised by the quality of students' own ideas that emerged from experimenting with kites and ramps. In addition, Cate acknowledged the opportunities in students' ideas that helped them make rich connections;

[3] From *Can you roll your tongue?* (Joyner et al., 1997).

it also highlighted how rarely school mathematics addresses everyday ambiguities (Zaslavsky, 2005).

As their initial instructional visions are disrupted by the challenges they experienced, a sense of uncertainty can derail teachers' vision. A number of moderating factors emerged from the teacher interviews that may point to where the teachers were able to respond positively to their first experiences.

Rebecca: A couple of times, I had to walk up to Mojo and I wanted to verify what I thought I was doing was the right thing. … and it just made me feel a lot better. … Maybe, as time goes on, maybe I'll know what to look for. I don't know yet. I don't know yet.

Bronte: The PD day we did with you was—opened our eyes, and plus we got to do one [inquiry]. I think doing one ourselves would have made a huge difference, because we knew what to expect. … You don't learn if you don't do. That's how we learnt.

Although the first professional development experience had come before the lessons they initially taught, Bronte's comment raises the possibility that the experience with their students may have helped them to re-interpret their earlier experiences. Rebecca was more open about her sense of uncertainty and how her colleague's validation had helped her manage her lack of confidence. This is where support networks can become important for reflecting on their practice and recalibrating their instructional vision to be less hazy (Munter & Wilhelm, 2021).

These experiences may have provided teachers with an explanatory narrative to help them make sense of the mismatch between expectations moulded by their instructional vision and their early experiences in the classroom. Natasha and Mojo's statements above created a narrative of lack of teacher planning or a lesson being too hard; Ruby, Bronte and Cate's excerpts suggest that their instructional vision was being re-calibrated to place more confidence in children's capacity to reason; and for Rebecca and Bronte, they used peer validation and experiences as a learner in the professional development day to reflect on and build their confidence in their practice in teaching mathematical inquiry. These varied scenarios add weight to how teachers' instructional visions for teaching mathematical inquiry were diverse, in flux and potentially influenced by many elements. In each case, however, the teachers likely created personal narratives to reflect on, make sense of, and try to align their instructional vision for teaching mathematical inquiry with their experience.

Confidence was mentioned by nearly all of the teachers in the interviews following their first inquiry. Fortunately, most teachers experienced heightened student enjoyment, engagement, and/or sense of relevance in their first inquiry sequence of lessons (Collie & Martin, 2017; Heyd-Metzuyanim, 2019). For example, Natasha followed her disappointment (above) with enthusiasm about her students' enjoyment.

Natasha: There was one day I could have thrown my hands up and said, "I'm not doing this!" But I could see that the children were enjoying it. … It was exciting for me as well as for the children just to see where they were going to go with it and what they were going to do and just see them getting so involved.

Nearly all of the teachers in their post-interviews talked about their students' enjoyment or engagement in inquiry. Student engagement is critical for their identity as a "good" teacher, which can be challenged by the uncertainty and frustration of inquiry (Voet & De Wever, 2019). Engagement, therefore, could moderate teachers' discomfort and help sustain them through and beyond their first attempt at teaching mathematical inquiry (Heyd-Metzuyanim, 2019).

As argued in the literature, teachers' beliefs and experiences have a profound effect on their instructional vision and its relationship with their fluency in responding to children's mathematical ideas (Voet & de Wever, 2019). The teachers' first experience teaching mathematical inquiry highlighted challenges in their instructional vision of the nature of inquiry. Following their first experience, teachers often blamed themselves when lessons did not go as planned, but their disappointment was buoyed by students' engagement and moderated by positive interactions with colleagues (Munter & Wilhelm, 2021, p. 351). Teachers' instructional vision influences their practice, the professional learning that they choose to engage with and their likelihood of growth within professional learning opportunities (Arbaugh et al., 2021). If teachers reflect on the alignment or gap between their ideal of where they wanted to be and their practice, they may adjust their instructional vision to allow it to remain achievable. Shifts in instructional vision take time, highlighting the need to support teachers beyond their first year in building opportunities to enrich the mathematical depth in their lessons (Land et al., 2019).

### Reflecting back after many years

The teacher participants interviewed in this paper remained in the project for 5 or more years. At the end of the project, they were asked to reflect on their experiences teaching mathematics through inquiry. The discussions included how their experiences had changed with time.

When looking back, the teachers spoke repeatedly about using students' ideas to guide the direction of lessons (see similar findings in Cowie et al., 2018; Franke et al., 2015; Land et al., 2019 and Stein et al., 2008). Their capacity to now envision where an inquiry was going gave them confidence in what to expect and to devise strategies that responded to students' ideas and maintained a desired focus (Dobber et al., 2017; Munter, 2014).

James: I'm a lot more confident with it. And I know now, alright, I expect it to be a little rowdy at the start, but I can see where it's going and I know when I am still in control. At the start, I felt like I had no control. Whereas now, I'm like, "Right, I'm starting to lose control. I can maybe rein it in a bit"

and then once I've—"Yep, alright, I am satisfied. Now I can let them go a bit more."

Mojo: I would give them the focus [inquiry] question, I would let them go for five minutes and then I would call them back and say, "Who is having problems? What are you having trouble with? I noticed you were doing that. Why were you doing that?" and then they hear everybody else's responses and then that gives them a little bit more of an idea. … You send them back out again and then go and do a little bit more, then you bring them back. … In the beginning, we thought inquiry meant that they did it on their own and we just sort of had to hope it would work out. Whereas now, we are much more comfortable directing.

James and Mojo described a strategy teachers created whereby students attempted a problem in their groups and then returned as a whole class to share issues, get feedback, refocus, then continued in their groups and repeated the process (which we called a "checkpoint", (Makar et al., 2015)). The teachers developed, practised, shared and refined strategies like this over time, which strengthened their instructional vision of teaching mathematical inquiry and sense of collective efficacy (Vaughn et al., 2021). These strategies became a critical way to respond when students were struggling, frustrated or losing momentum. They had been tested multiple times with their students in different scenarios over the years. This helped them to envision how they would respond to most issues that came up. They described these strategies in abstraction, where they encapsulated an instructional vision of teaching mathematical inquiry that could be translated into multiple scenarios yet-to-happen.

The teachers overwhelmingly spoke about how the collegial validation over the years had helped them persist through their own struggles (Arbaugh et al., 2021; Spillane et al., 2018).

Bronte: Just listening to other people, what other people have done, thinking, "Oh, yeah, I could try that".

Julia: Because it's over time, every time you go out and try it, you have more questions and then you come back and you share it and then everyone answers those questions.

In addition to valuing collegiality, students' engagement in inquiry remained a standout characteristic to the teachers over the years, particularly for students who normally didn't enjoy mathematics (Voet & De Wever, 2019). In some cases, a specific experience stood out as a powerful exemplar.

Carla: Some of those children who sat quietly and never really engaged [before], they were the ones who really made valuable contributions. … One of the boys, who had a few behaviour issues, at first it took him a little while to become involved because it was … "Oh, you are actually listening to me? Nobody's listened to me before. You are actually valuing what I have to

say?" And I thought that was really motivating. … For some kids who don't particularly like Maths, they are … just loving it.

Natasha: The students are really engaged, and that's very, very powerful. … I had one inquiry that we did about packaging [four snack] boxes, and one of the kids came in one day and her mum was a shelf stacker at Coles (the local supermarket). And she came in and she said, "Right, I went down and I spoke to the manager and I measured the shelves and we couldn't actually stack it this way (all four snack boxed stacked vertically) because the shelves aren't high enough. … My mum's job is just awesome!" So she got so much pride in what her mum did, because she suddenly saw importance in it. So it was that level of engagement where you get children that hate Maths going down to the local supermarket, interviewing the manager! (laughs)

At the beginning of their journey, student engagement was a pivotal force for them as teachers. However, Carla and Natasha's responses highlighted how they now saw engagement with an equity lens—bringing opportunities to students who may otherwise not experience success (Collie & Martin, 2017; Franke et al., 2015). Their insights about student engagement in inquiry suggest that their instructional vision for teaching mathematical inquiry may have been broadened to see mathematics as more inclusive.

## Discussion and conclusion

The research question under investigation in this article was *How do teachers shift their perception of the nature of inquiry as they experience teaching mathematical inquiry?* The paper reports on ten primary teachers' interviews before and after their first sequence of lessons teaching mathematics through inquiry, and again after several years of experience. The data are based on teachers' perspectives, rather than evaluating the researcher's perspective of their practice. The intent of the paper therefore was not to provide general "answers" for how to shift teachers' practice, but rather to understand and value teachers' experiences in context in light of their instructional vision for teaching mathematical inquiry. The open-ended nature of tasks that teachers encountered in mathematical inquiry were substantially different from how they had taught mathematics previously. This change in task highlighted how the nature of inquiry brought with it a broad array of new demands on their pedagogical practice and their instructional vision for teaching mathematical inquiry.

Table 2 summarises how these teachers collectively described their experiences in each stage of the study as it relates to the nature of inquiry. The diversity of the teachers' anticipated and initial perspectives speaks to the importance of acknowledging that teachers come into the experience of teaching mathematical inquiry with a wide spectrum of instructional visions and expectations about inquiry (Uiterwijk-Luijk et al., 2019; van Es et al., 2017).

**Table 2** Aspects of teachers' instructional vision related to the nature of inquiry across the study

| Pre-inquiry | After first inquiry | Through experience |
|---|---|---|
| Inquiry as discovery | Inquiry as discovery or highly structured | Inquiry as guided |
| Implementation hazy | Managing unanticipated issues | Envision possible directions |
| Challenge staying on task | Teacher self-blame | Strategies to structure, focus lessons |
| Relinquishing control | Awareness of student ideas | Valued and responsive to student ideas |
| Concerns about logistics (noise, resources, time) | Buoyed by student engagement | Collegiality critical<br>Inquiry as equity |
| Mixed teacher confidence | Low teacher confidence | High teacher confidence |

The nature of inquiry—initially conceived by many as unguided (Swain & Swan, 2007)—was an important change in perspective over time that allowed the teachers to become more aware of their role to support students and develop strategies to do so (Dobber et al., 2017; Hmelo-Silver et al., 2007). Managing logistics and uncertainties, relinquishing control, and staying on task were concerns that several teachers held as part of their initial instructional vision for teaching mathematical inquiry.

Following their initial mathematical inquiry lessons, most teachers blamed themselves for the challenges they faced or when lessons did not go to plan (Hallman-Thrasher, 2016). That is, when the enactment contradicted their early instructional vision for teaching mathematical inquiry, they often interpreted the misalignment as poor planning on their part. They also expressed a lack of confidence in how to respond to students' ideas or unanticipated issues (Cowie et al., 2018; Dobber et al., 2017; Stein et al., 2008). These challenges highlight the strategic and critical importance of supporting teachers early in their experience lest they decide to abandon inquiry as too difficult, or worse, continue without developing practical strategies to guide lessons (Krajcik et al., 1998). Fortunately, the teachers were buoyed by students' engagement in the lessons (Collie & Martin, 2017; Franke et al., 2015). This engagement likely boosted their sense of teacher efficacy (Tschannen-Moran & Hoy, 2001) in enacting their instructional vision.

After substantial experience, the teachers reconceived their instructional vision of inquiry as guided, and devised strategies to keep lessons focused (Munter, 2014). Their capacity to envision potential directions in an inquiry was an indicator that their instructional vision of teaching mathematical inquiry was more concrete. It also enabled them to see inquiry as equitable and respond more confidently to students' ideas, even those they didn't expect (Cowie et al., 2018; Hallman-Thrasher, 2016). The teachers unilaterally expressed how sharing and validation from colleagues sustained them over the years. This aligns with other findings that suggest peers help shape instructional vision (e.g., Spillane et al., 2018).

The key issues that teachers encountered throughout their experience were wrestling with productively balancing teacher-student control, anticipating and responding to student ideas, developing adept questioning and strategies to re-focus and mathematically enrich lessons. The teachers' evolving instructional vision for teaching mathematical inquiry alongside their new pedagogical skills took time, support

and experience to develop. Critical to sustaining them through the challenges they encountered were students' engagement in lessons, peer validation and opportunities to devise strategies that could be adapted locally (Spillane et al., 2018).

In this study, the teachers' instructional vision for teaching mathematical inquiry began hazy and idealised, but was not absent. The important early stages need to be taken seriously to allow them to be supported through this critical early stages. Teachers' practices and evolution were not uniform. Some simplified and structured their mathematical inquiries more and some gained more nuanced strategies for letting students struggle. Their instructional vision also changed and became more specific over time, particularly as they built confidence and developed strategies to scaffold students. The primary challenge was in bridging the gap between their evolving vision and strategies to get there, which required shifts in disposition as they remained positive and allowed themselves to take pedagogical risks. These changes required reflection, professional learning, collegial support, optimism and opportunities to develop in their classrooms over time.

## Implications

The reader is cautioned that this is a small exploratory study across two schools, so was not intended to generalise beyond these teachers. The experiences of the participants in the study suggests ways in which teachers could be vulnerable in the early stages of adopting mathematical inquiry. It also illustrates ways that teachers' instructional vision for teaching mathematical inquiry may be a useful lens to consider how teachers are experiencing the nature of inquiry over time. For example early instructional vision for some teachers included a presumption that inquiry was unguided; this perspective of inquiry was fortunately dismissed with experience. The implications are that in this early stage it may be helpful for teachers to observe (live or on video) classrooms with similar contexts prior to beginning to teach lessons so that they can be aware that teachers scaffold children in teaching mathematical inquiry. They also can be aware of studies like this and others (e.g., McCrone, 2005, where student talk took a year to develop) that point to the time it takes for changes in classroom practices.

The contrast in teachers' early anticipation (positive and negative), struggle with unexpected issues and initial self-blame could easily have derailed their early attempts, or worse, mutated as a hybrid discovery approach in which students were expected, but feared, to work autonomously. With experience, the capacity to anticipate and have confidence in managing unexpected challenges was a critical element of their instructional vision that appeared to change for these teachers over time. The implications for supporting teachers' early experiences could draw on points raised here by teachers as supporting them. For example, teachers mentioned collegial support and discussion as important to validate their experiences and professional learning where they could experience mathematical inquiry as learners. One idea of the latter is the use of Fermi problems, characterised as "open questions offering little or no specific information for the problem solvers to direct them in the solution process" (Ärlebäck & Albarracín, 2019, p. 980). For example, "how many balloons

fit in a car?" is a Fermi problem. Experience with Fermi problems as learners could help teachers to experience ambiguity in mathematical problem-solving that have no clear correct answer. Through experiencing these problems as learners, they may be able to understand and see as normal the grappling that learners engage in when addressing ambiguous problems, as well as the type of questioning that a facilitator would do to support them through the process.

Because the teachers in this study were experienced teachers at the start of the study, they already had established classroom practices. The benefit was in having confidence in their teaching identities and classroom routines, which may not be the case with preservice or early career teachers. Some researchers have therefore relied on rehearsals to learn to enact student-centered pedagogies such as mathematical inquiry (Jansen et al., 2020).

This study acknowledges how teachers envision the benefits of inquiry for students, but also highlights teachers' concerns and the frailty of teachers' confidence in the early stages (Marshall et al., 2009). For example, it is important to attend to teachers' beliefs early on in light of Choppin's (2011) work of how teachers' beliefs affect their adaptation of tasks when they observe student struggle. More research is needed to understand how anticipating issues, validating concerns, sustaining support, and providing time for reflection, collegial sharing and development of strategies may ease the burden for teachers in adopting mathematical inquiry over time.

Findings of this study align with substantial research in the field. However, much of the research to date has been either aspirational (exemplar cases) or has highlighted challenges (early obstacles). Studies examining how teachers' perspectives of their experiences evolve over many years and with support are rare. Understanding teachers' perceptions over time under these conditions can assist the field to anticipate and validate opportunities and obstacles that teachers encounter when adopting inquiry practices as a long-term goal. Given the high accountability pressures that teachers face, risks in adopting ambitious pedagogies are amplified. Many of the challenges confront their identity and sense of efficacy (Heyd-Metzuyanim, 2019; Voet & De Wever, 2019). The critical role of student engagement and collegiality arose in this study for facilitating teachers to persevere through challenges. Recognising unique difficulties in early stages of transition can enable school leaders to anticipate and provide additional support, resources and acknowledgement through this vulnerable period, and persist when inquiry may initially appear ineffective.

**Funding** Open Access funding enabled and organized by CAUL and its Member Institutions. The data in this article come from a study funded by the Australian Research Council (ARC LP0776703; LP0990184), Education Queensland and the University of Queensland. The Centre for Teachers and Teaching Research, Institute of Education, University College London hosted me as visiting scholar during the analysis and drafting of this paper. The author wishes to thank the generosity of the teachers in contributing to the project.

## Declarations

**Conflict of interest** I have no conflicts of interest to disclose.

**Ethical approval** This project has approved ethical clearance from the University of Queensland and Department of Education.



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