

Visually Dynamic Presentation of Proofs in Plane Geometry

Part 2. Automated Generation of Visually Dynamic Presentations with the Full-Angle Method and the Deductive Database Method

Zheng Ye · Shang-Ching Chou · Xiao-Shan Gao

Received: 30 November 2009 / Accepted: 6 December 2009 / Published online: 23 December 2009
© Springer Science+Business Media B.V. 2009

Abstract We present the method for automated generation of visually dynamic presentations of plane geometry proofs based on the full-angle method. The proof generated by the full-angle method is organized hierarchically, thus it is particularly suitable for visual presentations. We also present the method for automated generation of visually dynamic presentation of proofs for the deductive database method with an additional new visual feature: given a geometrical configuration or a diagram, the final database (the fixpoint) in the deductive database method has numerous geometric properties organized into a few categories. By clicking each category, all properties of the configuration in this category are listed. And by clicking each of these properties, the corresponding geometry elements in the diagram blink or animate and, if needed, the proof of this property is generated.

Keywords Geometry theorem proving · Proof with hierarchical structures · Visually dynamic presentation of proof · Dynamic geometry · Full-angle · Deductive database · Fixpoint · Unordered geometry · Hilbert geometry · Tarski geometry

The work reported here was supported by NSF Grant CCR-0201253.

Zheng Ye: On leave from ZJU and working at Wichita State University.

Z. Ye (✉)
College of Computer Science and Technology,
Zhejiang University, Hangzhou, China
e-mail: yezheng@gmail.com

S.-C. Chou
Department of Computer Science, Wichita State University,
Wichita, KS 67220, USA

X.-S. Gao
KLMM, Institute of Systems Science,
The Chinese Academy of Sciences, Beijing, China

1 Introduction

In Part 1 of this series [1], we have proposed a new presentation of geometry theorem proofs—visually dynamic presentation of proofs and presented our manual input method for creating the proofs. Whereas the manual input method is valuable for users to write and present proofs of geometry theorems with various dynamic visual effects, the focus of our work in Part 2 and sequels to Part 2 of this series is *automated generation* of visually dynamic presentations of proofs.

In Part 2, we will present the automated generation of visually dynamic presentation of proofs for the full-angle method and the deductive database method. These two methods can generate proofs that are readable and diagram independent. Before discussing full-angles, an exposition of unordered geometries in depth would be very helpful for readers who encounter diagram independent proofs and full-angles for the first time.

1.1 Algebraic Proofs in Ordered Geometries and Unordered Geometries

In 1978, Wen-Tsün Wu introduced an algebraic method for proving geometry theorems of the equality type [2]. Hundreds of difficult theorems have been proved by the computer programs based on this method.¹ Inspired by the success of Wu's method, around 1985–1986, three groups were successful in applying another algebraic method, the Gröbner basis method, to the *same* class of geometry theorems that Wu's method addresses [3–5].

The success of Wu's method is his key insight into the differences between unordered geometry and ordered geometry. For proving theorems in unordered geometry, i.e., theorems of the equality type, the real closed field quantifier elimination algorithms [6–8] are not needed. Instead, efficient algebraic methods for polynomial equality can be used. These methods are for unordered geometries. Since unordered geometry is new to novice, and some phenomena are new even to experts, and it is vitally important for our automated method, some discussions on unordered geometry in depth would be helpful. The novice can find supplement discussions in, e.g., books [9] or [10].

1.2 Unordered Geometries and Diagram-Independent Proofs

It is a very common belief that the proofs taught in high school geometry are based on the Euclidean axiom system and thus are rigorous. Actually the Euclidean axiom system in common textbooks is very incomplete and the proofs are not rigorous as they look to be. Here we cite a passage from the book [11]:

One of the main defects in the traditional Euclidean proof is its almost complete disregard of such notions as the *two sides of a line* and the *interior of an angle*. Without clarification of these ideas, absurd consequences result.

¹For his pioneering work and outstanding contribution in geometry theorem proving, Wu received the 1997 Herbrand Award in Automated Deduction.

Traditionally one use the order relation to prove theorems of equality type. However, the statements of these theorems do not involve the order relation. These geometry theorems are in unordered geometry, whose proofs are immune of order relations.

First we will use three examples to explain these phenomena. The first example shows the diagram-dependent nature of traditional proofs. The next two examples show implicit facts or assumptions that are often used in the traditional proofs in common geometry books.

Example 1 (The diagram-dependent nature in ordered geometry) Let points C and D be the intersections of two circles (A) and (B), line HF be passing through C and meeting circles (A) and (B) at H and F , and line GE be passing through D and meeting circles (A) and (B) at G and E . Show that GH is parallel to EF .

Figure 1 shows four diagrams of this theorem. For each diagram, the traditional proof is different. For example, in the left most diagram, we assume that points G and C are on the opposite sides of line DH and points E and C are on the opposite sides of line DF . These additional assumptions should be **a part of** the hypotheses, although geometry textbooks do not include these hypotheses explicitly. Authors generally use a diagram, such as the left most of Fig. 1, in addition to the hypothesis text. This diagram is intended to give these additional order relation assumptions. This common practice is generally accepted. The proof is complete with these additional assumptions as a part of the hypotheses, but it might be diagram-dependent (for a rigorous diagram independent proofs in unordered geometry see Fig. 6).

However, there are other kinds of implicit assumptions which should be **a part of** the proof, not a part of the hypotheses. Without proving these implicit assumptions the proof is not rigorous and is incomplete. This is the kind of defect that the book [11] refers to. Let us use the following two examples to illustrate our point of view.

Example 2 Let $ABCD$ be a parallelogram, i.e., $AB \parallel CD$ and $BC \parallel AD$, E be the intersection of the two diagonals AC and BD . Show $AE = CE$ (the left of Fig. 2).

The traditional proof of this theorem is first to prove $\triangle ACB \cong \triangle CAD$ (hence $AB = CD$), then to prove $\triangle AEB \cong \triangle CED$ (hence $AE = CE$). In proving the congruence of these triangles, we have repeatedly used the fact $\angle BAC = \angle DCA$. This fact is quite evident because the two angles are a pair of the inner alternative angles with respect to the parallel lines AB and CD . However, here we have implicitly assumed the “trivial fact” that points D and B are on the opposite sides of line AC . This “trivial fact” is harder to prove than the original statement.

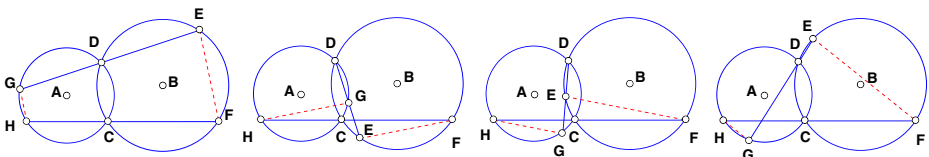


Fig. 1 A two circle theorem

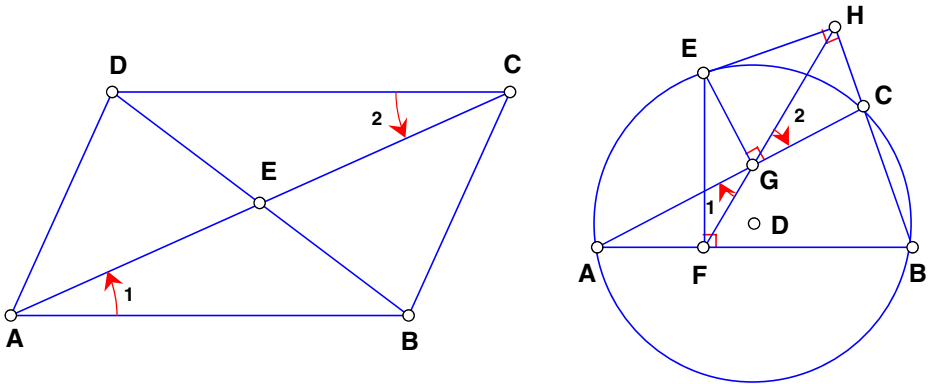


Fig. 2 Two examples

Example 3 (Simson’s theorem) Let E be a point on the circumscribed circle (D) of a triangle ABC . From E three perpendiculars are drawn to the three sides AB , AC and BC of $\triangle ABC$. Let F , G and H be the three feet respectively. Show that points F , G and H are collinear (the right of Fig. 2).

The proof in common textbooks is to prove $\angle 1 = \angle 2$; thus F , G and H are collinear. However, here we have at least one implicit assumption that points H and F are on the opposite sides of line AC . Without proving this assumption, the proof is incomplete or not rigorous (see Fig. 9 for a rigorous diagram-independent proof of this theorem.)

Now the question arises that do we have to use the order relation to prove these theorems of the equality type? The answer is ‘no’, and is already in Wu’s algebraic method itself that can prove theorems of the equality type without using the order relation.

The proofs generated by Wu’s method are not synthetic proofs. Are there synthetic proofs of theorems of the equality type without using the order relation at all? The answer is ‘yes’ and is related to *unordered geometries*.

In Hilbert’s first 4 groups of axioms, Group 2 (axioms of order), is introduced before Group 3 (axioms of congruency) which are dependent on the order relation. The important notions, such as angles, etc., defined in Group 3, use the order relation.

Hilbert’s book itself and the development after it have indicated that it is possible to introduce axioms of congruency before axioms of order. See Chapter V “Desargues’ Theorem” and Chapter VI “Pascal’s Theorem” [12] or Wu’s book [10]. This is unordered geometries, or unordered metric geometries. For this kind of geometries, axioms of congruency need to be revised in a way that does not use the order relation. Notions such as perpendicularity, angle (full-angle) or triangle congruency, etc., are defined without using the order relation.

Whatever emphasis on the order-independent or the diagram-independent nature of unordered geometry might not be enough partially because we are so accustomed to the real plane geometry. The following example (Fig. 3a) given by A. Nevins [13] was used by us ([14]: Example 6.4).

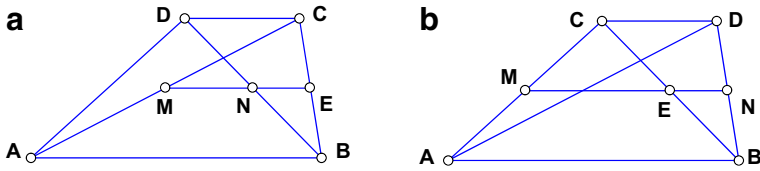


Fig. 3 An example by Nevins

Example 4 Let $ABCD$ be a trapezoid with $AB \parallel DC$, M and N be the midpoints of its two diagonals, and E be the intersection of MN and BC . Show that E is the midpoint of BC .

For a trapezoid in unordered geometry, the pair of two sides and the pair of two diagonals are interchangeable. Thus for unordered geometry, Fig. 3a and b are two different diagrams of the same theorem. The sum or difference of two segments generally needs the order relation, as the following example illustrates.

Example 5 Let BD be the altitude on the side AC of an isosceles triangle ABC , E be any point on the base side AB . From E draw two lines perpendicular to the two sides with the feet F and G . Show that $BD = EG + EF$ (Fig. 4a).

Because there is an addition of two segments in *different orientations*, the representation of this theorem is beyond unordered geometry. Also the theorem implicitly assumes that E is between A and B . In unordered geometry, we can prove a weaker version using Wu’s method: one is the sum of the other two among the three segments. See three diagrams in Fig. 4.

However, the addition of two orientated segments on the same line or on two parallel lines is in affine geometry. In Fig. 3, we always have $\overline{MN} + \overline{NE} = \overline{ME}$ and $\overline{MN} = 1/2(\overline{AB} + \overline{CD})$, where \overline{AB} , etc., are signed, oriented segments. These equalities are valid for both Fig. 3a and b. A single synthetic proof without the order relation of the unordered version of Example 5 is a very good **challenging** exercise for full understanding of unordered geometries discussed in this section.

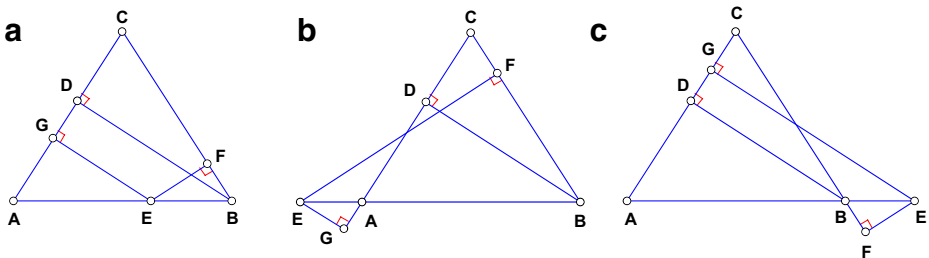


Fig. 4 A theorem on isosceles triangles

2 The Full-Angle Method

The definition of angles in common geometry textbooks uses the order relation. In unordered geometries, we need to use a concept similar to angles that we will call full-angles which do not involve the order relation. Since the goal of this paper is the automated generation of visual presentation of proofs for the full-angle method [15] and the deductive database method which uses full-angles [14], in this section we will give a brief review of the basic concepts of full-angles and the full angle method, so that the readers can understand this paper, and when needed, they can resort to the original paper [15]. Section 2.1 presents the definition and the basic properties of full-angles. While the properties listed here do not cover all properties in our original paper [15], this section includes additional insights into full-angles other than discussed in our original paper [15]. Section 2.2 discusses the full-angle method briefly.

2.1 Full-Angles

- R1.** A full-angle is defined as an ordered pair of two lines u and v denoted by $\angle[u, v]$.
- R2.** $\angle[u, v] + \angle[v, u] = \angle[0]$ or $\angle[u, v] = -\angle[v, u]$.
- R3.** Constant full-angle $\angle[u, u]$ is denoted by $\angle[0]$. We have $\angle[0] + \angle[u, v] = \angle[u, v] + \angle[0] = \angle[u, v]$. Also $\angle[u, v] = \angle[0]$ iff $u \parallel v$.
- R4.** If $u \perp v$, then we define $\angle[u, v]$ to be a constant full-angle $\angle[1]$. $\angle[u, v] + \angle[u, v] = \angle[0]$, or $\angle[1] + \angle[1] = \angle[0]$.
- R5.** For two full-angles $\angle[u, v]$ and $\angle[u_1, v_1]$, there are three lines l_1, l_2 , and l_3 such that $\angle[u, v] = \angle[l_1, l_2]$ and $\angle[u_1, v_1] = \angle[l_2, l_3]$. We define $\angle[l_1, l_3]$ to be the sum of $\angle[u, v]$ and $\angle[u_1, v_1]$, denoted by $\angle[u, v] + \angle[u_1, v_1] = \angle[l_1, l_3]$.

Remark This definition of the two full-angle congruence follows the style of Hilbert's book [12]. More informally, two full-angles $\angle[l, m]$ and $\angle[u, v]$ are equal if there exists a rotation K such that $K(l) \parallel u$ and $K(m) \parallel v$. For the geometric meaning of the addition of full-angles, let l, m, u , and v be four lines and K be a rotation such that $K(l) \parallel v$. Then $\angle[u, v] + \angle[l, m] = \angle[u, K(m)]$.

As a special case, $\angle[u, l] + \angle[l, v] = \angle[u, v]$. Thus the addition of two full-angles does not use the order relation and is diagram-independent. The set of all full-angles with the addition operator '+', is an additive (Abelian) group.

The geometries with the definition of the addition of full-angles are Euclidean, as opposed to non-Euclidean geometries. For example, the assertion that the sum of three angles of a triangle ABC equals 180° , which is equivalent to Euclidean fifth hypothesis, can be expressed as

$$\angle[CA, AB] + \angle[AB, BC] + \angle[BC, CA] = \angle[CA, CA] = \angle[0].$$

- R6.** If $l \perp v$, we have $\angle[u, v] = \angle[u, l] + \angle[l, v] = \angle[u, l] + \angle[1]$.

Note that this rule is the combination of R4 and R5.

- R7.** If $AB = AC$, then $\angle[AB, BC] = \angle[BC, CA]$.

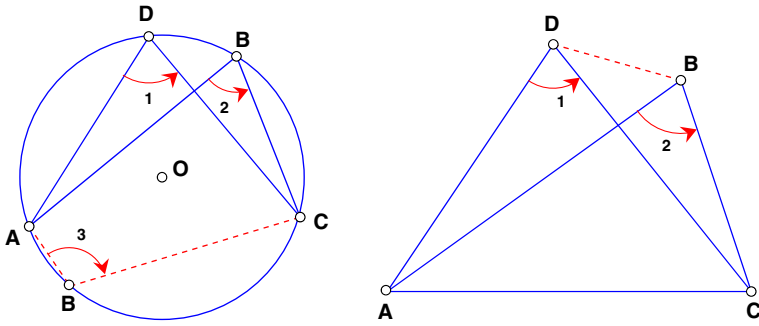


Fig. 5 The inscribed angle theorem

The full-angle method works particularly well for theorems involving circles and angles.

Let $A, B, C,$ and D be four non-collinear points.² We use $\angle[ABC]$ to denote the full-angle $\angle[AB, BC]$ and use $\angle ABC$ to denote the traditional angle. In ordered geometry, $\angle[ABC] = \angle[DEF]$ if and only if $\angle ABC = \angle DEF$ and the two angles have the same orientation, or $\angle ABC = 180^\circ - \angle DEF$ and the two angles have the opposite orientations.

R8. (*The Inscribed Angle Theorem*) Points $A, B, C,$ and D are cyclic iff $\angle[ABC] = \angle[ADC]$ (the left of Fig. 5.)

Applying this twice we can have a combined rule, say, $\angle[BAC] = \angle[BDC]$ iff $\angle[ABC] = \angle[ADC]$ (the right of Fig. 5), etc. Here the four points can be in any order. These diagram-independent properties of full-angles make the proofs not only much simpler and diagram-independent, but also rigorous.

2.2 An Overview of the Full-Angle Method

The full angle method uses a fix set of rules based on the full-angle definition and its properties. The rules can be considered as lemmas or axioms of our system. The independence of these rules is not our concern at all. Actually we have a set of combined rules based on our basic rules. Our main concern, beside the consistency, is how to select or combine these rules so that the system can effectively generate short proofs.

In [15] we use 14 basic rules and 7 combined rules. We rewrite these rules in the current system JGEX for generation of visually dynamic presentation of full-angle proofs.

²Whenever we discuss a geometry theorem in this paper we assume certain non-degenerate conditions to be valid which have been well discussed in [16] for a class of geometry statements of constructive type. For example, in Simson’s theorem (Example 3) we need to add a condition among others that line AB is non-isotropic. In the real plane, this is equivalent to $A \neq B$.

The full-angle method is a combination of the forward chaining followed by the backward chaining. Beginning with the hypotheses of the theorem as the initial set of properties (facts), we repeatedly apply these rules to this set of properties to get new properties until no new properties can be added. The final set of properties is called Geometry Information Bases (GIB) [15].

If the conclusion is in the GIB, then we prove the theorem. Otherwise we do a backward chaining beginning from the conclusion, using the technique of elimination of points or lines from the full-angles. For details see [15].

3 Automated Generation of Visually Dynamic Presentations of Proofs for the Full-Angle Method

Before going into the detail of visually dynamic presentation of proofs for the full-angle method, we need to further discuss the full angle method in Sections 3.1 to 3.4.

3.1 Hierarchical Structure of Full-Angle Proofs

For a relatively complicated proof, it is crucial that the proof be presented in a hierarchical way. Automated generation of proofs with hierarchical structures is an important topic in theorem proving in general. In some cases, it is closely related to lemma applications in mathematical proofs (See Miquel's theorem for $n = 5$ in Example 12.)

If a machine-generated proof with hierarchical structures is printed in a paper, then the presentation is static and the advantage is not so evident. However, if the proof is represented with a tree in a computer display window as we shown in examples of Section 3.5, the advantage is evident. Depending on options, the proof can be presented with only main (top-level) steps so that the user can concentrate on main steps, and when a particular step is needed for further investigation, the user can expand the proof of this step to a subtree.

On one extreme end, the text pane displays a proof text only with the main proof steps. On the other extreme end, the proof text is displayed in its full expansion, including substeps of the main steps, and further substeps of each substep, etc.

Theoretically, the backward chaining gives a natural way to generate a proof with hierarchical structures. We have tried this approach and the program generates a proof tree. However, for relatively complicated proofs, this is not a natural way for presenting proofs of geometry theorems similar to those presented in geometry textbooks. In addition, some leafs are nested too unnecessarily deep and make the proof hard to read. We need a better way to do this in addition to backward chaining.

In our full angle method, we achieve automated generation of proofs with hierarchical structures by a combination of the GIB, obtained by the forward chaining, and the backward chaining. This provides a natural way to generate full-angle proofs with hierarchical structures more similar to lemma applications: whenever a non-initial property in the GIB is used in the backward chaining, the proof of this property can be expanded with the forward chaining. This proof can be considered as a substep of the main proof. If the proof of this property has sufficient number of steps, we can consider this property and its proof as a lemma application.

Let us use the full-angle proof of the Miquel theorem for $n = 5$ to see the proof with hierarchical structures and instances of the lemma application.

Example 6 (Miquel's Theorem for $n = 5$. See Example 12) The number of main (top-level) steps of this proof is 12. The number of total steps is 46. In particular, two main steps, Step 2 and Step 11, use a property that is actually Miquel's theorem for $n = 4$ found in the GIB with the forward chaining. We can expand its forward chaining proof which consists of 19 steps. So Steps 2 and 11 can be considered as applications of Miquel's theorem for $n = 4$. See Fig. 11.

From the examples in Section 3.5, we can see that most main steps in the proofs have a subtree. It is very convenient to expand or to shrink a proof subtree with mouse clicks.

3.2 Automated Generation of Auxiliary Geometry Elements

One of the main features of the full-angle method is that auxiliary geometry elements such as points, segments, angles, and circles are automatically added in some proof steps, as these elements are required by the rules of the full-angle method in that proof step [15, 18].

For example, in the proof of the Miquel theorem for $n = 5$ (Fig. 11), Step 2

$$\angle[M_2M_5P_5] = \angle[M_2P_3P_5]$$

uses Rule 8, the full-angle version of the inscribed angle theorem (see **R8** in Section 2.1). This rule requires three segments P_5M_5 , M_5M_2 , M_2P_3 , and the 4-point circle $M_5P_3M_2P_5$. Thus three auxiliary segments and one circle are added in Step 2. See the proof of this theorem in Fig. 11 for detail.

3.3 Automated Generation of Visual Effects for the Proof Text

When we step through the proof line by line, the auxiliary lines, segments, angles, and circles required for the proof are added automatically, as these elements are required by the rules of the full-angle method in that proof step [15, 18]. In geometry textbooks, all these geometry elements are generally put in one diagram for the proof. The diagram with all auxiliary elements is already somewhat cumbersome for reading a proof of even a moderately complicated theorem.

With our dynamic presentation of proofs, when segments, angles, or circles are needed in a proof step, they appear only when this step is clicked. When another step is clicked, these auxiliary elements disappear.

Not only each line of the proof is clickable with expected dynamic visual effects, but also some part of a proof line, e.g., $\angle[ABC]$ in the line, is clickable with the corresponding visual effects. In this way, the angles in the proofs text is related to the visual effects in the diagram.

JGEX also provides several other options for adding auxiliary elements dynamically.

- (1) Put all auxiliary elements in the final diagram for the proof when the number of auxiliary elements is relatively small, say, is less than 6.
- (2) When auxiliary elements are required in a step, they appear. Afterward, we only keep those that the prover considers ‘essential’ for reading the proof.

3.4 Visualization of the Rule Applications

It is very crucial for the reader to understand the full-angle rules before going to the detail proof text. However, the visualization of these rules is considered to be not easy [17]. In JGEX, we provide a method to visualize these rules and believe that it can help the reader a lot in understanding full-angle proofs.

When a full-angle rule is applied, it displays the text, e.g., ‘Rule 8’ in Fig. 9. By clicking the text ‘Rule 8’, it creates another window to show the portion of diagram where the rule is applied (see the right window of Fig. 9). In this way, we get rid of the unnecessary elements while keep the key elements in the diagram so that the user can understand the rule much easier. This method is especially useful for complicate theorems such as the Miquel Theorem for $n = 5$ in Fig. 11.

In the popup window, all objects in the new window are mouse-sensitive just as the main diagram pane. For example, we can drag a point in the new window to change the shape of the diagram so that the rule application can be seen more clearly. Also the point and angle labels are movable with the mouse. Moreover, whenever an object is changed in this window with mouse, the corresponding object of the diagram in the main pane is changed accordingly, and vice versa.

If we want to know further about the rule application, JGEX provides each rule with an example to further explain it. For example, the left popup window of Fig. 9 shows the built-in example for Rule 8 (the inscribed angle theorem) of the full-angle method. In this way, the rule application is vividly seen.

3.5 More Examples

We have tested JGEX with 110 theorems in the collection [18] and it took about 12 seconds to generate the proofs of all theorems on a Pentium IV 2.8G machine. We choose to dump the screen, because this might be the best way to give some features of visually dynamic presentation of proofs in a static piece of paper: the proof text tree pane and the diagram pane are side by side and we can see whether a node in the proof tree is expanded or not. More examples can be found on our webserver [19].

Example 7 The same as Example 1. Figure 6 is the proof of this theorem: the left pane is for the proof text tree and the right pane is for the diagram.

The proof is organized into a hierarchical tree and currently the tree is in its full expansion. There are two ways to see the visual effects. (1) One can click each line of the proof tree and the corresponding visual effects will be shown in the diagram in the right pane. Each geometry element in a proof line is generally clickable. For example, $\angle[GDC]$ in the highlighted line in Fig. 6 is clickable with its own visual

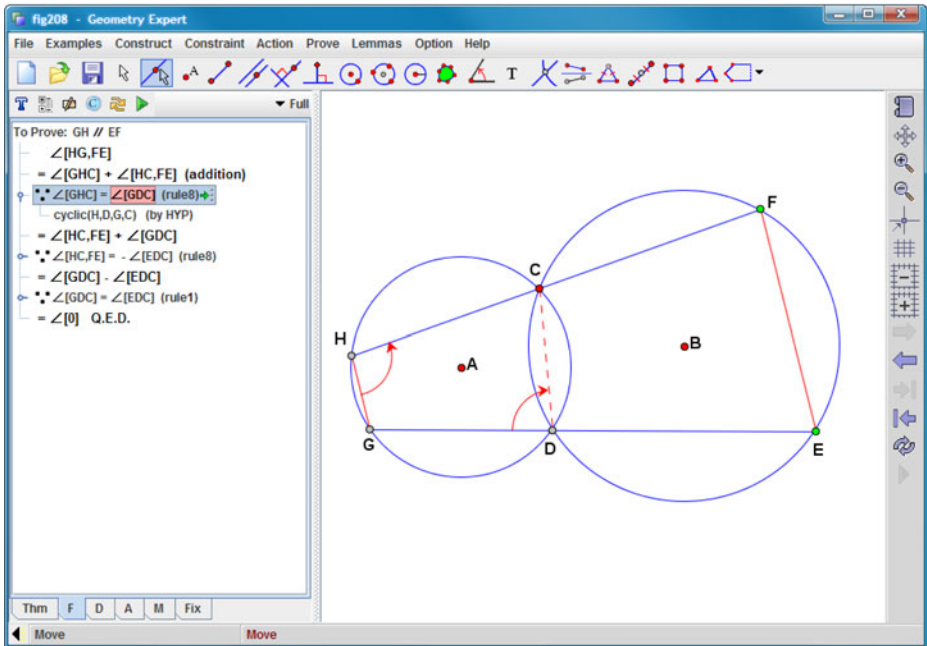


Fig. 6 A two circle theorem

effects: the mark of $\angle[GDC]$ is blinking. (2) One can step the proof line by line and each line is highlighted with the same visual effects as if clicking the line itself.

Example 8 (The Orthocenter Theorem) The three altitudes of a triangle are concurrent. Let F be the intersection of the two altitudes BD and CE of triangle ABC . Show that $AF \perp BC$.

The proof adds three auxiliary elements: one segment and two circles. Note that the proof is diagram-independent, it is valid for acute triangles, as well as for obtuse triangles. However, a right triangle is a degenerate case for this proof, although the theorem is still valid.

The contents in the left text pane can be switched by clicking the menu items in the bottom bar. When a theorem file is opened or a diagram is drawn, it contains the text of the theorem and the menu item ‘Thm’ is highlighted. If a proof with the full-angle method is generated, the pane contains the text of the proof tree and the menu item ‘F’ is highlighted. We can switch back to the text of the theorem by clicking ‘Thm’. The menu item ‘D’ is for switching to the proof text with deductive database method if it is already generated. ‘A’ is for switching proof text of the area method (not implemented yet). ‘M’ is for the manual inputs. ‘Fix’ is to generate the contents of the fixpoint in the pane.

Example 9 (The Nine Point Circle Theorem) In a triangle ABC , the feet of the three altitudes, the midpoints of the three sides, and the midpoints from the three vertexes to the orthocenter are on the same circle.

Let F be the orthocenter of the triangle ABC (Fig. 7 of the preceding example). Then A is the orthocenter of $\triangle FBC$, B is the orthocenter of $\triangle FAC$, and C is the orthocenter of $\triangle FAB$. Thus we only need to prove that the midpoints F, G , and E of the three sides and the foot D of one altitude, AD , are on the same circle. Figure 8 is its proof. Here we choose the option of putting all auxiliary elements, one circle, one segment and two angles, in the diagram.

In Fig. 8, the current step is highlighted which uses Rule 7. When this step is clicked (hence is highlighted), there appears a green arrow to the right of ‘(rule 7)’. By clicking this arrow, it creates another floating window showing the portion of the diagram to which this Rule 7 is applied.

Here the proof consists of five main steps corresponding to the five equal signs ‘=’. Since we separate the application of the addition of two full-angles from the combined rules as a new step, there are generally more steps in proofs generated by JGEX than those in our previous publications [15, 18].

Example 10 (Simson’s Theorem) Let D be a point on the circumscribed circle (O) of a triangle ABC . Let P, R , and Q be the feet of the three perpendicular lines from point D to the three sides BC, AC , and AB , respectively. Show that P, R , and Q are collinear.

The proof is given in Fig. 9. The left proof text pane contains the proof tree. There are two auxiliary segments and two auxiliary circles added to the diagram.

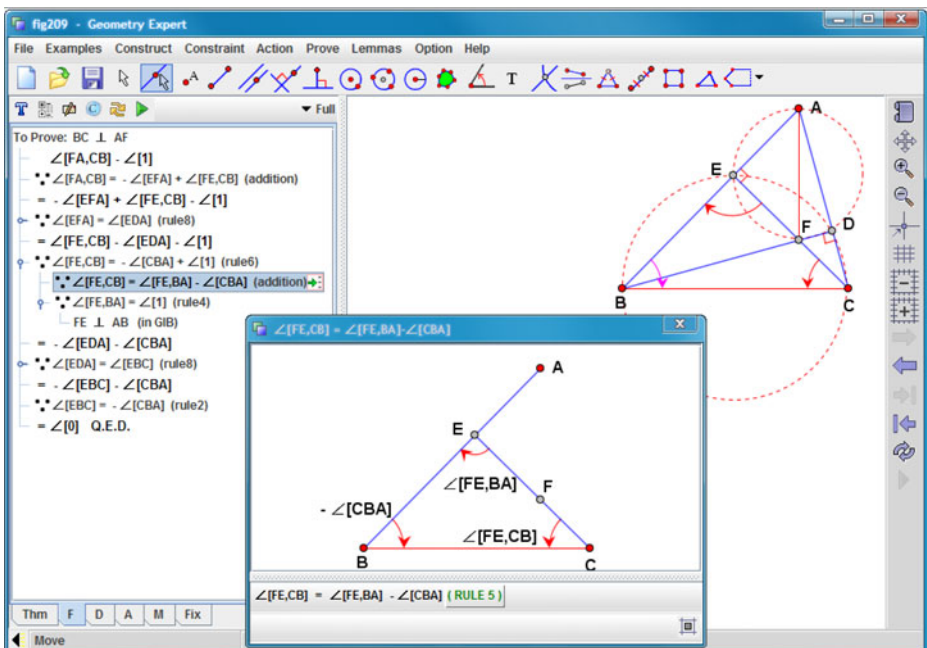


Fig. 7 The orthocenter theorem

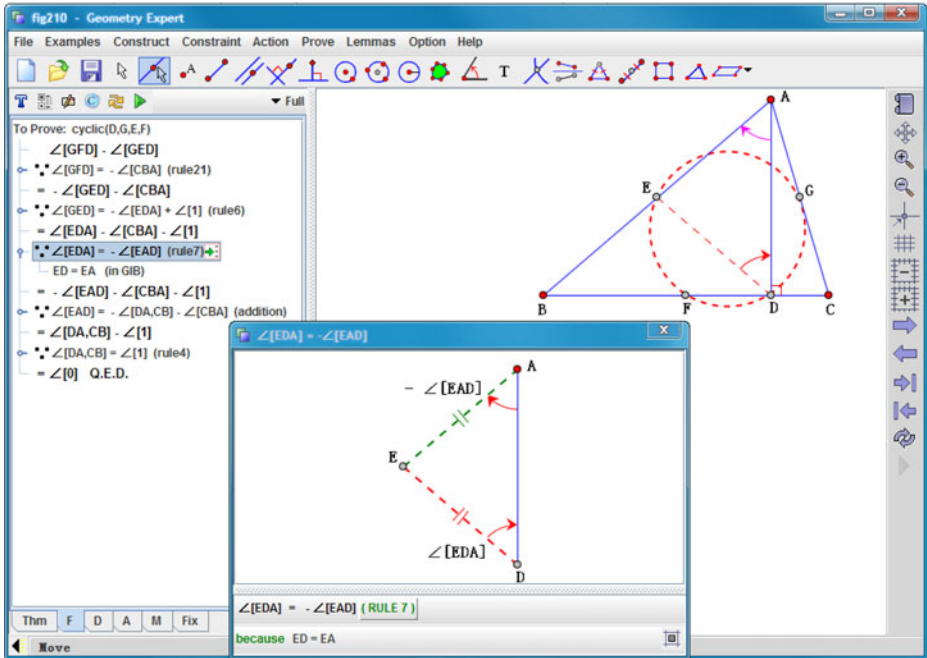


Fig. 8 The nine point circle theorem

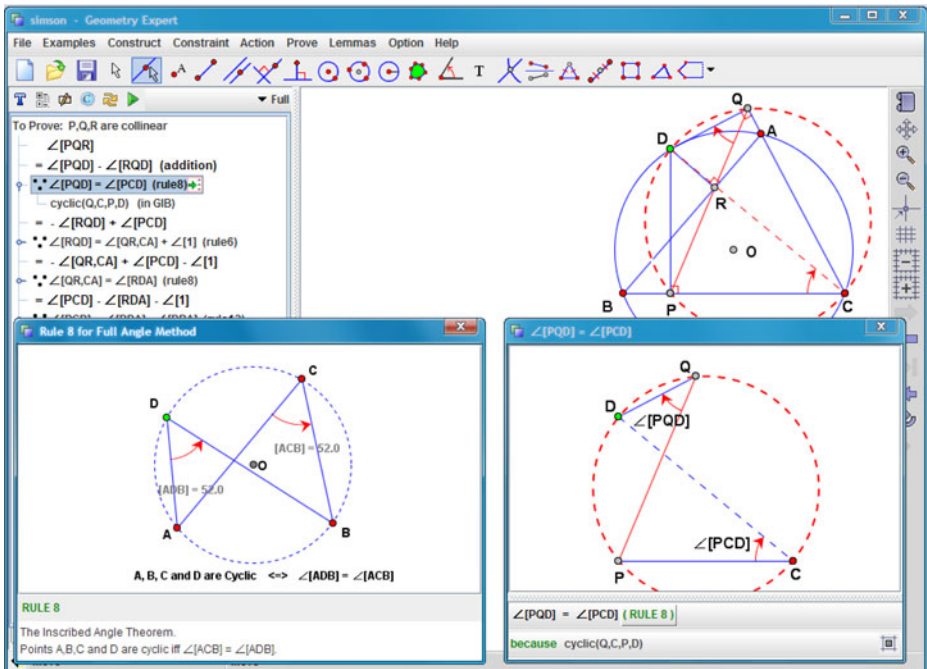


Fig. 9 Simson's theorem

Alternatively, we can start to prove $\angle[ARQ] = \angle[BRP]$ similar to the traditional proof in common geometry textbooks. However, here we are using full-angle rules and thus the proof is rigorous—no additional assumptions of the order relation, such as points D and P are on the opposite sides of line AC , are required.

The current step is highlighted that uses Rule 8. The floating window on the right of the diagram shows that Rule 8 is the full-angle version of the inscribed angle theorem (Fig. 9). The window on the right of the diagram shows the built-in example for this theorem.

Example 11 (A Two Parallelogram Theorem) Let $ABCD$ be a parallelogram, AH and CG be perpendicular to the diagonal BD at feet H and G , BF and DE be perpendicular to the diagonal AC at feet F and E . Show that $HF \parallel EG$ (Fig. 10).

There are two auxiliary circles added. The segments HE and FG are added by the user (the authors) to explain that the original problem is to prove that the quadrilateral $EGFH$ is a parallelogram. This is actually a hint for human to prove the theorem: the traditional proof in the common geometry textbooks is to prove that the two diagonal HG and EF bisect each other with the triangle congruence method.

However, to understand this hint is hard for the computer program. Furthermore, the computer program generally does not know to generalize from $HF \parallel EG$ to $HE \parallel FG$. So using the traditional method to prove this theorem with the computer is expected to be hard.

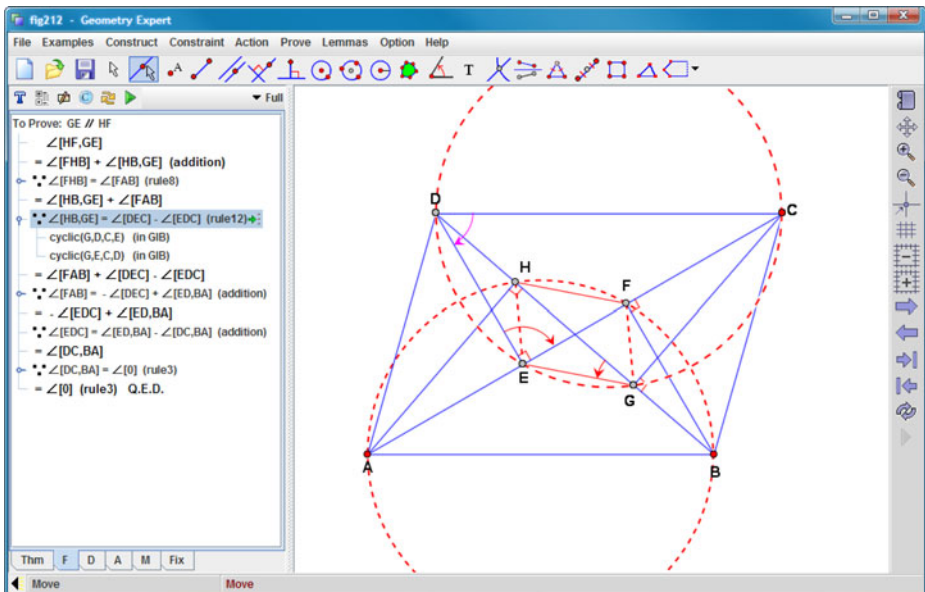


Fig. 10 A two parallelogram theorem

This is a pure line figure theorem and it is a surprise that the proof uses two auxiliary circles.

For all examples so far, the sizes of the proof text pane and the diagram pane are large enough to hold the text and the diagram. The next example shows that the sizes of the panes are the most valuable computer resource, while the proof is generated virtually with minimal CPU time. In that example, we have to choose the option to use small fonts for the proof text. However, the two panes barely hold the proof text and the diagram.

Example 12 (The Miquel Theorem for $n = 5$) Let $P_1, P_2, P_3, P_4,$ and P_5 be five points. Let $Q_1 = P_2P_5 \cap P_1P_3, Q_2 = P_1P_3 \cap P_2P_4, Q_3 = P_2P_4 \cap P_3P_5, Q_4 = P_3P_5 \cap P_1P_4,$ and $Q_5 = P_1P_4 \cap P_2P_5.$ Let the other intersections of the consecutive circumscribed circles of triangles $Q_5Q_1P_1, Q_1Q_2P_2, Q_2Q_3P_3, Q_3Q_4P_4,$ and $Q_4Q_5P_5$ be $M_1, M_2, M_3, M_4,$ and M_5 respectively. Show that points $M_1, M_2, M_3, M_4,$ and M_5 are cyclic.

The 12 machine-generated main proof steps are shown in Fig. 11 with Step 2 being clicked (and thus is highlighted). It uses Rule 8 with the condition $\text{cyclic}(M_5, P_3, M_2, P_5)$ which is a fact found by the forward chaining, i.e., a fact in the GIB. This fact is the Miquel theorem for $n = 4$. By clicking this step, circle (M_5, P_3, M_2, P_5) , together with three segments and two angle marks, appear and blink (Fig. 11).

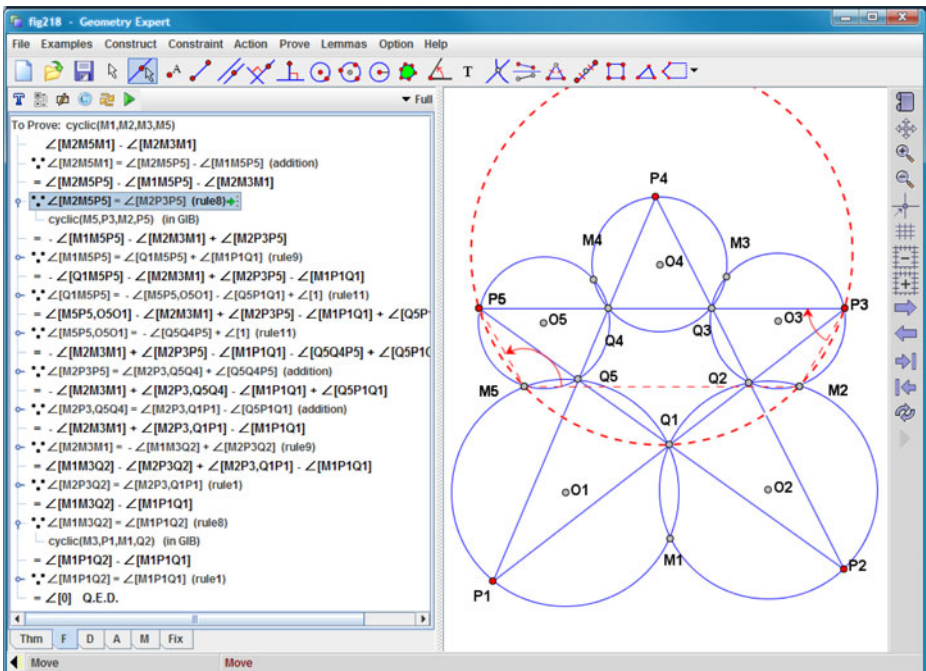


Fig. 11 The Miquel theorem for $n = 5$

By clicking $\text{cyclic}(M_5, P_3, M_2, P_5)$ with the right mouse button, a selection menu is opened. By selecting “prove”, a proof of $\text{cyclic}(M_5, P_3, M_2, P_5)$ with 19 steps is expanded and inserted into the proof.

The Miquel theorem for $n = 4$ with the point labeling in this example states as following: Let P_5Q_1, P_5P_3, P_1P_3 and P_1Q_4 be four lines that form four triangles $P_1Q_1Q_5, P_1P_3Q_4, P_5Q_4Q_5$, and $P_5P_3Q_1$. Show that the circumscribed circles of the four triangles intersect at the same point.

Likewise, Step 11 in the text pane of Fig. 11 is another instance of the Miquel Theorem for $n = 4$: the fact $\text{cyclic}(M_3, P_1, M_1, Q_2)$ is shown in the 4th line from the bottom.

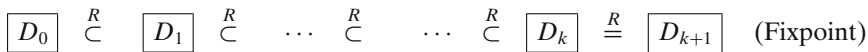
4 The Automated Generation of Visually Dynamic Presentations of Proofs for the Deductive Database Method

4.1 An Overview of the Deductive Database Method

During 1994–1995, we used a general-purpose deductive database program based on Prolog to experiment with the full-angle rules, and soon found that general-purpose deductive databases had two major problems for our purpose: (1) infinite loops, e.g., $\text{coll}(A, B, C) :- \text{coll}(B, A, C) :- \text{coll}(A, B, C) :- \dots$; (2) high inefficiency.

We then implemented our own structured database and a rule set R consisting 75 rules for full-angles, some of which are built-in into the structured database [14].

Given a geometry statement of the equality type, each hypothesis is considered as a fact (a property). All hypotheses of the statement form the initial database D_0 of facts. Then we apply the rule set R to D_0 to get new facts. We expand the database D_0 by adding the new facts to D_0 to have the enlarged database D_1 . We repeat the process for D_1 to have an enlarged database D_2 . Since the number of points is finite, the numbers of segments, of angles, of triangles, and of circles are also finite. This process will finally terminate, i.e., there is $k \geq 0$ such that $D_k = D_{k+1}$. Then we call D_k the fixpoint of the given geometric configuration. This process can be expressed by the following diagram:



Since the rule set R and the structured database are built into the source code, the prover is very efficient and the fixpoint can be generally reached within a fraction of a second even for an antique Pentium machine. If the conclusion (a fact also) is in the fixpoint, then we prove the statement to be true and the proof can be generated. The visualization of the rule application for deductive database method is similar to the rule application for the full-angle method, except that they have different rule sets.

As the full-angle method, the deductive database method also automatically adds auxiliary segments, angles, and circles as the rule applications in the proof process require. However, we generally do not add new points, e.g., by intersecting two lines. In our deductive database method we add 18 rules for adding auxiliary points [14]. But we do not do this recursively, i.e., we will not add a new auxiliary point if this point depends on an already added auxiliary point.

4.2 Visualization of the Proof Generated by the Deductive Method

The deductive database is achieved by forward-chaining. However, a straightforward print of the deduction steps is not easy to understand. We choose to construct the proof from the conclusion to the hypotheses. Each step of the proof can be clicked to see the corresponding visual effects. The effects for the deductive database method are similar to those generated by the full-angle method.

Generally, it requires more steps for the deductive database method than those for the full-angle method. There are three reasons for this phenomenon:

(1) The full-angle method uses the technique of elimination of points and lines similar to the area method [21], thus the proof is more direct; (2) the full-angle method uses combined rules, and each of the combined rule consists several rules in the deductive database method; (3) the full-angle method uses facts found by the forward chaining, i.e., in the GIB, and if the proofs of these facts are expanded, there are more steps.

However, the deductive database method has its own advantage: (1) The proof generated by the deductive database method is more similar to the traditional proof, and thus easier to understand. (2) In comparison of the fixpoint with the GIB in the full-angle method, the fixpoint in the deductive database method is structurally organized and there are more rules in the rule set R . So the forward-chaining for deductive database method is much more powerful and can be used to discover new facts.

4.3 Visualization of the Deductive Database

Generally, the fixpoint has surprisingly rich amounts of information even for a very simple configuration. We can classify the facts in the fixpoint into several categories according to the geometric predicates: collinear, parallel, perpendicular, segment equal, cyclic, (full) angle equal, triangle congruence, etc. By clicking each category, all properties of the configuration in this category are listed. And by clicking each of these properties, the corresponding geometry elements in the diagram blink or animate and, if needed, the proof of this property is generated.

For example, one fact is circle($P_3P_5Q_1M_2M_5$) in Fig. 15. By clicking this fact, the circle $P_3P_5Q_1M_2M_5$ appears and blinks in the diagram. By clicking with left mouse button, it creates another window for selecting which four points, since there are five points and each time we can only prove four points being cyclic. After selecting the four points, the prover generates the proof of this fact in the text pane with the menu item 'D' highlighted. We can switch back to the content of the fixpoint by clicking the menu item 'Fix'.

Some of the facts in the fixpoint are really unexpected and visualizing them is enjoyable. For example, in Fig. 8 (the nine point circle theorem), if we click the menu item 'Fix', we find there are five groups of congruent triangles. One group has five congruent triangles: actually we might expect there are only 4 congruent triangles because the three midpoint connecting segments FE , EG , GF divides the triangle ABC into four triangles $\triangle FEB$, $\triangle GAE$, $\triangle CGF$, and $\triangle EFG$. However, " $-\triangle GDE$ " is another one!

The minus sign ‘-’ before the triangle sign Δ means that the orientation of ΔGDE is clockwise. The fact “ $\Delta FEB \cong \Delta DGE$ ” is not obvious and its proof takes 7 steps with the deductive database method.

The visual effects are as follows: First ΔGAE is filled with a color and moves on the fly, and drops in the position of ΔCGF . The ΔCGF is filled with another color and moves ..., and so on. At the end, ΔEFG is filled with a different color and moves on the fly with a flip or reflection (because it has the opposite orientation to that of ΔGDE), and drops to the position of ΔGDE . Finally, all triangles appear with different colors and we use the transparent technique to solve the problem of overlap of triangles.

4.4 More Examples

We choose two examples: one for visually dynamic presentation of proofs with the deductive database method and the other ones for visualization of the fixpoint.

Example 13 (A two Altitude Theorem) Let BE and CD be two altitudes of triangle ABC and points G and F be the midpoints of BC and DE respectively. Show that $DE \perp FG$.

Figure 12 is the proof of this theorem. Here we choose the option to put all auxiliary elements, one circle and two segments, in the diagram.

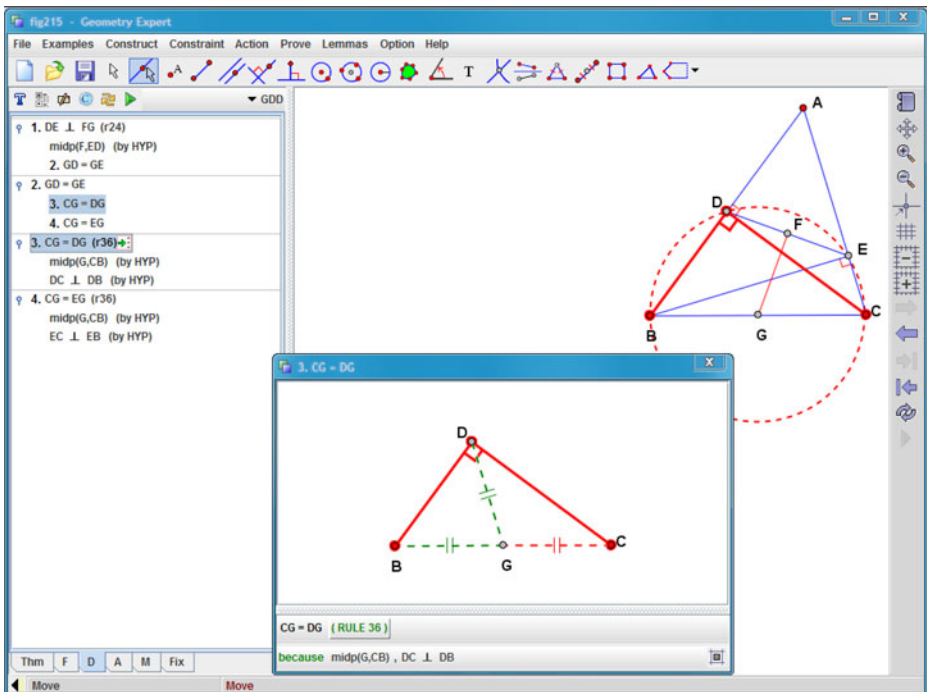


Fig. 12 The proof of two altitude theorem

The current step (Step 3) is highlighted and it uses the deductive database Rule 36. By clicking the green arrow to the right of “(r36)”, it creates a floating window to show how this rule is applied. Rule 36 states that:

The hypotenuse of a right triangle is the diameter of the circumscribed circle of the triangle, and the midpoint of the hypotenuse is the center of the circle.

This is an easy problem in junior high schools. However, to add the two auxiliary segments and one auxiliary circle is hard for computer programs.

The input of the deductive database method does not have to have a conclusion: we could input all conditions (hypotheses) of a geometry configuration and let the program to find all facts (the fixpoint) with the method. The next three examples show the fixpoints of three configurations.

Example 14 (The Orthocenter Configuration 1) Let $A, B,$ and C be three points, CD be perpendicular to AB at the foot D, BE be perpendicular to AC at the foot $E, .$

We can click the “Fix” menu item to switch the contents in the left pane to contain the fixpoint (the final database). The pane now contains 5 icons corresponding to 5 categories of properties. By clicking each icon, e.g., the perpendicular lines icon, it lists all facts of this property, i.e., all pairs of perpendicular lines. By clicking the left button of the mouse, the two corresponding perpendicular lines, BC and $AF,$ appear and blink in the diagram. By clicking the right mouse button, it creates another menu

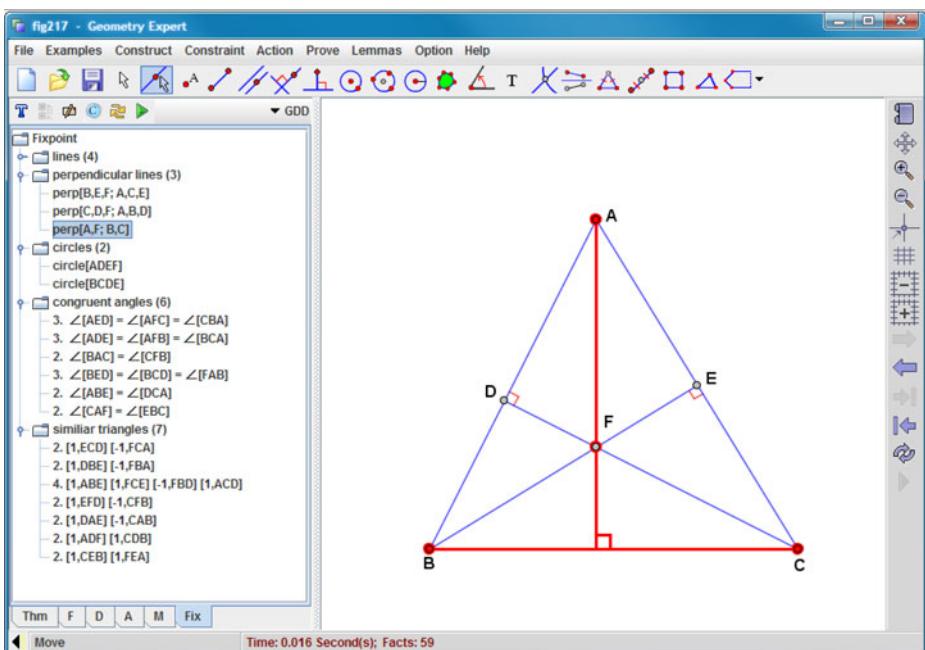


Fig. 13 The orthocenter configuration 2

with two items “Prove” and “Prove in a New Window”. By choosing “Prove”, the proof based on the deductive database method is generated in the left pane.

Note that there are three pairs of perpendicular lines. Two are a part of the hypotheses, the third one, $BC \perp AF$, is the orthocenter theorem. By clicking this property, two auxiliary red color lines appear and blink (Fig. 13).

Example 15 (The Orthocenter Configuration 2) Let CD and BE be two altitudes of triangle ABC , F be the intersection of CD and BE , G be the intersection of AF and BC .

The differences between this diagram and the diagram in the preceding example are a new intersection point G of lines AF and BC . However, the properties in the fixpoint are surprisingly richer, especially for equal angles (Fig. 14).

With 7 groups of equal full-angles, it is expected that there are a lot of similar triangles: there are 7 groups of similar triangles. Each group has four triangles. When we click a fact of one of the 7 groups (not shown in the figure), the visual effects are as following: beginning with one triangle moving on the fly, with a possible flip (reflection) if the second triangle is in the orientation opposite to the first. At the same time the triangle is dynamically scaled to the size of the second one and drops into its position. Then repeat the second one, and so on. Finally, all triangles appear and overlap with different transparent colors.

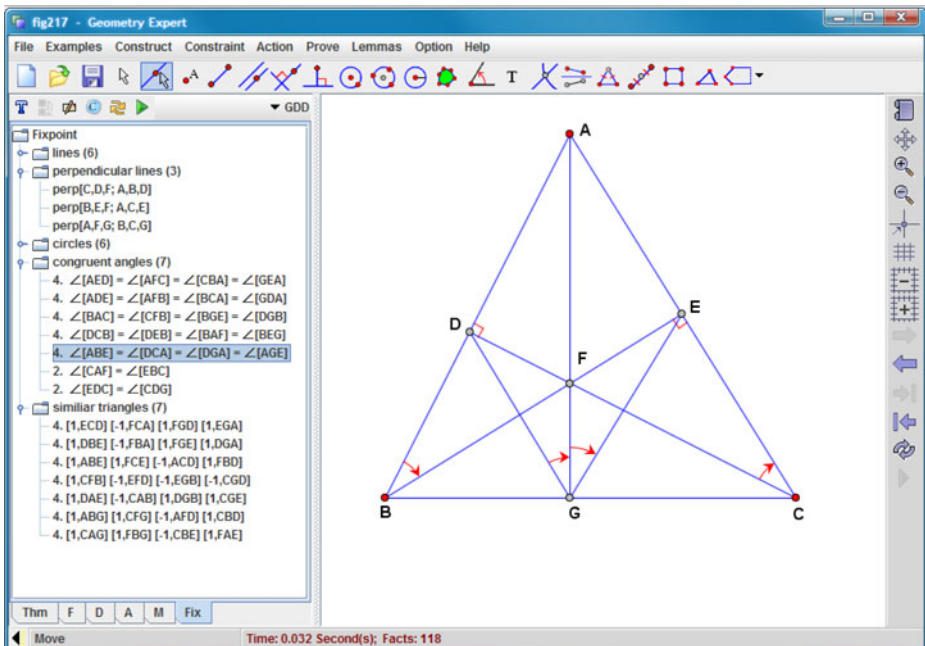


Fig. 14 The orthocenter configuration 2

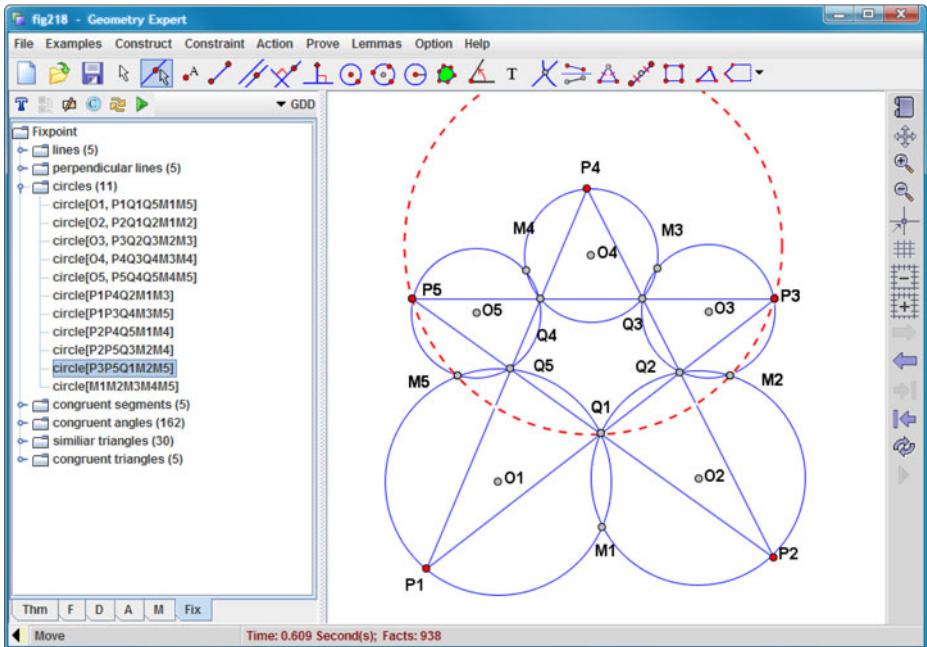


Fig. 15 The Miquel configuration with $n = 5$

Example 16 (The Miquel Configuration with $n = 5$) Let $P_1, P_2, P_3, P_4,$ and P_5 be five points. Let $Q_1 = P_2P_5 \cap P_1P_3, Q_2 = P_1P_3 \cap P_2P_4, Q_3 = P_2P_4 \cap P_3P_5, Q_4 = P_3P_5 \cap P_1P_4,$ and $Q_5 = P_1P_4 \cap P_2P_5.$ Let the other intersections of the consecutive circumscribed circles of triangles $Q_5Q_1P_1, Q_1Q_2P_2, Q_2Q_3P_3, Q_3Q_4P_4,$ and $Q_4Q_5P_5$ be $M_1, M_2, M_3, M_4,$ and M_5 respectively.

The fixpoint contains 11 circles: five are given in the hypotheses, one is the original conclusion circle($M_1M_2M_3M_4M_5$) of Miquel’s theorem, the rest five 5-point circles are new and are ‘discovered’ by the prover. By clicking one of the new facts (highlighted in the text pane), circle($P_3P_5Q_1M_2M_5$), a circle with red dash-lines shows and blinks in Fig. 15. The proof of this fact needs 6 steps with the deductive database method.

Furthermore, if we allow to introduce new intersection points in the configuration, then there are another 10 points that are on the circle $M_1M_2M_3M_4M_5,$ e.g., if we add point $N_1 = P_5M_4 \cap P_3M_3$ in the input, then a fact circle($N_1M_1M_2M_3M_4M_5$) is in the fixpoint.

The proof of the original conclusion, cyclic(M_1, M_2, M_3, M_4, M_5), based on the deductive database method consists of 21 steps.

5 Related Work

S. Wilson and J. Fleuriot [17] reported the success in implementing our full-angle method [15]. Their program, *Geometry Explorer* (GE), has proved 100 out of 110

theorems in our collection [18]. *Geometry Explorer* can create ‘purely diagrammatic proofs’ for visualizing full-angle proofs. According to the authors, “... it is hard to relate textual geometry facts to a separate diagram.” However, the internal relationship of a clickable part of the proof text and its corresponding geometric elements is the most important and distinctive feature of the visually dynamic presentation of proofs in the work reported here.

The goals of GE and JGEX are very different. The goal of our work is to make it easier for the user to read the proof of a geometry theorem with various dynamic visual effects that the computer display and the mouse can offer. Whereas GE emphasizes visualizing the full-angle method *itself*, such as visualizing the forward chaining and backward chaining used in the full-angle method, and visualizing multiple proofs generated by the full-angle method.

Also GE uses many essentially identical diagrams of the theorem on the computer display for a proof, whereas we use a **single**, but dynamically changed diagram. The duplication of the same diagram makes a proof harder to read, especially when the reader is comparing the proof text with the diagram. For complicate proof trees, dealing with a large number of nodes (hence a large number of essentially identical diagrams) is unmanageable for both the program and the reader.

6 Conclusions and Future Work

In Part 1 of this series, we have proposed an entirely new presentation of proofs in plane geometry—visually dynamic presentation of proofs. The focus of Part 2 is the automation of these proofs as we have demonstrated: automated generation of visually dynamic presentation of proofs for the full-angle method and for the deductive database method. The new feature of visualization of all rules and geometric properties in the fixpoint is another addition to our method.

The system JGEX is an ongoing developing system: the current version is Beta 0.80 which is available on our server [19]. In addition, JGEX can be run online with a large number of examples and the gif files of these examples saved by JGEX can be seen in the collection [20] without Java.

Ongoing work and future work include:

6.1 Visual Dynamic Presentation of Proofs for the Area Method

The area method [21], is a combination of synthetic method with algebraic computation. For area method proofs with simple algebraic computations, the corresponding visually dynamic presentation of proofs are expected to have impressive dynamic visual effects. However, the area method sometimes generates relatively large algebraic expressions. To make long algebraic expressions visually meaningful and appealing is under investigation.

6.2 Visually Dynamic Presentation of Proofs for the Traditional Methods

So far the success in using the traditional methods for automated proving geometry theorems is very limited. Currently we are working on this topic. Based on our

experience with the full-angle method and on the new ideas we have worked out, we believe that we can prove not only more theorems but also more difficult theorems with computers.

Comparing with the full-angle method, angles used in the traditional methods are more intuitive. For a full angle $\angle[AB, CD]$, lines AB and CD even do not have to be intersected. However, for a traditional angle, it always has a vertex and its interior (and exterior). Thus a traditional proof would have a wider range of visual effects.

6.3 Adding Auxiliary Elements

For both full-angle method and deductive database method, an auxiliary segment joining the two points or an auxiliary circle passing through four points is naturally added if a rule requires so. However, we do not add a new point by intersecting two lines.

For the deductive database method, based on the patterns of many examples, we have 18 heuristic rules for adding new auxiliary points in our deductive database method [14].

In our opinion, human traditional proofs of geometry theorems require enormous amounts of human intelligence. Even more intelligence is required for adding auxiliary elements. Thus heuristics for adding auxiliary elements are necessary. A thorough investigation is being carried out and hundreds of theorems in our large collection of books are the best guide to the traditional proof work.

Acknowledgements Professor Robert S. Boyer have been giving us valuable advice and suggestions from the early work on JGEX to the final touch of this paper. The authors also wish to thank the referees for their helpful criticisms and suggestions. The NSF grant provides Zheng Ye with an unique opportunity to work at WSU. X.S. Gao is supported by NSFC.

References

1. Ye, Z., Gao, X.S., Chou, S.C.: Visually dynamic presentation of proofs in plane geometry part 1. Basic features and the manual input method. *J. Autom. Reason.* (2009). doi:[10.1007/s10817-009-9162-5](https://doi.org/10.1007/s10817-009-9162-5)
2. Wu, W.T.: On the decision problem and the mechanization of theorem in elementary geometry. *Sci. Sin.* **21**, 159–172 (1978); Automated theorem proving: after 25 years. *Contemp. Math.-Am. Math. Soc.* **29**, 213–234 (1984)
3. Kutzler, B., Stifter, S.: Automated geometry theorem proving using Buchberger's algorithm. In: Proc. of SYMSAC'86, Waterloo, pp. 209–214 (1986)
4. Chou, S.C., Schelter, W.F.: Proving geometry theorem with rewrite rules. *J. Autom. Reason.* **4**, 253–273 (1986)
5. Kapur, D.: Geometry theorem proving using Hilbert's Nullstellensatz. In: Proc. of SYMSAC'86, Waterloo, pp. 202–208 (1986)
6. Collins, G.E.: Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition. In: Brakhage, H. (ed.) *Automata Theory and Formal Languages*. 2nd GI Conference. Lecture Notes in Computer Science, vol. 33, pp. 134–183. Springer, Berlin (1975)
7. Arnon, D., Mignotte, M.: On mechanical quantifier elimination for elementary algebra and geometry. *J. Symb. Comput.* **5**, 237–259 (1988)
8. Collins, G.E., Hong, H.: Partial cylindrical algebraic decomposition for quantifier elimination. *J. Symb. Comput.* **12**, 299–328 (1991)
9. Chou, S.C.: *Mechanical Geometry Theorem Proving*. D.Reidel, Dordrecht (1988)
10. Wu, W.T.: *Basic Principles of Mechanical Theorem Proving in Geometries*. Press, Beijing (in Chinese) (1984). English Version. Springer, New York (1993)

11. Adler, C.F.: Modern Geometry. McGraw-Hill Book, Sydney (1958)
12. Hilbert, D.: Foundations of Geometry, 1st edn. (in Germany) was published in 1899. Open Court, La Salla (1971)
13. Nevins, A.J.: Plane geometry theorem proving using forward chaining. *Artif. Intell.* **6**(1), 1–23 (1975)
14. Chou, S.C., Gao, X.S., Zhang, J.Z.: A deductive database approach to automated geometry theorem proving and discovering. *J. Autom. Reason.* **25**(3), 219–246 (2000)
15. Chou, S.C., Gao, X.S., Zhang, J.Z.: Automated generation of of readable proofs with geometric invariants, II. Proving theorems with full-angles. *J. Autom. Reason.* **17**, 349–370 (1996)
16. Chou, S.C., Gao, X.S.: Proving geometry statements of constructive type. In: Proceedings of CADE–11. Lecture Notes in Computer Science, vol. 607, pp. 20–34. Springer, New York (1992)
17. Wilson, S., Fleuriot, J.D.: Combining dynamic geometry, automated geometry theorem proving, and diagrammatic proofs. In: Workshop on User Interfaces for Theorem Provers (UITP). <http://www.dai.ed.ac.uk/homes/jdf/entcs-geom-05.pdf> (2005)
18. Chou, S.C., Gao, X.S., Zhang, J.Z.: A collection of 110 geometry theorems and machine produced proofs for them, TR-94-5. CS Dept., WSU (1994)
19. Chou, S.C., Gao, X.S., Ye, Z.: Java geometry expert server. <http://woody.cs.wichita.edu> (2009) (One can run the most part of JGEX with a browser)
20. Chou, S.C., Gao, X.S., Ye, Z.: A collection of GIF files created with JGEX. <http://woody.cs.wichita.edu/collection/index.html> (2009)
21. Chou, S.C., Gao, X.S., Zhang, J.Z.: Machine Proofs in Geometry. World Scientific, Singapore (1994)