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On Protocols for the Automated Discovery of Theorems in Elementary Geometry

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Abstract In this paper we consider the problem of dealing automatically with arbitrary geometric statements (including, in particular, those that are generally false) aiming to find complementary hypotheses for the statements to become true. Our approach proceeds within the framework of computational algebraic geometry. First we argue and propose a plausible protocol for automatic discovery, and then we present some algorithmic criteria, as well as the meaning (regarding the algebraic geometry of the varieties involved in the given statement), for the protocol success/failure. A detailed collection of examples in also included.

Keywords Automatic theorem proving · Automatic theorem discovery · Elementary geometry · Computational algebraic geometry

1 Introduction

In this paper we will deal with automatic *discovery* of elementary geometry theorems. We address this issue within the algebraic geometry framework that has already shown its success for automatic theorem proving.

Roughly speaking, the algebraic geometry scheme towards theorem proving, proceeds translating theses and hypotheses about geometric entities into systems of polynomial equations, say $T = \{t_1 = 0, \dots, t_s = 0\}$ and $H = \{h_1 = 0, \dots, h_r = 0\}$.

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Solutions[1] for H and T, can be interpreted as geometric instances verifying the hypotheses (respectively, the theses). In this scheme, a statement $\mathcal{T} : \{H \Rightarrow T\}$ is to be declared true if and only if the algebraic set[2] defined by H is included in the solution set for T, i.e. if $V(H) \subseteq V(T)$, since this inclusion expresses that all instances verifying the hypotheses satisfy, as well, the theses. Further refinements of this method allow considering the case when such inclusion is valid only on an (Zariski) open set of $V(H)$. Here the algebraic procedure automatically generates a set of inequations that $V(H)$ has to fulfill in order to be contained in $V(T)$.

The above is just a very sketchy picture of the current state of the art. In fact, it is well known that some problems—we will not deal with in this paper—may arise on the geometric/algebraic translation process (see the references to this issue that appear, for instance, at the book [10], or at the papers [1, 13, 21], or in the introduction to [3]). Moreover, many different concepts of truth, diverse protocols to achieve it and methods to accomplish them, have been considered, all of them dealing, in an algebraic geometry manner, with automated theorem proving (see, for its specific consideration of the relativeness of truth in this context, the papers [6, 14]).

It is not the goal of this paper to present a survey of this flourishing topic, started several decades ago by professor Wen-tsun Wu, who has received, in 1997, the CADE (Conference on Automated Deduction) Herbrand Award and, in 2006, the Shaw Prize for his contribution to the mechanization of mathematics (we refer the reader to the impressive bibliography on the subject kept by prof. D. Wang in http://www-calfor.lip6.fr/~wang/). Yet we can summarize that automatic theorem proving deals, in general, with confirming or refuting statements that are true in a majority of instances verifying the given hypotheses.

On the other hand, automatic discovery of theorems addresses—in our opinion—the case of statements that are false in most relevant cases. Even so, it aims to produce, automatically, complementary hypotheses for the statement to be proved as correct. For example, imagine we draw a quadrilateral $[ABCD]$ and we want to place a point E on the line supporting the side AD, such that the area of the triangle ΔABE equals the area of the quadrilateral. Obviously this is false in general for an arbitrary position of E on the given line, but we want to discover, automatically, the location of such point for the equality of areas to hold (see Example 8). We would like to proceed by placing point E arbitrarily on that line and stating as thesis the equality of areas. Then, the automatic discovering protocol we will describe in our paper should take care of this situation, finding the correct location(s) of E for the thesis to hold.

Another example could be the following: we draw a triangle and, then, the feet of the corresponding altitudes. These feet are the vertices of a new triangle, the so called *orthic* triangle for the given triangle. We want this orthic triangle to be equilateral, but it is not so, in general. It seems quite obvious that this will be the case if the original triangle is itself equilateral, but, only in this case? Deciding if there are other possibilities is, again, the task of our protocol for automatic discovery of theorems.

[1] In a suitable field: there will be different interpretations for different choices of this field, but in this paper we will adhere to the algebraically closed case, so we will miss, for instance, oriented geometry.

[2] We denote by $V(H)$ or $V(T)$ the solution set of the corresponding system of equations.

We would like to impose as a thesis that the orthic triangle is equilateral and we expect our method to find the more general situations in which this is going to happen (see Example 6).

The interest of developing such automatized discovery procedure is quite obvious. For instance, in the context of CAD, it could be used as an auxiliary tool for determining the specific positions of the components of a drawing that are *a priori* required to verify some geometric restrictions (an observation already remarked in [15]). Automated discovery could be also useful in the educational context, since it could allow a dynamic geometry program (provided with a link to a computer algebra program, as shown in [4] or [22]) to act as an intelligent agent, being able to *know* in advance the response for most (right or wrong) conjectures made by a user attempting to construct a certain figure on the screen; in this way, the dynamic geometry program could act as a tutor, guiding in the right direction the efforts of the user towards the assigned task.

Although not as popular as automatic proving, automatic discovery of elementary geometry theorems is not a new idea. It can be traced back to the work of Chou (see [8, 9, 24] and [11]), generally under the form of "automatic derivation of formulas", a primitive version of automatic discovery where the goal consists in deriving results that always occur under some given hypotheses, but that can be formulated in terms of some specific set of variables (such as expressing the area of a triangle in terms of the lengths of its sides). For instance, "Discovering Theorems" (the title of Section 5.2 in [11]) actually deals with finding some properties that hold under the given hypotheses. Describing the geometric locus of a point defined through some geometric constraints, say, finding the locus of a point when its projection on the three sides of a given triangle form a triangle of given constant area ([10], Example 5.8) can be considered as another variant of this "automatic derivation" approach. Although quite similar in methodology, automatic derivation is somehow different from automatic discovery—as we understand it—in that the former does not include a priori a specific thesis (although one could always consider as thesis the trivial one $\{0 = 0\}$) and does not pretend to modify substantially the given hypotheses, which is crucial in our approach.

The idea of dealing with generally false statements by adding new, equality type, hypotheses, has also precedents, such as [15], where it is explicitly stated that "...the objective here is to find the missing hypotheses so that a given conclusion follows from a given incomplete set of hypotheses...". In the same vein, the paper of [25] or the book of [10] refer to "...finding new geometry theorems...Suppose that we are trying to prove a theorem...and the final remainder... R_0 is nonzero. If we add a new hypotheses $R_0 = 0$, then we have a theorem..." [10], page 72. We must also mention [26] and [19] (a book written in Spanish for secondary education teachers, with circa one hundred pages devoted to this topic and with many worked out examples), and the papers of [20], [21] or [16]. Examples achieved through a specific software for discovery, named *GDI* (the initials of *Geometría Dinámica Inteligente*), of Botana-Valcarce, appear in [4] or [22], such as the automatic discovery of the celebrated Maclane 8_3-Theorem, or the automatic solving of some items on a test posed by Richard [23], on proof strategies in mathematics courses, for students 14–16 years old. See [18] for a most recent contribution and further details and examples of theorem discovery (see, also, [7], a closely related contribution to the techniques of [18]; differences and similarities are discussed in the introduction of [18]).

Our present paper arises from the approach to discovery in [21], that proceeds, roughly speaking, first identifying a set of independent variables (those ruling the construction of the hypothesis variety), and, then, the corresponding *privileged* components of this variety, where these variables are independent (and maximally independent, as well). It is shown that the elimination ideal (over the independent variables), of the *Saturation*[3] ideal $H : T^{\infty}$ of the hypothesis ideal by the thesis ideal, is not zero if and only if the theorem is true over all these components (and then the theorem is called "generally true" [10]). When the given theorem is not generally true, it turns out that the elimination ideal of the ideal generated by the hypotheses plus the thesis is not zero if and only if the thesis does not hold over any privileged component (the so called "generally false" case, the one suitable for discovery). In this latter situation, [21] considers adding, as new hypotheses, the equations provided by the elimination of the old hypotheses plus the thesis, and proceeds further on, identifying a subset of the privileged variables that remain maximally independent over the new hypothesis variety. It turns out that this new set of hypotheses and the given thesis yields a non-generally false theorem, and, in many interesting examples, it is generally true (but not always: the method is incomplete, as shown in [21]).

Our approach in this paper is different to that of [21] in several directions. First of all, we do not start providing a method, but stating a general strategy for discovery, a kind of general protocol (see Section 2.1). Namely, we aim towards collecting the conditions that a couple of ideals, each one belonging to different set of variables, should verify in order to express necessary and sufficient conditions for the given thesis to hold. This couple of ideals (named here as a *full set for discovering interesting conditions*, or an **FSDIC**) will contribute for some extra hypotheses of equality (respectively, of inequality) type that are required for the theorem to hold. The precise statement is provided in the next Section (see Definition 1).

It is shown, through examples, that such couple does not always exist (and it is, in general, not unique, although it has some uniqueness properties, see Theorem 2). But, if it exists, then we can identify (see Theorem 1) and construct a particular **FSDIC**, that turns out to be very close to the output of the method presented in [21]. A test for the existence of an **FSDIC** is then stated (Theorem 3). Next Section 3 explores the meaning of **FSDIC** for wise choices of variables (ie. when we deal with "interesting" variables), but also remarks its limitations in other situations. Section 4 works out in detail an example taken (after adapting it to the discovery framework) from [10].

Section 5 addresses an issue that seems forgotten in the mechanization of theorem discovery, that of discussing the different ways of introducing inequations as part of the hypotheses provided by the user. We believe it is quite useful in practice (and there is an ancient tradition to do so, converting inequations into equations by adding some auxiliary variables), but also quite subtle. In fact, we show in that Section that there are two different (but related) possibilities, and the decision about which one is more suitable should be left to the user, since it has consequences in the discovery process. The discussion requires the analysis in detail of the saturation of one ideal by another one, a summary of which is provided in an Appendix. Finally, Section 6

[3]The definition of this standard notion in Commutative Algebra (see [17]) and a collection of its properties, that will be required in Section 5, are included in an Appendix to this paper.

provides three more examples, fully discussed, showing that, still, human intervention could be crucial in automatic discovery... but also that automatic discovery can contribute fundamentally to human understanding of geometric problems.

2 The Search for Interesting Conditions for Discovering Theorems

2.1 Rationale

Let us assume as given an (algebraically translated) statement of the kind $\{H \Rightarrow T\}$, where[4] $H = (h_1, \ldots, h_r)$ and $T = (t_1, \ldots, t_s)$) are ideals of polynomials in a ring $K[X]$, $X = \{x_1, \ldots, x_n\}$, with coefficients in a fixed field K, with algebraic closure $\overline{K}$. Then the algebraic sets $V(H)$, $V(T)$, defined as the solutions of the equations generated by H, T over the affine space $\overline{K}^n$ can be interpreted as geometric instances verifying the hypotheses (respectively, the theses) of the statement.

A natural goal could be searching for complementary hypotheses, both of equality type R' and inequations—provided by another set of polynomials R''—so that[5] $\{(H \wedge R' \wedge \neg R'') \Rightarrow T\}$ and $\{H \wedge R' \wedge \neg R''\}$ is not empty. In fact, except for the inclusion of new, equality type, hypotheses, this is the traditional formulation for automatic proving, where the R'' represents the—so called—degeneracy conditions. But in our context it is clear that, taking $R' = T$ and a trivial R'', we will have, in general, a (useless) solution. So we should reconsider the formulation of the goal, taking into account that we actually want to find the complementary hypotheses in terms of some specific set of variables ruling our statement.

For example, if we want to find out the conditions for the orthic triangle of a given one to be equilateral, our intuition expects the answer to provide some extra constraints to be fulfilled for the given triangle. That is, we want to learn about new hypotheses in terms of the variables assigned to the vertices of the given triangle (and not, for example, in terms of the variables naming the vertices of the orthic triangle). In fact, if we are given a triangle through the three vertices and then construct its orthic triangle, the algebraic translation will be a system of 6 equations (two for each vertex of the orthic triangle, stating that this vertex belongs to one side of the given triangle and to an altitude) and 12 variables (two per vertex of both the triangle and the orthic triangle). The system solution set has dimension six (six degrees of freedom), as one expects, since the given triangle can be arbitrarily parametrized, through the two-times-three coordinates of its vertices. These should be considered as the independent variables ruling our given construction and, thus, the search for the extra constraints yielding to an equilateral orthic triangle should proceed finding R', R'' as polynomials in these variables.

Let us remark that the mere consideration of the algebraic system does not allow to highlight such meaningful set of variables: there are, as well, other sets of six free variables for the algebraic solution set, such as those corresponding to the

[4]By abuse of notation, H, T can be thought, as in the introduction, as some sets of equations, but also as representing the ideals generated by the polynomials defining the equations.

[5]Again, by abuse of notation, the following formulas are to be understood as "for all points in $\overline{K}^n$ that are simultaneous solutions of the equations in H and in R' and are not solutions of R'', then they are also solutions of the equations defining T", etc...

coordinates of the orthic triangle vertices (since they determine, conversely, the given triangle). Different examples show that it is impossible to determine the meaningful variables in an automatic way by relying on heuristics, such as considering those variables that are not involved in the thesis, etc. It should be human intuition (i.e. the user) who has to point out the concrete collection of variables that will turn meaningful the discovery process. Thus, our goal, as stated above, should be modified by referring to some specific set of variables for the complementary hypotheses.

Next, to get things a little more complicated, we should notice that, once the equality type extra hypotheses R' are found, the degenerate conditions R'' should be expressed in terms of some subset of the selected variables, since the whole construction, after adding R', could possess, then, less degrees of freedom (for instance, in the example above, if R' is found and it states that the given triangle must be equilateral, then the degree of freedom, of the new system of hypotheses, will be reduced from six to four, since an equilateral triangle is determined by just two vertices). In summary, our goal should be to look for the existence of two subsets of variables $U' \subseteq U \subseteq X$, and two ideals (R', R''), in $K[U]$ and $K[U']$, respectively, such that $(H \wedge R' \wedge \neg R'') \Rightarrow T$ and $\{H \wedge R' \wedge \neg R''\}$ is not contradictory.

But this framework is still not sound enough. It is true that, if we could find such couple (R', R''), we would have found a true statement, keeping the given theses T but adding some extra hypotheses $\{H \wedge R' \wedge \neg R''\}$. But nothing guarantees that such statement really covers all possible discoveries related to the given T, H. In the example above, imagine that some R' is found expressing that the given triangle should be equilateral; then it will yield to a true statement (in fact, the orthic triangle of an equilateral triangle is also equilateral), but there are other possibilities (far more interesting) for the given statement to hold. That the given triangle is equilateral is, indeed, a sufficient condition for the orthic triangle to be equilateral, but it is not a necessary condition. So, if we want to avoid discovering just some trivial statements, what we really need to find out is a collection of non contradictory (i.e. such that there is at least one instance of the given hypotheses were they actually hold) extra hypotheses R', R'',

a) expressed in the right variables,
b) which are, when added to H, sufficient for T, so that $\{(H \wedge R' \wedge \neg R'') \Rightarrow T\}$,
c) which also verify that $\{H \wedge T\} \Rightarrow R'$

The last requirement expresses that R' represents a conjunction of necessary, equality-type, additional conditions for the theses to hold under the given hypotheses H.

Actually, one could also think about imposing, instead of c), some stricter condition, such as $c')$: $\{H \wedge T\} \Rightarrow \{R' \wedge \neg R''\}$. But then (jointly with a) and b)) it will mean that $\{H \wedge T\} \equiv \{H \wedge R' \wedge \neg R''\}$. Let P be the formula (in $\overline{K}[U]$) expressing the elimination of existential quantifiers on the variables $X \setminus U$ in H (that is, the projection of $V(H)$ over the affine space described by the U-variables). It is easy to show that, if we impose condition c'), then we will get $P \wedge R' \wedge \neg R''$ as the result of projecting $H \wedge T$ over the U-space. On the other hand, since U should be a set of variables that rule H, it is quite plausible that, in many instances, P will not introduce any restriction on the U variables (i.e. every assignment of the U variables can be lifted to a value, of the remaining variables, verifying H). In conclusion, if c'

is regarded, we will have in many cases that the projection of $V(H) \cap V(T)$ over the affine space described by the selected U variables will be equal to $V(R') \backslash V(R'')$.

Now, this is a particularly strong requirement, since it is known that the projection of an algebraic variety is a general constructible set, that is, a finite union of sets, each one being the intersection of an algebraic variety and the complement of another one, such as $V(R') \backslash V(R'')$. It is a finite union, and not, in general, just one of the terms of such union (take, for example, $V(R'_1)$= a plane, $V(R''_1)$= a line on the plane, $V(R'_2)$=a point on this line; then $(V(R'_1) \setminus V(R''_1)) \cup V(R'_2)$ can not be expressed as $V(R'_3) \backslash V(R''_3)$, for whatever sets of polynomials R'_3, R''_3.

This means that imposing c') as a condition will yield to a discovery protocol that would fail in several instances, due to the lack of an appropriate language to express all necessary conditions. At this point two possibilities arise. One, that of reformulating the whole approach to discovery, allowing, from the beginning, the introduction of a finite union of equations and inequations. In some sense, this is what has been achieved in [18] or [7]) and can be seen as quite close to performing a certain kind of quantifier elimination procedure. But let us remark that, in the theorem *proving* context, this formulation (i.e. requiring that $\{H \wedge T\} \Rightarrow \{\neg R''\}$ for non-degeneracy conditions) has not been followed in most works, perhaps due to its complexity.

A second possibility is keeping condition c), at the risk of losing some necessary inequality-type conditions. Since these conditions, in general, only describe the degeneracy cases that should be avoided for the statement to become true, we think it is quite safe to keep condition c) in our approach, as we will not miss any interesting results just because of not paying attention to some degenerate cases.

With this rationale in mind, next section contains a formal description of the proposed discovery protocol (finding R', R'' such that...). An extra, technical, condition has been added (items d) or d') in the definition below), in order to achieve some kind of unicity (see Theorem 2). Anyway, it is easy to prove (see Remark 3) that an equivalent theory could be established deleting this last item d). That is, Theorem 1 will also hold with an alternate definition of **FSDIC** consisting just of items a), b), c) and e) in Definition 1.

2.2 A Full Set of (Discovering) Interesting Conditions

As above, we will consider some subsets of a main set of variables $X = \{x_1, \ldots, x_n\}$, namely $U' \subseteq U \subseteq X$. Then, we will often deal with the extension $K[U'] \hookrightarrow K[U] \hookrightarrow K[X]$ of polynomial rings on the corresponding variables, with coefficients in a fixed field K. Let A be an ideal in $K[U']$, B an ideal in $K[U]$, and C an ideal in $K[X]$. We will denote—as it is standard in Commutative Algebra—by $A^{e'} = AK[U]$, the extended ideal; by $A^e = AK[X]$, and by $B^e = BK[X]$. Clearly $(A^{e'})^e = A^e$. Moreover we will denote by $C^{c'} = C \cap K[U']$, its contraction ideal; by $C^c = C \cap K[U]$, and by $B^{c'} = B \cap K[U']$. Again, it is clear that $(C^c)^{c'} = C^{c'}$. Finally, if I is an ideal in $K[X]$, we will denote by $V(I) = \{(x_1, \ldots, x_n) \in \overline{K}^n \mid f(x_1, \ldots, x_n) = 0,\ \forall f \in I\}$ the algebraic set defined by I in $\overline{K}^n$, where $\overline{K}$ is the algebraic closure of K.

Definition 1 Let $\mathcal{T}$ be a statement, of the kind $H \Rightarrow T$, where the ideals H, $T \subseteq K[x_1, \ldots, x_n]$ will be the corresponding hypothesis ideal and thesis ideal. Let $U' \subseteq U \subseteq \{x_1, \ldots, x_n\} = X$.

Then a couple (R', R'') of ideals, respectively in $K[U]$ and $K[U']$, will be called a **Full Set of (Discovering) Interesting Conditions (FSDIC)** for $\mathcal{T}$ with respect to U and U' if the following conditions hold:

a) $R' \subseteq K[U]$ and $R'' \subseteq K[U']$;
b) $V(H + R'^e)\backslash V(R''^e) \subseteq V(T)$;
c) $V(H + T) \subseteq V(R'^e)$;
d) if $f \in K[U']$ is such that $V(H + R'^e)\backslash V((f)^e) \subseteq V(T)$, then $f \in \sqrt{R''}$;
e) $V(H + R'^e)\backslash V(R''^e) \neq \emptyset$.

Remark 1 Condition d) is equivalent to the following:

d′) if $R''' \subseteq K[U']$ is an ideal such that $V(H + (R')^e)\backslash V((R''')^e) \subseteq V(T)$, then $\overline{K}^n \backslash V((R''')^e) \subseteq \overline{K}^n \backslash V((R'')^e)$.

In fact suppose that d) holds and that $R''' = (g_1, \ldots, g_l) \subseteq K[U']$ is such that $V(H + (R')^e)\backslash V((R''')^e) \subseteq V(T)$. Then for any $i = 1, \ldots, l$, we have $V(H + R'^e)\backslash V((g_i)^e) \subseteq V(H + R'^e)\backslash V(R'''^e) \subseteq V(T)$, therefore $g_i \in \sqrt{R''}$, and so $g_i \in \sqrt{R''^e}$. This last condition is equivalent to $V((g_i)^e) \supseteq V(R''^e)$ and so to $\overline{K}^n \backslash V((g_i)^e) \subseteq \overline{K}^n \backslash V(R''^e)$. Then $\overline{K}^n \backslash V(R'''^e) = \cup_{i=1}^{l} \overline{K}^n \backslash V((g_i)^e) \subseteq \overline{K}^n \backslash V(R''^e)$.

Viceversa, let $f \in K[U']$ be such that $V(H + R'^e)\backslash V((f)^e) \subseteq V(T)$. Then, if we take $R''' = (f)$, we obtain $\overline{K}^n \backslash V((f)^e) \subseteq \overline{K}^n \backslash V(R''^e)$, so $V((f)^e) \supseteq V(R''^e)$, i.e $f \in \sqrt{R''^e}$. Since $f \in K[U']$ and $R''^{ec'} = R''$ we obtain $f \in \sqrt{R''}$.

Example 1 For instance, suppose $H = (x \cdot y \cdot z)K[x, y, z] \Rightarrow T = (y)K[x, y, z]$ be a(n) (obviously false) statement. Let $U' = \{z\} \subseteq U = \{y, z\} \subseteq \{x, y, z\}$. Then the couple $(R' = (y \cdot z), R'' = (z))$ of ideals, respectively in $K[U]$ and $K[U']$, will be an **FSDIC**, meaning

a) that the theorem will be true under the new hypotheses $\{x \cdot y \cdot z = 0, y \cdot z = 0, z \neq 0\}$ (as stated in condition b) above),
b) that if the thesis holds on $x \cdot y \cdot z = 0$, then necessarily $y \cdot z = 0$, (condition c) above),
c) that any polynomial $f \in K[z]$, such that it is a non degeneracy condition $f \neq 0$ for the thesis to hold under the new hypotheses $\{x \cdot y \cdot z = 0, \; y \cdot z = 0\}$, necessarily it must be a multiple of a power of (z) (as stated in condition d) above)
d) and that, moreover, there are points verifying $x \cdot y \cdot z = 0, \; y \cdot z = 0, \; z \neq 0$ (condition e)).

But it is easy to check that also the couple $(L' = (y), L'' = (1))$ is another **FSDIC** for H, T, and for the same variables.

Therefore, the natural question to ask is when these pairs of ideals exist and how can we compute some of them. The following propositions give us a complete answer to these questions:

Theorem 1 *Let* $H' = (H + T) \cap K[U]$ *and* $H'' = ((H + H'^e) : (T)^\infty) \cap K[U']$ (*see the Appendix for the notation and main properties of this operation of saturation by* T). *Then there exist two ideals* R', R'' *such that* (R', R'') *is an* **FSDIC** *for* $\mathcal{T}$ *with respect to* U *and* U' *if and only if* (H', H'') *is* **FSDIC** *for* $\mathcal{T}$ *with respect to* U *and* U'.

Proof Obviously, we have just to prove that if there exist two ideals R', R'' such that (R', R'') is **FSDIC** for $\mathcal{T}$ with respect to U and U', then (H', H'') is also an **FSDIC**.

a) $H' \subseteq K[U]$ and $H'' \subseteq K[U']$ by definition.
b) First we observe that $T \subseteq \sqrt{H + H'^e} : (H''^e)^\infty$. In fact let $g \in H''^e$, and let $T = (t_1, \ldots, t_s)$, so

$$g \in \sqrt{H''^e} \subseteq \sqrt{(H + H'^e) : (T)^\infty} \subseteq \sqrt{H + H'^e} : (T)^\infty$$
$$= \cap_{j=1}^s \sqrt{H + H'^e} : (t_j)^\infty \subseteq \sqrt{H + H'^e} : (t_i)^\infty$$

for any $i = 1, \ldots, s$. Then $t_i \in \sqrt{H + H'^e} : (g)^\infty$ for any $g \in H''^e$, i.e. $t_i \in \sqrt{H + H'^e} : (H''^e)^\infty$ for any i. Therefore we have

$$V(H + H'^e)\backslash V(H''^e) = V(\sqrt{H + H'^e})\backslash V(H''^e) \subseteq$$
$$V(\sqrt{(H + H'^e)} : (H''^e)) = V(\sqrt{\sqrt{(H + H'^e)} : (H''^e)^\infty}) =$$
$$V(\sqrt{(H + H'^e)} : (H''^e)^\infty) \subseteq V(T).$$

c) $H' = (H + T) \cap K[U]$, so $H'^e \subseteq (H + T)$ and then $V(H + T) \subseteq V(H'^e)$. Notice that, if (R', R'') is an **FSDIC**, we always have that $V(H + H'^e) \subseteq V(H + R'^e)$, since $V(H + T) \subseteq V(R'^e)$ implies $R'^e \subseteq \sqrt{H + T}$; thus $R' = R'^{ec} \subseteq \sqrt{H + T}^c = \sqrt{H'}$, concluding that $V(H'^e) \subseteq V(R'^e)$. We will use this fact in the proof of *e*).
d) Let $f \in K[U']$ be such that $V(H + H'^e)\backslash V((f)^e) \subseteq V(T)$. Then we have that $\overline{V(\sqrt{H + H'^e})\backslash V((f)^e)} \subseteq V(T)$. But we work in an algebraically closed field, so $V(\sqrt{H + H'^e} : (f)^e) \subseteq V(T)$.
Since, $V(\sqrt{H + H'^e} : (f)^e) = V(\sqrt{(H + H'^e) : ((f)^e)^\infty})$, then

$$T \subseteq \sqrt{(H + H'^e) : ((f)^e)^\infty} \subseteq \sqrt{H + H'^e} : ((f)^e)^\infty$$

and this implies $f \in \sqrt{H + H'^e} : (T)^\infty \cap K[U'] = \sqrt{H''}$.
e) It holds $V(H + R'^e)\backslash V(R''^e) \subseteq V(H) \cap V(T)$ by definition and property *b*). Since $H' = (H + T) \cap K[U]$, we have that $H'^e \subseteq (H + T)$. Thus $V(H + R'^e)\backslash V(R''^e) \subseteq V(H) \cap V(T) \subseteq V(H'^e)$. On the other hand $V(H + R'^e)\backslash V(R''^e) \subset V(H)$, thus $V(H + R'^e)\backslash V(R''^e) \subseteq V(H + H'^e)$. Moreover $R'' \subseteq \sqrt{H''}$, since if $f \in R'' \subseteq K[U']$, then $V(H + H'^e)\backslash V((f)^e) \subseteq V(H + R'^e)\backslash V((f)^e) \subseteq V(T)$ and this implies $f \in \sqrt{H''}$ by condition *d*). Since $R'' \subseteq \sqrt{H''}$ if and only if $R''^e \subseteq \sqrt{H''^e}$, then $V(R''^e) \supseteq V(H''^e)$. Thus

$$\emptyset \neq V(H + R'^e)\backslash V(R''^e) \subseteq V(H + H'^e)\backslash V(H''^e)$$

Notice that the hypothesis on (R', R'') as an **FSDIC** has only been used to prove property *e*). □

Remark 2 As remarked above in the proof of *c*), if (R', R'') is an **FSDIC**, we always have that $V(H + H'^e) \subseteq V(H + R'^e)$. On the other hand we have also shown, along the above proof, that it holds the relation $V(H + R'^e)\backslash V(R''^e) \subseteq V(H + H'^e)\backslash V(R''^e)$. Therefore we have that

$$V(H + R'^e)\backslash V(R''^e) = V(H + H'^e)\backslash V(R''^e)$$

Remark 3 Suppose, as motivated in the previous subsection, that an alternate definition of **FSDIC** is given, in which we drop property *d*). Then the statement of the theorem above will still hold. In fact, the only point that requires the hypothesis of (R', R'') as **FSDIC** is property *e*), but in its proof property *d*) for (R', R'') is not used. Rather, it uses only that (H', H'') verifies property *d*), which holds in general, as observed. Thus the existence of an **FSDIC**, with or without condition *d*), is always equivalent to (H', H'') being an **FSDIC** in the stronger sense we have formally introduced in Definition 1.

2.3 Existence and Unicity

The above theorem tells us that, if an **FSDIC** exists, then the couple (H', H'') is indeed one such full set of conditions, providing an extra algebraic set of equality-type constraints that is the smallest one in terms of the variety given by the first ideal of the couple (since $V(H'^e) \subseteq V(R'^e)$, see Remark 2) and also providing the largest set of non degeneracy conditions in terms of the complement of the variety given by the second ideal of the couple (as we have shown in the proof that always $V(R''^e) \supseteq V(H''^e)$).

Moreover, the above Remark 2 shows that the hypotheses of equality type $H + R'^e$ arising from whatever **FSDIC** will be always geometrically equivalent to $H + H'^e$ (after adding the non-degeneracy hypotheses), and in this sense we can conclude that our protocol yields, essentially, to a unique solution (when it exists one) on the additional hypotheses of equality-type for the statement to become true.

But, algebraically speaking, there are, in general, several possible **FSDIC**'s.

Theorem 2 *Given an* **FSDIC** (R', R'')*, take any ideals* $\tilde{R}', \tilde{R}''$*, such that* $R' \subseteq \tilde{R}' \subseteq \sqrt{H'} \subseteq K[U]$ *and* $\tilde{R}'' \subseteq K[U']$*, with* $\sqrt{\tilde{R}''^e} = \sqrt{H''^e} \subseteq K[X]$*. Then* $(\tilde{R}', \tilde{R}'')$ *is also an* **FSDIC**.

Proof

a) Condition *a*) holds by definition.
b) By hypothesis $V(H'^e) \subseteq V(\tilde{R}'^e) \subseteq V(R'^e)$ and $V(H''^e) = V(\tilde{R}''^e)$. Thus

$$V(H + \tilde{R}'^e)\backslash V(\tilde{R}''^e) \subseteq V(H + R'^e)\backslash V(\tilde{R}''^e)$$

Now we apply Remark 2, concluding that

$$V(H + R'^e)\backslash V(\tilde{R}''^e) = V(H + H'^e)\backslash V(\tilde{R}''^e) = V(H + H'^e)\backslash V(H''^e) \subseteq V(T)$$

c) Next, since by hypothesis $V(H'^e) \subseteq V(\tilde{R}'^e)$ and $V(H + T) \subseteq V(H'^e)$, we have $V(H + T) \subseteq V(\tilde{R}'^e)$.
d) Moreover, if for some ideal $R''' \subseteq K[U']$ we have that $V(H + \tilde{R}'^e)\backslash V(R'''^e) \subseteq V(T)$, then also $V(H + H'^e)\backslash V(R'''^e) \subseteq V(H + \tilde{R}'^e)\backslash V(R'''^e) \subseteq V(T)$. Thus $V(H''^e) \subseteq V(R'''^e)$, by condition *d'*) on (H, H'). But, by hypothesis, $V(\tilde{R}''^e) = V(H''^e)$.
e) Finally, we see that $\emptyset \neq V(H + H'^e)\backslash V(H''^e) \subseteq V(H + \tilde{R}'^e)\backslash V(\tilde{R}''^e)$ □

Now let us see how to check for the existence of an **FSDIC**, determining some necessary and sufficient algorithmic conditions for (H', H'') to be an **FSDIC**.

Theorem 3 *(H', H'') is* **FSDIC** *for $\mathcal{T}$ with respect to U and U' if and only if $1 \notin (H')^{c'} : H''^{\infty}$ (equivalently, iff $H'' \nsubseteq \sqrt{(H')^{c'}}$).*

Proof From the proof of the previous theorem we have that the pair of ideals (H', H'') always verify *a*), *b*), *c*) and *d*).

Therefore we just have to prove that $V(H + H'^e)\backslash V(H''^e) \neq \emptyset$ if and only if $1 \notin (H')^{c'} : H''^{\infty}$. This condition is equivalent to $H'' \nsubseteq \sqrt{(H')^{c'}}$, since

$$H'' \subseteq \sqrt{(H')^{c'}} \text{ iff } (1) = \sqrt{(H')^{c'}} : H'' = \sqrt{(H')^{c'} : H''^{\infty}} \text{ iff } (H')^{c'} : H''^{\infty} = (1)$$

Suppose $H'' \subseteq \sqrt{(H')^{c'}}$. Then $H''^e \subseteq \sqrt{(H')^{c'}}^e = \sqrt{(H')^{c'e}} \subseteq \sqrt{(H')^e} \subseteq \sqrt{H'^e + H}$, so $V(H + H'^e) \subseteq V(H''^e)$.

Now suppose $V(H + H'^e) \subseteq V(H''^e)$. Then $H''^e \subseteq \sqrt{H'^e + H}$ and so we conclude $H'' = H''^{ec'} \subseteq \sqrt{H'^e + H}^{c'} = \sqrt{(H'^e + H)^{c'}} \subseteq \sqrt{((H + T) + H)^{c'}} = \sqrt{H'^{c'}}$. □

Example 2 In the example $H = (x \cdot y \cdot z)K[x, y, z] \Rightarrow T = (y)K[x, y, z]$, we can now check that it actually has an **FSDIC**, since $H' = (y)K[y, z]$, $H'' = (1)K[z]$ do verify that $1 \notin (H')^{c'} : H''^{\infty}$.

Example 3 Let us consider the following geometric situation. Let A, B, C be three points in the plane, with coordinates $A = (0, 0)$, $B = (x[1], 0)$, $C = (x[2], x[3])$. Consider as hypothesis the only condition $x[4] \cdot x[3] - 1 = 0$ and as thesis $T = x[3] = 0$. Then $H + T$ is clearly (1), so for any choice of subsets of variables from $\{x[1], \ldots, x[4]\}$, we will have $H' = (1)$ and H'' will be also (1). So there is no couple (R', R'') that will be an **FSDIC** for this example.

Corollary 1 *Moreover, if U' is a set of algebraically independent variables for H', then (H', H'') is an* **FSDIC** *for $\mathcal{T}$ with respect to U and U' if and only if $H'' \neq (0)$.*

Proof Note the new hypothesis implies $H'^{c'} = 0$ and thus (H', H'') is an **FSDIC** for $\mathcal{T}$ iff $1 \notin (0) : H''^{\infty}$. □

3 The Protocol

3.1 Interpretation

It should be clear by now that we propose, as discovery protocol, the search for an **FSDIC**, for some pair of sets of variables. Let us recall that a couple $(R' \subset K[U], R'' \subset K[U'])$), which is an **FSDIC**, it is supposed to provide, as discussed in Section 2.1,

- some necessary (as expressed by item *c*) of the definition of **FSDIC**)
- and sufficient (as expressed by items *b*) and *d*) of the definition of **FSDIC**)
- non-trivial (as expressed by item *e*) of the definition of **FSDIC**)

conditions of equality kind (given by R') and of non degeneracy type (given by R'') for the given theses to hold under the given hypotheses. And, of course, such conditions are meant to be *meaningful* for the geometric situation we are dealing with. For instance, R'' should be given in terms of variables that rule the new hypotheses $H + R'^e$. But, formally speaking, in the definition of an **FSDIC** we have not imposed the independence of the variables U and U'. In particular, if we take $U = U' = X$, then $H' = H + T$ and $H'' = (1)$ is (unless $H' = H + T = (1)$) always an **FSDIC** but it does not add real information to learn that the theses will hold if we merely consider it as part of the hypotheses. The choice of taking U and U' as sets of independent variables is exclusively related to the elusive concept of "interesting" variables, that we have already discussed in Section 2.1 and that will be analyzed in the different examples.

But, what does it mean for an statement $H \Rightarrow T$ to have an **FSDIC**? To start with, the following propositions give us some interpretation in different cases:

Proposition 1 *Notation as in the previous section. Suppose that $U' \subset U$ is a set of algebraically independent variables for $H + H'^e$. Then T is contained in all the minimal primes of $H + H'^e$ where U' are independent if and only if $1 \notin (H')^{c'} : H''^{\infty}$ (and this is equivalent to the couple (H', H'') being an* **FSDIC***).*

Proof Let $\sqrt{H + H'^e} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_l \cap \mathfrak{p}_{l+1} \cap \cdots \cap \mathfrak{p}_m$ be the unique primary decomposition of $\sqrt{H + H'^e}$, where U' are independent over $\mathfrak{p}_i$ for any $i = 1, \ldots, l$ and dependent over the remaining.

Since U' are independent over $H + H'^e$, it follows that $(H + H'^e)^{c'} = 0$ and, in particular, $0 = (H + H'^e)^{c'} \supseteq (H'^e)^{c'} = ((H'^e)^c)^{c'} = H'^{c'}$, so $H'^{c'} = 0$ and the U' are also independent over H'. In this case, as remarked above, the condition $1 \notin (H')^{c'} : H''^{\infty}$ is equivalent to showing that $H'' \neq 0$.

Suppose there exists $0 \neq g \in H'' = ((H + H'^e) : (T)^{\infty}) \cap K[U']$. Then $T \subseteq \sqrt{(H + H'^e)} : ((g)^e)^{\infty} = \bigcap_{g \notin \mathfrak{p}_j} \mathfrak{p}_j \subseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_l$ since $g \in K[U']$ and so $g \notin \mathfrak{p}_j$ for $j = 1, \ldots, l$.

Conversely, assume T is contained in all primes $\mathfrak{p}_i$, $i = 1, \ldots, l$, and take for any $j = l + 1, \ldots, m$, some $0 \neq g_j \in \mathfrak{p}_j \cap K[U']$ (that should exist because U' is dependent on each of the remaining primes). Let $g := \prod_{j=l+1}^{s} g_j$. Then $g \cdot T \subseteq \sqrt{(H + H'^e)}$, i.e. $0 \neq g \in ((H + H'^e) : T)^{\infty} \cap K[U'] = H''$. □

Lemma 1 *Let $\mathfrak{p}$ be a prime ideal in the polynomial ring $R = K[x_1, \ldots, x_n]$ and let $U' = \{x_{i_1}, \ldots, x_{i_k}\} \subset \{x_1, \ldots, x_n\}$ be such that $\mathfrak{p} \cap K[U'] = (0)$ and $\dim(R/\mathfrak{p}) = k$. If $t \notin \mathfrak{p}$ then there exist $q \in R$ and $0 \neq r \in K[U']$ such that $tq + r \in \mathfrak{p}$.*

Proof Since $t \notin \mathfrak{p}$, then $\bar{t} \neq 0$ in the fraction field $FF(R/\mathfrak{p})$, which is an algebraic extension of $K(U')$ (because $[FF(R/\mathfrak{p}) : K] = [K(U') : K] = k$). Therefore there exist $a_0, \ldots, a_{m-1} \in K(U')$ such that $\bar{t}^m + a_{m-1}\bar{t}^{m-1} + \cdots + a_0 = 0$, where $a_0 \neq 0$. Clearing denominators, we obtain $b_m\bar{t}^m + b_{m-1}\bar{t}^{m-1} + \cdots + b_0 = 0$ in $R/\mathfrak{p}$, where $b_i \in K[U']$ and $b_0 \neq 0$. Then we can take $q = b_m t^{m-1} + \cdots + b_1$ and $r = b_0$. □

Proposition 2 *Suppose that $U' \subset U \subset X$ is a set of algebraically independent variables for $H + H'^e$ and, moreover, suppose it is maximal among the subsets of X with this property (ie. $K[\tilde{U}] \cap (H + H^e) \neq (0)$ for any $U' \subset \tilde{U} \subset X$).*

Then, the couple (H', H'') is not an **FSDIC** *is equivalent to the fact that T is not contained in all the minimal primes of $H + H'^e$ where U' are independent, but also that it is contained in at least one of them.*

Proof Notation as in the previous proposition. By the maximality of U', we can suppose $\dim K[x_1,\ldots,x_n]/\mathfrak{p}_i = k$ =cardinal of U', for any $i = 1,\ldots,l$.

By the independence hypothesis we have that $H'^{c'} = 0$, and thus, (H, H') not being an **FSDIC** is equivalent to $H'' = 0$. By the precedent proposition this implies T is not contained in all the minimal primes of $H + H'^e$ where the variables in U' are independent.

So let us show that in this case it is also true that T is contained in at least one of the minimal primes of $H + H'^e$ where the variables in U' are independent, Otherwise, for every $j = 1,\ldots,l$, let $t_j \in T$ be such that $t_j \notin \mathfrak{p}_j$. Then, by the Lemma 1, there exist $q_1,\ldots,q_l \in K[x_1,\ldots,x_n]$ and $r_1,\ldots,r_l \in K[U']$, not zero, such that $q_j t_j + r_j \in \mathfrak{p}_j$. Moreover, for any $j = l+1,\ldots,m$, let $0 \neq g_j \in K[U'] \cap \mathfrak{p}_j$ (since U' is dependent over these components). Take $g = (q_1t_1 + r_1)(q_2t_2 + r_2)\cdots(q_lt_l + r_s)g_{l+1}\cdots g_m$. Then $g \in \sqrt{(H + H'^e)}$, i.e. there exists $d \in \mathbb{N}$ such that $g^d \in (H + H'^e)$. But $g^d = t + (r_1 \cdots r_l g_{l+1} \cdots g_m)^d$ for some $t \in T$. Therefore $0 \neq (r_1 \cdots r_l g_{l+1} \cdots g_m)^d \in ((H + H'^e) + T) \cap K[U']$. But $(H + H'^e + T) \cap K[U'] \subseteq (H + T) \cap K[U'] = H' \cap K[U'] = H'^{c'} = (0)$. Contradiction.

The converse is trivial, since if T is not contained in all the minimal primes of $H + H'^e$ where U' are independent, then, by the previous proposition, $H'' = 0$. □

Therefore we believe we have got a good translation of the idea of **FSDIC** in the case we choose U' as a set of algebraically independent variables for $H + H'^e$, since it means that we have obtained a description of all the relevant primes for the theses to hold. On the other hand we think we have achieved a more complete description if the set of variables U' is maximally independent, since in this case, having an **FSDIC** allows us to identify all the relevant components where the theses holds and, in the case an **FSDIC** does not exists, we know it is due to the fact that the theses holds over some relevant component (but not over all).

Example 4 Things are quite subtle. Notice that such set of variables U' with good independence properties may not exist: consider $H = (a+1) \cap (b+1)$ in $K[a,b,c]$, $T = (a+b+1, c)$, and $U = \{b,c\}$. Then $H' = (H+T) \cap K[b,c] = (c, b^2 + b)$ and there is not $U' \subset U$ a set of algebraically independent variables for $H + H'^e$.

Remark 4 On the other hand, what happens if U' are dependent on $H + H'^e$, in particular if $U' = U$? Assume for this remark that we have an **FSDIC** for $U' = U$ and that U is independent on H. Then $H' \neq (0)$. Also, if $1 \notin (H' \cap K[U']) : H''^{\infty}$, then T is contained in some minimal prime of $H + H'^e$, but we do not have more information about these components. We can produce examples where there does not exist $V \subset U$ a set of algebraically independent variables for $H + H'^e$, examples where $V \subset U$ are independent (but not a maximal set!!), and T is contained just in the components where all the variables V are dependent. Only in the case $V \subset U$ is a maximal set of algebraically independent variables for $H + H'^e$, then T results contained in some minimal prime of $H + H'^e$ where V are independent. But also in this case we cannot conclude (as shown by different examples) that T is contained in all the minimal primes of $H + H'^e$ where V are independent, even if $1 \notin H' : H''^{\infty}$.

3.2 Applying the Protocol

After this analysis, we consider reasonable to adopt, in the examples, a protocol that can be summarized as follows: Given a statement $H \Rightarrow T$, consider some couple of "suitable" sets of variables U, U', compute the corresponding pair of ideals H', H'' and check whether there is an **FSDIC** in this context. If so, the above results explain what it does mean, in geometric terms, to add H' as new set of equality type hypotheses and the negation of H'' as inequation hypotheses. If there is not an **FSDIC**, the above propositions give some hints about what is happening, yielding, in general, to a deeper understanding of the geometric statement. Anyway, either you start again, with some other couple of variables or you should consider finding a decomposition of the hypothesis variety in terms of prime components.

The following are some finer hints about this process and, in particular, about the selection of variables.

a) Check for the dimension of $V(H)$. If it is -1 (i.e. if $V(H)$ is empty), the statement is trivial.
b) If $dim(V(H))$ is zero, take $X = U = U'$. It is easy to prove—through Theorem 3—that there is an **FSDIC** if and only if T holds over some point of $V(H)$. Check the existence of an **FSDIC** (that here consists just in verifying that $H + T \neq (1)$). If this is the case, the couple of ideals $H' = H + T$, $H'' = (1)$ will describe precisely the points of H where T holds, i.e. it will hold true that $\{H \wedge H' \wedge \neg H''\} \Rightarrow T$ and this covers all possible cases.
c) If $dim(V(H)) > 0$, then it is practical (but not indispensable) to verify before hand that the statement is not one of a true theorem (except for degenerate conditions). Use your favorite proving protocol for this task. If it is not a true theorem, identify a maximal[6] subset of variables U such that $H \cap K[U] = 0$, ie. a set of variables "ruling" the relevant given data $V(H)$, so that it is reasonable to express the extra hypotheses H' and H'' in terms of these variables.
Then we should analyze two sub-cases, noticing that for any $U' \subseteq U$, we have $(H + H'^e) \cap K[U'] = H' \cap K[U']$:

  – For all nonempty subsets $U' \subseteq U$, we have $H'^c \neq 0$, ie. $(H + T) \cap K[U'] \neq 0$, so that all variables in U are algebraically dependent over H'. In this case it seems, again, reasonable to consider $U = U'$ as set of variables for checking the existence of an **FSDIC**.
  If there exists an **FSDIC**, then adding H' and $\neg H''$ to the given hypotheses will produce a valid statement, and we will have discovered some components of $V(H + H'^e)$ where T vanishes, precisely those which are separated by the non-degeneracy conditions given by H''. In fact, in this case $H'' = ((H + H'^e) : T^\infty) \cap K[U]$, and this saturation ideal gives the intersection with $K[U]$ of the all primary components of $(H + H'^e)$ associated to primes that do not contain T. If these were all the components of $H + H'^e$, then $H'' = H'$ and there will be no **FSDIC**, by Theorem 3. Contradiction.

[6]Notice that if $K[U] \cap H = (0)$ and the cardinality of U is equal to $\dim(V(H))$ then U is a maximal set of independent variables.

But the converse—which holds when $dim(V(H)) = 0$—is not true in this case:[7] it can happen that T vanishes over some components of $V(H + H'^e)$ and yet there will be no **FSDIC** for any pair $U \supseteq U'$.

- There is a nonempty subset $U' \subseteq U$, of independent variables for $H' = (H + T) \cap K[U]$. Then we should find a maximal[8] U' set of such independent variables for H', included in U. We proceed checking the existence of a **FSDIC** for such couple $(U,\ U')$, since then the equality-type new hypotheses will be expressed in terms of the variables U parametrizing our given construction H, while the inequality-type conditions will be given in terms of independent variables U' for the new hypotheses $H + H'^e$. In this situation, Proposition 1 shows that the existence of **FSDIC** is equivalent to the fact that T vanishes over all components of $V(H + H'^e)$ where the variables in U' remain independent.[9] So, if there is an **FSDIC** and $\{H \wedge H' \wedge \neg H''\} \Rightarrow T$ will be a true statement, holding over some components.
  Otherwise, check with different couples of variables, with the same properties. Finally, if there is no **FSDIC** at all, the method fails to identify such components, if there are any (and, if U' is a maximal set of independent variables for $H + H'^e$, Proposition 2 shows there will be at least one such component, yielding a warning sign for the need to factorize).

The examples below show this procedure is quite satisfactory, in our opinion, but a different protocol, that yields results even when T vanishes just over some—not all—independent components, or that considers expressing inequality conditions in terms of variables not contained in U, could be also interesting and subject of future work.

[7] In fact, it is enough to show examples of such behavior when there is no **FSDIC** for $U = U'$, since it will imply, as well, there will be no **FSDIC** for any couple $U \supset U''$. In fact, suppose there is no **FSDIC** for $U = U'$; then $((H + H'^e) : T^\infty) \cap K[U] \subseteq \sqrt{(H')}$. This implies $((H + H'^e) : T^\infty) \cap K[U''] \subseteq \sqrt{H'} \cap K[U''] = \sqrt{H \cap K[U'']}$ and, thus, there is no **FSDIC** for any $U \supset U''$.

For instance, we can consider $H = ((a + 1) * (a + 2) * (b + 1)) \subset K[a, b, c]$, $T = (a + b + 1, c) \subset K[a, b, c]$. Take $U = \{b, c\} = U'$, a set of $dim(H)$-variables, independent over H. Then $H' = (c, b^3 - b) = H''$, so there is no **FSDIC**. But $H + H'^e = (a + 1, b, c) \cap (a + 2, b, c) \cap (a + 1, b - 1, c) \cap (a + 2, b - 1, c) \cap (b + 1, c)$ and T vanishes over some components, such as $(a + 2, b - 1, c)$ (and does not vanish over some other ones, such as $(a + 1, b - 1, c)$: the existence of a **FSDIC** requires that T vanishes simultaneously over all the components that have a common projection on the $\{b, c\}$-plane, that cannot be separated by H''. Otherwise, even if there are components over which T vanishes, they can not be detected by a **FSDIC** and this could be regarded as a limitation of the method, the price for not attempting to find a complete factorization of $V(H)$.

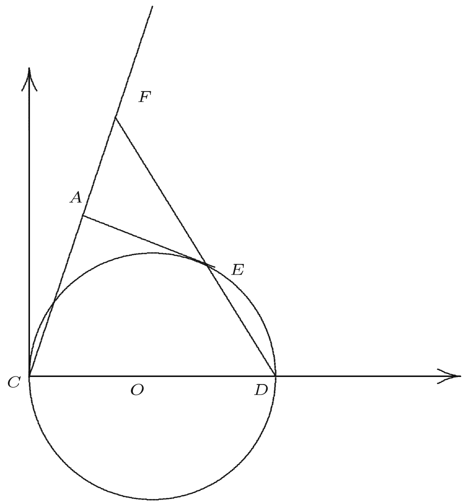
[8] But U' could be not maximal for $H + H'^e$. Take $H = ((a + 1) * (b + 1)) \subset K[a, b, c, d]$, $T = (a + b + 1, c) \subset K[a, b, c, d]$, $U = \{b, c, d\}$, $U' = \{d\}$. Then $H' = (c, b^2 + b) \subset K[U]$ is of dimension 1, with d as only independent variable, but $dim(H + H'^e) = 2$, since $\{a, d\}$ are independent. Here there is no **FSDIC**, since $H'' = 0$.

[9] The previous footnote provides an example where T vanishes over some, but not all, components where U' is independent. Moreover, taking $H = ((a + 1) * (b + 1) * b) \subset K[a, b, c, d]$, $T = (a + b + 1, c) \subset K[a, b, c, d]$, $U = \{b, c, d\}$, $U' = \{d\}$, we have $H' = (c, b^3 + b^2)$ of dimension 1, $dim(H + H'^e) = 2$, $H'' = 0$—hence there is no **FSDIC**—and T vanishes on none of the components of $H + H'^e = (a + 1, b^2, c) \cap (b, c) \cap (b + 1, c)$. This cannot happen if the cardinal of U' equals the dimension of $(H + H'^e)$, since Proposition 2 above claims, in this situation, that if there is no **FSDIC**, then T must both hold and fail over some independent component.

4 An Example

Next, we will develop the above introduced notions over an example from [10] (Example 91 in his book), suitably adapted to the discovery framework.

Example 5 Let us consider as given data a circle and two diametrically opposed points on it (say, take a circle centered at $(1,0)$ with radius 1, and let $C=(0,0)$, $D=(2,0)$ the two ends of a diameter), plus an arbitrary point $A=(u_1,u_2)$. Then trace a tangent from A to the circle and let $E=(x_1,x_2)$ be the tangency point. Let $F=(x_3,x_4)$ be the intersection of DE and CA. Then we claim that the unoriented lengths $[AE]=[AF]$ are equal. Moreover, in order to be able to define the lines DE, CA, we require, as hypotheses, that $D\neq E$ (ie. $u_1\neq 2$) and that $C\neq A$ (ie. $u_1\neq 0$ or $u_2\neq 0$).



Now, using CoCoA[10] [12] and its package TP (for Theorem Proving, see [2]), we translate the given situation as follows

```
Alias  TP   := $contrib/thmproving;

Use R::=Q[x[1..4],u[1..2]];

A:=[u[1],u[2]];
E:=[x[1],x[2]];
D:=[2,0];
F:=[x[3],x[4]];
C:=[0,0];
O:=[1,0];
```

[10]The default ordering is degrevlex; to compute contracted ideals (by the command Elim) the program uses a predefined term-ordering, which is an ordering of elimination for the selected variables, and degrevlex over the remaining.

```
Ip1:=TP.Perpendicular([E,A],[E,O]);
Ip2:=TP.LenSquare([E,O])-1;
Ip3:=TP.Collinear([0,0],A,F);
Ip4:=TP.Collinear(D,E,F);

H:=Saturation(Ideal(Ip1,Ip2,Ip3,Ip4),Ideal(u[1]-2)*
                                      Ideal(u[1], u[2]));

T:=Ideal(TP.LenSquare([A,E])-TP.LenSquare([A,F]));
```

where T is the thesis and H describes the hypothesis ideal. Notice that $Ip1$ expresses that the lines EA, EO are perpendicular; $Ip2$ states that the square of the length of $[EO]$ is 1 (so $Ip1$, $Ip2$ imply E is the tangency point from A); and the next two hypotheses express that the corresponding three points are collinear. The hypothesis ideal H is here constructed by using the *saturation* command, since (see Appendix, Remark 3) it is a compact form of stating that the hypothesis variety is the closure of the set defined by all the conditions $Ip[i] = 0, i = 1 \ldots 4$ minus the union $\{u[1] = 2\} \cup \{u[1] = 0, u[2] = 0\}$, as declared in the formulation of this example (see also Proposition 6 below). Finally, the thesis expresses that the two segments $[AE]$, $[AF]$ have equal non oriented length.

Now we check that the statement $H \Rightarrow T$ is not algebraically true in any conceivable way. For instance, applying the protocol in [14], it turns out that (notice that input lines in CoCoA end with a semicolon; therefore, the lines without it, are the output)

```
Saturation(H, Saturation(H,T));
Ideal(1)
-------------------------------
```

and this computation shows that all possible non-degeneracy conditions (those polynomials $p(\mathbf{u}, \mathbf{x})$ that could be added to the hypotheses as conditions of the kind $p(\mathbf{u}, \mathbf{x}) \neq \mathbf{0}$) lie in the hypothesis ideal, yielding, therefore to an empty set of conditions of the kind $p \neq 0 \wedge p = 0$. This implies, in particular, that the same negative result would be obtained if we restrict the computations to some subset of variables, since the thesis does not vanish on any irreducible component of the hypothesis variety.

Thus we must switch on to the discovery protocol, checking before hand that $u[1], u[2]$ actually is a (maximal) set of independent variables for our construction:

```
Dim(R/H);
2
-------------------------------
Elim([x[1],x[2],x[3],x[4]],H);
Ideal(0)
-------------------------------
```

Then we add the thesis to the hypothesis ideal and we eliminate all variables except $u[1], u[2]$

```
H':=Elim([x[1],x[2],x[3],x[4]],H+T);
H';
Ideal(-1/2u[1]^5 - 1/2u[1]^3u[2]^2 + u[1]^4)
-------------------------------
Factor(-1/2u[1]^5 - 1/2u[1]^3u[2]^2 + u[1]^4);
[[u[1]^2 + u[2]^2 - 2u[1], 1], [u[1], 3], [-1/2, 1]]
-------------------------------
```

yielding as complementary hypotheses the conditions $u[1]^2 + u[2]^2 - 2u[1] = 0 \vee u[1] = 0$ that can be interpreted by saying that either point A lies on the given circle or (when $u[1] = 0$) triangle $\Delta(A, C, D)$ is rectangle at C. In the next step of the discovery procedure we consider as new hypothesis ideal the set $H + H'^e$, which is of dimension 1 and where both $u[2]$ or $u[1]$ can be taken as independent variables ruling the new construction.

```
Dim(R/(H+H'^e));
1
-------------------------------
Elim([x[1],x[2],x[3],x[4],u[1]],H+H'^e);
Ideal(0)
-------------------------------
Elim([x[1],x[2],x[3],x[4],u[2]],H+H'^e);
Ideal(0)
```

Choosing, for example, $u[2]$ as relevant variable, we check that the new statement $H \wedge H'^e \Rightarrow T$ is correct under the non-degeneracy condition $u[2] \neq 0$:

```
H'':=Elim([x[1],x[2],x[3],x[4],u[1]], Saturation(H+H'^e,T));
H'';
Ideal(u[2]^3)
--------------------------------
```

Thus we have arrived to the following statement: Given a circle of radius 1 and centered at $(1, 0)$, and a point A not in the X-axis and not in the line $X = 2$, the segments $[AE]$, $[AF]$ (where E is the tangency point from A to the circle and F is the intersection of the lines passing by $(2, 0)$, E and A, $(0, 0)$) are of equal length if A is on the Y-axis or on the circle. The latter case is quite trivial, since it means that $A = E = F$.

Remark that if we choose $u[1]$ as the privileged variable, what we get is

```
H'':=Elim([x[1],x[2],x[3],x[4],u[2]], Saturation(H+H'^e,T));
H'';
Ideal(2u[1]^4)
-------------------------------
```

a more trivial statement, since here A is subject to the conditions of being “not in the Y-axis and not in the line $X = 2$ and A is not the origin, plus A on the Y-axis or in the circle”; which can be summarized as A is in the circle and A is neither the origin nor the point $(0, 2)$ (for the equality of lengths of the segments AE, AF).

All the computations required 0.52 CPU-seconds and 1908 kb of memory, and they were done with a processor AMD Athlon(tm) XP 2000+ CPU at 1.66 GHz and 1 GB RAM.

5 Introducing Non-degeneracy Hypotheses

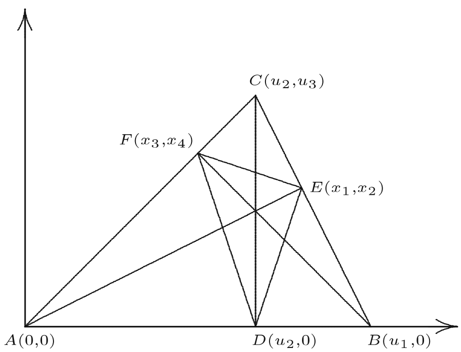
The protocol we have developed in the previous sections, although it seems the natural way to deal with the automatic discovery, can give different answers depending on the way we add non-degenerate conditions as hypotheses.

For example, take three points A, B, C, $A = (0,0)$, $B = (u_1, 0)$, $C = (u_2, u_3)$ and let $T = (u_3)$ be the thesis ideal. We want to exclude, as our only hypothesis, the case $u_3 = 0$. One way of doing so, as in the example above, is to consider the saturation of the ideal H of equality type hypotheses (the zero ideal in this case) by the ideal of inequality type hypotheses (the ideal (u_3) in this instance), that is, to consider the ideal $(0) : (u_3)^\infty = (0)$. This ideal represents (see Appendix) the Zariski closure of the set defined by the equations in the hypothesis minus the set $u_3 \neq 0$. Let $U = \{u_1, u_2, u_3\}$. Then our protocol yields $H' = (u_3)$ and $H'' = (1)$ (for whatever subset of variables), therefore H', H'' is an **FSDIC** for our problem. It seems that we have proved $\{u_3 \neq 0 \wedge u_3 = 0\} \Rightarrow u_3 = 0$, but a closer look reveals that the new hypothesis ideal is $H + H'^e = (u_3)$ and, thus, we have just discovered that $u_3 = 0 \Rightarrow u_3 = 0$. So, in this case, adding H' we have lost the essential information about $u_3 \neq 0$.

On the other hand, we can proceed in the following way, obtaining a different answer. Consider the following hypothesis ideal $HH = (0) + (u_3 z - 1)$ in $K[u_1, u_2, u_3, z]$. Apparently, HH can be "read" as another simple way of stating that $u_3 \neq 0$. Then our protocol gives[11] $R' = ((u_3 z - 1) + T) \cap K[u_1, u_2, u_3] = (1)$, so there is not an **FSDIC** for our problem since $V(HH + R'^e) = \emptyset$.

This example seems quite artificial, but we are going to see that this situation is not rare.

Example 6 Given an arbitrary triangle, we construct the associated orthic triangle, that is, the triangle whose vertices are the endpoints of the altitudes of the given triangle. Then we conjecture the orthic triangle is equilateral.



[11] We use different notation (i.e. R' instead H', etc.) to distinguish the two ways to solve the problem.

As in the previous example, we translate the given situation. Thus, let h_1, h_2, h_3 and h_4 be the polynomials in $R = Q[x[1..4], u[1..3], z]$, which translate the hypotheses $EA \perp BC$, $FB \perp AC$, $F \in AC$ and $E \in CB$. Let $HH = (h_1, h_2, h_3, h_4, u[1]u[3]z - 1)$ and let $T = (t_1, t_2)$ be the thesis ideal, where t_1 is given by the equality of the square of the lengths of the segments $[DE]^2 = [EF]^2$ and t_2 is given by $[DE]^2 = [DF]^2$.

Notice we have added, as hypothesis, that $u[1]u[3]$ should be different from zero (by introducing the equation $u[1]u[3]z - 1 = 0$ and by enlarging the set of variables). Then we check that $u[1], u[2], u[3]$ actually is a (maximal) set of independent variables for our construction since $HH \cap K[u[1], u[2], u[3]] = (0)$ and $Dim(R/HH) = 3$.

As the theorem is obviously false, we turn over to discovery conditions, adding the theses to the hypothesis ideal, ie. considering $R' = (HH + T) \cap K[u[1], u[2], u[3]]$ and checking that R' is not the zero ideal. Then we take $U' = u[1]$ and we check that U' is a maximal set of independent variables for $HH + R'$. Following our protocol, we compute next the ideal of conditions $R'' = ((HH + R') : T^\infty) \cap K[u[1]]$ and we obtain that $R'' = Ideal(1)$. Therefore R', R'' is an **FSDIC** for our problem.[12]

But, what do we have discovered? In order to look for an answer, we find a decomposition[13] of the new set of hypotheses $V(HH + R'^e)$. We can easily check that the ideal R' is the intersection of the prime ideals $P_1 = (u[1]-2u[2], u[2]^2 - 3u[3]^2)$, $P_2 = (3u[1]-2u[2], u[2]^2-3u[3]^2)$, $P_3 = (u[1]+2u[2], 3u[2]^2-u[3]^2)$ and $P_4 = (u[1]-2u[2], 3u[2]^2-u[3]^2)$, which means that the angles of the given triangle should verify one of the following set of degrees: $HH1$: $\{A=30, B=30, C=120\}$, $HH2$: $\{A=30, B=120, C=30\}$, $HH3$: $\{A=120, B=30, C=30\}$ or $HH4$: $\{A=60, B=60, C=60\}$.

Therefore we have discovered that the orthic triangle would be equilateral if the given triangle is itself equilateral or isosceles (and, then, of the particular type with angles equal to 120, 30 and 30 degrees, respectively). We have not been able to find a reference in the literature to this, somehow surprising, result.

Now let us turn to solve the previous example in a different way, by adding the hypothesis of non degeneracy for $u[1], u[3]$ by means of a *saturation*, and so now we consider $H = (h_1, h_2, h_3, h_4) : (u[1]u[3])^\infty$. Following our protocol, we first compute $H' = (H + T) \cap K[u[1], u[2], u[3]]$, then $H'' = ((H + H') : T^\infty) \cap K[U[1]]$; in this case we obtain that $H'' = (0)$. Since $1 \in (H')^{c'} : H''^\infty$, it turns out that (H', H'') does not result to be an **FSDIC** for our problem.[14] A confirmation for this different answer can be found in the prime decomposition of $\sqrt{H + H'^e}$, that is, the intersection of $P_1 = (u[1]-2u[2], u[2]^2-3u[3]^2, 2x[4] - 3u[3], 2x[3] - 3u[2], 2x[2] - 3u[3], 2x[1] - u[2])$, $P_2 = (3u[1] - 2u[2], u[2]^2 - 3u[3]^2, 2x[4] - u[3], 2x[3] - u[2], 2x[2]+u[3], 2x[1]-u[2])$, $P_3 = (u[1]+2u[2], 3u[2]^2-$

[12]The computations required 0.41 CPU-seconds and 1726 kb of memory.

[13]Notice that if $P_1, \ldots, P_s$ are the minimal primes of R'^e, then $V(HH + R'^e) = V(HH) \cap V(\sqrt{R'^e}) = V(HH) \cap V(\bigcap_{i=1}^s P_i) = V(HH) \cap (\bigcup_{i=1}^s V(P_i)) = \bigcup_{i=1}^s (V(HH) \cap V(P_i)) = \bigcup_{i=1}^s (V(HH + P_i))$.

[14]The computations in this setting required 0.88 CPU-seconds and 3388 kb of memory.

$u[3]^2, 2x[4]+u[3], 2x[3]+u[2], 2x[2]-u[3], 2x[1]+u[2])$, $P_4=(u[1]-2u[2], 3u[2]^2-u[3]^2, 2x[4]-u[3], 2x[3]-u[2], 2x[2]-u[3], 2x[1]-3u[2])$, $P_5=(u[3], u[2], u[1], x[3]^2+x[4]^2, x[2]x[3]+x[1]x[4], x[1]x[3]-x[2]x[4], x[1]^2+x[2]^2)$, $P_6=(u[3], u[2], x[3]^2+x[4]^2-x[3]u[1], x[2], x[1])$, $P_7=(u[3], u[1]-u[2], x[4], x[3]-u[2], x[1]^2+x[2]^2-x[1]u[2])$, $P_8=(u[2], u[1], x[4], x[3], x[2], x[1])$, $P_9=(u[3], u[2], u[1], x[4], x[3], x[1]^2+x[2]^2)$, $P_{10}=(u[3], u[2], u[1], x[3]^2+x[4]^2, x[2], x[1])$, $P_{11}=(u[3], u[1]-u[2], x[4], x[3]-u[2], x[2], x[1]-u[2])$, $P_{12}=(u[3], u[2], x[4], x[3], x[2], x[1])$.

We can check that the thesis ideal is not contained in P_7. So we observe that the thesis ideal does not belong to some minimal primes of $H+H'^e$ where the initial degenerate conditions lie and where $u[1]$ is independent.

In view of these examples it is natural to ask for the relations between the ideals computed after introducing the non-degeneracy hypotheses in the two different ways, i.e. to understand the relations between H' and R', and between H'' and R''. Let E represent the ideal of the equations in the hypotheses and let $G=(g_1,\ldots,g_l)$ be the ideal associated to a collection of inequations $\{g_1\neq 0\vee\cdots\vee g_l\neq 0\}$ that we have also as hypotheses. We remark that there is no loss of generality in considering only the disjunction of inequations, since a conjunction $\{g_1\neq 0\wedge\cdots\wedge g_l\neq 0\}$ can be replaced by a product $g_1\cdots g_l\neq 0$. Moreover, any conjunction of disjunctions of inequations can be expressed (by the distributive law) as a disjunction of conjunctions of inequations.

So we have E, G and T in $K[X]$. Let H be the hypothesis ideal built up according to the saturation method (ie. $H=(E:G^\infty)$) and let HH be the hypothesis ideal constructed by adding a slack variable, ie. $HH=(EK[X,s]+((g_1s-1)\cdots(g_ls-1))$.

Let $U'\subset U$ be the sets of our privileged variables. Then we proceed in both cases with our protocol, yielding, respectively, $H'=((E:G^\infty)+T)\cap K[U]$ and $R'=(EK[X,s]+((g_1s-1)\cdots(g_ls-1))+TK[X,s])\cap K[U]$, where s is an extra variable. By Proposition 6 in the Appendix, we have that

$$(EK[X,s]+((g_1s-1)\cdots(g_ls-1)))\cap K[X]=HH\cap K[X]=E:G^\infty=H,$$

so both represent, after the elimination of the slack variable, the same set of initial hypotheses. On the other hand we have seen in previous examples that $H+H'K[X]$ and $HH+R'K[X,s]$ may represent different sets of new (that is, obtained after applying the discovery protocol) equality type hypotheses. This is clarified through the following statements.

Proposition 3 *Suppose G contained in $K[U]$. Then $R'=H':G^\infty$. Moreover, for the new (equality type) hypothesis ideals we have the following relation:*

$$(HH+R'K[X,s])\cap K[X]=(H+H'K[X]):(GK[X])^\infty$$

Proof Let $S_G = (g_1 s - 1) \cdots (g_l s - 1) \subseteq K[X, s]$. By Proposition 6 and by the Lemmas 2 and 3 we have

$$\begin{aligned}
R' &= (HH + TK[X, s]) \cap K[U] \\
&= (EK[X, s] + S_G + TK[X, s]) \cap K[U] \\
&= (EK[X, s] + S_G + TK[X, s]) \cap K[X] \cap K[U] \\
&= ((E + T) : (GK[X])^\infty) \cap K[U] \\
&= (((E : GK[X]^\infty) + T) : (GK[X])^\infty) \cap K[U] \\
&= ((H + T) : (GK[X])^\infty) \cap K[U] \\
&= ((H + T) \cap K[U]) : G^\infty = H' : G^\infty
\end{aligned}$$

Analogously, we can prove the relation holding between the two possible new hypothesis ideals:

$$\begin{aligned}
&(EK[X, s] + S_G + R'K[X, s]) \cap K[X] \\
&\quad = (E + R'K[X]) : (GK[X])^\infty \\
&\quad = (E + (H' : G^\infty)K[X]) : (GK[X])^\infty \\
&\quad = (E + (H'K[X] : (GK[X])^\infty)) : (GK[X])^\infty \\
&\quad = (E + H'K[X]) : (GK[X])^\infty
\end{aligned}$$

□

Therefore, the zero sets of the two new hypothesis ideals could be different because $V(H + H'^e)$ can contain some components where the inequality conditions vanish simultaneously, and these components are taken away by the saturation by G, so what remains agrees—as shown in the preceding proposition—with the projection over the X-variables of $V(HH + R'^e)$.

Next, let study the relation between R'' and H''.

Proposition 4 *Suppose* $G \subseteq K[U']$. *Then* $R'' = H'' : G^\infty$

Proof As above, let $S_G = (g_1 s - 1) \cdots (g_l s - 1) \subseteq K[X, s]$. Again, by Proposition 6, by the Lemma 2 and by the previous proposition, we have

$$\begin{aligned}
R'' &= ((EK[X, s] + S_G + R'K[X, s]) : (TK[X, s])^\infty) \cap K[U'] \\
&= ((EK[X, s] + S_G + R'K[X, s]) : (TK[X, s])^\infty) \cap K[X] \cap K[U'] \\
&= (((EK[X, s] + S_G + R'K[X, s]) \cap K[X]) : T^\infty) \cap K[U'] \\
&= (((E + H'K[X]) : (GK[X])^\infty) : T^\infty) \cap K[U'] \\
&= ((((E : (GK[X])^\infty) + H'K[X]) : (GK[X])^\infty) : T^\infty) \cap K[U'] \\
&= ((((E : (GK[X])^\infty) + H'K[X]) : T^\infty) : (GK[X])^\infty) \cap K[U'] \\
&= ((((E : (GK[X])^\infty) + H'K[X]) : T^\infty) \cap K[U']) : G^\infty = H'' : G^\infty
\end{aligned}$$

□

Hence, in the case $G \subseteq K[U']$, we have

$$(R' \cap K[U']) : R''^{\infty} = ((H' \cap K[U']) : H''^{\infty}) : G^{\infty}.$$

So, if R', R'' is an **FSDIC** (i.e. $1 \notin R'^{c'} : R''^{\infty}$) for $HHK[X,s]$, $TK[X,s]$, then H', H'' is an **FSDIC** for H, T. The converse holds only in the case U' is a set of independent variables for H'. In fact, suppose $H'^{c'} = (0)$. Then, by Lemma 2, $R'^{c'} = H'^{c'} : G^{\infty} = (0)$. Suppose R', R'' is not an **FSDIC**, i.e. $R'^{c'} : R''^{\infty} = (1)$. Then $((0) : H''^{\infty}) : G^{\infty} = (1)$, i.e. $((0) : G^{\infty}) : H^{\infty} = (1)$ and this implies $(0) : H''^{\infty} = (1)$, i.e. H', H' is not an **FSDIC** for H, T.

We can summarize the precedent analysis in the following result:

Theorem 4 *Let $\mathcal{T}$ be a statement, let $E \subseteq K[x_1, \ldots, x_n]$ be the ideal corresponding to the hypotheses given by equations, $G = (g_1, \ldots, g_l) \subseteq K[U']$ the ideal corresponding to a disjunction of inequations and T the thesis ideal. Let $U' \subseteq U \subseteq \{x_1, \ldots, x_n\}$. Let H', H'', R', R'' as before, and suppose that R', R'' is an* **FSDIC** *for $HH = (EK[X,s] + ((g_1 s - 1) \cdots (g_l s - 1)))$, $TK[X,s]$. Then H', H'' is a* **FSDIC** *for $H = (E : G^{\infty})$, T.*

Moreover, if U' is a set of independent variables for H', then R', R'' is an **FSDIC** *if and only if H', H'' is an* **FSDIC**.

In conclusion, we can say our protocol does not suggest a specific way the user has to deal with in order to include inequations as part of the hypothesis, to eliminate known before hand degenerate cases. In fact the user faces here a philosophical problem: either from the beginning decides to eliminate some cases, and wants to continue to exclude these situations when considering the new hypothesis ideal; or—since the user is trying to discover theorems—the user could prefer to find out that the answer is that the theorem holds if he/she does not eliminate the previously considered as degenerate cases. The first situation corresponds to the choice of HH as hypothesis; the second, to the election of H to describe the hypothesis through saturation.

6 Further Examples

Example 7 This is an example of automatic discovery concerning Euler's formula relating the radii of the ex-circle and of the in-circle of a triangle, as well as the distance between the centers of these circles. This problem has been approached by Recio [19] and by Botana and Recio [4] in the automatic discovery framework; and also by Wang and Zhi [27], but for automatic proving, giving explicitly the formula of Euler (arguably named Poncelet's theorem by them, for example see http://mathworld.wolfram.com/EulerTriangleFormula.html or see the web page http://enriques.mathematik.uni-mainz.de/intgeo/poncelet.html) and requiring the machine to prove its validity.

We consider, for simplicity, a triangle with vertices at $A = (-1, 0)$, $B = (1, 0)$, $C = (u[1], u[2])$. Then take the circumscribed circle of this triangle, of radius, say, c; the inscribed circle, with radius r; and the distance d between the two centers of these circles. We wonder if, in this case, there are some constraints that c, r, d will have to fulfill; or if, on the contrary, these three quantities can take any arbitrary values. Notice that in this case we should apply the protocol with thesis $T = Ideal(0)$ and $U = \{d, c, r\}$, so that the "complementary" hypotheses are expressed just in terms of these variables.

Thus we proceed constructing the circumcenter $O = (x[1], x[2])$ as a point equidistant to the three vertices; ditto for the incenter $I = (x[3], x[4])$, as the center of a circle which is tritangent to the sides of the triangle (that is, passing through every side and having a perpendicular radius at the corresponding point of contact $(y[1], y[2])$). So the polynomial h_1 which translates the incenter hypothesis will be determined in the following way:

```
Elim(y,Ideal((y[1]-x[3])^2+(y[2]-x[4])^2-r^2,
                       TP.Collinear([y[1],y[2]],A,C),
                       TP.Perpendicular([I,[y[1],y[2]]],[A,C])));

Ideal(-u[2]^2x[3]^2 + 2u[1]u[2]x[3]x[4] - u[1]^2x[4]^2 +
u[1]^2r^2 + u[2]^2r^2- 2u[2]^2x[3] + 2u[1]u[2]x[4] +
2u[2]x[3]x[4] - 2u[1]x[4]^2 + 2u[1]r^2 - u[2]^2 +
2u[2]x[4] - x[4]^2 + r^2)
```

Analogously, one gets, by considering the tangency condition to the side BC, the second hypothesis polynomial h_2:

```
Elim(y,Ideal((y[1]-x[3])^2+(y[2]-x[4])^2-r^2,
                       TP.Collinear([y[1],y[2]],B,C),
                       TP.Perpendicular([I,[y[1],y[2]]],[B,C])));
Ideal(
    -u[2]^2x[3]^2+2u[1]u[2]x[3]x[4]-u[1]^2x[4]^2+u[1]^2r^2
    +u[2]^2r^2+2u[2]^2x[3]-2u[1]u[2]x[4]-2u[2]x[3]x[4]+
    2u[1]x[4]^2-2u[1]r^2-u[2]^2+2u[2]x[4]-x[4]^2+r^2)
```

The remaining hypotheses: h_3, h_4, h_5, h_6, h_7, as presented below, are self evident: $h_3 = r^2 - x[4]^2, h_4 = [AO]^2 - [BO]^2, h_5 = [AO]^2 - [CO]^2, h_6 = [AO]^2 - c^2, h_7 = d^2 - [IO]^2$. Now we should search for a consequence from $H = (h_1, \ldots, h_7)$ that just relates d, r, c. But

```
Use R::=Q[u[1..2],x[1..4],d,c,r];

Elim([u[1],u[2],x[1], x[2], x[3], x[4]],H);
Ideal(0)
-------------------------------
Dim(R/H);
3
-------------------------------
```

that means that, if $\{d, r, c\}$ are constrained to be, respectively, the distance between the in and ex-centers, and the corresponding radii, then they must verify the equation $0 = 0$, which is obvious and not very interesting. Therefore $\{d, r, c\}$ are independent variables for H. But the expected dimension should be 2, i.e. the configuration should be fixed by $u[1], u[2]$. We conclude that it is possible we have, without noticing it, introduced some degenerate cases.

It is perhaps remarkable to study in this case what happens if we add the hypothesis $u[2]z - 1$, that is $u[2] \neq 0$, in order to prevent the degeneration of the given triangle. After all, one might easily verify by hand that if $u[2] = 0$ and the triangle collapses to a line, the equations in H do not impose any constraint on $\{d, r, c\}$. Thus one could, perhaps, eliminate, in the given statement, this degenerate case. This is attempted below, using the *saturation* mode.

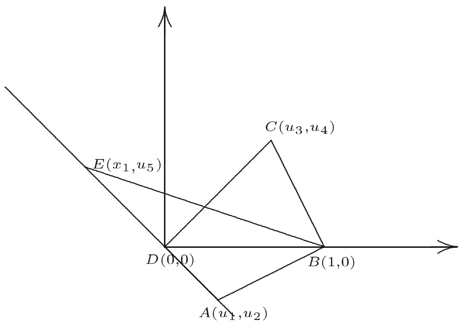
```
H:=Saturation(Ideal(Ip1,Ip2,Ip3,Ip4,Ip5,Ip6,Ip7),
              Ideal(u[2]));

Elim([u[1],u[2],x[1], x[2], x[3], x[4]],H);
Ideal(d^4 - 2d^2c^2 + c^4 - 4c^2r^2)
-------------------------------
Factor(d^4 - 2d^2c^2 + c^4 - 4c^2r^2);
[[d^2 - c^2 + 2cr, 1], [d^2 - c^2 - 2cr, 1]]
-------------------------------
```

yielding now that the considered variables must verify $\{d^2 - 2rc - c^2 = 0\} \vee \{d^2 + 2rc - c^2 = 0\}$, which is, essentially, Euler's formula (extended to cover the case of the different possible tritangent circles to a triangle).

All the computations in the previous example required 0.4 CPU-seconds and 2292 kb of memory.

Example 8 Let $ABCD$ be a quadrilateral and let E be a point on the line AD. We wonder where we have to place E in order that the triangle ABE and the quadrilateral have same area.

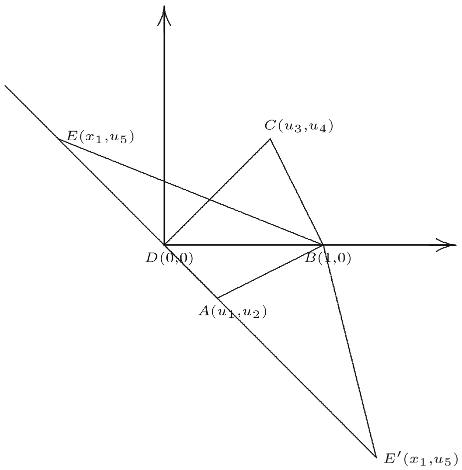


This is a very common exercise in the elementary geometry classroom, were the teacher expects the student to find "the" answer, namely, that point E should be placed in the intersection of the line supporting side AD with the parallel to DB through C, because all the triangles with common base BD and vertex on this parallel line will have same height and, therefore, same area. Let us see what our protocol for automatic discovery yields in this case.

First, we (naively) observe that the area of the quadrilateral could be expressed as $u_4/2 - u_2/2$. Then, in order to compute the area of ABE, let $F = (x_2, x_3)$ be the feet of the altitude on the side AB. Let $[AB] = b$ and $[EF] = h$, so that the area of the triangle ABE becomes $bh/2$. Moreover, we introduce (this time adding one slack variable) the non-degeneracy condition $u_2u_4 \neq 0$. So let $h_1, \ldots, h_6$ be the polynomials which translate the conditions $E \in AD$, $F \in AB$, $EF \perp AB$, $[EF]^2 = h^2$ and $[AB]^2 = b^2$. Let $HH = (h_1, \ldots, h_6, zu[2]u[4] - 1)$, and $T = (u[4]/2 - u[2]/2 - bh/2)$.

Then we check that the dimension of the whole system is 5 (as it is ruled by the two free vertices of the quadrilateral and the gliding point E on the line, ie. $\{u[1], \ldots, u[5]\}$ is a maximal set of independent variables). Finally we apply our discovery protocol, searching for an **FSDIC** and we obtain $R' = (-2u[2]u[4] + u[4]^2 + 2u[2]u[5] - u[5]^2)$. It is easy to check that $-2u[2]u[4] + u[4]^2 + 2u[2]u[5] - u[5]^2 = -(u[4] - u[5])(2u[2] - u[4] - u[5])$. We verify that these complementary hypotheses still leave free the two free vertices of the triangle, but involve now one coordinate of point E, as expected. Therefore we set $U' = \{u[1], \ldots, u[4]\}$. So, now, we search for the complementary non-degeneracy hypotheses and we obtain that $R" = (0)$, but they are contradictory, since it introduces as hypotheses the expression $0 \neq 0$.

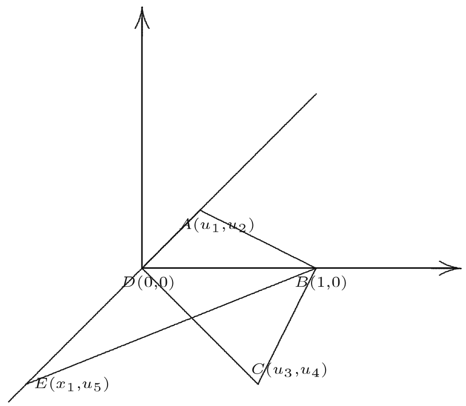
Therefore we do not have an **FSDIC** for our problem. Still, let us analyze the answer we have obtained. It is clear what the condition $u[4] - u[5]$ means, i.e. E has to be taken on the parallel to DB passing through C. The condition $2u[2] - u[4] - u[5]$ gives a point E' such that $[AE] = [AE']$, as in the following picture:



But what the reason is for not having an **FSDIC** in this context? Let us try to find out the relations between the various involved areas.

```
Elim([x[1],x[2],x[3],z,u[5],u[1],u[3]],HH+R');
Ideal(b^2h^2 - u[2]^2 + 2u[2]u[4] - u[4]^2)
------------------------------
Factor(b^2h^2 - u[2]^2 + 2u[2]u[4] - u[4]^2);
[[bh - u[2] + u[4], 1], [bh + u[2] - u[4], 1]]
------------------------------
```

The second relation means that the area of the triangle, as computed, is the opposite of the area of the quadrilateral. In fact, we could have the following picture:



and in this case we would have $bh/2 = -u[2]/2 + u[4]/2$. So our approach, although unsuccessful, tells us that the sign of the area plays a fundamental role in the formulation of this situation. Then we try to solve the problem using, to express the area of a triangle, the well known determinantal formula and expressing the area of the quadrilateral as the sum of the areas of two triangles. The following is a self-explanatory transcription of the computer session:

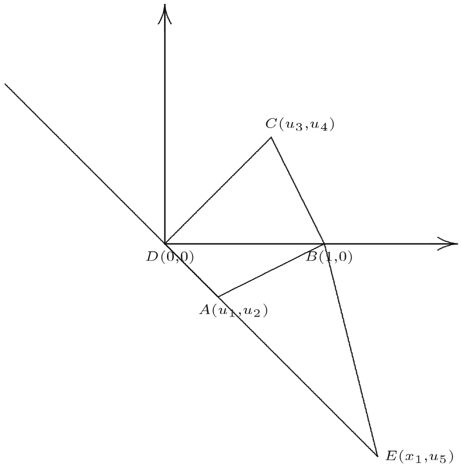
```
H1:=TP.Collinear(A,D,E);
HH:=Ideal(H1,zu[2]u[4]-1);

AreaABD:=1/2Det(Mat([[1,u[1],u[2]],[1,1,0],[1,0,0]]));
AreaBCD:=1/2Det(Mat([[1,1,0],[1,u[3],u[4]],[1,0,0]]));
AreaABE:=1/2Det(Mat([[1,u[1],u[2]],[1,1,0],[1,x[1],u[5]]]));

T:=Ideal(AreaABE-AreaBCD-AreaABD);

R':=Elim([x[1],z],HH+T);
R';
Ideal(-u[4] + u[5])
-------------------------------
R'':=Elim([x[1],z,u[5]],Saturation(HH+R',T));
R'';
Ideal(1)
-------------------------------
```

So now we have an **FSDIC** and the complementary hypothesis described by R' determines the location of E by tracing a parallel to DB passing through C. Notice that in this way we do not obtain the condition $2u[2] - u[4] - u[5] = 0$ that appeared before. Intrigued by this fact, we pay attention to the following picture



where we observe that here $AreaABE = -AreaBCD - AreaABD$. So a more complete formulation could to consider as the thesis polynomial all the possible combinations of the signed sum of the areas, as in the following **FSDIC** session:

```
T:=Ideal((AreaABE-AreaBCD-AreaABD)(AreaABE+AreaBCD-AreaABD)
         (AreaABE+AreaBCD+AreaABD)(AreaABE-AreaBCD+AreaABD));

R':=Elim([x[1],z],HH+T);
R';
Ideal(4u[2]^2u[4]^2 - u[4]^4 - 4u[2]u[4]^2u[5] -
        4u[2]^2u[5]^2 + 2u[4]^2u[5]^2 + 4u[2]u[5]^3 - u[5]^4)
-------------------------------

Factor(4u[2]^2u[4]^2 - u[4]^4 - 4u[2]u[4]^2u[5] -
       4u[2]^2u[5]^2 + 2u[4]^2u[5]^2 + 4u[2]u[5]^3 - u[5]^4);

[[u[4] - u[5], 1], [u[4] + u[5], 1],
[2u[2] + u[4] - u[5], 1], [2u[2] - u[4] - u[5], 1]]
-------------------------------
R'':=Elim([x[1],z,u[5]],Saturation(HH+R',T));
R'';
Ideal(1)
-------------------------------
```

that provides, through the factors of the new hypothesis generating R', four possible locations for point E. Of course, this procedure can be used to explore, as well, other situations (such as introducing as thesis polynomial just the product of the first and third factors, etc.).

All the computations in the previous example required 1.23 CPU-seconds and 3604 kb of memory.

Example 9 This final example has been brought to our attention by one of the referees, to whom we want to express our thanks. The example deals with a generalization of the Steiner-Lehmus Theorem on the equality of lengths of the angle bisectors on a given triangle, an issue which has attracted along the years a considerable interest. We refer to http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/steiner-lehmus for a large collection of references (sometimes with comments) on the original statement, concerning the equality of two internal angle bisectors, and to [28] or [5] for automatic approaches dealing with its generalization, regarding internal as well as external angle bisectors.

Without loss of generality we will consider a triangle of vertices $A(0, 0)$, $B(1, 0)$, $C(x, y)$. Then at each vertex we can determine two bisectors (one internal, another one external) for the angles described by the lines supporting the sides of the triangle meeting at that vertex. We want to discover what kind of triangle has, say, one bisector at vertex A and one bisector at vertex B, of equal length. Recall that the Steiner-Lehmus Theorem states that this is the case, for internal bisectors, if and only if the triangle is isosceles. So the question here is about the equality of lengths when we consider external bisectors, too.

Algebraically we translate the construction of a bisector, say, at vertex A, as follows. We take a point (p, q) at the same distance as $C = (x, y)$ from A, so it verifies $p^2 + q^2 - (x^2 + y^2) = 0$. Then, we place this point at the line AB, by adding the equation $q = 0$. Then the midpoint from (p, q) and C will be $((x + p)/2, (y + q)/2)$ and the line defined by A and by this midpoint intersects the opposite side BC (or its prolongation) at point (a, b), verifying $\{p^2 + q^2 - (x^2 + y^2) = 0, q = 0, -a(y + q)/2 + b(x + p)/2 = 0, -ay + b(x - 1) + y = 0\}$. Finally, distance from (a, b) to A is given as $a^2 + b^2$, and this quantity provides (the square of) the length of the bisector(s) associated to A. Notice that by placing (p, q) at different positions in the line AB, the previous construction provides both the internal and the external bisector through A. There is no way of distinguishing both bisectors, without introducing inequalities, something alien to our setting (since we work on algebraically closed fields).

Likewise, we associate a set of equations to determine the length of the bisector(s) at B, introducing a point (r, s) in the line AB, so that its distance to B is equal to that of vertex C. Then we consider the midpoint of (r, s) and C and place a line through it and B. This line intersects side AC at a point (m, n), which is defined by the following set of equations: $\{(r - 1)^2 + s^2 - ((x - 1)^2 + y^2) = 0, s = 0, -m((y + s)/2) + n((x + r)/2 - 1) + (y + s)/2 = 0, -my + nx = 0\}$. The length of this bisector will be $(m - 1)^2 + n^2$.

Finally, we apply our discovery protocol to the hypotheses H given by the two sets of equations and having as thesis T the equality $(a^2 + b^2) - ((m - 1)^2 + n^2) = 0$. It is clear the that the only two (geometrically meaningful for the construction) independent variables are $\{x, y\}$, so we eliminate in $H + T$ all variables except these two, getting in this way the ideal H'. The result is a polynomial that factors as the product of y^3 (a degenerate case), $2x - 1$ (triangle is isosceles) and the degree 10 polynomial $14x^2y^4 + y^2 + 246y^2x^6 + 76x^8 - y^6 + 8x^{10} + 9y^{10} - 164y^2x^5 + 12y^4x - 10x^2y^2 - 4x^4 - 44y^8x - 136y^4x^3 + 278y^4x^4 - 64x^7 - 164x^7y^2 + 122y^6x^2 - 6y^4 + 8x^5 -$

$36y^6x+20y^2x^3+84y^4x^6+86x^4y^6+44x^2y^8+16x^6+41y^2x^8+31y^2x^4-40x^9-252y^4x^5-172y^6x^3+14y^8$ (cf. [28], page 150, also [5] for a picture of the curve given by this polynomial).

Next, in order to compute H'' we must choose one of the variables x, y, say, variable x, and eliminate y in the saturation of $H+H'$ by T. The result is (0), so there is no **FSDIC**, according to Corollary 1. In fact it is hard to expect that for almost all triangles with vertex C placed at the locus of H' and for any interpretation of the bisectors at A and B, they will all have simultaneously an equal length. But it also means (by Proposition 2) that adding H' to the set of hypotheses, for instance, placing vertex C at any point on the degree 10 curve, there will be an interpretation for the bisectors such that the equality of lengths follow. It is easy to deduce that this is so (except for some degenerate cases) considering internal/external, external/internal and external/external bisectors (since the internal/internal case holds only for isosceles triangles). Moreover, intersecting this curve with the line $2x-1=0$ we can find out two points $x=1/2$, $y=(1/2)RootOf(-1+3Z^2)$ (aprox. $x=0.5000000000$, $y=+-0.2886751346$) where all four bisectors (the internal and external ones of A and B) have equal length. The other two points of intersection correspond to the case of equilateral triangles, where the two internal bisectors and the two infinite external bisectors of A, B have pairwise equal length, but the length is not equal for the internal and external bisectors.

A further question can be considered in this setting, regarding the equality of lengths of all three bisectors in a triangle. Here the hypotheses include the algebraic description of the bisector(s) for A, B, C and the two theses describe the equality of the lengths of the bisector(s) of A, B and of A, C. As above, we eliminate all variables except $\{x, y\}$, yielding an ideal H' generated by several polynomials (here presented as product of irreducible factors):

1. $y^3(2x-1)(136x^2y^4+115021x^2y^2-23136x^2+23136x-115021xy^2-136xy^4-21504+95149y^2+116789y^4)$,
2. $y^3(2x-1)(17x^2-17x-2+19y^2)(x^2-x+1+y^2)$,
3. $-y^3(2x-1)(103155x^2y^2-20960x^2-103155xy^2+20960x+85459y^2-136y^6-19328+104787y^4)$,
4. $y^3(377084y^2x^4-17856-61088x^4-148192x^2-381436xy^4+87104x-412545xy^2+80325y^2+544y^8+96005y^4-754168x^3y^2+122176x^3-2856y^6+789629x^2y^2+381436x^2y^4)$

The solution set of this system of two-variable polynomials is the x-axis (a degenerate case) plus a finite number of real and complex points. A detailed case study shows that these real points correspond to the following situations:

a) The triangle is equilateral.
b) $x=2/17-(2/17)RootOf(4Z^4+349Z^2-64)^2$, $y=RootOf(4Z^4+349Z^2-64)$ (aprox. $x=0.09611796796$, $y=+-0.4277818044$). These two points correspond to the equality of lengths for the external bisectors of A and C and the internal bisector of B.
c) Likewise, we have the two points $x=15/17+(2/17)RootOf(4Z^4+349Z^2-64)^2$, $y=RootOf(4Z^4+349Z^2-64)$ (aprox. $x=0.09038820320$, $y=+-0.4277818044$). These points correspond to the equality of lengths for the external bisectors of B and C and the internal bisector of A.

d) $x = 1/2$, $y = RootOf(4Z^4 - 19Z^2 - 4)$ (aprox.$x = .5000000000$, $y = \pm 2.225295714$). These two points correspond to the equality of lengths for the external bisectors of A and B and the internal bisector of C.

In particular we remark that there are no triangles where two internal bisectors and one external bisector (for different vertices) have equal length, and that there are no triangles where the three external bisectors (one for each of the three vertices) are equal (except for the case of infinite length).

7 Conclusions

The precedent examples, written in a dialectical style: user and machine both contributing their part towards discovery ... show, in our opinion, that human reflection is still crucial in automatic discovery, but also that automatic discovery, through the proposed protocol, is already an enlightening tool for human understanding of geometric situations. It is by no means evident (at least in the etymological sense of obvious, immediate) to a trained human, the finding of some of the conditions that have mechanically appeared (performed with a laptop and a non-commercial software) in the Examples 5 or 6. On the other hand, Examples 7, 8 and 9 show how human cooperation is badly needed in some occasions (more or less trivial to overcome in the case of Example 7; more tangled—but also more rewarding, as the difficulties lead us to some unexpected positions for our query point—in the case of Example 8 or Example 9).

The above examples also show how the inclusion "a priori", as part of the hypotheses, of some non-degeneracy conditions, could help the discovery protocol we have presented. And, then, the discussion carried on Section 5 implies that this inclusion can be done through two (apparently similar) methods, but that the choice of one or the other has, sometimes, different consequences and require human decision. We think that this remark has been overlooked until now and we have presented examples using, randomly, one of the methods, and, occasionally, both of them, showing its crucial role in some occasions.

The framework for discovery that we have called **FSDIC** aimed to be a general approach towards discovery (stating in general terms what we wanted to achieve, rather that how we wanted to achieve it). It turned out that despite the fact that there are, if any, several possible **FSDIC**'s, its existence is equivalent to the existence of a very concrete couple of ideals, verifying some simple to check conditions, close to the ones presented in [21]. In this sense the present paper can be thought, after an exhaustive formalization and analysis, as a definitive "closure" of the main ideas originated there.

Appendix: Some Properties of the Saturation

Definition 2 I, J ideals of $K[X]$. Then $I : J = \{x, xJ \subset I\}$. The *saturation of I by J* is defined as $I : J^\infty = \cup_n(I : J^n)$. By abuse of notation, for a principal ideal $J = (f)$, $I : (f)^\infty$ will be denoted as $I : f^\infty$.

Remark 5 The saturation of I by J gives the intersection of all primary components Q associated to prime ideals of a minimal decomposition of I such that there is an f in J with f not in such primes, i.e. *the saturation of I by J is the intersection of the primary components associated to the primes such that J is not contained in them.* Clearly this implies that $\sqrt{I} : J^\infty = \sqrt{I} : J = \sqrt{I} : \sqrt{J} = \sqrt{I : J^\infty}$ (see also Lemma 3.3 of [1]).

Remark 6 Let X be a set of variables, U a subset of the X-variables and $Y = X \setminus U$. First we remark that, for any ideal J in $K[Y]$, its extension to $K[Y, U]$ is exactly the collection of polynomials that can be written as polynomials in the U-variables with coefficients in $JK[Y]$.

It follows that, given two ideals $J_1, J_2 \subset K[Y]$, the extension of their intersection is the intersection of their extension. It is also true that $(J_1 : J_2)K[Y, U] = (J_1K[U, Y] : J_2K[U, Y])$, i.e. the extension of the quotient of two ideals is the quotient of the extended ideals. Moreover, since $J_1 : J_2^\infty = J_1 : J_2^n$ for some n and $(J_1K[U, Y] : J_2K[U, Y]^\infty) = (J_1K[U, Y] : (J_2K[U, Y])^m)$ for some m, we have we $(J_1 : J_2^\infty)K[Y, U] = (J_1K[U, Y] : J_2K[U, Y]^\infty)$ taking the $\max\{n, m\}$.

The following properties have been used through the paper and are collected here, since for some of them is difficult to find precise references.

Proposition 5 *Let I, J be ideals of $K[X]$. Assume J is generated by $f_1, \ldots, f_r$. Then $(I : J^\infty) = \{x \in K[X], \forall f \in J, \exists n \geq 0, xf^n \in I\}$. Moreover, $(I : J^\infty) = \{x \in K[X], \forall l = 1, \ldots, r, \exists n \geq 0, x{f_l}^n \in I\}$.*

Proof By definition, if $x \in (I : J^\infty)$ then $\exists n, x \in (I : J^n)$. Obviously, for every $f \in J$, $f^n \in J^n$. Then $\forall f \in J$, $\exists n \geq 0$ (independent of f), such that $xf^n \in I$. The same applies for f_l.

Conversely, we want to prove that if x is such that $\forall f \in J$ there is some $n = n(f)$ with $xf^n \in I$, then $x \in (I : J^m)$ for some m. Since every element of J^m is a sum of products of m elements in J (in particular, a product of m generators f_l), if it is enough to prove that x times each one of these products lies in I. Consider the power m_l such that $x{f_l}^{m_l} \in I$ and let $m = \sum_l m_l$. Then, in the product of m factors of f_l, each f_l will be repeated α_l times, so that $\sum \alpha_l = m$. But this implies for some l, $\alpha_l \geq m_l$. Thus the multiplication by x of this product will lie in I. □

Proposition 6 *Let I and $J = (f_1, \ldots, f_r)$ be ideals in $K[X]$ and s an extra variable, then $I : J^\infty = (I^e + ((f_1s - 1) \cdots (f_rs - 1))) \cap K[X]$, where I^e denotes here the extended ideal $IK[X, s]$.*

Proof In fact $(I : J^\infty) = \bigcap_{i=1}^r (I : (f_i)^\infty)$, so if $f \in I : J^\infty$ then $f \in (I : (f_i)^\infty)$, i.e. $ff_i^{a_i} \in I$ for some a_i, hence $ff_i^{a_i}s^{a_i} \in I^e$. But $(1 - (sf_i)^{a_i}) = (1 - sf_i)(1 + sf_i + \cdots + (sf_i)^{a_i - 1})$, so $f \cdot \prod_{i=1}^r (1 - (sf_i)^{a_i}) \in ((f_1s - 1) \cdots (f_rs - 1))$. Performing the product in the first term of this equality, and noticing that it is f plus a collection of summands in which there is always a product $f_i^{a_i}s^{a_i}$ times f, we conclude that $f \in I^e + ((f_1s - 1) \cdots (f_rs - 1))$.

Viceversa, if $f \in I^e + ((f_1 s - 1) \cdots (f_r s - 1))$ and $f \in K[X]$, then $f = \sum_{j=1}^m p_j h_j + h$, where $h_j \in I \subseteq K[X]$, $p_j \in K[X, s]$ and $h \in ((f_1 s - 1) \cdots (f_r s - 1))$. Let $s = \frac{1}{f_i}$, then clearing denominators we have $f f_i^{a_i} \in I$ for any i. □

Corollary 2 *Let I and $J_i = (f_{i1}, \ldots, f_{il_i})$ be ideals in $K[X]$ for $i = 1, \ldots, r$ and let $s_1 \cdots s_r$ be some auxiliary, independent, variables. Then it holds that $I : (\prod_{1\ldots r} J_i)^\infty = (IK[X, s_1, \ldots, s_r] + ((f_{11}s_1 - 1) \cdots (f_{1l_1}s_1 - 1), \ldots, (f_{r1}s_r - 1) \cdots (f_{rl_r}s_r - 1))) \cap K[X]$.*

Proof It follows from Proposition 6 and from the previous remarks. □

Lemma 2 *Let J_1 be an ideal in $K[U, Y]$ and let J_2 an ideal in $K[U]$. Then $(J_1 : J_2K[U, Y]^\infty) \cap K[U] = (J_1 \cap K[U]) : J_2^\infty$.*

Proof Clearly if $f \in K[U]$ is such that $fb^n \in J_1$ for any $b \in J_2K[U, Y]$ and for some n, then $fb^n \in J_1 \cap K[U]$ for any $b \in J_2$. Viceversa, let $f \in (J_1 \cap K[U]) : J_2^\infty$. Let $b \in J_2K[U, Y]$, so $b = \sum c_i b_i$ with $b_i \in J_2K[U]$ and $c_i \in K[U, Y]$. Then $fb = \sum c_i b_i f \in J_1$. □

Lemma 3 *Let J_1, J_2, J_3 ideals in $K[X]$. Then*

$$(J_1 : J_2^\infty + J_3) : J_2^\infty = (J_1 + J_3) : J_2^\infty$$

Proof Clearly $(J_1 : J_2^\infty + J_3) : J_2^\infty \supseteq (J_1 + J_3) : J_2^\infty$. Let $f \in (J_1 : J_2^\infty + J_3) : J_2^\infty$ and let $b \in J_2$. We claim that there exists n such that $fb^n \in J_1 + J_3$. In fact there exists m such that $fb^m \in J_1 : J_2^\infty + J_3$, so $fb^m = s + c$, with $sb^k \in J_1$ for some k and $c \in J_3$. Therefore $fb^{m+k} \in J_1 + J_3$. □

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