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Semisymmetric graphs of order twice prime powers with the same prime valency

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Abstract
A simple undirected graph is said to be *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. Folkman (J Combin Theory Ser B 3, 215–232, 1967) proved that a graph of order $2p$ or $2p^2$ is not semisymmetric for any prime p. Wang and Guo (J Algebra Comb 54, 49–73, 2021) proved that there is only one semisymmetric graph of order $2p^3$ with valency p. In this paper, we give a necessary condition for semisymmetric graphs of order $2p^n$ with valency p, where p is an odd prime, and construct an infinite family of such graphs. As an application, a classification of semisymmetric graphs of order $2p^4$ with valency p is given.

Keywords Permutation group · Semisymmetric graph · Bi-coset graph

Mathematics Subject Classification 05C25 · 20B25.

1 Introduction

Throughout this paper, all graphs are finite, simple, connected and undirected. Let $X = (V(X), E(X))$ be a graph with vertex set $V(X)$ and edge set $E(X)$. For any vertex $v \in V(X)$, denote by $X_1(v)$ the set of vertices which are adjacent to v. The valency of v is the size of the set $X_1(v)$. A graph X is said to be *regular* if all the vertices have the same valency. A graph X is a bipartite graph if $V(X)$ can be partitioned into two subsets $U(X)$ and $W(X)$, called partite sets, such that every edge of X joins a

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vertex of $U(X)$ and a vertex of $W(X)$. Denote, by $K_{m,n}$, the complete bipartite graph with partite sets of size m and n, respectively.

Let X_1 and X_2 be graphs with vertex sets $V(X_1)$ and $V(X_2)$, respectively. An isomorphism σ from X_1 to X_2 is a bijection from $V(X_1)$ to $V(X_2)$ such that for any $u, v \in V(X_1)$, u and v are adjacent in X_1 if and only if u^σ and v^σ are adjacent in X_2. Two graphs X_1 and X_2 are said to be isomorphic, denoted by $X_1 \cong X_2$, if there is an isomorphism from X_1 to X_2. Let X be a graph, an automorphism of X is an isomorphism from X to itself. All automorphisms of X form a group under the composition of maps. This group is denoted by $\mathrm{Aut}(X)$ and is called the full automorphism group of X. The graph X is said to be *vertex-transitive* or *edge-transitive* if $\mathrm{Aut}(X)$ acts transitively on $V(X)$ or on $E(X)$, respectively. It is well known that a connected graph, which is edge-transitive but not vertex-transitive, is bipartite (see [9]). The complete bipartite graph $K_{m,n}$, where $m \neq n$, is a simple example of such graphs. But it is interesting to find such regular graphs. A graph X is said to be *semisymmetric* if it is regular and edge-transitive but not vertex-transitive.

Let X be a bipartite graph with the bipartition $V(X) = U(X) \cup W(X)$ and $A = \mathrm{Aut}(X)$. Suppose that A^+ is the subgroup of A preserving both $U(X)$ and $W(X)$. The connectedness of the graph X implies that either $|A : A^+| = 2$ or $A = A^+$, depending on whether or not there exists an automorphism, which interchanges $U(X)$ and $W(X)$. For $G \leq A^+$, X is said to be *G-semitransitive* if G acts transitively on both $U(X)$ and $W(X)$, while an A^+-semitransitive graph is simply called *semitransitive*. It can be checked easily that every semisymmetric graph is a semitransitive bipartite graph with two partite sets having the same size. Thus, the order of a semisymmetric graph must be even.

Semisymmetric graphs were first investigated by Folkman [9] in 1967. He gave some characterizations of semisymmetric graphs and constructed several infinite families of such graphs. Meanwhile, Folkman put forward 8 open problems, which spurred the interest in this topic. Whereafter, many semisymmetric graphs were constructed which nearly solved all Folkman's open problems (see [2, 3, 8, 14, 15]). More recently, some new results on semisymmetric graphs have appeared by some group-theoretical methods, graph coverings and computer searching (see [4, 6, 10, 12, 18, 19, 25]).

One of Folkman's problems is "for which pairs of integers v and d is there a connected semisymmetric graph with v vertices and d valency?" In response to this problem, Parker [20] studied semisymmetric cubic graphs of twice odd order. Li and Lu [16] classified pentavalent semisymmetric graphs of square-free order. But there are a few known semisymmetric graphs with twice prime powers vertices. Let p be a prime, Folkman [9] proved that there is no semisymmetric graph of order $2p$ or $2p^2$. Malnič, et al. [17] showed that cubic semisymmetric graph of order $2p^3$ is the Gray graph. Du and Wang et al. [7, 21–24] gave a partial classification of semisymmetric graphs of order $2p^3$. Especially, semisymmetric graphs of order $2p^3$ with prime valency have been completely classified [24]. As a natural process, for any pair of positive integer n and prime p, whether there are semisymmetric graphs of order $2p^n$ with valency p is an interesting problem. In this paper, we give a necessary condition for semisymmetric graphs of order $2p^n$ with valency p. Applying this condition, an infinite family of such graphs is constructed and semisymmetric graphs of order $2p^4$ with valency p are classified. Note that when $p = 2$, the graph of order $2p^n$ with valency p is the cycle graph and vertex-transitive. It is also known that there is no semisymmetric graph of order $2p$ or $2p^2$. Throughout this paper, let p be an odd prime and $n \geq 3$. The main results in this paper are listed in the following three theorems.

Theorem 1.1 *Let p be an odd prime, $n \geq 3$ and X be a semisymmetric graph of order $2p^n$ with valency p. Let P be a Sylow p-subgroup of $\mathrm{Aut}(X)$. If $\mathrm{Aut}(X)$ acts primitively on one partite set at least, then $p \mid n$ or P contains an elementary abelian group as a maximal subgroup.*

Theorem 1.2 *Let p be an odd prime, $n \geq 3$ and X be a semisymmetric graph of order $2p^n$ with valency p. Let P be a Sylow p-subgroup of $\mathrm{Aut}(X)$. If P contains an elementary abelian group as a maximal subgroup, then $n \leq p$ and X is isomorphic to the graph $X_{p,n}$, which is described in Construction 2.6.*

It was shown that there is only one semisymmetric graph of order $2p^3$ with valency p. The graph $X_{p,3}$ is just such the graph [24]. Moreover, $X_{3,3}$ is the Gray graph and $\mathrm{Aut}(X_{3,3})$ acts primitively on one partite set and imprimitively on the other. If $p \geq 5$, $\mathrm{Aut}(X_{p,3})$ acts imprimitively on both partite sets.

Theorem 1.3 *Let $p \geq 5$ be a prime and X be a semisymmetric graph of order $2p^4$ with valency p. Then, X is isomorphic to the graph $X_{p,4}$ or $\widetilde{X}_{p,4}$, which are described in Constructions 2.6 and 2.7, respectively.*

2 Preliminaries

Denote by $\mathbb{Z}_n$ the cyclic group of order n, and by A_n and S_n the alternating group and the symmetric group of degree n, respectively. An elementary abelian p-group is a direct product of several cyclic groups of order p, where p is a prime. For a transitive group G on Ω and a subset Ω_1 of Ω, denote by G_{Ω_1} and $G_{(\Omega_1)}$ the setwise stabilizer and the pointwise stabilizer of G relative to Ω_1, respectively. For a group G and a subgroup H of G, we use $Z(G)$, $C_G(H)$ and $N_G(H)$ to denote the center of G, the centralizer and the normalizer of H in G, respectively. Denote by $[G : H]$ and $|G : H|$ the set of right cosets and the index of H in G, respectively. The action of G on $[G : H]$ is always assumed to be the right multiplication action. For a group G, denote by G' the derived group of G. A semidirect product of a group N by a group H is denoted by $N \rtimes H$, where N is normal in $N \rtimes H$. For any two sets A and B, denote by $A \backslash B = \{a \mid a \in A, a \notin B\}$ the difference set of A and B. For group-theoretical concepts and notations not defined here, the reader is refereed to [5, 13].

Let G be a group with subgroups L and R and $D = \bigcup_i Rg_iL$ be a union of double cosets of R and L in G, where $g_i \in G$. Define the bipartite graph $X = \mathbf{B}(G, L, R; D)$ with bipartition $V(X) = [G : L] \cup [G : R]$ and edge set $E(X) = \{\{Lg_1, Rg_2\} \mid g_1, g_2 \in G, g_2g_1^{-1} \in D\}$. This graph is called the bi-coset graph of G with respect to L, R and D. Clearly, the graph $X = \mathbf{B}(G, L, R; D)$ is well-defined, i.e., the adjacency relation is independent of the choice of representatives of right cosets. Under the right multiplication action of G on $V(X)$, the graph X is G-semitransitive.

Proposition 2.1 *[8] Let $X = \mathbf{B}(G, L, R; D)$ be the bi-coset graph of G with respect to L, R and D. Then,*

(i) X is G-edge-transitive if and only if D is a single double coset of R and L in G, i.e., $D = RgL$ for some $g \in G$;
(ii) the valency of any vertex in $[G : L]$ (resp. $[G : R]$) is equal to the number of right cosets of R (resp. L) in D (resp. D^{-1}), so X is regular if and only if $|L| = |R|$;
(iii) X is connected if and only if G is generated by $D^{-1}D$;
(iv) $X \cong \mathbf{B}(G, L^a, R^b; D')$, where $D' = \bigcup_i R^b(b^{-1}d_i a)L^a$, for any $a, b \in G$, and $d_i \in D$;
(v) $X \cong \mathbf{B}(\widetilde{G}, L^\sigma, R^\sigma; D^\sigma)$, where $\widetilde{G}$ is a group and σ is an isomorphism from G to $\widetilde{G}$.

Proposition 2.2 *[8] Suppose that X is a G-semitransitive graph with $V(X) = U(X) \cup W(X)$. Take u in $U(X)$, w in $W(X)$ and set $D = \{g \in G \mid w^g \in X_1(u)\}$. Then D is a union of double cosets of G_u and G_w in G, and $X \cong \mathbf{B}(G, G_u, G_w; D)$. Especially, any semisymmetric graph is a bi-coset graph.*

In the following, a sufficient condition for judging the vertex transitivity of an edge-transitive bi-coset graph is given.

Lemma 2.3 *Let $X = \mathbf{B}(G, L, R; D)$ be a bi-coset graph, where $D = RgL$ for some $g \in G$. If there exists $\sigma \in \mathrm{Aut}(G)$ such that $L^\sigma = R$, $R^\sigma = L$ and $(D^{-1})^\sigma = D$, then X is vertex-transitive.*

Proof Set $[G : L] = \{Lg_l \mid g_l \in G\}$ and $[G : R] = \{Rg_r \mid g_r \in G\}$. By definition of a bi-coset graph, $V(X) = [G : L] \cup [G : R]$. Let $\overline{\sigma}$ induced by σ be a mapping on $V(X)$ as follows:

$$(Lg_l)^{\overline{\sigma}} = Rg_l^\sigma \text{ and } (Rg_r)^{\overline{\sigma}} = Lg_r^\sigma,$$

where $g_l, g_r \in G$. We claim that $\overline{\sigma} \in \mathrm{Aut}(X)$.

Firstly, we prove that $\overline{\sigma}$ is a permutation on $V(X)$. It is clear that $\overline{\sigma}$ is a surjection. For any $g_l, g_r \in G$,

$$Rg_{l_1}^\sigma = Rg_{l_2}^\sigma \Leftrightarrow (g_{l_1})^\sigma (g_{l_2}^\sigma)^{-1} \in R \Leftrightarrow (g_{l_1} g_{l_2}^{-1})^\sigma \in R \Leftrightarrow g_{l_1} g_{l_2}^{-1} \in R^{\sigma^{-1}} = L \Leftrightarrow Lg_{l_1} = Lg_{l_2}.$$

Secondly, we prove that $\overline{\sigma}$ preserves edge set. Since $(D^{-1})^\sigma = D$, it follows that $g_l^\sigma (g_r^\sigma)^{-1} = (g_l g_r^{-1})^\sigma = ((g_r g_l^{-1})^{-1})^\sigma \in D$, for any $g_r g_l^{-1} \in D$. Therefore, for any $\{Lg_l, Rg_r\} \in E(X)$, $\{Lg_l, Rg_r\}^{\overline{\sigma}} = \{Rg_l^\sigma, Lg_r^\sigma\} = \{Lg_r^\sigma, Rg_l^\sigma\} \in E(X)$. Thus $\overline{\sigma} \in \mathrm{Aut}(X)$.

Since G acts transitively on both $[G : L]$ and $[G : R]$, respectively, $\langle G, \overline{\sigma} \rangle$ acts transitively on $V(X)$. Thus, X is vertex-transitive. □

The following lemma gives some sufficient conditions for a regular bipartite edge-transitive graph to be semisymmetric.

Lemma 2.4 *Let X be a regular edge-transitive bipartite graph with $V(X) = U(X) \cup W(X)$, and P be a Sylow subgroup of $\mathrm{Aut}(X)$. Then, the graph X is semisymmetric if P satisfies one of the following conditions:*

(i) $P_u \ncong P_w$, *for any* $u \in U(X)$ *and* $w \in W(X)$;
(ii) $N_P(P_u) \ncong N_P(P_w)$, *especially,* $|N_P(P_u)| \neq |N_P(P_w)|$, *for any* $u \in U(X)$ *and* $w \in W(X)$;
(iii) $C_P(P_u) \ncong C_P(P_w)$, *especially,* $|C_P(P_u)| \neq |C_P(P_w)|$, *for any* $u \in U(X)$ *and* $w \in W(X)$.

Proof In the following, we prove the result by a contradiction. Suppose that the graph is vertex-transitive. Then, there exists $\sigma \in \mathrm{Aut}(X)$ such that $A_u^\sigma = A_w$, where $u \in U(X)$ and $w \in W(X)$. Now $P_u = P \cap A_u$ and $P_w = P \cap A_w$ are Sylow subgroups of A_u and A_w, respectively. Then, $P_u^\sigma \leq A_w$ and so $P_u^\sigma = P_w^\eta$ for some $\eta \in A_w$. Therefore, $\tau = \sigma\eta^{-1} \in \mathrm{Aut}(X)$ satisfies $P_u^\tau = P_w$. Thus, $N_A(P_u)^\tau = N_A(P_w)$ and $C_A(P_u)^\tau = C_A(P_w)$. Note that $P \cap N_A(P_u) = N_P(P_u)$ and $P \cap N_A(P_w) = N_P(P_w)$ are Sylow subgroups of $N_A(P_u)$ and $N_A(P_w)$, respectively. Then, $N_P(P_u)$ and $N_P(P_w)$ are conjugate in $\mathrm{Aut}(X)$. Similarly, $C_P(P_u)$ and $C_P(P_w)$ are conjugate in $\mathrm{Aut}(X)$. Especially, $|N_P(P_u)| = |N_P(P_w)|$ and $|C_P(P_u)| = |C_P(P_w)|$. Therefore, if P satisfies one of the conditions (i), (ii) or (iii), the graph X is not vertex-transitive. This implies that X is a semisymmetric graph. □

Since any semisymmetric graph is a bi-coset graph, in order to study semisymmetric graphs of twice prime powers order, groups with subgroups of prime powers index are needed. The following proposition is the classification of nonabelian simple groups with a subgroup of prime powers index.

Proposition 2.5 *[11] Let S be a nonabelian simple group with a subgroup $H < S$ satisfying $|S : H| = p^a$, for p a prime. Then one of the following holds:*

(i) $S = A_n$ *and* $H = A_{n-1}$ *with* $n = p^a$;
(ii) $S = \mathrm{PSL}(n, q)$, *H is the stabilizer of a projective point or a hyperplane in* $\mathrm{PG}(n-1, q)$ *and* $|S : H| = (q^n - 1)/(q - 1) = p^a$;
(iii) $S = \mathrm{PSL}(2, 11)$ *and* $H = A_5$;
(iv) $S = M_{11}$ *and* $H = M_{10}$;
(v) $S = M_{23}$ *and* $H = M_{22}$;
(vi) $S = \mathrm{PSU}(4, 2)$ *and H is a subgroup of index 27.*

At the end of this section, we construct two families of bi-coset graphs.

Construction 2.6 *Let $p \geq 5$ be a prime and n be a positive integer with $n \leq p$. Define the group P by*

$$P = \langle x_1, x_2, \cdots, x_n, y \mid x_1^p = \cdots = x_n^p = y^p = 1, [x_i, y] = x_{i+1}, [x_n, y] = [x_j, x_i] = 1\rangle,$$

$i = 1, \cdots, n-1$, $j = 1, 2, \cdots, n$. *Then, the order of P is p^{n+1}. Define the bi-coset Graph* $X_{p,n} = \mathbf{B}(P, \langle y\rangle, \langle x_1\rangle; \langle x_1\rangle\langle y\rangle)$.

Construction 2.7 *Let $p \geq 5$ be a prime. Define the group Q by*

$$Q = \langle a, b, c, d, e | a^p = b^p = c^p = d^p = e^p = 1, [b, a] = c, [c, a] = d, [d, a] = [c, b] = e, [d, b] = [d, c] = [e, a] = [e, b] = [e, c] = [e, d] = 1 \rangle.$$

Define the bi-coset Graph $\widetilde{X}_{p,4} = \mathbf{B}(Q, \langle a \rangle, \langle b \rangle; \langle b \rangle \langle a \rangle)$.

3 Some results on *p*-groups

Lemma 3.1 *Let p be a prime, n be a positive integer, and let P be a nonabelian group of order p^{n+1}, which is generated by two elements of order p. If P contains a maximal subgroup H which is an elementary abelian p-group, then*

(i) *$n \leq p$, and*

$$P = \langle x_1, x_2, \cdots, x_n, y \mid x_1^p = \cdots = x_n^p = y^p = 1, [x_i, y] = x_{i+1}, [x_n, y] = [x_j, x_i] = 1 \rangle$$

where $i = 1, \cdots, n-1$ and $j = 1, \cdots, n$;

(ii) *$C_P(y) = Z(P) \times \langle y \rangle \cong \mathbb{Z}_p^2$ and $C_P(h) = H \cong \mathbb{Z}_p^n$, for any $h \in H \backslash Z(P)$;*

(iii) *For any $g = x_1^{i_1} \cdots x_n^{i_n} y^{i_{n+1}} \in P$ with $0 \leq i_1, \cdots, i_n, i_{n+1} \leq p-1$, $o(g) = p^2$ if and only if $n = p$ and $i_1 i_{n+1} \neq 0$.*

Proof (i) Let $P = \langle x, y \rangle$ with $o(x) = o(y) = p$. Since H is a maximal subgroup of P, it follows that $H \trianglelefteq P$, and at least one of x, y is not in H. Without loss of generality, we assume that $y \notin H$. Then, $P = H \rtimes \langle y \rangle$. Since $x \in P$ and $x \neq y$, there exists $h \in H$ such that $x = hy^i$ for some $i \in \{0, 1 \cdots, p-1\}$. Since $P = \langle hy^i, y \rangle = \langle h, y \rangle$ and $o(h) = p$, without loss of generality, let $x = h \in H$. Note that P is a nonabelian group, then $[x, y] \neq 1$, that is $x^y \neq x$. Thus, $\langle x^{H\langle y \rangle} \rangle = \langle x^{\langle y \rangle} \rangle \leq H$.

For any $g \in P = \langle x, y \rangle$, we have $g = x^{i_1} y^{j_1} x^{i_2} y^{j_2} \cdots x^{i_s} y^{j_s}$, where $i_1, i_2, \cdots, i_s, j_1, j_2, \cdots, j_s \in \{0, 1, \cdots, p-1\}$. Then there exist integers $k_1, k_2, \cdots, k_{s+1}$ such that

$$g = y^{k_1} (x^{i_1})^{y^{k_2}} \cdots (x^{i_s})^{y^{k_{s+1}}} \in \langle y \rangle \langle x^{\langle y \rangle} \rangle.$$

Therefore, $P = \langle y \rangle \langle x^{\langle y \rangle} \rangle = \langle y \rangle H$. It is clear that $\langle y \rangle \cap H = 1$ and $\langle y \rangle \cap \langle x^{\langle y \rangle} \rangle = 1$. Thus, $|\langle x^{\langle y \rangle} \rangle| = |H|$. Since $\langle x^{\langle y \rangle} \rangle \leq H$, it follows that $H = \langle x^{\langle y \rangle} \rangle$. Since $|\langle y \rangle| = p$, we have $|\langle x^{\langle y \rangle} \rangle| \leq p^p$. But $|\langle x^{\langle y \rangle} \rangle| = |H| = p^n$, which implies that $n \leq p$.

For any $g \in P$, we have $g = x^{i_0} (x^y)^{i_1} \cdots (x^{y^{p-1}})^{i_{p-1}} y^{i_p}$. From

$$g^y = (x^{i_0} (x^y)^{i_1} \cdots (x^{y^{p-1}})^{i_{p-1}} y^{i_p})^y = x^{i_{p-1}} (x^y)^{i_0} (x^{y^2})^{i_1} \cdots (x^{y^{p-1}})^{i_{p-2}} y^{i_p},$$

we know that $g \in C_P(y)$ if and only if $i_0 = i_1 = \cdots = i_{p-1}$. Hence, $C_P(y) = \langle xx^y \cdots x^{y^{p-1}} \rangle \times \langle y \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $Z(P) = \langle xx^y \cdots x^{y^{p-1}} \rangle \cong \mathbb{Z}_p$. It is clear that $C_P(C_P(y)) = C_P(y)$. Thus, P is a p-group of maximal class. Since $|P| = p^{n+1}$, the nilpotent class of P is n.

Suppose that $x_1 = x$, $x_{i+1} = [x_i, y]$ with $i = 1, \cdots, n-1$. Since the nilpotent class of the group P is n, one has $[x_n, y] = 1$. Clearly, $x_i \in H$, for any $i \in \{1, 2, \cdots, n\}$. Then, $\langle x_1, x_2, \cdots, x_n, y\rangle \leq P$. It is clear that $|\langle x_1, x_2, \cdots, x_n, y\rangle| = p^{n+1} = |P|$. Thus,

$$P = \langle x_1, \cdots, x_n, y \mid x_1^p = \cdots = x_n^p = y^p = 1, [x_i, y] = x_{i+1}, [x_n, y] = [x_j, x_i] = 1\rangle,$$

where $i = 1, \cdots, n-1$, $j = 1, \cdots, n$.

(ii) We have already shown $C_P(y) = \mathbb{Z}_p^2$ in (i). Since H is an elementary abelian p-group, it is clear that $H \subseteq C_P(h)$ for any $h \in H \setminus Z(P)$. From $[h, y] \neq 1$ and H is a maximal subgroup of P, we have $C_P(h) = H$.

(iii) For $j \in \{1, 2, \cdots, p-1\}$, we show first that $x_1^{y^j} = \prod_{k=1}^{j+1} x_k^{C_j^{k-1}}$, where C_j^{k-1} is the binomial coefficient, by induction on j. It is clear that $x_1^y = x_1x_2$, so the statement is true when $j = 1$. Suppose that it is true for $j = m$. Now let $j = m+1$, we have

$$\begin{aligned} x_1^{y^{m+1}} &= (x_1^{y^m})^y = \left(\prod_{k=1}^{m+1} x_k^{C_m^{k-1}}\right)^y = \prod_{k=1}^{m+1} (x_k^y)^{C_m^{k-1}} = \prod_{k=1}^{m+1} (x_kx_{k+1})^{C_m^{k-1}} \\ &= x_1x_{m+2}\prod_{k=2}^{m+1} x_k^{C_m^{k-2}+C_m^{k-1}} = x_1x_{m+2}\prod_{k=1}^{m+1} x_k^{C_{m+1}^{k-1}} \\ &= \prod_{k=1}^{m+2} x_k^{C_{m+1}^{k-1}}, \end{aligned}$$

so the statement is true for $j = m+1$. By induction principle, we have $x_1^{y^j} = \prod_{k=1}^{j+1} x_k^{C_j^{k-1}}$ for $j \in \{1, 2, \cdots, p-1\}$.

It can be checked easily that the derived group P' of P is $\langle x_2\rangle \times \cdots \times \langle x_n\rangle$. If $n \leq p-1$, then $|P| \leq p^p$. Consequently, P is a regular p-group. For any $g = x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y^{i_{n+1}} \in P$, one get that

$$g^p = (x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y^{i_{n+1}})^p = x_1^{pi_1}x_2^{pi_2}\cdots x_n^{pi_n}y^{pi_{n+1}}d_1^pd_2^p\cdots d_s^p,$$

for some $d_i \in P'$, $i = 1, 2, \cdots, s$. Since P' is an elementary abelian p-subgroup, we have $g^p = 1$ which follows $o(g) = p$ for any $1 \neq g \in P$.

Now suppose that $n = p$. Note that $P' \trianglelefteq P$, it follows that $P'\langle y\rangle \leq P$ and $|P'\langle y\rangle| = p^n = p^p$. Thus, $P'\langle y\rangle$ is a regular p-group. Similarly, we have $g_1^p = 1$ for any $g_1 \in P'\langle y\rangle$. For any $g = x_1^{j_1}x_2^{j_2}\cdots x_p^{j_p}y^{j_{p+1}} \in P$, where $j_1j_{p+1} \neq 0$, let $g_1 = x_2^{j_2}\cdots x_p^{j_p}y^{j_{p+1}}$, then $g_1 \in P'\langle y\rangle$. Therefore,

$$\begin{aligned} 1 = g_1^p &= (x_2^{j_2}\cdots x_p^{j_p}y^{j_{p+1}})^p = (x_2^{j_2}\cdots x_p^{j_p})^{1+y^{j_{p+1}}+y^{2j_{p+1}}+\cdots+y^{(p-1)j_{p+1}}} \\ &= (x_2^{j_2}\cdots x_p^{j_p})^{\sum_{l=0}^{p-1}(y^{j_{p+1}})^l}. \end{aligned}$$

Thus,

$$\begin{aligned} g^p &= (x_1^{j_1} x_2^{j_2} \cdots x_p^{j_p} y^{j_{p+1}})^p = (x_1^{j_1} x_2^{j_2} \cdots x_p^{j_p})^{\sum_{l=0}^{p-1} (y^{j_{p+1}})^l} \\ &= (x_1^{j_1})^{\sum_{l=0}^{p-1} (y^{j_{p+1}})^l} (x_2^{j_2} \cdots x_p^{j_p})^{\sum_{l=0}^{p-1} (y^{j_{p+1}})^l} \\ &= (x_1^{j_1})^{\sum_{l=0}^{p-1} (y^{j_{p+1}})^l} \\ &= \prod_{k=0}^{p-1} \left(\prod_{i=1}^{k+1} x_i^{C_k^{i-1}} \right) = \prod_{i=1}^{p} x_i^{i_1 \sum_{j=1}^{p-i+1} C_j^{i-1}} \\ &= x_p^{j_1} \prod_{i=1}^{p-1} x_i^{j_1 \sum_{k=1}^{p-i+1} C_k^{i-1}} \end{aligned} .$$

Note that $o\left(x_p \prod_{i=1}^{p-1} x_i^{\sum_{j=1}^{p-i+1} C_k^{i-1}} \right) = p$ and $j_1 \neq 0$, it follows that $o(g^p) = p$. Therefore, $o(g) = p^2$. □

Lemma 3.2 *Let* $3 \leq n \leq p$ *and*

$$P = \langle x_1, x_2, \cdots, x_n, y \mid x_1^p = \cdots = x_n^p = y^p = 1, [x_i, y] = x_{i+1}, [x_n, y] = [x_j, x_i] = 1 \rangle,$$

(i) *If* $n = p$, *then every automorphism of* P *is of the form*

$$\varphi : x_1 \mapsto x_1^{i_1} x_2^{i_2} \cdots x_p^{i_p}, \quad y \mapsto x_2^{j_2} \cdots x_p^{j_p} y^{j_{p+1}},$$

where $0 \leq i_l, j_k \leq p-1$, $1 \leq l \leq p, 2 \leq k \leq p+1$, *and* $i_1, j_{p+1} \neq 0$. *Hence,* $|\mathrm{Aut}(P)| = p^{2p-2}(p-1)^2$.

(ii) *If* $n < p$, *then every automorphism of* P *is of the form*

$$\phi : x_1 \mapsto x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \quad y \mapsto x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} y^{j_{n+1}},$$

where $0 \leq i_l, j_k \leq p-1$, $1 \leq l \leq n, 1 \leq k \leq n+1$, *and* $i_1, j_{n+1} \neq 0$. *Hence,* $|\mathrm{Aut}(P)| = p^{2n-3}(p-1)^2$.

Proof It is clear that P can be generated by x_1 and y, so it is only need to give the images of x_1 and y for any automorphism of P. Define the following maps on P via the generators x_1 and y and preserving multiplications by

$$\begin{aligned} &\sigma_i : x_1 \mapsto x_1 x_i, \quad y \mapsto y; \quad &&\rho_i : x_1 \mapsto x_1, \quad y \mapsto x_i y; \\ &\mu : x_1 \mapsto x_1^2, \quad y \mapsto y; \quad &&\tau : x_1 \mapsto x_1, \quad y \mapsto y^2; \end{aligned}$$

where $i = 2, \cdots, n$. It can be checked easily that all of them are automorphisms of the group P.

(i) Suppose that $n = p$. In the follows, we will show that $\mathrm{Aut}(P) = \langle \sigma_i, \rho_i, \mu, \tau \rangle$.

If $i_1 i_{p+1} \neq 0$, then from Lemma 3.1 we know that $o(x_1^{i_1} x_2^{i_2} \cdots x_p^{i_p} y^{i_{p+1}}) = p^2$. Thus, there is no φ in $\mathrm{Aut}(P)$ such that $x_1^{\varphi} = x_1^{i_1} x_2^{i_2} \cdots x_p^{i_p} y^{i_{p+1}}$ or $y^{\varphi} =$

$x_1^{i_1}x_2^{i_2}\cdots x_p^{i_p}y^{i_{p+1}}$, for any $x_1^{i_1}x_2^{i_2}\cdots x_p^{i_p}y^{i_{p+1}} \in P$ and $i_1 i_{p+1} \neq 0$. Since $C_P(h) = H$ for any $h \in H\backslash Z(P)$, and $C_P(y) = \mathbb{Z}_p^2$, there is no φ in $\mathrm{Aut}(P)$ such that $y^\varphi \in H$. Thus, under the automorphisms of P, y might map to an element of the form $x_2^{j_2}\cdots x_p^{j_p}y^{j_{p+1}}$. By $\rho_i, \tau \in \mathrm{Aut}(G)$, it follows

$$y^{\langle\rho_i,\tau\rangle} = \{x_2^{j_2}\cdots x_p^{j_p}y^{j_{p+1}} \mid 0 \leq j_k \leq p-1, k = 2, \cdots, p+1, j_{p+1} \neq 0\} = y^{\mathrm{Aut(P)}}.$$

Note that $P' = \langle x_2, x_3, \cdots, x_p\rangle$ and $P'\langle y\rangle < P$, we have $x_1^\varphi \neq x_2^{i_2}\cdots x_p^{i_p}$ for any $\varphi \in \mathrm{Aut}(P)$, since otherwise, $\langle x_1^\varphi, y^\varphi\rangle < P$. There is no φ in $\mathrm{Aut}(P)$ such that $x_1^\varphi = y$, because of $|C_P(x_1)| \neq |C_P(y)|$ from Lemma 3.1. Under the automorphism of P, x_1 might map to an element of the form $x_1^{i_1}x_2^{i_2}\cdots x_p^{i_p}$ with $i_1 \neq 0$. Note that $\sigma_i, \mu \in \mathrm{Aut}(P)$, it follows

$$x_1^{\langle\sigma_i,\mu\rangle} = \{x_1^{i_1}x_2^{i_2}\cdots x_p^{i_p} \mid 0 \leq i_l \leq p-1,\ l = 1, 2, \cdots, p, i_1 \neq 0\} = x_1^{\mathrm{Aut}(P)}.$$

Therefore, $\mathrm{Aut}(P) = \langle\sigma_i, \rho_i, \mu, \tau\rangle$, and hence $|\mathrm{Aut}(P)| = p^{2p-2}(p-1)^2$.

(ii) Suppose that $n < p$. Similarly, the map π on P defined via the generators x_1 and y and preserving multiplications by

$$\pi : x_1 \mapsto x_1, \quad y \mapsto x_1 y,$$

is an automorphism of the group P. In the follows, we will show $\mathrm{Aut}(P) = \langle\sigma_i, \rho_i, \mu, \tau, \pi\rangle$. Note that $C_P(y) = \mathbb{Z}_p \times \mathbb{Z}_p$ and $C_P(h) = H$, for any $h \in H\backslash Z(P)$. Since $n \geq 3$, it follows that $|H| \geq p^3$ and $p^2 = |C_P(y)| \neq |H|$. Thus, $y^\phi \notin H$ for any $\phi \in \mathrm{Aut}(P)$. Therefore, under the automorphism of P, y might map to an element of the form $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y^{j_{n+1}}$ with $j_{n+1} \neq 0$. By $\rho_j, \tau, \pi \in \mathrm{Aut}(P)$, we have

$$y^{\langle\rho_j,\tau,\pi\rangle} = \{x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}y^{j_{n+1}} \mid 0 \leq j_k \leq p-1, k = 1, \cdots, n+1, j_{n+1} \neq 0\} = y^{\mathrm{Aut}(P)}.$$

Note that $y^\phi \notin H$, for any $\phi \in \mathrm{Aut}(P)$. Since there exists ϕ in $\mathrm{Aut}(P)$ such that $y^\phi = x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y^{i_{n+1}}$ where $0 \leq i_k \leq p-1, k = 1, 2, \cdots, n+1$ and $i_{n+1} \neq 0$, it follows that $x_1^\phi \neq x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y^{i_{n+1}}$ for any $\phi \in \mathrm{Aut}(P)$. Since P' is the characteristic subgroup of P and $x_1 \notin P'$, it follows that $x_1^\phi \notin P'$ for any $\phi \in \mathrm{Aut}(P)$. Thus, under the automorphisms of P, x_1 might map to an element of the form $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ with $i_1 \neq 0$. Note that $\sigma_i, \mu \in \mathrm{Aut}(P)$, then

$$x_1^{\langle\sigma_i,\mu\rangle} = \{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \mid i_k = 0, 1, \cdots, p-1,\ k = 1, 2, \cdots n, i_1 \neq 0\} = x_1^{\mathrm{Aut}(P)}.$$

Therefore, $\mathrm{Aut}(P) = \langle\sigma_i, \rho_i, \mu, \tau, \pi\rangle$, and hence $|\mathrm{Aut}(P)| = p^{2n-3}(p-1)^2$. □

4 Semisymmetric graphs of order $2p^n$ with valency p

In this section, let X be a p-valent semisymmetric graph with bipartitions $V(X) = U(X) \cup W(X)$ of order $2p^n$, $A = \mathrm{Aut}(X)$, and P be a Sylow p-subgroup of $\mathrm{Aut}(X)$.

Lemma 4.1 *Let X be a semisymmetric graph with prime valency. Then* $\mathrm{Aut}(X)$ *is faithful on both partite sets of X.*

Proof We will prove the result by contradiction. Assume that $\mathrm{Aut}(X)$ is unfaithful on at least one partite set of X. Without loss of generality, suppose that $\mathrm{Aut}(X)$ is unfaithful on $U(X)$. Let K be the kernel of A on $U(X)$. In this case, $K \neq 1$ and K is not transitive on $W(X)$, since otherwise the graph is the complete bipartite graph. Now let $\mathcal{B}_{\mathcal{W}} = \{B_1, B_2, \cdots, B_m\}$ be the complete imprimitive block system induced by K, where $m \bigm| |W(X)|$. For any $u \in U(X)$, if $w \in X_1(u)$, then $w^K \subseteq X_1(u)$. Since $|X_1(u) \cap B_j| \bigm| |X_1(u)|$ and $|X_1(u)|$ is a prime, we have $|X_1(u) \cap B_j| = |X_1(u)|$ or 1. If $|X_1(u) \cap B_j| = |X_1(u)|$, then $X_1(u) \subseteq B_j$ and the graph is unconnected. Thus, $|X_1(u) \cap B_j| = 1$. Since B_j is an orbit of K on $W(X)$, it follows that $w^K = w$, where $w \in X_1(u)$. According to the arbitrariness of u, this implies that K fixes any vertex in $W(X)$. Thus, K fixes any vertex in $V(X)$. This contradicts with the faithfulness of A on $V(X)$. The result then follows. □

Lemma 4.2 *Let X be a semisymmetric graph of order $2p^n$ with valency p and P be a Sylow p-subgroup of A, where $n \geq 3$, and p is an odd prime. Then, $|P| = p^{n+1}$.*

Proof Since A is transitive on both $U(X)$ and $W(X)$, $A = A_u P = A_w P$ for any $u \in U(X)$ and $w \in W(X)$. By Frattini's argument, P is transitive on both $U(X)$ and $W(X)$, and it needs only to show that $|P_u| = p$ for some $u \in U(X)$. Since the bipartite graph X is edge-transitive, one has that P_u is transitive on $X_1(u)$. In other words, we need only to prove that P_u is regular on $X_1(u)$. For $w \in X_1(u)$, if $g \in P_u \cap P_w$, then g fixes every vertex in $X_1(u)$ because of $|X_1(u)| = p$. Since $|X_1(w)| = p$ and $(u, w) \in E(X)$, g fixes every vertex in $X_1(u) \cup X_1(w)$. By the connectivity of X, we can get that g fixes every vertex in $V(X)$. This forces $g = 1$. Consequently, $|P_u| = p$ and $|P| = p^{n+1}$. □

Proof of Theorem 1.1 Let X be a semisymmetric graph of order $2p^n$ with valency p. Set $V(X) = U(X) \cup W(X)$. In [21], the conclusion has been proved to be correct when $n = 3$. Next, let $n \geq 4$. Without loss of generality, suppose that A acts primitively on $U(X)$. From O'Nan-Scott Theorem [5], A is of almost simple type, product type or affine group type.

(1) We claim that A cannot be of almost simple type. Conversely, suppose that A is of almost simple type, i.e., the socal S of A is a nonabelian simple group. By Proposition 2.5, S is A_{p^n} or $\mathrm{PSL}(m, q)$, where $\frac{q^m-1}{q-1} = p^n$ and $q = p_1^l$ for some prime p_1 and positive integer l. We distinguish the following two cases.

(a) A is primitive on $W(X)$.

Assume that two representations of S on both partite sets are equivalent. Consider the action of S_u on $[S : S_w]$, where $u \in U(X)$ and $w \in W(X)$. Then, the lengths of the orbits are 1 and $p^n - 1$, respectively. It is impossible as $p^n - 1 \neq p$. Now Assume that two representations of S on the partite sets are not equivalent. Then, $S = \mathrm{PSL}(m, q)$ and $\frac{q^m-1}{q-1} = p^n$. Consider the action of S_u on $[S : S_w]$, there are two orbits with different lengths $\frac{q^{m-1}-1}{q-1}$ and q^{m-1}, respectively. If $\frac{q^{m-1}-1}{q-1} = p$, then

$$q^{m-1} = \frac{q^m - 1}{q - 1} - \frac{q^{m-1} - 1}{q - 1} = p^n - p = p(p^{n-1} - 1).$$

And so $p \mid q$. This forces $p_1 = p$ and $q = p^l$. Since $p \nmid p^{n-1} - 1$, it follows that $q^{m-1} = p^{l(m-1)} \neq p^n - p$, which is a contradiction. If $q^{m-1} = p$, then $q = p$ and $m = 2$. In this case, $\frac{q^m-1}{q-1} = p + 1 \neq p^n$. This contradicts the conditions.

(b) A is imprimitive on $W(X)$.

Let $\mathcal{B}_{\mathcal{W}}$ be a maximal imprimitive block system of A on $W(X)$ and $|\mathcal{B}_{\mathcal{W}}| = p^k$, where $k < n$. Let K be the kernel of A on $\mathcal{B}_{\mathcal{W}}$. Since $S \trianglelefteq A$, it follows that S is transitive on $\mathcal{B}_{\mathcal{W}}$ or $S \leq K$. Note that S is a simple group, it is impossible since S has no faithful permutation representation of degree p^t, where $t < n$.

(2) A is of product type.

In this case, there is a nonabelian simple group R of degree p^s such that $A = R^k \rtimes M$, where M is a transitive permutation group of degree k and $sk = n$. Let P and Q be Sylow p-subgroups of A and R, respectively. Since Q^k is a Sylow p-subgroup of R^k and R^k is transitive on $U(X)$, we have Q^k is transitive on $U(X)$, and thus, $p^n \mid |Q^k|$. By the action of R^k on $U(X)$ and $|P| = p^{n+1}$, we get $|Q^k| = p^n$ and $|Q| = p^s$. Thus, every Sylow p-subgroup of M is $\mathbb{Z}_p$. If $s = 1$, then $Q \cong \mathbb{Z}_p$ and $P = \mathbb{Z}_p^k \rtimes \mathbb{Z}_p$. So P contains an elementary abelian group as a maximal subgroup. If $s > 1$, then $|Z(Q)| > p$. Let P_1 be a subgroup of order p of $Z(Q)$. By the structure of A, $P_1^k \rtimes \mathbb{Z}_p = \mathbb{Z}_p^k \rtimes \mathbb{Z}_p$ is a subgroup of A. From Lemma 3.2 (1), we can get that $k \leq p$. Since $M \leq S_k$, it follows that $p \leq k$. This forces $p = k$ and then $p \mid n$.

(3) A is of affine group type.

In this case, the socal of A is $\mathbb{Z}_p^n$ and is regular on $U(X)$. Note that $|P| = p^{n+1}$. Thus, $\mathbb{Z}_p^n \leq P$ and $\mathbb{Z}_p^n$ is a maximal subgroup of P, i.e., P contains an elementary abelian group as a maximal subgroup. □

Next, we will classify semisymmetric graphs X whose one Sylow p-subgroup of $\mathrm{Aut}(X)$ contains an elementary abelian p-subgroup as a maximal subgroup.

Proof of Theorem 1.2 Every semisymmetric graph is a semitransitive bipartite graph, and every semitransitive bipartite graph is a bi-coset graph. In the follows, we study semisymmetric graphs by means of bi-coset graphs. Since P is transitive on both partite sets $U(X)$ and $W(X)$, from Proposition 2.2, let $X = \mathbf{B}(P, P_u, P_w; D)$, where $u \in U(X)$, $w \in W(X)$ and $D = P_w d P_u$ for some $d \in P$. From Proposition 2.1(iv), $\mathbf{B}(P, P_u, P_w; P_w d P_u) \cong \mathbf{B}(P, P_u, P_w^d; P_w^d P_u)$. So we need only to consider the graph $X = \mathbf{B}(P, P_u, P_w; P_w P_u)$ and its vertex transitivity.

Since the graph X is connected, we can get $P = \langle P_u, P_w\rangle$, which implies that P is generated by two elements of order p. Thus,

$$P = \langle x_1, x_2, \cdots, x_n, y \mid x_1^p = \cdots = x_n^p = y^p = 1, [x_i, y] = x_{i+1}, [x_n, y] = [x_j, x_i] = 1\rangle,$$

where $i = 1, \cdots, n-1$ and $j = 1, \cdots, n$, from Lemma 3.1(i). Let $P_u = \langle x_1^{j_1} \cdots x_n^{j_n} y^{j_{n+1}} \rangle$ and $P_w = \langle x_1^{i_1} \cdots x_n^{i_n} y^{i_{n+1}} \rangle$, where $0 \le i_k, j_l \le p-1, 1 \le k, l \le n+1$. Now, $(i_{n+1}, j_{n+1}) \ne (0, 0)$, since otherwise $\langle P_u, P_w \rangle < P$. Similarly, $(i_1, j_1) \ne (0, 0)$. Since $x_2^{i_2} \cdots x_n^{i_n} \in P'$ and $P' \le \Phi(P)$, where $\Phi(P)$ is the Frattini subgroup of P, i_1 and i_{n+1} cannot be equal to 0. Similarly, $j_1 = 0$ and $j_{n+1} = 0$ cannot simultaneously occur.

Case 1: $n = p$.

Note that $|P_w| = |P_u| = p$, then $i_1 i_{p+1} = 0$ and $j_1 j_{p+1} = 0$, by Lemma 3.1(iii). Without loss of generality, let $i_1 \ne 0$, $j_{p+1} \ne 0$ and $j_1 = i_{p+1} = 0$. For any $x_1^{i_1} x_2^{i_2} \cdots x_p^{i_p} \in P$, there exists $\varphi \in \mathrm{Aut}(P)$ such that $x_1^{\varphi} = x_1^{i_1} x_2^{i_2} \cdots x_p^{i_p}$ and $y^{\varphi_1} = x_2^{j_2} \cdots x_p^{j_p} y^{j_{p+1}}$. Thus, under the automorphism of P, one can set $P_u = \langle y \rangle$ and $P_w = \langle x_1 \rangle$. From Proposition 2.1, we can get

$$X = \mathbf{B}(P, P_u, P_w; P_w P_u) \cong \mathbf{B}(P, \langle y \rangle, \langle x_1 \rangle; \langle x_1 \rangle \langle y \rangle).$$

From Lemma 3.1(ii), $C_P(P_u) = \mathbb{Z}_p^2$ and $C_P(P_w) = \mathbb{Z}_p^p$. Note that p is an odd prime, the graph X is not vertex transitive by Lemma 2.4 and so X is semisymmetric.

Case 2: $n < p$.

Without loss of generality, let $j_{n+1} \ne 0$. Then, there exists ϕ in $\mathrm{Aut}(P)$ such that $y^{\phi} = x_1^{j_1} \cdots x_n^{j_n} y^{j_{n+1}}$ and $(x_1^{i_1} \cdots x_n^{i_n} y^{i_{n+1}})^{\phi} = x_1^{i_1} x_2^{l_2} \cdots x_n^{l_n} y^{l_{n+1}}$, where $l_{n+1} \ne 0$. In this case, $i_1 \ne 0$, since otherwise X is disconnected. If $i_{n+1} = 0$, there exists ϕ_1 in $\langle \sigma_i \rangle$, where $i = 2, \cdots, n$, such that $x_1^{\phi_1} = x_1^{i_1} x_2^{l_2} \cdots x_n^{l_n}$ and $y^{\phi_1} = y$. Without loss of generality, let $P_u = \langle y \rangle$ and $P_w = \langle x_1 \rangle$. If $i_{n+1} \ne 0$, there exists ϕ_2 in $\langle \sigma_i, \tau \rangle$ such that $(x_1 y)^{\phi_2} = x_1^{i_1} x_2^{l_2} \cdots x_n^{l_n} y^{l_{n+1}}$ and $\langle y \rangle^{\phi_2} = \langle y \rangle$. In this case, we can set $P_u = \langle y \rangle$ and $P_w = \langle x_1 y \rangle$. By Proposition 2.1, the connected edge-transitive regular graph $\mathbf{B}(P, P_u, P_w; P_w P_u)$ is isomorphic to one of the following two bi-coset graphs

$$X = \mathbf{B}(P, \langle y \rangle, \langle x_1 \rangle; \langle x_1 \rangle \langle y \rangle) \ \text{ and } \ X_1 = \mathbf{B}(P, \langle y \rangle, \langle x_1 y \rangle; \langle x_1 y \rangle y).$$

Similar to the case 1, we have $C_P(P_u) = \mathbb{Z}_p^2$ and $C_P(P_w) = \mathbb{Z}_p^n$, where $n \ge 3$. Then, the graph X is semisymmetric by Lemma 2.4.

For the graph X_1, take $\sigma \in \mathrm{Aut}(P)$ such that

$$\sigma : \ y \mapsto x_1 y, \ \ x_1 \mapsto x_1^{p-1}.$$

Then $P_w^{\sigma} = P_u$, $P_u^{\sigma} = P_w$ and $D^{\sigma} = (P_w P_u)^{\sigma} = P_u P_w = D^{-1}$. By Proposition 2.3, the graph X_1 is vertex transitive, and thus, X_1 is not a semisymmetric graph. □

5 A classification of semisymmetric graphs of order $2p^4$ with valency p

In this section, we will classify semisymmetric graphs of order $2p^4$ with valency p. For $p = 3$, there is no cubic semisymmetric graph of order 162 (see [4]). In the follows, we assume that $p \ge 5$. Let P be a Sylow p-subgroup of $\mathrm{Aut}(X)$. By Lemma 4.2 and the proof of Theorem 1.2, we need only to consider the graph $\mathbf{B}(P, P_u, P_w; P_w P_u)$ where $P = \langle P_u, P_w\rangle$ and $|P| = p^5$. In [1, 26], groups of order p^5 which generated by two elements of order p are as follows:

$P_1 = \langle a, b, c, d, e | a^p = b^p = c^p = d^p = e^p = 1, [b, a] = c, [c, a] = d, [d, a] = e, [c, b] = [d, b] = [d, c] = [e, a] = [e, b] = [e, c] = [e, d] = 1\rangle$;

$P_2 = \langle a, b, c, d, e | a^p = b^p = c^p = d^p = e^p = 1, [b, a] = c, [c, a] = d, [c, b] = e, [d, a] = [d, b] = [d, c] = [e, a] = [e, b] = [e, c] = [e, d] = 1\rangle$;

$P_3 = \langle a, b, c, d, e | a^p = b^p = c^p = d^p = e^p = 1, [b, a] = c, [c, a] = d, [d, a] = [c, b] = e, [d, b] = [d, c] = [e, a] = [e, b] = [e, c] = [e, d] = 1\rangle$.

It is clear that the group P_1 is just the group in Lemma 3.1 for $n = 4$ and its automorphism group has already been given in Lemma 3.2. In the follows, we determine the automorphic groups of P_2 and P_3.

Lemma 5.1 *Every element in* $\mathrm{Aut}(P_2)$ *is of the form*

$$\varphi : a \mapsto a^{i_1} b^{i_2} c^{i_3} d^{i_4} e^{i_5}, \quad b \mapsto a^{j_1} b^{j_2} c^{j_3} d^{j_4} e^{j_5},$$

where $0 \le i_l, j_k \le p-1, 1 \le l, k \le 5$ *and* $i_1 j_2 - i_2 j_1 \not\equiv 0 (\mathrm{mod}\, p)$. *Especially,* $|\mathrm{Aut}(P_2)| = p^7(p-1)^2(p+1)$.

Proof It is clear that P_2 can be generated by a and b, and the derived group of P_2 is $P_2' = \langle c\rangle \times \langle d\rangle \times \langle e\rangle$. Note that $|P_2| = p^5 \le p^p$ because of $p \ge 5$. So P_2 is a regular p-group. For any $a^{i_1} b^{i_2} c^{i_3} d^{i_4} e^{i_5} \in P_2$, $(a^{i_1} b^{i_2} c^{i_3} d^{i_4} e^{i_5})^p = a^{pi_1} b^{pi_2} d_1^p d_2^p \cdots d_s^p = 1$, where $d_1, d_2, \cdots, d_s \in P_2'$. It follows that $\exp(P_2) = p$. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_{i_1 i_2 j_1 j_2}$ be the following maps on P_2 defined via generators a and b and preserving multiplications by

$$\begin{array}{lll}
\sigma_1 : \quad a \mapsto ac, b \mapsto b; & \sigma_2 : a \mapsto ad, b \mapsto b; & \sigma_3 : a \mapsto ae, b \mapsto b;\\
\sigma_4 : \quad a \mapsto a, b \mapsto bc; & \sigma_5 : a \mapsto a, b \mapsto bd; & \sigma_6 : a \mapsto a, b \mapsto be;\\
\sigma_{i_1 i_2 j_1 j_2} : a \mapsto a^{i_1} b^{j_1}, b \mapsto a^{i_2} b^{j_2}. & &
\end{array}$$

It is easy to prove that $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6 \in \mathrm{Aut}(P_2)$. Then, every element σ in $\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\rangle$ is of the form:

$$\sigma : a \mapsto ac^{i_3} d^{i_4} e^{i_5}, \; b \mapsto bc^{j_3} d^{j_4} e^{j_5},$$

where $1 \le i_l, j_k \le p-1, 3 \le l, k \le 5$. Consider the induced action of $\mathrm{Aut}(P_2)$ on $P_2/P_2' \cong \mathbb{Z}_p \times \mathbb{Z}_p$, the kernel of this action is $K = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\rangle$. Hence $\mathrm{Aut}(P_2)/K \lesssim \mathrm{GL}(2, p)$.

In the follows, we consider the maps $\sigma_{i_1 i_2 j_1 j_2}$. If $\sigma_{i_1 i_2 j_1 j_2} \in \mathrm{Aut}(P_2)$, then

$$\begin{aligned}
c^{\sigma_{i_1 i_2 j_1 j_2}} &= [b^{\sigma_{i_1 i_2 j_1 j_2}}, a^{\sigma_{i_1 i_2 j_1 j_2}}] = [a^{i_2} b^{j_2}, a^{i_1} b^{j_1}]\\
&= c^{i_1 j_2 - i_2 j_1} d^{j_2 C_{i_1}^2 - j_1 C_{i_2}^2} e^{i_1 C_{j_2+1}^2 + i_2 C_{p-j_1+1}^2 + j_1(i_1 j_2 - j_1 i_2 - i_2 j_2)},\\
d^{\sigma_{i_1 i_2 j_1 j_2}} &= [c^{\sigma_{i_1 i_2 j_1 j_2}}, a^{\sigma_{i_1 i_2 j_1 j_2}}] = [c^{i_1 j_2 - i_2 j_1}, a^{i_1} b^{j_1}] = d^{i_1(i_1 j_2 - i_2 j_1)} e^{j_1(i_1 j_2 - i_2 j_1)},
\end{aligned}$$

and

$$e^{\sigma_{i_1i_2j_1j_2}}=[c^{\sigma_{i_1i_2j_1j_2}},b^{\sigma_{i_1i_2j_1j_2}}]=[c^{i_1j_2-i_2j_1},a^{i_2}b^{j_2}]=d^{i_2(i_1j_2-i_2j_1)}e^{j_2(i_1j_2-i_2j_1)}.$$

Since $Z(P_2)=\langle d,e\rangle$ and $c\notin Z(P_2)$, it follows that $c^{\sigma_{i_1i_2j_1j_2}}\notin Z(P_2)$. This forces that $i_1j_2-i_2j_1\not\equiv 0(\bmod p)$. It is clear that $\sigma_{i_1i_2j_1j_2}\notin K$, so $K\cap\langle\sigma_{i_1i_2j_1j_2}\rangle=1$. Thus

$$|\langle\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_{i_1i_2j_1j_2}\rangle|=|K||\mathrm{GL}(2,p)|.$$

Note that $|\mathrm{Aut}(P_2)|\le|K||\mathrm{GL}(2,p)|$. Therefore, $\mathrm{Aut}(P_2)=\langle\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_{i_1i_2j_1j_2}\rangle$, where $i_1j_2-i_2j_1\not\equiv 0(\bmod p)$. Hence $|\mathrm{Aut}(P_2)|=p^7(p-1)^2(p+1)$. □

Lemma 5.2 *Every element in* $\mathrm{Aut}(P_3)$ *is of the form*

$$\phi: a\mapsto a^{i_1}b^{i_2}c^{i_3}d^{i_4}e^{i_5},\quad b\mapsto b^{i_1^2}c^{j_3}d^{j_4}e^{j_5},$$

where $0\le i_l,j_k\le p-1$, $1\le l\le 5$, $3\le k\le 5$ *and* $i_1\neq 0$. *Especially,* $|\mathrm{Aut}(P_3)|=p^7(p-1)$.

Proof By the same argument as in Lemma 5.1, P_3 is a regular p-group and $\exp(P_3)=p$. Let $\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_7,\sigma_{ij}$ be the follows maps on P_3 defined via generators a and b and preserving multiplications by

$$\begin{array}{lll}\sigma_1: a\mapsto ac, b\mapsto b; & \sigma_2: a\mapsto ad, b\mapsto b; & \sigma_3: a\mapsto ae, b\mapsto b;\\ \sigma_4: a\mapsto a, b\mapsto bc; & \sigma_5: a\mapsto a, b\mapsto bd; & \sigma_6: a\mapsto a, b\mapsto be\\ \sigma_7: a\mapsto ab, b\mapsto b; & \sigma_{ij}: a\mapsto a^i, b\mapsto b^j; & \end{array}$$

where $1\le i,j\le p-1$. It can be checked easily that $\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_7\in\mathrm{Aut}(P_3)$. Note that $P_3'=\langle c,d,e\rangle\trianglelefteq P_3$. Consider the induced action of $\mathrm{Aut}(P_3)$ on $P_3/P_3'\cong\mathbb{Z}_p\times\mathbb{Z}_p$. The kernel of this action is $K=\langle\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6\rangle$. Thus, $\mathrm{Aut}(P_3)/K\lesssim\mathrm{GL}(2,p)$.

Now consider the maps σ_{ij}, we have

$$\begin{aligned}c^{\sigma_{ij}}&=[b^{\sigma_{ij}},a^{\sigma_{ij}}]=[b^j,a^i]=c^{ij}d^{jC_i^2}e^{iC_j^2+j\Sigma_{k=1}^{j-1}C_k^2},\\ d^{\sigma_{ij}}&=[c^{\sigma_{ij}},a^{\sigma_{ij}}]=[c^{ij}d^{jC_i^2}e^{iC_j^2+j\Sigma_{k=1}^{j-1}C_k^2},a^i]=[c^{ij}d^{jC_i^2},a^i]=d^{i^2j}e^{2ijC_i^2},\\ e^{\sigma_{ij}}&=[c^{\sigma_{ij}},b^{\sigma_{ij}}]=[c^{ij}d^{jC_i^2}e^{iC_j^2+j\Sigma_{k=1}^{j-1}C_k^2},b^j]=[c^{ij},b^j]=e^{ij^2},\end{aligned}$$

and

$$e^{\sigma_{ij}}=[d^{\sigma_{ij}},a^{\sigma_{ij}}]=[d^{i^2j}e^{2ijC_i^2},a^i]=[d^{i^2j},a^i]=e^{i^3j}.$$

Thus if $\sigma_{ij}\in\mathrm{Aut}(P_3)$, then $e^{ij^2}=e^{i^3j}$. It follows that $ij(j-i^2)\equiv 0(\bmod p)$. Since $i,j\neq 0$, one has $j\equiv i^2(\bmod p)$. That is, $\sigma_{ii^2}\in\mathrm{Aut}(P_3)$.

Note that $C_{P_3}(a) = \langle a\rangle \times \langle e\rangle$ and $C_{P_3}(b) = \langle b\rangle \times \langle d\rangle \times \langle e\rangle$. Thus, a and b are not conjugate in $\mathrm{Aut}(P_3)$. By the above argument, for any $a^{i_1}b^{i_2}c^{i_3}d^{i_4}e^{i_5} \in P_3$, there exists $\sigma \in \mathrm{Aut}(P_3)$ such that $a^\sigma = a^{i_1}b^{i_2}c^{i_3}d^{i_4}e^{i_5}$, where $0 \le i_k \le p-1, k = 1, \cdots, 5$, and $i_1 \ne 0$. Thus, there is no ϕ in $\mathrm{Aut}(P_3)$ such that $b^\phi = a^{i_1}b^{i_2}c^{i_3}d^{i_4}e^{i_5}$, where $i_1 \ne 0$. Since $b \notin P_3'$, it follows that $b^\phi \ne c^{i_3}d^{i_4}e^{i_5}$, for any $\phi \in \mathrm{Aut}(P_3)$. Therefore, under the automorphisms of P_3, b might map to an element of the form $b^{j_2}c^{j_3}d^{j_4}e^{j_5}$, where $j_2 \ne 0$. From the above discussions on σ_{ij}, we have

$$b^{\langle\sigma_4,\sigma_5,\sigma_6,\sigma_{i_1 i_1^2}\rangle} = \{b^{i_1^2}c^{j_3}d^{j_4}e^{j_5} \mid 0 \le i_1, j_3, j_4, j_5 \le p-1, i_1 \ne 0\} = b^{\mathrm{Aut}(P_3)}.$$

Therefore $\mathrm{Aut}(P_3) = \langle\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_{ii^2}\rangle$. Hence, $|\mathrm{Aut}(P_3)| = p^7(p-1)$. □

Proof of Theorem 1.3 It is clear that P_1 has a subgroup $\langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$ which is an elementary abelian p-group of order p^4. It has already been proved that the semisymmetric graph of order $2p^4$ with valency p which is a bi-coset graph of P_1 is $X_{p,4}$ in Theorem 1.2. Now we consider two groups P_2 and P_3. By the proof of Theorem 1.2, we need only to consider the bi-coset graph $\mathbf{B}(P, P_u, P_w; P_w P_u)$, where $P = P_2$ or P_3.

Case 1: $P = P_2$.

Firstly, we determine the structures of P_u and P_w. From $|P_u| = |P_w| = p$, let $P_u = \langle a^{i_1}b^{i_2}c^{i_3}d^{i_4}e^{i_5}\rangle$ and $P_w = \langle a^{j_1}b^{j_2}c^{j_3}d^{j_4}e^{j_5}\rangle$, where $0 \le i_k, j_l \le p-1$, $k, l = 1, \cdots, 5$. Note that $P' = \langle c, d, e\rangle \trianglelefteq P$, then $(i_1, j_1) \ne (0, 0)$, since otherwise $\langle P_u, P_w\rangle < P$. Without loss of generality, we assume that $i_1 \ne 0$. By Lemma 5.1, for any $a^{i_1}b^{i_2}c^{i_3}d^{i_4}e^{i_5} \in P_2$ with $i_1 \ne 0$, there exists ϕ in $\mathrm{Aut}(P_2)$ such that $a^\phi = a^{i_1}b^{i_2}c^{i_3}d^{i_4}e^{i_5}$. By Proposition 2.1, set $P_u = \langle a\rangle$. Since $P = \langle P_u, P_w\rangle = \langle a, a^{j_1}b^{j_2}c^{j_3}d^{j_4}e^{j_5}\rangle$, it follows that $j_2 \ne 0$. Note that there exists φ in $\langle\sigma_4, \sigma_5, \sigma_6, \sigma_{11 j_1 j_2}\rangle$ such that $a^\varphi = a$ and $b^\varphi = a^{j_1}b^{j_2}c^{j_3}d^{j_4}e^{j_5}$, where $j_2 \ne 0$. From Proposition 2.1, we have

$$\mathbf{B}(P, P_u, P_w; P_w P_u) \cong \mathbf{B}(P, \langle a\rangle, \langle b\rangle; \langle b\rangle\langle a\rangle).$$

Secondly, we can show that the graph $\mathbf{B}(P, \langle a\rangle, \langle b\rangle; \langle b\rangle\langle a\rangle)$ is vertex-transitive. Take $\sigma \in \mathrm{Aut}(P_2)$ such that

$$\sigma : a \mapsto b,\ b \mapsto a.$$

Then it is clear that $P_u^\sigma = \langle a\rangle^\sigma = \langle b\rangle = P_w$, $P_w^\sigma = \langle b\rangle^\sigma = \langle a\rangle = P_u$ and $(D^{-1})^\sigma = ((P_w P_u)^{-1})^\sigma = (P_u P_w)^\sigma = P_w P_u = D$. From Lemma 2.3, the graph $\mathbf{B}(P_2, \langle a\rangle, \langle b\rangle; \langle b\rangle\langle a\rangle)$ is vertex-transitive and thus the graph $\mathbf{B}(P_2, \langle a\rangle, \langle b\rangle; \langle b\rangle\langle a\rangle)$ is not semisymmetric.

Case 2: $P = P_3$.

Similarly, we first determine the structures of P_u and P_w. Let $P_u = \langle a^{i_1}b^{i_2}c^{i_3}d^{i_4}e^{i_5}\rangle$ and $P_w = \langle a^{j_1}b^{j_2}c^{j_3}d^{j_4}e^{j_5}\rangle$, where $0 \le i_k, j_l \le p-1$ with $k, l = 1, \cdots, 5$. By the same argument as in Case 1, let $i_1 \ne 0$. By Lemma 5.2, there exists ψ in $\mathrm{Aut}(P_3)$ such that $a^\psi = a^{i_1}b^{i_2}c^{i_3}d^{i_4}e^{i_5}$, where $i_1 \ne 0$. Without loss of generality, let

$P_u = \langle a\rangle$. Since $P = \langle P_w, P_u\rangle$, one has $j_2 \neq 0$. If $j_1 = 0$, for any $b^{j_2}c^{j_3}d^{j_4}e^{j_5} \in P_3$, there exists $\omega \in \langle\sigma_4, \sigma_5, \sigma_6\rangle$ such that $a^\omega = a$ and $(b^{j_2})^\omega = b^{j_2}c^{j_3}d^{j_4}e^{j_5} \in P_3$. Thus $P = \langle a, b\rangle$. If $j_1 \neq 0$, for any $a^{j_1}b^{j_2}c^{j_3}d^{j_4}e^{j_5} \in P_3$, there exists ν in $\langle\sigma_4, \sigma_5, \sigma_6, \sigma_{ii^2}\rangle$ such that $\langle a\rangle^\nu = \langle a\rangle$ and $(ab^k)^\nu = a^{j_1}b^{j_2}c^{j_3}d^{j_4}e^{j_5}$, where $kj_1^2 \equiv j_2 (\mathrm{mod}\, p)$. Therefore $P = \langle a, ab^k\rangle$. By Proposition 2.1, the connected edge-transitive regular graph $\mathbf{B}(P_3, P_u, P_w; P_wP_u)$ is isomorphic to one of the following two bi-coset graphs

$$X = \mathbf{B}(P_3, \langle a\rangle, \langle b\rangle; \langle b\rangle\langle a\rangle), \quad \text{and} \quad X_k = \mathbf{B}(P_3, \langle a\rangle, \langle ab^k\rangle; \langle ab^k\rangle\langle a\rangle).$$

We claim that the graph X_k is vertex-transitive. Take $\sigma_k \in \mathrm{Aut}(P_3)$ such that

$$\sigma_k : a \mapsto (ab^k)^{-1}, \ b \mapsto bc^{-1}de^j,$$

where $j \equiv (k-1)(\mathrm{mod}\, p)$. Then,

$$\begin{aligned}(ab^k)^{\sigma_k} &= b^{-k}a^{-1}(bc^{-1}d^{-1}e^j)^k \\ &= b^{-k}a^{-1}b^kc^{-k}d^ke^{-C_k^2+jk} \\ &= a^{-1}c^kd^{-k}e^{-C_k^2}c^{-k}d^ke^{-C_k^2+jk} \\ &= a^{-1}e^{jk-k(k-1)} \\ &= a^{-1}.\end{aligned}$$

Hence, we have

$$\begin{aligned}P_u^{\sigma_k} &= \langle a\rangle^{\sigma_k} = \langle (ab^k)^{-1}\rangle = \langle ab^k\rangle = P_w, \\ P_w^{\sigma_k} &= \langle ab^k\rangle^{\sigma_k} = \langle a^{-1}\rangle = \langle a\rangle = P_u.\end{aligned}$$

Thus, $(D^{-1})^{\sigma_k} = ((P_wP_u)^{-1})^{\sigma_k} = (P_uP_w)^{\sigma_k} = P_wP_u = D$. By Lemma 2.3, the graph X_k is vertex-transitive.

Now consider the graph X. Note that $C_{P_3}(a) = \mathbb{Z}_p \times \mathbb{Z}_p$ and $C_{P_3}(b) = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. By Lemma 2.4, the graph X is not vertex-transitive and so it is semisymmetric.

Therefore, from Theorem 1.2 and the above argument, the semisymmetric graph of order $2p^4$ with valency p is isomorphic to $X_{p,4}$ or $\widetilde{X}_{p,4}$, which are described in Constructions 2.6 and 2.7. Furthermore, $\mathrm{Aut}(\widetilde{X}_{p,4})$ is imprimitive on both partite sets by Theorem 1.1. □

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