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The v-number of monomial ideals

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Abstract

We show that the v-number of an arbitrary monomial ideal is bounded below by the v-number of its polarization and also find a criteria for the equality. By showing the additivity of associated primes of monomial ideals, we obtain the additivity of the v-numbers for arbitrary monomial ideals. We prove that the v-number $\mathrm{v}(I(G))$ of the edge ideal $I(G)$, the induced matching number $\mathrm{im}(G)$ and the regularity $\mathrm{reg}(R/I(G))$ of a graph G, satisfy $\mathrm{v}(I(G)) \leq \mathrm{im}(G) \leq \mathrm{reg}(R/I(G))$, where G is either a bipartite graph, or a (C_4, C_5)-free vertex decomposable graph, or a whisker graph. There is an open problem in Jaramillo and Villarreal (J Combin Theory Ser A 177:105310, 2021), whether $\mathrm{v}(I) \leq \mathrm{reg}(R/I) + 1$, for any square-free monomial ideal I. We show that $\mathrm{v}(I(G)) > \mathrm{reg}(R/I(G)) + 1$, for a disconnected graph G. We derive some inequalities of v-numbers which may be helpful to answer the above problem for the case of connected graphs. We connect $\mathrm{v}(I(G))$ with an invariant of the line graph $L(G)$ of G. For a simple connected graph G, we show that $\mathrm{reg}(R/I(G))$ can be arbitrarily larger than $\mathrm{v}(I(G))$. Also, we try to see how the v-number is related to the Cohen–Macaulay property of square-free monomial ideals.

Keywords v-number · Monomial ideals · Induced matching number · Castelnuovo–Mumford regularity

Mathematics Subject Classification Primary 13F20 · 13F55 · 05C70 · 05E40 · 13H10

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1 Introduction

Let $R = K[x_1 \dots, x_n] = \bigoplus_{d=0}^{\infty} R_d$ denote the polynomial ring in n variables over a field K, with the standard gradation. Given a graph G, we assume $V(G) = \{x_1, \dots, x_n\}$ and all graphs are assumed to be simple graphs.

For a graded ideal I of R, the set of associated prime ideals of I, denoted by $\mathrm{Ass}(I)$ or $\mathrm{Ass}(R/I)$, is the collection of prime ideals of R of the form $(I : f)$, for some $f \in R_d$. A prime ideal $\mathfrak{p} \in \mathrm{Ass}(R/I)$ is said to be a *minimal prime* of I if for all $\mathfrak{q} \in \mathrm{Ass}(R/I)$, with $\mathfrak{p} \neq \mathfrak{q}$, we have $\mathfrak{q} \nsubseteq \mathfrak{p}$. If an associated prime ideal of I is not minimal, then I is called an *embedded prime* of I.

Definition 1.1 ([8, Definition 4.1]) Let I be a proper graded ideal of R. Then v-*number* of I is denoted by $\mathrm{v}(I)$ and is defined by

$$\mathrm{v}(I) := \min\{d \geq 0 \mid \exists\, f \in R_d \text{ and } \mathfrak{p} \in \mathrm{Ass}(I), \text{ with } (I : f) = \mathfrak{p}\}.$$

For each $\mathfrak{p} \in \mathrm{Ass}(I)$, we can locally define v-number as

$$\mathrm{v}_{\mathfrak{p}}(I) := \min\{d \geq 0 \mid \exists\, f \in R_d, \text{ with } (I : f) = \mathfrak{p}\}.$$

Then $\mathrm{v}(I) = \min\{\mathrm{v}_{\mathfrak{p}}(I) \mid \mathfrak{p} \in \mathrm{Ass}(I)\}$.

The v-number of I was introduced as an invariant of the graded ideal I, in [8], in the study of Reed–Muller-type codes. This invariant of I helps us understand the behaviour of the generalized minimum distance function δ_I of I, in the said context. See [8, 18, 24, 26] for further details on this.

Procedure A1 in [13] helps us compute the v-number of monomial ideals using *Macaulay2* [12]. In [18], Jaramillo and Villarreal have discussed some properties of $\mathrm{v}(I)$ and have proved combinatorial formula of $\mathrm{v}(I)$, where I is a square-free monomial ideal. They have proved that $\mathrm{v}(I) \leq \mathrm{reg}(R/I)$ is satisfied for several cases of square-free monomial ideals I. In the same article, the authors have also disproved [26, Conjecture 4.2] by giving an example [18, Example 5.4] of a connected graph G, with $3 = \mathrm{v}(I(G)) > \mathrm{reg}(R/I(G)) = 2$. They have proposed an open problem in [18], whether $\mathrm{v}(I) \leq \mathrm{reg}(R/I) + 1$, for any square-free monomial ideal I. In this paper, we give a counter-example (Example 5.1) to this open problem and modify the question (Question 5.2) for edge ideals of those clutters which cannot be written as a disjoint union of two clutters. We try to give a partial answer to this question. We find the relation between the v-number of an arbitrary monomial ideal and the v-number of its polarization, along with some criteria for equality.

Bounds of Castelnuovo–Mumford regularity of edge ideals (see [2, 3, 9, 11, 14, 20, 30]) and bounds of induced matching number of graphs (see [5–7, 19, 23, 31]) are two trending topics in the research of commutative algebra and combinatorics, respectively. Also, obtaining induced matching number in general is NP-hard. So it would be an interesting problem to find the bounds of regularity and induced matching number by the v-number. Considering $I(G)$ as the edge ideal of a graph G, we give a relation between $\mathrm{v}(I)$, $\mathrm{reg}(R/I)$ and $\mathrm{im}(G)$ for bipartite graphs (Theorem 4.5) , (C_4, C_5)-free vertex decomposable graphs (Theorem 4.11), whisker graphs (Theorem 4.12), etc. We also obtain some results on the v-number and propose some problems. The paper is arranged in the following manner.

In Sect. 2, we discuss the Preliminaries. We recall some definitions, notations, basic concepts pertinent to Graph theory and Commutative Algebra and results from [18]. In Sect. 3, our main result is the following:
Theorem 3.4. Let I be a monomial ideal. If there exists

$$\mathfrak{p} = \langle x_{s_1,b_{s_1}}, \ldots, x_{s_k,b_{s_k}} \rangle \in \mathrm{Ass}(I(\mathrm{pol})),$$

such that $\mathrm{v}(I(\mathrm{pol})) = \mathrm{v}_{\mathfrak{p}}(I(\mathrm{pol}))$, and if there is no embedded prime of I properly containing $\langle x_{s_1}, \ldots, x_{s_k} \rangle$, then

$$\mathrm{v}(I) = \mathrm{v}(I(\mathrm{pol})).$$

In general, we get $\mathrm{v}(I(\mathrm{pol})) \leq \mathrm{v}(I)$ (Corollary 3.5). Also, in this section, we generalize some results proved in [18], related to the v-number of square-free monomial ideals to arbitrary monomial ideals with special type, which includes the monomial ideals having no embedded primes. The additive property of associated primes is known for square-free monomial ideals (see [22, Lemma 2.14]). We prove that the additive property of associated primes holds for arbitrary monomial ideals in Lemma 3.8 and this result has been used to show the additivity of v-numbers for monomial ideals, which is the following:
Proposition 3.9. Let $I_1 \subseteq R_1 = K[\mathbf{x}]$ and $I_2 \subseteq R_2 = K[\mathbf{y}]$ be two monomial ideals and consider $R = K[\mathbf{x}, \mathbf{y}]$. Then we have

$$\mathrm{v}(I_1R + I_2R) = \mathrm{v}(I_1) + \mathrm{v}(I_2).$$

In addition, we derive some properties (see Proposition 3.13) of $\mathrm{v}(I)$ for any square-free monomial ideal I. For any graph G, we show that $\mathrm{v}(I(G)) \leq \alpha_0(G)$ (Proposition 3.14), where $\alpha_0(G)$ is the vertex covering number of G. In Sect. 4, we relate $\mathrm{v}(I(G))$ with an invariant of $L(G)$, the line graph of G (see Proposition 4.1). We derive some properties of the v-number for edge ideals of graphs (see Proposition 4.2), which could be helpful in finding the relation between the v-number and the regularity of edge ideals. We find the following relations between $\mathrm{v}(I(G))$, $\mathrm{reg}(R/I(G))$, $\mathrm{im}(G)$ for certain classes of graphs G:
Theorems 4.5, 4.11, 4.12. If G is a bipartite graph or (C_4, C_5)-free vertex decomposable graph or whisker graph, then

$$\mathrm{v}(I(G)) \leq \mathrm{im}(G) \leq \mathrm{reg}(R/I(G)).$$

Also, we show that for a graph G, the difference between $\mathrm{v}(I(G))$ and $\mathrm{reg}(R/I(G))$ may be arbitrarily large (see Corollary 4.15). In Proposition 4.18, we try to relate the Cohen–Macaulay property of (R/I) with $\mathrm{v}(I^\vee)$, where $I = I(\mathcal{C})$ is an edge ideal of a clutter $\mathcal{C}$, such that $\mathcal{C}$ cannot be written as a union of two disjoint clutters and $I^\vee$ denotes the Alexander dual ideal (Definition 4.17) of I. In Sect. 5, we give a counter-example (Example 5.1) to the problem given in [18] and pose the modified question (see Question 5.2). We also propose some open problems related to the v-number in terms of regularity, depth and induced matching number.

2 Preliminaries

In this section, we recall some basic definitions, results, and notations of graph theory and commutative algebra. Also, we mention some results, concepts, and notations from [18].

In R, we denote a monomial $x_1^{a_1} \cdots x_n^{a_n}$ by $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $\mathbb{N}$ denotes the set of all non-negative integers. An ideal $I \subseteq R$ is called a monomial ideal if it is minimally generated by a set of monomials in R. The set of minimal monomial generators of I is unique and it is denoted by $G(I)$. If $G(I)$ consists of only square-free monomials, then we say I is a square-free monomial ideal.

Definition 2.1 A *clutter* $\mathcal{C}$ is a pair of two sets $(V(\mathcal{C}), E(\mathcal{C}))$, where $V(\mathcal{C})$ is called the vertex set and $E(\mathcal{C})$ is a collection of subsets of $V(\mathcal{C})$, called edge set, such that no two elements (called edges) of $E(\mathcal{C})$ contains each other. A clutter is also known as *simple hypergraph*. A simple graph is an example of a clutter, whose edges are of cardinality two.

Let $\mathcal{C}$ be a clutter on a vertex set $V(\mathcal{C})$. An edge $e \in E(\mathcal{C})$ is said to be *incident* on a vertex $x \in V(\mathcal{C})$ if $x \in e$. A subset $C \subseteq V(\mathcal{C})$ is called a *vertex cover* of $\mathcal{C}$ if any $e \in E(\mathcal{C})$ is incident to a vertex of C. If a vertex cover is minimal with respect to inclusion, then we call it a *minimal vertex cover*. The cardinality of a minimum (smallest) vertex cover is known as the *vertex covering number* of $\mathcal{C}$ and is denoted by $\alpha_0(\mathcal{C})$. Also a subset $A \subseteq V(\mathcal{C})$ is said to be *stable* or *independent* if $e \nsubseteq A$ for any $e \in E(\mathcal{C})$ and A is said to be *maximal independent set* if it is maximal with respect to inclusion. The number of vertices in a maximum (largest) independent set, denoted by $\beta_0(\mathcal{C})$, is called the *independence number* of $\mathcal{C}$. Note that a vertex cover C is a minimal vertex cover of $\mathcal{C}$ if and only if its complement $V(\mathcal{C}) \setminus C$ is a maximal independent set.

Let $\mathcal{C}$ be a clutter on the vertex set $V(\mathcal{C}) = \{x_1, \ldots, x_n\}$. Then for $A \subseteq V(\mathcal{C})$, we consider $X_A := \prod_{x_i \in A} x_i$ as a square-free monomial in the polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. The edge ideal of the clutter $\mathcal{C}$, denoted by $I(\mathcal{C})$, is the ideal in R defined by

$$I(\mathcal{C}) = \big\langle X_e \mid e \in E(\mathcal{C}) \big\rangle.$$

Set of square-free monomial ideals are in one to one correspondence with the set of clutters. For a simple graph G, the edge ideal $I(G)$ is generated by square-free quadratic monomials. It is a well-known fact that

$$\mathrm{ht}(I(\mathcal{C})) = \alpha_0(\mathcal{C}) \text{ and } \dim(R/I(\mathcal{C})) = \beta_0(\mathcal{C}),$$

where $\mathrm{ht}(I(\mathcal{C}))$ is the height of $I(\mathcal{C})$ and $\dim(R/I(\mathcal{C}))$ is the Krull dimension of $R/I(\mathcal{C})$. Note that $\alpha_0(\mathcal{C}) + \beta_0(\mathcal{C}) = n$.

Let A be a stable set of a clutter $\mathcal{C}$. Then the *neighbour* set of A in $\mathcal{C}$, denoted by $\mathcal{N}_{\mathcal{C}}(A)$, is defined by

$$\mathcal{N}_{\mathcal{C}}(A) = \{x_i \in V(\mathcal{C}) \mid \{x_i\} \cup A \text{ contains an edge of } \mathcal{C}\}.$$

We denote $\mathcal{N}_{\mathcal{C}}[A] := \mathcal{N}_{\mathcal{C}}(A) \cup A$. Now we recall some notations and results from [18]. Let $\mathcal{F}_{\mathcal{C}}$ denote the collection of all maximal stable sets of $\mathcal{C}$ and $\mathcal{A}_{\mathcal{C}}$ denote the collection of those stable sets A of $\mathcal{C}$, such that $\mathcal{N}_{\mathcal{C}}(A)$ is a minimal vertex cover of $\mathcal{C}$. The following theorems in [18] gives the combinatorial formula for $\mathrm{v}(I(\mathcal{C}))$.

Lemma 2.2 *([18, Lemma 3.4]) Let $I = I(\mathcal{C})$ be the edge ideal of a clutter $\mathcal{C}$. Then the following hold:*

(a) *For $A \in \mathcal{A}_{\mathcal{C}}$, we have $(I : X_A) = \langle \mathcal{N}_{\mathcal{C}}(A) \rangle$.*
(b) *If A is stable and $\mathcal{N}_{\mathcal{C}}(A)$ is a vertex cover, then $\mathcal{N}_{\mathcal{C}}(A)$ is a minimal vertex cover.*
(c) *If $(I : f) = \mathfrak{p}$ for some $f \in R_d$ and some $\mathfrak{p} \in \mathrm{Ass}(I)$, then there is $A \in \mathcal{A}_{\mathcal{C}}$, with $|A| \leq d$, such that $(I : X_A) = \langle \mathcal{N}_{\mathcal{C}}(A) \rangle = \mathfrak{p}$.*
(d) *If $A \in \mathcal{F}_{\mathcal{C}}$, then $\mathcal{N}_{\mathcal{C}}(A) = V(C) \setminus A$ and $(I : X_A) = \langle \mathcal{N}_{\mathcal{C}}(A) \rangle$.*

Theorem 2.3 *([18, Theorem 3.5]) Let $I = I(\mathcal{C})$ be the edge ideal of a clutter $\mathcal{C}$. If I is not prime, then $\mathcal{F}_{\mathcal{C}} \subseteq \mathcal{A}_{\mathcal{C}}$ and*

$$\mathrm{v}(I) = \min\{|A| : A \in \mathcal{A}_{\mathcal{C}}\}.$$

In this paper, we use Lemma 2.2 and Theorem 2.3 frequently.

Let $V = \{x_1, \ldots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V, with the following properties:

(i) $\{x_i\} \in \Delta$ for all $x_i \in V$;
(ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$.

An element $F \in \Delta$ is called a *face* of Δ. A maximal face of Δ is called a *facet* of Δ. For a vertex $v \in V$, $\mathrm{del}_{\Delta}(v)$ is a subcomplex, called *deletion* of v, on the vertex set $V \setminus \{v\}$ given by

$$\mathrm{del}_{\Delta}(v) := \{F \in \Delta \mid v \notin F\}$$

and the $\mathrm{lk}_{\Delta}(v)$, called the *link* of v, is the subcomplex of $\mathrm{del}_{\Delta}(v)$ given by

$$\mathrm{lk}_{\Delta}(v) := \{F \in \Delta \mid v \notin F \text{ and } F \cup \{v\} \in \Delta\}.$$

If V is the only facet of Δ, then Δ is called a *simplex*.

Definition 2.4 A simplicial complex Δ is called *vertex decomposable*, if either Δ is a simplex, or $\Delta = \phi$, or Δ contains a vertex v such that

(a) both of $\mathrm{del}_{\Delta}(v)$ and $\mathrm{lk}_{\Delta}(v)$ are vertex decomposable, and
(b) every facet of $\mathrm{del}_{\Delta}(v)$ is a facet of Δ.

A vertex v satisfying condition (b) is called a *shedding* vertex of Δ.

The *independence complex* $\Delta_{\mathcal{C}}$ of a clutter $\mathcal{C}$ is a simplicial complex whose faces are the stable sets of $\mathcal{C}$. Note that the Stanley–Reisner ideal $I_{\Delta_{\mathcal{C}}}$ is equal to $I(\mathcal{C})$.

Definition 2.5 Let I and J be ideals of a ring R. The *colon ideal* of I with respect to J is an ideal of R, denoted by $(I : J)$ and is defined as

$$(I : J) = \{u \in R \mid uv \in I \text{ for all } v \in J\}.$$

For an element $f \in R$, $(I : f) := (I : (f))$. If I is a monomial ideal and $f \in R$ is a monomial, then by [16, Proposition 1.2.2], we have

$$(I : f) = \left\langle \frac{u}{\gcd(u, f)} \mid u \in G(I) \right\rangle.$$

Let I be an ideal in a ring R. Then a presentation $I = \bigcap_{i=1}^{k} \mathfrak{q}_i$, where each $\mathfrak{q}_i$ is a primary ideal, is called a primary decomposition of I. A primary decomposition is irredundant if no $\mathfrak{q}_i$ can be omitted in the presentation and $\mathfrak{p}_i \neq \mathfrak{p}_j$ for $i \neq j$, where $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Each $\mathfrak{p}_i$ is said to be an associated prime ideal of I and the set of associated prime ideals of I is denoted by $\mathrm{Ass}(I)$ or $\mathrm{Ass}(R/I)$. From [1, Theorem 4.5] and [16, Corollary 1.3.10], we can say that the associated prime ideals of a monomial ideal I are precisely the prime ideals of the form $(I : f)$, for some monomial $f \in R$. If a monomial ideal cannot be written as a proper intersection of two other monomial ideals, then we say it is irreducible. For a monomial ideal I, a presentation of the form $I = \bigcap_{i=1}^{k} \mathfrak{q}_i$, where each $\mathfrak{q}_i$ is irreducible, is called an irredundant irreducible decomposition if no $\mathfrak{q}_i$ can be omitted in the decomposition. By [29, Theorem 6.1.17] and [16, Corollary 1.3.2], any monomial ideal can be written as an unique irredundant intersection of irreducible monomial ideals, and the irreducible components are generated precisely by pure powers of the variables.

Lemma 2.6 *([29, Lemma 6.3.37]) Let $I = I(\mathcal{C})$ be an edge ideal of a clutter $\mathcal{C}$. Then $\mathfrak{p} \in \mathrm{Ass}(I)$ if and only if $\mathfrak{p} = \langle C \rangle$ for some minimal vertex cover C of $\mathcal{C}$.*

Definition 2.7 [[27, Construction 21.7]] The *polarization* of monomials of the type $x_i^{a_i}$ is defined as $x_i^{a_i}(\mathrm{pol}) = \prod_{j=1}^{a_i} x_{i,j}$ and the *polarization* of $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ is defined as

$$\mathbf{x}^{\mathbf{a}}(\mathrm{pol}) = x_1^{a_1}(\mathrm{pol}) \cdots x_n^{a_n}(\mathrm{pol}).$$

For a monomial ideal $I = \langle \mathbf{x}^{\mathbf{a_1}}, \ldots, \mathbf{x}^{\mathbf{a_n}} \rangle \subseteq R$, the *polarization* $I(\mathrm{pol})$ is defined to be the square-free monomial ideal

$$I(\mathrm{pol}) = \langle \mathbf{x}^{\mathbf{a_1}}(\mathrm{pol}), \ldots, \mathbf{x}^{\mathbf{a_n}}(\mathrm{pol}) \rangle,$$

in the ring $R(\mathrm{pol}) = K[x_{i,j} \mid 1 \leq i \leq n,\ 1 \leq j \leq r_i]$, where r_i is the power of x_i in the lcm of $\{\mathbf{x}^{\mathbf{a_1}}, \ldots, \mathbf{x}^{\mathbf{a_n}}\}$.

Definition 2.8 Let **F.** be a minimal graded free resolutions of R/I as R module such that

$$\mathbf{F.}\ 0 \to \bigoplus_{j} R(-j)^{\beta_{k,j}} \to \cdots \to \bigoplus_{j} R(-j)^{\beta_{1,j}} \to R \to R/I \to 0,$$

where I is a graded ideal of the graded ring R. The *Castelnuovo–Mumford regularity* of R/I (in short *regularity* of R/I) is denoted by $\text{reg}(R/I)$ and defined as

$$\text{reg}(R/I) = \max\{j - i \mid \beta_{i,j} \neq 0\}.$$

The *projective dimension* of R/I is defined to be

$$\text{pd}(R/I) = \max\{i \mid \beta_{i,j} \neq 0 \text{ for some } j\} = k.$$

For a clutter $\mathcal{C}$ and $A \subseteq V(\mathcal{C})$, we define the *induced clutter* $\mathcal{C} \setminus A$ on the vertex set $V(\mathcal{C}) \setminus A$ with $E(\mathcal{C} \setminus A) = \{e \in E(\mathcal{C}) \mid e \cap A = \phi\}$. If $A = \{x_i\}$, $\mathcal{C} \setminus \{x_i\}$ is the clutter, called deletion of x_i, and in this case $\langle I(\mathcal{C} \setminus \{x_i\}), x_i\rangle = \langle I(\mathcal{C}), x_i\rangle$. We often denote the ideal generated by I and f by (I, f) instead of $\langle I, f\rangle$.

Definition 2.9 Let G be a graph. A set $M \subseteq E(G)$ is said to be a *matching* in G if no two edges in M are adjacent, i.e., no two edges in M share a common vertex. A matching $M = \{e_1, \ldots, e_k\}$ is called an *induced matching* in G if the induced subgraph on the vertex set $\bigcup_{i=1}^{k} e_i$ contains only M as the edge set, i.e., no two edges in M are joined by an edge. The cardinality of a maximum (largest) induced matching in G is known as the *induced matching number* of G, denoted by $\text{im}(G)$.

3 v-number of monomial ideals via polarization

The v-number of square-free monomial ideals has been discussed broadly in [18]. In this section, we study the v-number of arbitrary monomial ideals using the technique of polarization and generalize some results of [18].

Proposition 3.1 *Let I be a monomial ideal and $f = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial such that $(I : f) = \langle x_{s_1}, \ldots, x_{s_k}\rangle$, where $a_i \leq$ highest power of x_i appears in $G(I)$. Then*

$$(I(\text{pol}) : f(\text{pol})) = \langle x_{s_1,b_{s_1}}, \ldots, x_{s_k,b_{s_k}}\rangle,$$

where $b_{s_i} - 1$ is the power of x_{s_i} in f.

Proof We know that $(I : f) = \langle \frac{u}{\text{gcd}(u, f)} \mid u \in G(I)\rangle$. Therefore, $x_{s_i} = \frac{u_i}{\text{gcd}(u_i, f)}$, for some $u_i \in G(I)$. Consider the ring $R(\text{pol})$ corresponding to the ideal $I(\text{pol})$. By the given condition on f, we have $f(\text{pol}) \in R(\text{pol})$. Let $u_i = x_1^{b_1} \cdots x_n^{b_n}$. Then

$\gcd(u_i, f) = x_1^{b_1} \cdots x_{s_i}^{b_{s_i}-1} \cdots x_n^{b_n}$ and we get

$$\frac{u_i(\text{pol})}{\gcd(u_i(\text{pol}), f(\text{pol}))} = x_{s_i, b_{s_i}},$$

where $b_{s_i} - 1$ is the power of x_{s_i} in f. Now suppose for some $u \in G(I)$, we have $\frac{u}{\gcd(u, f)} \in \langle x_m \rangle$, where $m \in \{s_1, \ldots, s_k\}$. Let $u = x_1^{r_1} \cdots x_n^{r_n}$ and $\gcd(u, f) = x_1^{p_1} \cdots x_n^{p_n}$. Then we have $r_m - p_m \geq 1$ and $r_i - p_i \geq 0$ for all $i \in [n] \setminus \{m\}$. Therefore, we can write

$$\begin{aligned}\frac{u(\text{pol})}{\gcd(u(\text{pol}), f(\text{pol}))} =& \frac{u(\text{pol})}{\gcd(u, f)(\text{pol})}\\ =&(x_{1,p_1+1} \cdots x_{1,r_1}) \cdots (x_{m,p_m+1} \cdots x_{m,r_m})\\ &\cdots (x_{n,p_n+1} \cdots x_{n,r_n}).\end{aligned}$$

Since $r_m \geq p_m + 1$, it follows that $\frac{u(\text{pol})}{\gcd(u(\text{pol}), f(\text{pol}))} \in \langle x_{m,p_m+1} \rangle$. Now $x_m^{p_m} \mid f$ but $x_m^{p_m+1} \nmid f$ imply p_m is the power of x_m in f. Therefore, $x_{m,p_{m+1}} \in (I(\text{pol}) : f(\text{pol}))$, and hence,

$$(I(\text{pol}) : f(\text{pol})) = \langle x_{s_1, b_{s_1}}, \ldots, x_{s_k, b_{s_k}} \rangle,$$

where $b_{s_i} - 1$ is the power of x_{s_i} in f. □

Lemma 3.2 *Let $I \subseteq R$ be a monomial ideal and $f \notin I$ be a monomial in R. If $\langle x_{s_1}, \ldots, x_{s_k} \rangle \subseteq (I : f)$, where all s_i are distinct, then there exists a monomial $g \in R$ such that*

$$(I : g) = \langle x_{s_1}, \ldots, x_{s_r} \rangle \text{ and } f \mid g,$$

for some $r \geq k$.

Proof We know that $(I : f) = \langle \frac{u}{\gcd(u, f)} \mid u \in G(I) \rangle$. If we have $\langle x_{s_1}, \ldots, x_{s_k} \rangle = (I : f)$, then take $g = f$ and we are done. So we may assume $\langle x_{s_1}, \ldots, x_{s_k} \rangle \subsetneq (I : f)$. Then for each $1 \leq i \leq k$, there exists $u_i \in G(I)$, such that $\frac{u_i}{\gcd(u_i, f)} = x_{s_i}$. Let $G(I) = \{u_1, \ldots, u_k, u_{k+1}, \ldots, u_{k+m}\}$. If $\frac{u_{k+1}}{\gcd(u_{k+1}, f)}$ is divided by any of $x_{s_1}, \ldots, x_{s_k}$, then $\frac{u_{k+1}}{\gcd(u_{k+1}, f)} \in \langle x_{s_1}, \ldots, x_{s_k} \rangle$ and set $f_1 = f$. If $\frac{u_{k+1}}{\gcd(u_{k+1}, f)} = h_1$ is not divided by any of $x_{s_1}, \ldots, x_{s_k}$, then h_1 is a non-constant monomial in $K[x_{s_{k+1}}, \ldots, x_{s_n}]$ as $f \notin I$. Without loss of generality, we assume $x_{s_{k+1}} \mid h_1$ and set $f_1 = \dfrac{fh_1}{x_{s_{k+1}}}$. Then $\dfrac{u_i}{\gcd(u_i, f_1)} = x_{s_i}$ is true for each $1 \le i \le k$. Now

$$\begin{aligned}\frac{u_{k+1}}{\gcd(u_{k+1}, f_1)} &= \frac{u_{k+1}}{\gcd\big(u_{k+1}, \frac{fh_1}{x_{s_{k+1}}}\big)}\\ &= \frac{u_{k+1}}{\gcd(u_{k+1}, f)\gcd\big(\frac{u_{k+1}}{\gcd(u_{k+1}, f)}, \frac{h_1}{x_{s_{k+1}}}\big)}\\ &= \frac{h_1}{\gcd\big(h_1, \frac{h_1}{x_{s_{k+1}}}\big)}\\ &= x_{s_{k+1}}.\end{aligned}$$

Therefore, we get $\langle x_{s_1}, \ldots, x_{s_{k+1}}\rangle \subseteq (I : f_1)$. Continue this process with the remaining elements of $G(I)$. Finally, we get $f_m = g$ such that

$$(I : g) = \langle x_{s_1}, \ldots, x_{s_r}\rangle \text{ and } f \mid g,$$

for some $r \ge k$. □

Proposition 3.3 *Let I be a monomial ideal and consider*

$$\mathfrak{p} = \langle x_{s_1,b_{s_1}}, \ldots, x_{s_k,b_{s_k}}\rangle \in \operatorname{Ass}(I(\mathrm{pol})),$$

such that there exists no embedded prime of I containing $\langle x_{s_1}, \ldots, x_{s_k}\rangle$. Let $D = \{d \mid \exists\, M \in R_d$ with $(I(\mathrm{pol}) : M) = \mathfrak{p}\}$. Then to find $\min D$ *we can choose M in such a way that $(I(\mathrm{pol}) : M) = \mathfrak{p}$, and for that M we get a monomial f with* $\deg f \le \deg M$, *such that $(I : f) = \langle x_{s_1}, \ldots, x_{s_k}\rangle$.*

Proof Since $\mathfrak{p} \in \operatorname{Ass}(I(\mathrm{pol}))$, by [10, Proposition 2.5], there exists an irredundant irreducible primary component of I such that $\mathfrak{q} = \langle x_{s_1}^{a_{s_1}}, \ldots, x_{s_k}^{a_{s_k}}\rangle$, where $a_{s_i} \ge b_{s_i} \ge 1$ for $i = 1, \ldots, k$. Let $\mathcal{C}$ be the clutter corresponding to the ideal $I(\mathrm{pol})$, i.e., $I(\mathcal{C}) = I(\mathrm{pol})$. By Lemma 2.2, there exists a stable set A of $\mathcal{C}$ such that

$$(I(\mathrm{pol}) : X_A) = \langle \mathcal{N}_{\mathcal{C}}(A)\rangle = \mathfrak{p}.$$

Now there exists $e_i \in E(\mathcal{C})$ such that $e_i \subseteq A \cup \{x_{s_i,b_{s_i}}\}$, for $1 \le i \le k$. For each $1 \le i \le k$, we have $X_{e_i} = u_i(\mathrm{pol})$ for some $u_i \in G(I)$. Again $u_i \in \mathfrak{q}$ implies $x_{s_{j_i}}^{a_{s_{j_i}}}$ divides u_i, for some $1 \le j_i \le k$. Now $x_{s_{j_i},b_{s_{j_i}}} \notin A$ and $e_i \subseteq A \cup \{x_{s_i,b_{s_i}}\}$ imply $s_{j_i} = s_i$. Let c_{s_i} be the power of x_{s_i} in u_i. Then $c_{s_i} \ge a_{s_i}$. Now consider the prime ideal $\mathfrak{p}' = \langle x_{s_1,a_{s_1}}, \ldots, x_{s_k,a_{s_k}}\rangle \in \operatorname{Ass}(I(\mathrm{pol}))$. Let

$$B = A \cup_{i=1}^{k} \{x_{s_i,b_{s_i}}\} \setminus \cup_{i=1}^{k}\{x_{s_i,a_{s_i}}\}.$$

For any $u \in G(I)$, there exists some $x_{s_j}^{a_{s_j}}$ which divides u, where $1 \leq j \leq k$. Therefore, $x_{s_j,a_{s_j}} \mid u(\text{pol})$, which implies that corresponding edge of $u(\text{pol})$ in $E(\mathcal{C})$ is not contained in B, and hence, B is a stable set in $\mathcal{C}$. Also it is clear that

$$|A| = |B| \text{ and } (I(\text{pol}) : X_B) = \langle \mathcal{N}_{\mathcal{C}}(B) \rangle = \mathfrak{p}'.$$

Take the stable set $B' = \cup_{i=1}^{k}(e_i \setminus \{x_{s_i,a_{s_i}}\}) \subseteq B$. Again using Lemma 2.2, we can write

$$|B'| \leq |B| \text{ and } (I(\text{pol}) : X_{B'}) = \langle \mathcal{N}_{\mathcal{C}}(B') \rangle = \mathfrak{p}'.$$

Now consider the monomial $f = \text{lcm}\left\{\frac{u_1}{x_{s_1}}, \ldots, \frac{u_k}{x_{s_k}}\right\}$. Then we have $\deg(f) = |B'|$ and $x_{s_i} f \in I$, for all $1 \leq i \leq k$, which imply $\langle x_{s_1}, \ldots, x_{s_k} \rangle \subseteq (I : f)$. Since

$$X_{B'} = \text{lcm}\left\{\frac{u_1(\text{pol})}{x_{s_1,a_{s_1}}}, \ldots, \frac{u_k(\text{pol})}{x_{s_k,a_{s_k}}}\right\} \text{ and } \frac{u_i(\text{pol})}{\gcd(u_i(\text{pol}), X_{B'})} = x_{s_i,a_{s_i}},$$

we have $\frac{u_i}{\gcd(u_i, f)} = x_{s_i}$, for all $1 \leq i \leq k$. Let u be a minimal generator of I other than $u_1, \ldots, u_k$. If $\frac{u}{\gcd(u, f)} \notin \langle x_{s_1}, \ldots, x_{s_k} \rangle$, then by Lemma 3.2 there exists an associated prime ideal of I properly containing $\langle x_{s_1}, \ldots, x_{s_k} \rangle$. This gives a contradiction to our assumption and so $\frac{u}{\gcd(u, f)} \in \langle x_{s_1}, \ldots, x_{s_k} \rangle$. Hence, $(I : f) = \langle x_{s_1}, \ldots, x_{s_k} \rangle$ and $\deg(f) = |B'| \leq |B| = |A|$. To find $\min D$ we can choose $M = X_A$, for some stable set A in $\mathcal{C}$, and this completes the proof. □

Theorem 3.4 *Let I be a monomial ideal. If there exists*

$$\mathfrak{p} = \langle x_{s_1,b_{s_1}}, \ldots, x_{s_k,b_{s_k}} \rangle \in \text{Ass}(I(\text{pol})),$$

such that $\text{v}(I(\text{pol})) = \text{v}_{\mathfrak{p}}(I(\text{pol}))$ *and if there is no embedded prime of I properly containing* $\langle x_{s_1}, \ldots, x_{s_k} \rangle$, *then*

$$\text{v}(I) = \text{v}(I(\text{pol})).$$

Proof Let $\mathfrak{p}' = \{x_{t_1}, \ldots, x_{t_r}\} \in \text{Ass}(I)$ and f be the monomial such that

$$(I : f) = \mathfrak{p}' \text{ with } \deg(f) = \text{v}(I).$$

Then power of any x_i in f is less than or equal to the highest power of x_i appearing in $G(I)$. Then by Proposition 3.1, we have

$$(I(\text{pol}) : f(\text{pol})) = \langle x_{t_1,b_{t_1}}, \ldots, x_{t_r,b_{t_r}} \rangle \in \text{Ass}(I(\text{pol})),$$

where $b_{t_i} - 1$ is the power of x_{t_i} in f for each $1 \leq i \leq r$. Thus, we have

$$\mathrm{v}(I(\mathrm{pol})) \le \mathrm{v}(I),$$

as $\deg(f) = \deg(f(\mathrm{pol}))$. Again there exists $\mathfrak{p} \in \mathrm{Ass}(I(\mathrm{pol}))$ and a square-free monomial M such that

$$(I(\mathrm{pol}) : M) = \mathfrak{p} \text{ with } \deg(M) = \mathrm{v}(I(\mathrm{pol})).$$

Then by Proposition 3.3, there exists a monomial g such that

$$(I : g) = \{x_{s_1}, \dots, x_{s_k}\} \in \mathrm{Ass}(I) \text{ with } \deg(g) \le \deg(M).$$

So we get $\mathrm{v}(I) \le \mathrm{v}(I(\mathrm{pol}))$, and hence, $\mathrm{v}(I) = \mathrm{v}(I(\mathrm{pol}))$. □

Corollary 3.5 *For a monomial ideal I, we have $\mathrm{v}(I(\mathrm{pol})) \le \mathrm{v}(I)$. Moreover, if I has no embedded prime, then $\mathrm{v}(I(\mathrm{pol})) = \mathrm{v}(I)$.*

The converse of the above Corollary 3.5 is not necessarily true, i.e., despite having an embedded prime of a monomial ideal I, it may happen that $\mathrm{v}(I(\mathrm{pol})) = \mathrm{v}(I)$.

Example 3.6 Let $I = \langle x_1x_2^2, x_2x_3^2, x_1^2x_3\rangle \subseteq \mathbb{Q}[x_1, x_2, x_3]$. Then

$$I = \langle x_2^2, x_3\rangle \cap \langle x_1, x_3^2\rangle \cap \langle x_1^2, x_2\rangle \cap \langle x_1^2, x_2^2, x_3^2\rangle.$$

Here $\mathrm{Ass}(I) = \{\langle x_2, x_3\rangle, \langle x_1, x_3\rangle, \langle x_1, x_2\rangle, \langle x_1, x_2, x_3\rangle\}$. With the help of [13, Procedure A1], we obtain $\mathrm{v}(I) = 3 = \mathrm{v}(I(\mathrm{pol}))$. In fact, we have $(I : x_1x_2x_3) = \langle x_1, x_2, x_3\rangle$, where $\langle x_1, x_2, x_3\rangle$ is an embedded prime of I. Also, we have $I(\mathrm{pol}) = \langle x_{1,1}x_{2,1}x_{2,2}, x_{2,1}x_{3,1}x_{3,2}, x_{3,1}x_{1,1}x_{1,2}\rangle$ and

$$(I(\mathrm{pol}) : x_{1,1}x_{2,1}x_{3,1}) = \langle x_{1,2}, x_{2,2}, x_{3,2}\rangle.$$

Note that $v(I(\mathrm{pol})) = 3 = \deg(x_{1,1}x_{2,1}x_{3,1})$ and this justifies our Theorem 3.4, as there is no associated prime ideals of I properly containing $\langle x_1, x_2, x_3\rangle$.

From Theorem 3.4, we get relations between the v-number of an arbitrary monomial ideal and the v-number of its polarization. The next result is the generalization of [18, Proposition 3.1] for a monomial ideal with some special properties.

For a graded module $M \neq 0$, we define $\alpha(M) := \min\{\deg(f) \mid f \in M \setminus \{0\}\}$.

Proposition 3.7 *Let I be a monomial ideal. Suppose there exists*

$$\mathfrak{p} = \langle x_{s_1,b_{s_1}}, \dots, x_{s_k,b_{s_k}}\rangle \in \mathrm{Ass}(I(\mathrm{pol}))$$

such that $\mathrm{v}(I(\mathrm{pol})) = \mathrm{v}_{\mathfrak{p}}(I(\mathrm{pol}))$ and there is no embedded prime of I properly containing $\langle x_{s_1}, \dots, x_{s_k}\rangle$. Then, we have

$$\mathrm{v}(I) = \min\{\alpha((I : \mathfrak{p})/I) \mid \mathfrak{p} \in \mathrm{Ass}(I)\}.$$

Proof If I is a prime ideal, then $(I : 1) = I$, $(I : I) = R$, and therefore, we have

$$\mathrm{v}(I) = \alpha((I : I)/I) = \alpha(R/I) = 0.$$

So we may assume I is not prime. Now there exists $\mathfrak{p}' \in \mathrm{Ass}(I)$ and $f \in R_d$ such that $(I : f) = \mathfrak{p}'$ with $\mathrm{v}(I) = \deg(f)$. Then $f \in (I : \mathfrak{p}')$ but $f \notin I$, and so $f \in (I : \mathfrak{p}') \setminus I$. Thus,

$$\mathrm{v}(I) \geq \alpha((I : \mathfrak{p}')/I) \geq \min\{\alpha((I : \mathfrak{p})/I) \mid \mathfrak{p} \in \mathrm{Ass}(I)\}.$$

Let us assume $\mathfrak{p}'' = \langle x_{s_1}, \ldots, x_{s_k} \rangle \in \mathrm{Ass}(I)$. Let $h \in (I : \mathfrak{p}'') \setminus I$ be a monomial. Then $hx_{s_i} \in I$ implies $u_i \mid hx_{s_i}$, for some $u_i \in G(I)$, where $1 \leq i \leq k$. Now $u_i \nmid h$ as $h \notin I$ and so $x_{s_i} \mid u_i$. We take $h' = \mathrm{lcm}\left\{\dfrac{u_1}{x_{s_1}}, \ldots, \dfrac{u_k}{x_{s_k}}\right\}$. Then $x_{s_i}h' \in I$ for all $1 \leq i \leq k$ and $\deg(h') \leq \deg(h)$. Each $\dfrac{u_i}{x_{s_i}} \mid h$ implies h' divide h, and hence, $h' \notin I$ as $h \notin I$. Therefore, we can say $h'(\mathrm{pol}) \in R(\mathrm{pol})$ and since $\dfrac{u_i}{\gcd(u_i, h')} = x_{s_i}$ for each $i \in \{1, \ldots, k\}$, we also have $(h'x_{s_1} \cdots x_{s_k})(\mathrm{pol}) \in R(\mathrm{pol})$. Now consider $g = \dfrac{(h'x_{s_1} \cdots x_{s_k})(\mathrm{pol})}{x_{s_1,1} \cdots x_{s_k,1}}$. Then $g \notin I(\mathrm{pol})$ as $h' \notin I$. Again $x_{s_i}h' \in \langle u_i \rangle$ implies $gx_{s_i,1} \in \langle u_i(\mathrm{pol}) \rangle$, for all $i \in \{1, \ldots, k\}$. Therefore, $g \in \left(I(\mathrm{pol}) : \langle x_{s_1,1}, \ldots, x_{s_k,1} \rangle\right) \setminus I(\mathrm{pol})$, and from [10, Proposition 2.5] we also have

$$\langle x_{s_1,1}, \ldots, x_{s_k,1} \rangle \in \mathrm{Ass}(I(\mathrm{pol})).$$

Note that $\deg(g) = \deg(h') \leq \deg(h)$. Since $h \in (I : \mathfrak{p}'') \setminus I$ is arbitrary, we have

$$\alpha\left((I(\mathrm{pol}) : \langle x_{s_1,1}, \ldots, x_{s_k,1} \rangle)/I(\mathrm{pol})\right) \leq \alpha((I : \mathfrak{p}'')/I).$$

Therefore, using [18, Proposition 3.1] we get $\mathrm{v}(I(\mathrm{pol})) \leq \min\{\alpha((I : \mathfrak{p})/I) \mid \mathfrak{p} \in \mathrm{Ass}(I)\}$ and also by Theorem 3.4, $\mathrm{v}(I) = \mathrm{v}(I(\mathrm{pol}))$. Hence, $\mathrm{v}(I) \leq \min\{\alpha((I : \mathfrak{p})/I) \mid \mathfrak{p} \in \mathrm{Ass}(I)\}$ and the result follows. □

Lemma 3.8 *Let $I_1 \subseteq R_1 = K[\mathbf{x}]$ and $I_2 \subseteq R_2 = K[\mathbf{y}]$ be two monomial ideals and K be a field. Consider $R = K[\mathbf{x}, \mathbf{y}]$ and $I = I_1R + I_2R$. Then $\mathfrak{p} \in \mathrm{Ass}(R/I)$ if and only if $\mathfrak{p} = \mathfrak{p}_1R + \mathfrak{p}_2R$, where $\mathfrak{p}_1 \in \mathrm{Ass}(R_1/I_1)$ and $\mathfrak{p}_2 \in \mathrm{Ass}(R_2/I_2)$.*

Proof Since I is the smallest ideal containing I_1R and I_2R, we have

$$G(I) = G(I_1) \sqcup G(I_2).$$

Let $\mathfrak{p} = \langle x_{s_1}, \ldots, x_{s_k}, y_{t_1}, \ldots, y_{t_l} \rangle \in \mathrm{Ass}(R/I)$. Then there exists a monomial $f \in R$, such that $(I : f) = \mathfrak{p}$. We can write $f = f_1f_2$, where $f_1 \in R_1$ and $f_2 \in R_2$. Now

$$\mathfrak{p} = (I : f) = \left\langle \frac{u}{\gcd(u, f)}, \frac{v}{\gcd(v, f)} \mid u \in G(I_1),\ v \in G(I_2) \right\rangle.$$

Also, we have

$$\frac{u}{\gcd(u, f)} = \frac{u}{\gcd(u, f_1)} \in \mathfrak{p} \cap R_1 \text{ and } \frac{v}{\gcd(v, f)} = \frac{v}{\gcd(v, f_2)} \in \mathfrak{p} \cap R_2,$$

for all $u \in G(I_1)$ and $v \in G(I_2)$. Therefore, we get

$$(I_1 : f_1) = \langle x_{s_1}, \ldots, x_{s_k} \rangle = \mathfrak{p}_1 \in \mathrm{Ass}(R_1/I_1)$$

and

$$(I_2 : f_2) = \langle y_{t_1}, \ldots, y_{t_l} \rangle = \mathfrak{p}_2 \in \mathrm{Ass}(R_2/I_2).$$

Hence, $\mathfrak{p} = \mathfrak{p}_1 R + \mathfrak{p}_2 R$, where $\mathfrak{p}_1 \in \mathrm{Ass}(R_1/I_1)$ and $\mathfrak{p}_2 \in \mathrm{Ass}(R_2/I_2)$.

Let $\mathfrak{p} = \mathfrak{p}_1 R + \mathfrak{p}_2 R$, where $\mathfrak{p}_1 \in \mathrm{Ass}(R_1/I_1)$ and $\mathfrak{p}_2 \in \mathrm{Ass}(R_2/I_2)$. Then clearly $\mathfrak{p}$ is a prime ideal in R containing I. We have monomials $f_1 \in R_1$ and $f_2 \in R_2$, such that $(I_1 : f_1) = \mathfrak{p}_1$ and $(I_2 : f_2) = \mathfrak{p}_2$. Setting $f = f_1 f_2$, we get for all $u \in G(I_1)$ and $v \in G(I_2)$,

$$\frac{u}{\gcd(u, f)} = \frac{u}{\gcd(u, f_1)} \in \mathfrak{p}_1 \text{ and } \frac{v}{\gcd(v, f)} = \frac{v}{\gcd(v, f_2)} \in \mathfrak{p}_2.$$

As $(I : f) = \left\langle \frac{w}{\gcd(w, f)} \mid w \in G(I) \right\rangle$ and $G(I) = G(I_1) \sqcup G(I_2)$, we have $(I : f) = \mathfrak{p}$, i.e., $\mathfrak{p} \in \mathrm{Ass}(R/I)$.

□

In [18, Proposition 3.8], the additivity of the v-number for square-free monomial ideals was shown. In the next proposition, we show that the v-number is additive for arbitrary monomial ideals.

Proposition 3.9 (v-number *is additive) Let* $I_1 \subseteq R_1 = K[\mathbf{x}]$ *and* $I_2 \subseteq R_2 = K[\mathbf{y}]$ *be two monomial ideals and consider* $R = K[\mathbf{x}, \mathbf{y}]$. *Then we have*

$$\mathrm{v}(I_1 R + I_2 R) = \mathrm{v}(I_1) + \mathrm{v}(I_2).$$

Proof Let $I = I_1 R + I_2 R$. Then there exists a monomial $f \in R$ and $\mathfrak{p} \in \mathrm{Ass}(R/I)$ such that

$$(I : f) = \mathfrak{p} \text{ and } \mathrm{v}(I) = \deg(f).$$

We can write $f = f_1 f_2$, such that $f_1 \in R_1$ and $f_2 \in R_2$. Then by Lemma 3.8, we have $\mathfrak{p} = \mathfrak{p}_1 R + \mathfrak{p}_2 R$, where

$$(I_1 : f_1) = \mathfrak{p}_1 \in \mathrm{Ass}(R_1/I_1) \text{ and } (I_2 : f_2) = \mathfrak{p}_2 \in \mathrm{Ass}(R_2/I_2).$$

By definition of v-number, $\mathrm{v}(I_1) + \mathrm{v}(I_2) \leq \deg(f_1) + \deg(f_2) = \mathrm{v}(I)$. For the reverse inequality, we choose monomials $f_i \in R_i$ and $\mathfrak{p}_i \in \mathrm{Ass}(R_i/I_i)$, such that

$(I_i : f_i) = \mathfrak{p}_i$ and $\mathrm{v}(I_i) = \deg(f_i)$, where $i \in \{1, 2\}$. Again by Lemma 3.8, we have $\mathfrak{p} = \mathfrak{p}_1 R + \mathfrak{p}_2 R \in \mathrm{Ass}(R/I)$ and $(I : f_1 f_2) = \mathfrak{p}$. Thus, $\mathrm{v}(I) \leq \deg(f_1 f_2) = \mathrm{v}(I_1) + \mathrm{v}(I_2)$. □

The next result is the generalization of [18, Proposition 3.9].

Proposition 3.10 *Let I be a complete intersection monomial ideal with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_k}\}$. If $d_i = \deg(\mathbf{x}^{\mathbf{a}_i})$ for all $i = 1, \ldots, k$, then we have*

$$\mathrm{v}(I) = d_1 + \cdots + d_k - k = \mathrm{reg}(R/I).$$

Proof I is complete intersection implies that $|G(I)| = \mathrm{ht}(I)$. By [10, Proposition 2.3], $\mathrm{ht}(I) = \mathrm{ht}(I(\mathrm{pol}))$, and therefore, we have

$$|G(I(\mathrm{pol}))| = |G(I)| = \mathrm{ht}(I) = \mathrm{ht}(I(\mathrm{pol})),$$

i.e., $I(\mathrm{pol})$ is a complete intersection. According to [16, Corollary 1.6.3], $\mathrm{reg}(R/I) = \mathrm{reg}(R(\mathrm{pol})/I(\mathrm{pol}))$. Since I is complete intersection, I has no embedded prime. Therefore, by Theorem 3.4, we have $\mathrm{v}(I) = \mathrm{v}(I(\mathrm{pol}))$. Again $\deg(\mathbf{x}^{\mathbf{a}_i}(\mathrm{pol})) = \deg(\mathbf{x}^{\mathbf{a}_i}) = d_i$, for $i = 1, \ldots, k$, and hence, by [18, Proposition 3.9], we have

$$\mathrm{v}(I) = \mathrm{v}(I(\mathrm{pol})) = d_1 + \cdots + d_k - k = \mathrm{reg}(R/I). \quad \square$$

Proposition 3.11 *Let I be a monomial ideal and f be a monomial such that $f \notin I$. Then $\mathrm{v}(I) \leq \mathrm{v}(I : f) + \deg(f)$.*

Proof Suppose $(I : f)$ is an associated prime of I. Then by definition of v-number, $\mathrm{v}(I) \leq \deg(f)$ and so the result follows as Proposition 3.10 implies $\mathrm{v}(I : f) = 0$. Now assume $(I : f) \notin \mathrm{Ass}(I)$. Then there exists an associated prime $\mathfrak{p}$ of $(I : f)$ and a monomial g such that $((I : f) : g) = \mathfrak{p}$ and $\mathrm{v}(I : f) = \deg(g)$. Note that $(I : fg) = \mathfrak{p}$, and hence, we get

$$\mathrm{v}(I) \leq \deg(fg) = \mathrm{v}(I : f) + \deg(f). \quad \square$$

Corollary 3.12 *Let I be a monomial ideal and x_i be a variable such that $x_i \notin I$. Then $\mathrm{v}(I) \leq \mathrm{v}(I : x_i) + 1$.*

Proof The result follows by taking $f = x_i$ in Proposition 3.11. □

Some properties of v-number of edge ideals of graphs were discussed in [18, proposition 3.12]. We extend some of those for edge ideals of clutters, i.e., for any square-free monomial ideal in Proposition 3.13.

Proposition 3.13 *Let $I = I(\mathcal{C})$ be an edge ideal of a clutter $\mathcal{C}$. Then the following results are true.*

(i) If $\{x_i\} \notin E(\mathcal{C})$, then $\mathrm{v}(I) \leq \mathrm{v}(I : x_i) + 1$, where $x_i \in V(\mathcal{C})$.
(ii) $\mathrm{v}(I : x_i) \leq \mathrm{v}(I)$, for some $x_i \in V(\mathcal{C})$.

(iii) *If* $\mathrm{v}(I) \geq 2$, *then* $\mathrm{v}(I : x_i) < \mathrm{v}(I)$ *for some* $x_i \in V(\mathcal{C})$.
(iv) $\mathrm{v}(I(\mathcal{C} \setminus \{x_i\})) \leq \mathrm{v}(I(\mathcal{C}))$ *for some* $x_i \in V(\mathcal{C})$.

Proof (i) Follows from Corollary 3.12.
(ii) By Lemma 2.2 and Theorem 2.3, we have a stable set A of $\mathcal{C}$ such that

$$(I : X_A) = \langle \mathcal{N}_{\mathcal{C}}(A) \rangle = \mathfrak{p} \in \mathrm{Ass}(I) \text{ and } \mathrm{v}(I) = |A|.$$

We are assuming $I \neq \mathfrak{m}$, otherwise, $(\mathfrak{m} : x_i) = R$ for any $x_i \in V(\mathcal{C})$. Then there exists some $x_i \in V(\mathcal{C})$, which is not in $\mathfrak{p}$. Note that $\mathfrak{p} \subseteq (I : x_i X_A)$. Let us take $f \in (I : x_i X_A)$. Then $f x_i \in \mathfrak{p}$ and $x_i \notin \mathfrak{p}$ together imply $f \in \mathfrak{p}$. Thus, $(I : x_i X_A) = \mathfrak{p}$, i.e., $((I : x_i) : X_A) = \mathfrak{p}$. Therefore, we have

$$\mathrm{v}(I : x_i) \leq |A| = \mathrm{v}(I).$$

(iii) Take a stable set A of $\mathcal{C}$ with

$$(I : X_A) = \langle \mathcal{N}_{\mathcal{C}}(A) \rangle = \mathfrak{p} \in \mathrm{Ass}(I) \text{ and } \mathrm{v}(I) = |A|.$$

Since $|A| \geq 2$, we have $A' = A \setminus \{x_i\} \neq \phi$ for any $x_i \in A$. Then

$$(I : X_A) = (I : x_i X_{A'}) = ((I : x_i) : X_{A'}) = \mathfrak{p},$$

which gives $\mathrm{v}(I : x_i) \leq |A'| < |A| = \mathrm{v}(I)$.
(iv) Note that if $I = I(\mathcal{C})$ then $\mathrm{v}(I, x_i) = \mathrm{v}(I(\mathcal{C} \setminus \{x_i\}))$. Take A and $\mathfrak{p}$ as in (ii). Pick $x_i \in V(\mathcal{C}) \setminus A$ and so A is a stable set of the clutter $\mathcal{C} \setminus \{x_i\}$ also. Let $e \in E(\mathcal{C} \setminus \{x_i\}) \subseteq E(\mathcal{C})$. Then there exists $y \in \mathcal{N}_{\mathcal{C}}(A)$ such that $y \in e$. Also, by definition of $\mathcal{N}_{\mathcal{C}}(A)$, there exists $e' \in E(C)$, such that $e' \subseteq A \cup \{y\}$. Now $x_i \notin e$ implies $y \neq x_i$, and therefore, $x_i \notin e'$. Then we have $e' \in E(\mathcal{C} \setminus \{x_i\})$, which implies $y \in \mathcal{N}_{\mathcal{C} \setminus \{x_i\}}(A)$. Thus, $\mathcal{N}_{\mathcal{C} \setminus \{x_i\}}(A)$ is a vertex cover of $\mathcal{C} \setminus \{x_i\}$ and A being a stable set of $\mathcal{C} \setminus \{x_i\}$, using Lemma 2.2, we have

$$(I(\mathcal{C} \setminus \{x_i\}) : X_A) = \langle \mathcal{N}_{\mathcal{C} \setminus \{x_i\}}(A) \rangle.$$

Indeed, it is easy to see that $\mathcal{N}_{\mathcal{C} \setminus \{x_i\}}(A) = \mathcal{N}_{\mathcal{C}}(A) \setminus \{x_i\}$. Hence, by Theorem 2.3, we get $\mathrm{v}(I, x_i) = \mathrm{v}(I(\mathcal{C} \setminus \{x_i\})) \leq |A| = \mathrm{v}(I)$. □

Proposition 3.14 *For a graph* G, *we have* $\mathrm{v}(I(G)) \leq \alpha_0(G)$.

Proof Let A be a minimal vertex cover of G with $|A| = \alpha_0(G)$. Since A is a minimal vertex cover for G, for each $x \in A$, there exists an edge $e_x \in E(G)$, which is not adjacent to any other vertex of A, i.e. $e_x \cap A = \{x\}$. Let $e_x = \{x, y_x\}$, for every $x \in A$ and $B = \{y_x \mid x \in A\}$. For different $x \in A$, some y_x may coincide and so $|B| \leq |A| = \alpha_0(G)$. By our choice of B, it is clear that $A \cap B = \phi$ and so B is a stable set in G. Also we have $\mathcal{N}_G(B) = A$, and hence, by Lemma 2.2, $(I(G) : X_B) = \langle \mathcal{N}_G(B) \rangle$. Thus, Theorem 2.3 gives $\mathrm{v}(I(G)) \leq |B| \leq \alpha_0(G)$. □

4 Bound of regularity and induced matching number by the v-number

The *line* graph of a graph G, denoted by $L(G)$, is a graph on the vertex set $V(L(G)) = E(G)$ and the edge set

$$E(L(G)) = \{\{e_i, e_j\} \subseteq E(G) \mid e_i \cap e_j \neq \phi \text{ in } G\}.$$

For a positive integer k, the *k-th power* of G, denoted by G^k, is the graph on the vertex set $V(G^k) = V(G)$ such that there is an edge between two vertices of G^k if and only if the distance between the corresponding vertices in G is less than or equal to k.

Finding a matching in a graph G is equivalent to finding an independent set in $L(G)$ (see [4]) and an induced matching in G is equivalent to an independent set in $L^2(G)$, the square of $L(G)$ (see [5]). Now we want to know the relation between $\mathrm{v}(I(G))$ and $\mathrm{im}(G)$, which might be a step forward towards answering Question 5.2. In the next proposition, we try to see $\mathrm{v}(I(G))$ in terms of some invariants in the graph $L(G)$. What is remaining is to see the connection between $\mathrm{v}(I(G))$ with invariants of the graph $L^2(G)$.

Proposition 4.1 *Let G be a simple graph and $L(G)$ be its line graph. Suppose that $c(L(G))$ denotes the minimum number of cliques in $L(G)$, such that any vertex of $L(G)$ is either a vertex of those cliques or adjacent to some vertices of those cliques. Then $\mathrm{v}(I(G)) = c(L(G))$.*

Proof Lemma 2.2 and Theorem 2.3 ensure that there exists a stable set A in G such that

$$(I(G) : X_A) = \langle \mathcal{N}_G(A) \rangle \text{ and } |A| = \mathrm{v}(I(G)).$$

For each $x_i \in A$, let $E_G(x_i) = \{e_{i1}, \dots, e_{im_i}\}$ be the set of edges incident to the vertex x_i. Then $E_G(x_i)$ forms a clique in $L(G)$, for each $x_i \in A$. Since A is stable, cliques corresponding to each $E_G(x_i)$, where $x_i \in A$, are disjoint to each other. Let $e \in V(L(G))$ be a vertex other than the vertices of the cliques $E_G(x_i)$, for $x_i \in A$. Let $e = \{u, v\}$ be the corresponding edge of e in G. Then, one of u or v should belong to $\mathcal{N}_G(A)$, as $\mathcal{N}_G(A)$ is a minimal vertex cover of G. Assume $u \in \mathcal{N}_G(A)$ and our choice of e ensures that $v \notin A$. Then $u \in \mathcal{N}_G(x_i)$ and so $\{x_i, u\} = e_{ik}$, for some $1 \leq k \leq m_i$. Therefore, e and e_{ik} being adjacent in G, we have e is adjacent to the vertex $e_{ik} \in E_G(x_i)$ in $L(G)$. Hence, we have $c(L(G)) \leq |A| = \mathrm{v}(I(G))$.

Now for the reverse inequality, let $r = c(L(G))$ and we can choose r disjoint cliques $\mathcal{C}_1, \dots, \mathcal{C}_r$ in $L(G)$, such that any vertex of $L(G)$ is either a vertex of $\mathcal{C}_i$ or adjacent to some vertices of $\mathcal{C}_i$, $1 \leq i \leq r$. Since each $\mathcal{C}_i$ is a clique in $L(G)$, corresponding edges in G of vertices of $\mathcal{C}_i$ either shares a common vertex or they form a triangle in G. Suppose corresponding edges of $\mathcal{C}_1, \dots, \mathcal{C}_k$ in G share a common vertex, say $x_1, \dots, x_k$, respectively, and corresponding edges in G of $\mathcal{C}_{k+1}, \dots, \mathcal{C}_r$ form triangles. Take one vertex from each triangle formed by the corresponding edges in G of $\mathcal{C}_{k+1}, \dots, \mathcal{C}_r$, say $x_{k+1}, \dots, x_r$. Since $\mathcal{C}_1, \dots, \mathcal{C}_r$ are disjoint in $L(G)$, $B =$

$\{x_1 \ldots, x_r\}$ is a stable set in G. We will show that $\mathcal{N}_G(B)$ forms a minimal vertex cover for G. Pick any $e = \{u, v\} \in E(G)$. Then, $e \in V(L(G))$ and if $e \in \mathcal{C}_i$ for $1 \leq i \leq r$ then one of u or v should belong to $\mathcal{N}_G(x_i)$. Suppose e is a vertex other than the vertices of $\mathcal{C}_1, \ldots, \mathcal{C}_r$ in $L(G)$. Then $e \cap B = \phi$ and e is adjacent to some vertex $e_{ij} \in \mathcal{C}_i$ in $L(G)$, $1 \leq i \leq r$. Therefore, e and e_{ij} share a common vertex, say u, in G. Then $u \in \mathcal{N}_G(x_i)$ and so $\mathcal{N}_G(B)$ is a vertex cover for G. Thus, using Lemma 2.2, we get

$$(I(G) : X_B) = \langle \mathcal{N}_G(B) \rangle,$$

and hence, $\mathrm{v}(I(G)) \leq |B| = r = c(L(G))$. □

Let G be a simple graph and $e \in E(G)$ be an edge.

- We define $G \setminus e$ as the graph on $V(G)$ just by removing the edge e from $E(G)$.
- By G_e, we mean the induced subgraph of G on the vertex set $V(G) \setminus \mathcal{N}_{G \setminus e}[e]$.
- The contraction of e on G (see [3, Definition 5.2]), denoted by G/e, is defined by $V(G/e) = (V(G) \setminus e) \cup \{w\}$, where w is a new vertex, and $E(G/e) = E(G \setminus e) \cup \{\{w, z\} : z \in \mathcal{N}_{G \setminus e}(e)\}$.

Let us first cite some results which give some bounds of $\mathrm{reg}(R/I(G))$ in terms of some graphs obtained from G:

(1) From [14, Theorem 3.5], we get

$$\mathrm{reg}(R/I(G)) \leq \max\{\mathrm{reg}(R/I(G \setminus e)), \mathrm{reg}(R/I(G_e)) + 1\}.$$

(2) In [3], Biyikoğlu and Civan proved that

$$\mathrm{reg}(R/I(G/e)) \leq \mathrm{reg}(R/I(G)) \leq \mathrm{reg}(R/I(G/e)) + 1.$$

(3) ([30, Theorem 3]). Let $J \subseteq V(G)$ be an induced clique in G. Then

$$\mathrm{reg}(R/I(G)) \leq \mathrm{reg}(R/I(G \setminus J)) + 1,$$

where $G \setminus J$ denotes the induced subgraph on $V(G) \setminus J$.

As a consequence of the above results, we prove the following Proposition 4.2, which might be helpful in finding a relation between the v-number and regularity using induction hypothesis.

Proposition 4.2 *Let G be a simple graph. Then*

(i) $\mathrm{v}(I(G \setminus e)) \leq \mathrm{v}(I(G)) + 1$*, for any $e \in E(G)$.*
(ii) $\mathrm{v}(I(G)) \leq \mathrm{v}(I(G \setminus J)) + 1$*, where J is a clique of G.*
(iii) There exists an edge $e \in E(G)$, such that $\mathrm{v}(I(G/e)) \leq \mathrm{v}(I(G))$.

Proof (i) By Lemma 2.2 and Theorem 2.3, there exists a stable set A of G such that

$$(I(G) : X_A) = \langle \mathcal{N}_G(A) \rangle \textit{ and } \mathrm{v}(I(G)) = |A|.$$

Clearly, A is a stable set too in $G \setminus e$.
Case I. Suppose $e \cap A = \phi$. Then $\mathcal{N}_{G\setminus e}(A) = \mathcal{N}_G(A)$ and it is also a vertex cover for $G \setminus e$. Thus, using Lemma 2.2, we have

$$(I(G \setminus e) : X_A) = \langle \mathcal{N}_{G\setminus e}(A) \rangle,$$

which implies $\mathrm{v}(I(G \setminus e)) \leq \mathrm{v}(I(G))$.
Case II. Let $e \cap A \neq \phi$ and $u \in e \cap A$, where $e = \{u, v\}$. Then $v \in \mathcal{N}_G(A)$. If $v \in \mathcal{N}_{G\setminus e}(A)$, then $\mathcal{N}_G(A) = \mathcal{N}_{G\setminus e}(A)$ is a vertex cover of $G \setminus e$. Therefore, by Lemma 2.2,

$$(I(G \setminus e) : X_A) = \langle \mathcal{N}_{G\setminus e}(A) \rangle,$$

and so $\mathrm{v}(I(G \setminus e)) \leq \mathrm{v}(I(G))$. If $v \notin \mathcal{N}_{G\setminus e}(A)$, then $A \cup \{v\}$ is a stable set in $G \setminus e$ and $\mathcal{N}_{G\setminus e}(A \cup \{v\})$ forms a vertex cover for $G \setminus e$. Again by Lemma 2.2, we have

$$(I(G \setminus e) : X_{A\cup\{v\}}) = \langle \mathcal{N}_{G\setminus e}(A \cup \{v\}) \rangle.$$

Hence, $\mathrm{v}(I(G \setminus e)) \leq |A| + 1 = \mathrm{v}(I(G)) + 1$.
(ii) From Lemma 2.2 and Theorem 2.3, we have a stable set A of $G \setminus J$ such that

$$(I(G \setminus J) : X_A) = \langle \mathcal{N}_{G\setminus J}(A) \rangle \text{ and } \mathrm{v}(I(G \setminus J)) = |A|.$$

Note that A is also a stable set in G. If all vertices of J is contained in $\mathcal{N}_G(A)$, then $\mathcal{N}_G(A)$ is a vertex cover of G and by Lemma 2.2, we have $(I(G) : X_A) = \langle \mathcal{N}_G(A) \rangle$. Thus, by Theorem 2.3, $\mathrm{v}(I(G)) \leq \mathrm{v}(I(G \setminus J))$. Suppose there is a vertex $x \in J$ such that $x \notin \mathcal{N}_G(A)$. Then $A \cup \{x\}$ is a stable set in G and $\mathcal{N}_G(A \cup \{x\})$ is a vertex cover of G. So by Lemma 2.2,

$$(I(G) : X_{A\cup\{x\}}) = \langle \mathcal{N}_G(A \cup \{x\}) \rangle.$$

Hence, by Theorem 2.3, $\mathrm{v}(I(G)) \leq \mathrm{v}(I(G \setminus J)) + 1$.
(iii) By Lemma 2.2 and Theorem 2.3, there is a stable set A of G such that

$$(I(G) : X_A) = \langle \mathcal{N}_G(A) \rangle \text{ and } \mathrm{v}(I(G)) = |A|.$$

Let $u \in A$ and by minimality of A there exists $v \in \mathcal{N}_G(u)$ such that $v \notin \mathcal{N}_G(A \setminus \{u\})$. Contract the edge $e = \{u, v\}$ in G and let after contracting e we get the vertex w in G/e instead of u and v. Then $B = (A \setminus \{u\}) \cup \{w\}$ is a stable set in G/e. It is clear that $\mathcal{N}_{G/e}(B)$ is a vertex cover for G/e. Using Lemma 2.2 we get

$$(I(G/e) : X_B) = \langle \mathcal{N}_{G/e}(B) \rangle.$$

Therefore, by Theorem 2.3, $\mathrm{v}(I(G/e)) \leq |B| = |A| = \mathrm{v}(I(G))$. □

In [21], Liu and Zhou gave formula for induced matching number of a graphs in terms of its induced bipartite subgraph. Using that formula, we show that $\mathrm{v}(I(G)) \leq \mathrm{im}(G)$ for any bipartite graph G (see Theorem 4.5).

Theorem 4.3 *([21, Theorem 2.1]) For a simple graph G,*

$$\mathrm{im}(G) = \max_H \min\{|X'| : X' \subseteq X \text{ and } Y \subseteq \mathcal{N}_H(X')\},$$

where H is an induced bipartite subgraph of G with partite sets X, Y and has no isolated vertices.

Theorem 4.4 *([21, Theorem 2.3]) Let G be a bipartite graph with partite sets X, Y and has no isolated vertices. Then*

$$\mathrm{im}(G) = \max_H \min\{|X'| : X' \subseteq H \subseteq X \text{ and } \mathcal{N}_G(X') = \mathcal{N}_G(H)\}.$$

Theorem 4.5 *Let G be a bipartite graph with partite sets X and Y. Then* $\mathrm{v}(I(G)) \leq \mathrm{im}(G)$. *Moreover, we have*

$$\mathrm{v}(I(G)) \leq \mathrm{reg}(R/I(G)).$$

Proof Let $X_1 \subseteq X$ be such that $\mathcal{N}_G(X_1) = \mathcal{N}_G(X)$ and

$$|X_1| = \min\{|X'| : X' \subseteq X \text{ and } \mathcal{N}_G(X') = \mathcal{N}_G(X)\}.$$

Then X_1 is a stable set in G and $\mathcal{N}_G(X)$ being a minimal vertex cover for G, we have by Theorem 2.3, $\mathrm{v}(I(G)) \leq |X_1|$. Now taking $H = X$ in Theorem 4.4, we get

$$\mathrm{v}(I(G)) \leq |X_1| \leq \mathrm{im}(G).$$

Therefore, by [14, Theorem 4.1] (or [20, Lemma 2.2]), we have

$$\mathrm{v}(I(G)) \leq \mathrm{im}(G) \leq \mathrm{reg}(R/I(G)).$$

□

Corollary 4.6 *Let G be a graph with a vertex $x \in V(G)$, such that any of the following holds:*

(i) The independent complex $\Delta(G \setminus \{x\})$ or $\Delta(G \setminus \mathcal{N}_G[x])$ is vertex decomposable.
(ii) The graph $G \setminus \{x\}$ or $G \setminus \mathcal{N}_G[x]$ is a bipartite graph.

Then $\mathrm{v}(I(G)) \leq \mathrm{reg}(R/I(G)) + 1$.

Proof Let $I = I(G)$. If the condition (i) or (ii) holds, then by Theorem 4.5 and [18, Theorem 3.13], we have

$$\mathrm{v}(I, x) \leq \mathrm{reg}(R/(I, x)) \text{ or } \mathrm{v}(I : x) \leq \mathrm{reg}(R/(I : x)).$$

Now by [18, Lemma 3.12], we have

$$\mathrm{v}(I) \le \mathrm{v}(I, x) + 1 \text{ and } \mathrm{v}(I) \le \mathrm{v}(I : x) + 1.$$

Also $G \setminus \{x\}$ and $G \setminus \mathcal{N}_G[x]$ being subgraphs of G [29, Proposition 6.4.6] implies that $\mathrm{reg}(R/(I, x)) \le \mathrm{reg}(R/I)$ and $\mathrm{reg}(R/(I : x)) \le \mathrm{reg}(R/I)$. Therefore, we have $\mathrm{v}(I) \le \mathrm{reg}(R/I) + 1$. □

Corollary 4.7 *If G is an unicyclic graph, i.e., a graph with only one induced cycle, then* $\mathrm{v}(I(G)) \le \mathrm{reg}(R/I(G)) + 1$.

Proof Choose a vertex x from the unique induced cycle of G. Then $G \setminus \{x\}$ is a bipartite graph, and hence, by Corollary 4.6, the result follows. □

Definition 4.8 ([5]) A *clique neighbourhood K_c* is the set of edges of a clique c in a graph G together with some edges which are adjacent to some edges of the clique c.

Theorem 4.9 *([5, Theorem 2]) Let G be a chordal graph. Then*

$$\mathrm{im}(G) = \min\{|\mathscr{N}| : \mathscr{N} \textit{ is a set of clique neighbourhoods in } G \textit{ which covers } E(G)\}.$$

We now prove that $\mathrm{v}(I(G)) \le \mathrm{im}(G) = \mathrm{reg}(R/I(G))$ is true for chordal graphs. This also follows from Theorem 4.11, where we prove the same inequality for a more general class. However, the proofs of Theorems 4.10 and 4.11 are of different flavours.

Theorem 4.10 *For a chordal graph G, we have*

$$\mathrm{v}(I(G)) \le \mathrm{im}(G) = \mathrm{reg}(R/I(G)).$$

Proof Let $\mathscr{N}$ be a set of clique neighbourhoods in G which covers $E(G)$. Let $\mathscr{N} = \{K_{c_1}, \dots, K_{c_m}\}$, where each K_{c_i}, for $1 \le i \le m$, is a clique neighbourhood containing the clique c_i such that every edge of K_{c_i} is adjacent to some edges of c_i. Now choose a maximal stable set from the set of vertices $\bigcup_{i=1}^m V(c_i)$, name it A. Since A is a maximal stable set in $\bigcup_{i=1}^m V(c_i)$, we have $\bigcup_{i=1}^m V(c_i) \setminus A \subseteq \mathcal{N}_G(A)$. Let $e \in E(G)$ be an edge. Then $e \in K_{c_i}$, for some $1 \le i \le m$. If $e \in E(c_i)$, then $e \cap \mathcal{N}_G(A) \neq \phi$. Suppose $e \notin E(c_i)$. Then e is adjacent to some edges of c_i, i.e., e is incident to some vertex $v \in V(c_i)$. Now if $v \notin \mathcal{N}_G(A)$, then $v \in A$ as A is a maximal stable set in $\bigcup_{i=1}^m V(c_i)$ and so $e \setminus \{v\} \in \mathcal{N}_G(A)$. As $e \in E(G)$ is an arbitrarily chosen edge, $\mathcal{N}_G(A)$ is a vertex cover of G. Therefore, by Lemma 2.2, we have

$$(I(G) : X_A) = \langle \mathcal{N}_G(A) \rangle,$$

and so Theorem 2.3 gives $\mathrm{v}(I(G)) \le |A|$. Since $A \subseteq \bigcup_{i=1}^m V(c_i)$ is a stable set and c_i's are cliques, $|A| \le m = |\mathscr{N}|$. This is true for any set of clique neighbourhoods in G which covers $E(G)$. Hence, by Theorem 4.9, $\mathrm{v}(I(G)) \le \mathrm{im}(G)$ and by [15, Corollary 6.9], $\mathrm{im}(G) = \mathrm{reg}(R/I(G))$. □

Theorem 4.11 *If G is a (C_4, C_5)-free vertex decomposable graph, then* $\mathrm{v}(I(G)) \leq \mathrm{im}(G) = \mathrm{reg}(I(G))$.

Proof If G is a (C_4, C_5)-free vertex decomposable graph, then by [2, Theorem 24], we get $\mathrm{im}(G) = \mathrm{reg}(I(G))$ and also G being a vertex decomposable graph, by [18, Theorem 3.13], we get $\mathrm{v}(I(G)) \leq \mathrm{reg}(I(G))$. □

Let G be a graph with $V(G) = \{x_1, \ldots, x_n\}$. Consider the graph W_G by adding a new set of vertices $Y = \{y_1, \ldots, y_n\}$ to G and attaching the edges $\{x_i, y_i\}$ to G for each $1 \leq i \leq n$. The graph W_G is known as the *whisker graph* of G and the attached edges $\{x_i, y_i\}$ are called the *whiskers*.

Theorem 4.12 *Let G be a simple graph and W_G be the whisker graph of G. Then* $\mathrm{v}(I(W_G)) \leq \mathrm{im}(W_G)$.

Proof Let A be a maximal stable set of G. Then the set of whiskers $M = \{\{x_i, y_i\} \mid x_i \in A\}$ forms an induced matching in W_G. Therefore, we have $\mathrm{im}(W_G) \geq |A|$. Now A is a stable set in W_G too and it is clear from the construction of W_G that $\mathcal{N}_{W_G}(A)$ is a vertex cover of W_G. Thus, applying Lemma 2.2 and Theorem 2.3, we get $\mathrm{v}(I(W_G)) \leq |A| \leq \mathrm{im}(W_G)$. □

Theorem 4.12 also follows from [13, Theorem 2 and Lemma 1].

Definition 4.13 ([17]) Let G be a simple graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$, without any isolated vertex. For an independent set $S \subseteq V(G)$, the S-suspension of G, denoted by G^S, is the graph given by

- $V(G^S) = V(G) \cup \{x_{n+1}\}$, where x_{n+1} is a new vertex;
- $E(G^S) = E(G) \cup \{\{x_i, x_{n+1}\} \mid x_i \notin S\}$.

Proposition 4.14 *Let G be a simple graph and G^S be a S-suspension of G with respect to an independent set $S \subseteq V(G)$. Then* $\mathrm{v}(I(G^S)) = 1$.

Proof Take $A = \{x_{n+1}\}$. Then we have

$$\mathcal{N}_{G^S}(A) = V(G) \setminus S = V(G^S) \setminus (S \cup \{x_{n+1}\}).$$

By construction of G^S, $S \cup \{x_{n+1}\}$ is an independent set of G^S, and hence, $\mathcal{N}_{G^S}(A)$ is a vertex cover of G^S. Therefore, from Lemma 2.2 it follows that

$$(I(G^S) : X_A) = \langle \mathcal{N}_{G^S}(A) \rangle.$$

Thus, by Theorem 2.3, we have $\mathrm{v}(I(G^S)) = |A| = 1$. □

Corollary 4.15 *Let n be any positive integer. Then we have a graph G such that* $\mathrm{reg}(R/I(G)) - \mathrm{v}(I(G)) = n$*, i.e., for a simple connected graph G,* $\mathrm{reg}(R/I(G))$ *can be arbitrarily larger than* $\mathrm{v}(I(G))$.

Proof We can choose a connected graph H with $\operatorname{reg}(R'/I(H)) = n + 1$, where $R' = K[V(H)]$. Consider the graph $G = H^S$, where S is a stable set of H. Then Proposition 4.14 gives $\mathrm{v}(I(G)) = 1$, and by [17, Lemma 1.5], we have $\operatorname{reg}(R/I(G)) = \operatorname{reg}(R'/I(H)) = n+1$, where $R = R'[x_{n+1}]$. Therefore, $\operatorname{reg}(R/I(G)) - \mathrm{v}(I(G)) = n$. □

Theorem 4.16 *(Terai, [28]) Let I be a square-free monomial ideal in a polynomial ring R. Then* $\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1 = \operatorname{pd}(R/I^\vee)$.

Definition 4.17 Let $I = I(\mathcal{C})$ be an edge ideal of a clutter $\mathcal{C}$. The *Alexander dual ideal* of I, denoted by $I^\vee$, is the ideal defined by

$$I^\vee = \big\langle \{X_C \mid C \text{ is a minimal vertex cover of } \mathcal{C}\} \big\rangle.$$

From [18, Lemma 3.16], we have $\mathrm{v}(I(\mathcal{C})^\vee) \geq \alpha_0(\mathcal{C}) - 1$.

Proposition 4.18 *Let $\mathcal{C}$ be a clutter that cannot be written as a union of two disjoint clutters. If the answer to Question 5.2 (see Sect. 5) is true and $\mathrm{v}(I^\vee) \geq \alpha_0(\mathcal{C}) + 1$, then $I = I(\mathcal{C})$ is not Cohen–Macaulay.*

Proof By the Auslander–Buchsbaum formula [27, Formula 15.3], we have

$$\operatorname{depth}(R/I) + \operatorname{pd}(R/I) = n.$$

By the given condition, Question 5.2 implies $\mathrm{v}(I^\vee) \leq \operatorname{reg}(R/I) + 1$. Therefore, using Theorem 4.16 we get

$$\begin{aligned} \operatorname{depth}(R/I) &= n - 1 - \operatorname{reg}(R/I^\vee) \\ &\leq n - \mathrm{v}(I^\vee) \\ &\leq n - 1 - \alpha_0(\mathcal{C}) = \dim(R/I) - 1. \end{aligned}$$

Hence, I is not Cohen–Macaulay as $\operatorname{depth}(R/I) < \dim(R/I)$. □

For a large class of square-free monomial ideals I, we have $\mathrm{v}(I) \leq \operatorname{reg}(R/I)$. The following Corollary gives a sufficient condition for the non-Cohen–Macaulayness of $I = I(\mathcal{C})$.

Corollary 4.19 *If $\alpha_0(\mathcal{C}) \leq \mathrm{v}(I^\vee) \leq \operatorname{reg}(R/I^\vee)$, then $I = I(\mathcal{C})$ is not Cohen–Macaulay.*

Proof Follows directly from the proof of Proposition 4.18. □

5 Some open problems on v-number

Jaramillo and Villarreal disproved [26, Conjecture 4.2] by giving an example [18, Example 5.4] of a graph G for which $\mathrm{v}(I(G)) > \operatorname{reg}(R/I(G))$. They also proposed an open problem, whether $\mathrm{v}(I) \leq \operatorname{reg}(R/I) + 1$ for any square-free monomial ideal I. The answer is no and we give the following example in support:

Example 5.1 Take $H = G_1 \sqcup G_2$, with $G_1 \simeq G_2 \simeq G$, where G is the graph in [18, Example 5.4]. It was given in [18, Example 5.4] that $\mathrm{v}(I(G)) = 3$ and $\mathrm{reg}(R'/I(G)) = 2$, where $R' = \mathbb{Q}[V(G)]$. Then by Proposition 3.9 and by [30, Lemma 7], we have $\mathrm{v}(I(H)) = 6$ and $\mathrm{reg}(R/I(H)) = 4$, where $R = \mathbb{Q}[V(H)]$. Hence, $\mathrm{v}(I(H)) > \mathrm{reg}(R/I(H)) + 1$.

In our example, the graph H is not connected. So we can modify the open problem by putting the condition of connectedness:

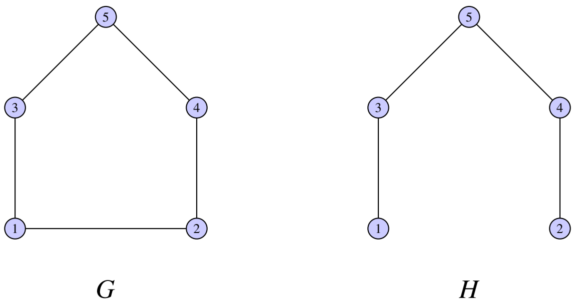
Question 5.2 *Let $\mathcal{C}$ be a clutter which cannot be written as a union of two disjoint clutters. Then is it true that*

$$\mathrm{v}(I(\mathcal{C})) \leq \mathrm{reg}(R/I(\mathcal{C})) + 1?$$

This question for graphs would be the following:
Let G be a simple connected graph. Is it true that

$$\mathrm{v}(I(G)) \leq \mathrm{reg}(R/I(G)) + 1?$$

For a simple graph G, we have from [14, Theorem 4.1] (or [20, Lemma 2.2]) that $\mathrm{im}(G) \leq \mathrm{reg}(R/I(G))$. So, we want to find a relation between $\mathrm{v}(I(G))$ and $\mathrm{im}(G)$ for a connected graph G, which might help us find an answer to Question 5.2 for connected graphs. In many cases, we have $\mathrm{v}(I(G)) \leq \mathrm{im}(G)$, for example, if G is a bipartite graph (see Theorem 4.5) or if G is a (C_4, C_5)-free vertex decomposable graph (see Theorem 4.11) or G is a whisker graph (see Theorem 4.12). Let us consider the following example:



We have $\mathrm{v}(I(G)) = 2$, $\mathrm{im}(G) = 1$ and $\mathrm{v}(I(H)) = 1$, $\mathrm{im}(H) = 2$. In view of this, we can ask the following question, which can answer Question 5.2 for edge ideals of graphs.

Question 5.3 *For a connected graph G, is it true that*

$$\mathrm{v}(I(G)) \leq \mathrm{im}(G) + 1?$$

Moreover, we can generalize Question 5.3 for edge ideals of a clutter $\mathcal{C}$, which cannot be written as a union of two disjoint clutters (see Question 5.4).

Let $\mathcal{C}$ be a clutter. A set $M \subseteq E(\mathcal{C})$ is called a *matching* in $\mathcal{C}$ if the edges in M are pairwise disjoint. The matching M is called an *induced matching* in $\mathcal{C}$ if the induced subclutter on the vertex set $(\bigcup_{e\in M} e)$ contains only M as the edge set. The maximum size of an induced matching in $\mathcal{C}$ is known as the *induced matching number* of $\mathcal{C}$, denoted by $\mathrm{im}(\mathcal{C})$.

Let $\mathcal{C}$ be a clutter and let $\{e_1, \ldots, e_k\}$ form an induced matching in $\mathcal{C}$. Then [14, Theorem 4.2] (or [25, Corollary 3.9]) gives

$$\sum_{i=1}^{k}(|e_i| - 1) \leq \mathrm{reg}(R/I(\mathcal{C})).$$

Question 5.4 *Let $\mathcal{C}$ be a clutter which cannot be written as a union of two disjoint clutters. Does there exist an induced matching $\{e_1, \ldots, e_k\}$ of $\mathcal{C}$ such that*

$$\mathrm{v}(I(\mathcal{C})) \leq \sum_{i=1}^{k}(|e_i| - 1) + 1?$$

An answer to Question 5.4, together with [14, Theorem 4.2] (or [25, Corollary 3.9]) can give an answer to Question 5.2.

The next problem is about our interest to know the relation between $\mathrm{depth}(R/I)$ and $\mathrm{v}(I)$ for any square-free monomial ideal. If R/I is Cohen–Macaulay, then by Theorem 2.3, $\mathrm{v}(I) \leq \mathrm{depth}(R/I)$.

Question 5.5 *For a square-free monomial ideal I, does $\mathrm{v}(I) \leq \mathrm{depth}(R/I)$ hold? Also can we say that*

$$\mathrm{v}(I) \geq \dim(R/I) - \mathrm{depth}(R/I)?$$

If we can relate $\mathrm{v}(I(G))$ with respect to some invariants of $L^2(G)$, then it would be easy to answer Question 5.3 because $\mathrm{im}(G) = \beta_0(L^2(G))$.

Question 5.6 *Find $\mathrm{v}(I(G))$ in terms of some invariants of $L^2(G)$, where G is a connected graph.*

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