



# Smallest graphs with given automorphism group

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## Abstract

For a finite group  $G$ , denote by  $\alpha(G)$  the minimum number of vertices of any graph  $\Gamma$  having  $\text{Aut}(\Gamma) \cong G$ . In this paper, we prove that  $\alpha(G) \leq |G|$ , with specified exceptions. The exceptions include four infinite families of groups, and 17 other small groups. Additionally, we compute  $\alpha(G)$  for the groups  $G$  such that  $\alpha(G) > |G|$  where the value  $\alpha(G)$  was previously unknown.

**Keywords** Graph · Automorphism group · Minimum order · Generalised dicyclic group · Generalised quaternion group

## 1 Introduction

In [4] it is shown that every finite group can be realised, up to isomorphism, as the automorphism group of a finite graph; in fact, for every finite group  $G$  there exist infinitely many finite graphs having automorphism group isomorphic to  $G$ . Given a finite group  $G$ , define  $\alpha(G)$  to be the smallest number of vertices of any graph  $\Gamma$  having  $\text{Aut}(\Gamma) \cong G$ . The problem of finding  $\alpha(G)$  has been considered by many authors. The value of  $\alpha(G)$  has been determined in [1] for abelian groups  $G$ , in [7, 9, 10, 15] for dihedral groups  $G$ , in [13] for quasi-dihedral groups and quasi-abelian groups  $G$  and in [8] for generalised quaternion groups  $G$ . The question has also been investigated for several families of finite simple groups in [14]. A recent survey on this problem can be found in [22]. In [2], Babai showed that  $\alpha(G) \leq 2|G|$ , for every finite group  $G$  that is not cyclic of order 3, 4 or 5. In this paper, we improve Babai's bound to  $|G|$ , with specified exceptions (including four infinite families of groups).

In Table 1, we let  $\text{Dic}_m = \langle a, b \mid a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1} \rangle$  for  $m = 3, 5, 6$ , which is a group of order  $4m$ , and  $G_{16} = \langle a, b \mid a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$ ,  $G'_{16} = \langle a, b \mid a^8 = b^2 = 1, bab^{-1} = a^5 \rangle$ , which are groups of order 16.

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**Theorem 1** *Let  $G$  be a finite group of order  $n$ . Then one of the following is true:*

- (i)  $\alpha(G) \leq n$ ,
- (ii)  $G$  is cyclic of order  $p^k$  or  $2p$ , where  $p$  is prime and  $k$  is positive integer ( $n \neq 2$ ),
- (iii)  $G$  is  $Q_{2^r}$  or  $Q_{2^r} \times C_2$ , where  $Q_{2^r}$  is the generalised quaternion group of order  $2^r, r \geq 3$ ,
- (iv)  $G$  is one of the 17 exceptional groups of order at most 25 shown in Table 1.

*If (ii), (iii) or (iv) holds, then  $\alpha(G) > n$ ; indeed, if (ii) holds,  $\alpha(G)$  is as described in Propositions 3.2 and 3.3; if (iii) holds,  $\alpha(G) = 2n$  or  $\alpha(G) = n + 2$  for  $G = Q_{2^r}$  or  $G = Q_{2^r} \times C_2$ , respectively; if (iv) holds,  $\alpha(G)$  is as shown in Table 1.*

As a consequence, we deduce when equality holds in Babai’s bound.

**Corollary 1.1** *Let  $G$  be a finite group of order  $n$ . Then  $\alpha(G) = 2n$  if and only if*

- (i)  $G$  is a generalised quaternion group of order  $2^r, r \geq 3$ , or
- (ii)  $G$  is cyclic of order  $p$ , where  $p$  is prime and  $p \geq 7$ , or
- (iii)  $G$  is the abelian group  $C_3 \times C_3$ .

The main tool in the proof of Theorem 1 is the GRR-Theorem (Theorem 2.3). It states that, with some specified families of exceptions, every finite group  $G$  has a graphical regular representation (GRR), i.e., a Cayley graph having full automorphism group isomorphic to  $G$ . If a group  $G$  has a GRR, then  $\alpha(G) \leq |G|$ . Therefore, in order to prove Theorem 1, it suffices to study the exceptions in the GRR-Theorem.

Making use of the preliminary results presented in Sect. 2, we prove Theorem 1 across Sects. 3, 4, and 5. Section 3 concerns the case of abelian groups; the key fact is that  $\alpha(G)$  has been determined for every abelian group  $G$ , in [1]. Sections 4 and 5 are devoted to the non-abelian exceptional groups of the GRR-Theorem. In Sect. 4, we address the non-abelian groups  $G$  for which the assertion of Theorem 1 is that  $\alpha(G) \leq |G|$ . For these groups, we construct a graph on at most  $|G|$  vertices having automorphism group isomorphic to  $G$ . In Sect. 5, we show that there exists no graph on at most  $|G|$  vertices with automorphism group isomorphic to  $G$ , for the non-abelian groups  $G$  for which the assertion of Theorem 1 is that  $\alpha(G) > |G|$ . The values of  $\alpha(Q_{2^r})$ ,  $\alpha(Q_{2^r} \times C_2)$  and  $\alpha(G)$  for the non-abelian groups  $G$  in Table 1 are also justified in this section.

**Table 1** The groups  $G$  mentioned in Theorem 1, (iv), and the values  $\alpha(G)$

	$G$	$\alpha(G)$
1–4	$C_{12}, C_{15}, C_{20}, C_{21}$	18, 21, 25, 23
5–8	$C_2 \times C_4, C_3 \times C_3, C_4 \times C_4, C_5 \times C_5$	12, 18, 20, 30
9–10	$C_2 \times C_2 \times C_3, C_2 \times C_3 \times C_3$	13, 20
11–13	$\text{Dic}_3, \text{Dic}_5, \text{Dic}_6$	17, 23, 25
14	$G_{16}$	18
15	$A_4$	16
16	$G'_{16}$	18
17	$Q_8 \times C_3$	25

## 2 Background

Throughout the paper, all groups and graphs mentioned are assumed to be finite.

Let us now present two families of groups that play an important role in our text.

**Definition 2.1** [21] Let  $A$  be an abelian group that contains an element of order  $2k$  for some  $k \geq 2$ . A group  $G$  of the form

$$G = \langle A, b \mid b^4 = 1, b^2 \in A \setminus \{1\}, bab^{-1} = a^{-1}, \forall a \in A \rangle$$

is called *generalised dicyclic* and is denoted by  $\text{Dic}(A, b^2)$ .

If  $A$  is cyclic,  $G$  is simply called *dicyclic*. It is denoted by  $\text{Dic}_m$ , where  $m = \frac{|G|}{4}$ .

A dicyclic group of order  $2^r$  is called generalised *quaternion* ( $r \geq 3$ ). We denote it by  $Q_{2^r}$ .

Note that the existence of the element of order  $2k$  in Definition 2.1 ensures that generalised dicyclic groups are non-abelian.

**Definition 2.2** Let  $A$  be an abelian group. A group  $G$  of the form

$$G = \langle A, b \mid b^2 = 1, bab^{-1} = a^{-1}, \forall a \in A \rangle$$

is called *generalised dihedral* and is denoted by  $\text{Dih}(A)$ .

If  $A$  is cyclic of order  $m$ ,  $G$  is the *dihedral* group of order  $2m$ , which we denote by  $D_{2m}$ .

A graph  $\Gamma$  consists of a vertex set, which we denote by  $V(\Gamma)$  and an edge set, denoted by  $E(\Gamma)$ ; we consider an edge to be an unordered pair of vertices of  $\Gamma$ . We denote an edge between  $v, w \in V(\Gamma)$  by  $v \sim w$  or we say that  $[v, w] \in E(\Gamma)$ . Moreover, if  $X$  is a subgraph of  $\Gamma$  and  $v \in V(X)$ , we denote by  $\rho_X(v)$  the *valency* of  $v$  in  $X$  and by  $\rho(v)$  the valency of  $v$  in the graph  $\Gamma$ . If a group  $G$  acts on a graph  $\Gamma$  and  $v \in V(\Gamma)$ , then we denote by  $\mathcal{O}_v$  the *orbit* containing  $v$ ,  $\mathcal{O}_v = \{gv \mid g \in G\}$ , and by  $G_v$  the *stabilizer* of  $v$ ,  $G_v = \{g \in G \mid gv = v\}$ .

Given a group  $G$  and a set  $S \subset G \setminus \{1\}$  that is inverse-closed, we define the *Cayley graph*  $\text{Cay}(G, S)$  to be the graph with vertex set  $G$  and edges  $\{x, sx\}$ , for all  $x \in G, s \in S$ .

A graph  $\Gamma$  is called a *Graphical Regular Representation* (GRR) of a group  $G$  if there exists some  $S \subset G$  such that  $\text{Cay}(G, S) = \Gamma$  and  $\text{Aut}(\Gamma) \cong G$ . The following theorem is known as the *GRR-Theorem*. It was proven by Godsil [6] for non-solvable groups and by Hetzel [11] for solvable groups using previous results of several authors including [12, 16–21].

**Theorem 2.3** [6] *A group admits a GRR if and only if it is not an abelian group of exponent greater than 2, a generalised dicyclic group, or one of the 13 exceptional groups shown in Table 2.*

**Corollary 2.4** *If  $G$  is a non-abelian, non-generalised dicyclic group that is not one of the 13 groups shown in Table 2, then  $\alpha(G) \leq |G|$ .*

**Table 2** The groups  $G$  mentioned in the GRR-Theorem (Theorem 2.3)

	$G$	$ G $
1–3	$C_2 \times C_2, C_2 \times C_2 \times C_2, C_2 \times C_2 \times C_2 \times C_2$	4, 8, 16
4–6	$D_6, D_8, D_{10}$	6, 8, 10
7	$A_4$	12
8	$\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$	16
9	$G'_{16}$	16
10	$\langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, (ac)^2 = (bc)^2 = 1 \rangle$	18
11	$\langle a, b, c \mid a^3 = c^3 = 1, ac = ca, bc = cb, b^{-1}ab = ac \rangle$	27
12–13	$Q_8 \times C_3, Q_8 \times C_4$	24, 32

Let us now state Babai’s theorem.

**Theorem 2.5** (Babai, [2]) *If  $G$  is a group different from the cyclic groups of order 3, 4, 5 then  $\alpha(G) \leq 2|G|$ .*

The values of  $\alpha(G)$  for cyclic groups  $G$  and graph constructions can be found in [1], which builds on the work of Sabidussi [18]. For the non-cyclic groups, we will use the following construction, given by Babai in [2]:

**Construction 2.6** *Let  $G$  be a non-cyclic group of order  $|G| \geq 6$  and let  $H = \{h_1, \dots, h_d\}$  be a minimal generating set of  $G$ . Let  $G'$  be an isomorphic copy of  $G$  with an isomorphism  $g \mapsto g'$  from  $G$  to  $G'$ . We define the graphs  $X_1$  and  $X_3$  to be such that*

$$V(X_1) = G, \quad E(X_1) = \{[gh_i, gh_{i+1}] \mid g \in G, i = 1, \dots, d - 1\},$$

$$V(X_3) = G', \quad E(X_3) = \{[g'h'_i, g'] \mid g' \in G'\}.$$

Let  $\rho_{X_s}$  be the valency of the vertices of  $X_s, s = 1, 3$ . We define the graph  $X_2$  to be

$$X_2 = \begin{cases} X_3, & \text{if } \rho_{X_1} \neq \rho_{X_3}, \\ \overline{X_3}, & \text{if } \rho_{X_1} = \rho_{X_3}, \end{cases}$$

where  $\overline{X_3}$  is the complement graph of  $X_3$ .

Finally, let us define the graph  $X$  such that

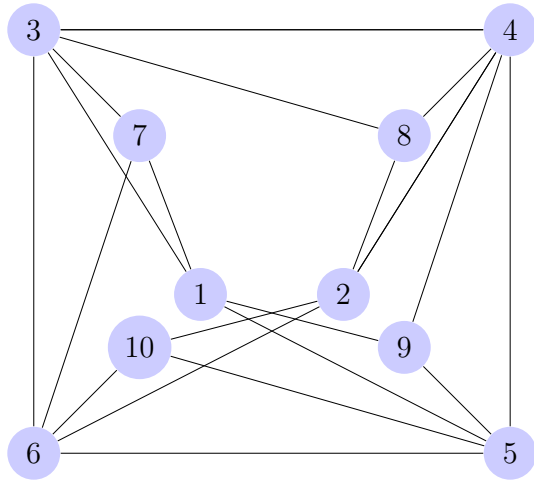
$$V(X) = V(X_1) \cup V(X_2),$$

$$E(X) = E(X_1) \cup E(X_2) \cup \{[g', g], [g', gh_i] \mid g \in G, i = 1, \dots, d\}.$$

The map  $g : V(X) \rightarrow V(X)$  such that

$$g(v) = \begin{cases} gv, & \text{if } v \in V(X_1), \\ g'v, & \text{if } v \in V(X_2), \end{cases}$$

**Fig. 1** A graph on 10 vertices that has automorphism group isomorphic to  $C_4$



is a graph automorphism for every  $g \in G$ , and  $\text{Aut}(X) \cong G$ ; the proof appears in [2].

The inequality in Babai’s Theorem 2.5 does not hold for the three cyclic groups excluded.

**Example 2.7** We will see shortly (Proposition 3.3) that  $\alpha(C_4) = 10$ . A graph on 10 vertices that has automorphism group isomorphic to  $C_4$  is shown in Fig. 1. In particular, the automorphism group of this graph can be realised as the subgroup  $\langle b \rangle$  of  $S_{10}$ , where  $b = (1\ 2)(3\ 4\ 5\ 6)(7\ 8\ 9\ 10)$  ([22, Lemma 2.1.3.3.]).

### 3 Proof of Theorem 1: abelian groups

The aim of this section is to prove that Theorem 1 holds for every abelian group  $G$ .

**Proposition 3.1** *Let  $G$  be an abelian group. Then one of the following holds:*

- (i)  $\alpha(G) \leq |G|$ ,
- (ii)  $G$  is cyclic of order  $p^k$  or  $2p$  for some prime number  $p$  ( $|G| \neq 2$ ),
- (iii)  $G$  is one of the 10 abelian groups shown in Table 1.

*If (ii) or (iii) is true then  $\alpha(G) > |G|$ .*

The value of  $\alpha(G)$  was determined for every cyclic group  $G$  by Sabidussi [18, 19], when  $|G|$  is a prime number, and by Meriwether (unpublished, see [19]), in general. However, Arlinghaus [1] was the first to present an algorithm to compute  $\alpha(G)$  when  $G$  is cyclic or, more generally, abelian. Table 3 contains the value of  $\alpha(G)$  for some small abelian groups, which we computed using Arlinghaus’ algorithm [1, Theorem 8.1].

**Table 3** The values of  $\alpha(G)$  for certain abelian groups  $G$

$p_1^{r_1} \backslash p_2^{r_2}$	2	3	4	5	7	8
2	4	11	12	17	16	16
3		18	18	21	23	22
4			20	25	24	24
5				30	29	29

$p_2^{r_2} \backslash p_3^{r_3}$	2	3	4	5	7	8	9	11	13
2	6	13	14	19	18	18	19	26	30
3		20	20	23	25	24	23	33	37
4			22	27	26	26	26	34	38

**Proposition 3.2** [1, Theorem 8.1] Consider the abelian group  $G = C_{q_1} \times C_{q_2} \times \dots \times C_{q_s}$ , where  $q_i$  is a prime power,  $i = 1, \dots, s$ . Then,

$$\alpha(G) \leq \alpha(C_{q_1}) + \alpha(C_{q_2}) + \dots + \alpha(C_{q_s}), \tag{1}$$

$$\alpha(C_2 \times C_{q_2}) = 2 + \alpha(C_{q_2}). \tag{2}$$

**Proposition 3.3** [1, Theorem 5.4] Let  $p$  be a prime number and  $r$  be a positive integer. Then

$$\alpha(C_{p^r}) = \begin{cases} 2, & \text{if } p^r = 2, \\ p^r + 2p, & \text{if } p = 3, 5, \\ p^r + 6, & \text{if } p = 2, r \geq 2, \\ p^r + p, & \text{if } p \geq 7. \end{cases}$$

Proposition 3.3 gives rise to the following inequalities that are essential for the proof of Proposition 3.1:

$$\alpha(C_{p^r}) \leq 3p^r, \tag{3}$$

$$\alpha(C_{p^r}) \leq 2p^r, \quad \text{if } p^r \geq 7, \tag{4}$$

$$\alpha(C_{p^r}) \leq p^r + \max\{6, 2p\}. \tag{5}$$

In preparation for proving Proposition 3.1, we establish the following lemma.

**Lemma 3.4** Proposition 3.1 holds when  $G$  is a direct product of two cyclic groups of prime-power order.

**Proof** Let  $G = C_{p_1^{r_1}} \times C_{p_2^{r_2}}$  for some prime powers  $p_1^{r_1}, p_2^{r_2}$  such that  $p_1^{r_1} \leq p_2^{r_2}$ . Using Table 3 we deduce that the Lemma holds when  $p_1^{r_1} \leq 5$  and  $p_2^{r_2} \leq 8$ . If  $p_1^{r_1} > 5$

then the inequalities (1) and (4) imply that

$$\alpha(G) \leq \alpha(C_{p_1^{r_1}}) + \alpha(C_{p_2^{r_2}}) \leq 2p_1^{r_1} + 2p_2^{r_2} \leq 4p_2^{r_2} < |G|.$$

Hence we make the assumption that  $p_1^{r_1} \leq 5$  and  $p_2^{r_2} > 8$ .

If  $|G| = 2p_2$  then (2) and Proposition 3.3 imply that  $\alpha(C_{2p_2}) = 2 + 2p_2$ , thus  $\alpha(G) > |G|$ . On the other hand, if  $|G| = 2p_2^{r_2}$  and  $r_2 > 1$ , then the inequality

$$2 + \max\{6, 2p_2\} < p_2^{r_2} \tag{6}$$

holds; indeed, we assumed that  $p_2^{r_2} > 8$ , so (6) holds in case  $p_2 = 2$ ; if  $p_2 \geq 3$  then  $2 + 2p_2 < 3p_2 \leq p_2^{r_2}$ . It follows from (2), (5) and (6) that

$$\alpha(C_{2p_2^{r_2}}) \leq 2 + p_2^{r_2} + \max\{6, 2p_2\} < 2p_2^{r_2} = |G|.$$

If  $p_1^{r_1} = 3$  then by (1), (4) and Proposition 3.3, we get that  $\alpha(G) \leq 9 + 2p_2^{r_2} \leq 3p_2^{r_2} = |G|$ .

Finally, if  $4 \leq p_1^{r_1} \leq 5$  then it is implied by (1), (4) and Proposition 3.3 that

$$\alpha(G) \leq 15 + 2p_2^{r_2} < 4p_2^{r_2} \leq |G|.$$

□

**Proof of Proposition 3.1** If  $|G| = 1$  then  $\alpha(G) = |G|$ . Let  $G = C_{p_1^{r_1}} \times C_{p_2^{r_2}} \times \dots \times C_{p_s^{r_s}}$ , where  $p_1^{r_1} \leq \dots \leq p_s^{r_s}$  are prime powers.

If  $s = 1$  or  $s = 2$  then the statements in Proposition 3.1 hold for  $G$  as a consequence of Proposition 3.3 or Lemma 3.4, respectively.

Let  $s = 3$ . If  $p_1^{r_1} p_2^{r_2} \geq 9$ , then using inequalities (1) and (3) we conclude that

$$\alpha(G) \leq 3p_1^{r_1} + 3p_2^{r_2} + 3p_3^{r_3} \leq 9p_3^{r_3} \leq |G|.$$

Assume now that  $p_1^{r_1} p_2^{r_2} < 9$ ; thus,  $p_1^{r_1} = 2$  and  $2 \leq p_2^{r_2} \leq 4$ . If  $p_3^{r_3} < 16$  then using Table 3 we verify that the claim in Proposition 3.1 holds for  $G$ . If  $p_3^{r_3} \geq 16$  instead, then (1) and (3) together with Proposition 3.3 imply that

$$\alpha(G) \leq 2 + 3p_2^{r_2} + 3p_3^{r_3} < 4p_3^{r_3} \leq |G|.$$

Let  $s = 4$ . If  $p_1^{r_1} p_2^{r_2} p_3^{r_3} \geq 12$ , then by (1) and (3) we have

$$\alpha(G) \leq 3p_1^{r_1} + 3p_2^{r_2} + 3p_3^{r_3} + 3p_4^{r_4} \leq 12p_4^{r_4} \leq |G|.$$

Otherwise  $p_1^{r_1} = p_2^{r_2} = p_3^{r_3} = 2$ , in which case (1), (3) and Proposition 3.3 show that

$$\alpha(G) \leq 2 + 2 + 2 + 3p_4^{r_4} < 8p_4^{r_4} = |G|.$$

Finally, let us assume that  $s \geq 5$ . Then, using (1) and (3) we conclude that

$$\alpha(G) \leq 3p_1^{r_1} + 3p_2^{r_2} + \dots + 3p_s^{r_s} \leq 3s p_s^{r_s}.$$

Furthermore, since  $3s < 2^{s-1}$  and  $p_i^{r_i} \geq 2$  for every  $i \in \{1, \dots, s - 1\}$ , we have that

$$3s p_s^{r_s} < 2^{s-1} p_s^{r_s} \leq p_1^{r_1} p_2^{r_2} \dots p_s^{r_s} = |G|.$$

Hence  $\alpha(G) < |G|$ . □

### 4 Proof of Theorem 1: the bound $\alpha(G) \leq |G|$

In this section we prove the bound  $\alpha(G) \leq |G|$  for groups  $G$  that are non-abelian and do not satisfy (iii), (iv) in Theorem 1, as summarised in the following theorem.

**Theorem 4.1** *Let  $G$  be a non-abelian group such that*

- (i)  $G$  is not a generalised quaternion group,
- (ii)  $G$  is not a generalised dicyclic group of the form  $Q_{2^r} \times C_2$ ,
- (iii)  $G$  is not one of the groups shown in Table 1.

*Then  $\alpha(G) \leq |G|$ .*

By the GRR-Theorem (Theorem 2.3), in order to prove Theorem 4.1, we only need to consider the cases when  $G$  is generalised dicyclic and when  $G$  is one of the non-abelian groups that appear in Table 2 but not in Table 1. We will do this in Propositions 4.9 and 4.10. In particular, in Proposition 4.9, we consider the groups  $D_6, D_8, D_{10}$ , and

$$\begin{aligned} G_1 &= \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle, \\ G_2 &= \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, (ac)^2 = (bc)^2 = 1 \rangle, \\ G_3 &= \langle a, b, c \mid a^3 = c^3 = 1, ac = ca, bc = cb, b^{-1}ab = ac \rangle, \\ G_4 &= Q_8 \times C_4, \\ G_{r+2} &= Q_{2^r} \times C_2 \times C_2 \times C_2, \quad r \geq 3; \end{aligned} \tag{7}$$

the remaining groups are addressed in Proposition 4.10.

Let us start with two lemmas that will be used in the proof of Proposition 4.10.

**Lemma 4.2** *Let  $G = \text{Dih}(X)$  be a generalised dihedral group of order  $2k$ , where  $k \geq 6, k \neq 9$ , that is not the group  $C_2 \times C_2 \times C_2 \times C_2$ . Then there exists a GRR for  $G$ .*

**Proof** By the GRR-Theorem, it suffices to prove that  $G$  is non-generalised dicyclic and not one of the groups appearing in Table 2.



Let  $G = \langle X, b \rangle$ ,  $b^2 = 1$ . Suppose that  $G = \text{Dic}(A, c^2)$ , for some  $A \leq G$ ,  $c \in G$ . Then the order of  $c$  is 4 and the order of  $b$  is 2, hence  $c \in X$ ,  $b \in A$ . It follows from the properties of generalised dicyclic and generalised dihedral groups that

$$bcb^{-1} = c^{-1} \quad \text{and} \quad cbc^{-1} = b^{-1}.$$

The equalities given above imply that  $c^2 = 1$ , which is a contradiction.

The restriction  $k \neq 9$ , implies that  $G$  is not the group of order 18 in Table 2. Moreover, since  $|G| = 2k \geq 12$ ,  $G$  is not among the groups  $C_2 \times C_2$ ,  $C_2 \times C_2 \times C_2$ ,  $D_6$ ,  $D_8$ ,  $D_{10}$  or the group of order 27 in Table 2. On the other hand, the group  $A_4$  has no abelian subgroup of index 2, hence it is not generalised dihedral. The remaining 4 suitable groups given in Table 2 contain a central element of order 3 or 4. However, if  $g$  is in the center of  $G$  and  $G$  is non-abelian then  $g \in X$ , hence  $bgb = g^{-1}$ . Furthermore,  $bgb = g$ , as  $g$  is central. Therefore,  $g$  has order 2. □

The following lemma can be proven by elementary group theory arguments.

**Lemma 4.3** *Let  $G$  be an abelian 2-group and let  $c \in G$  be an element of order 2. Then there exists some  $y \in G$ ,  $A < G$  such that*

$$G = \langle y \rangle \oplus A \quad \text{and} \quad c \in \langle y \rangle.$$

Let us now define a collection of graphs, one for each group appearing in (7).

**Construction 4.4** *Let us first define the graph  $\Gamma_1$  on 16 vertices and 52 edges. Let  $V(\Gamma_1) = V_1 \cup V_2$ , where  $V_1 = \{1, 2, \dots, 8\}$  and  $V_2 = \{1', 2', \dots, 8'\}$ . Let  $E(\Gamma_1)$  be such that, for  $v, w \in V_1$ ,*

$$v \sim w \iff (v, w) \in \bigcup_{i, j \in \{0, 1\}} (\{1+i+4j, 3+i+4j\} \times \{5+i-4j, 7+i-4j\});$$

$$v' \sim w' \iff v, w \in \bigcup_{i \in \{0, 1\}} \{1+4i, 2+4i, 3+4i, 4+4i\}, \quad v \neq w;$$

$$v \sim w' \iff \begin{cases} w - v = 0, 4, & \text{or} \\ v - w \equiv 2 \pmod{4}, v > 4, w \leq 4, & \text{or} \\ v - w \equiv \pm 1 \pmod{4} \text{ and } (v > 4 \iff w > 4). \end{cases}$$

**Construction 4.5** *Let us now define the graph  $\Gamma_2$ , which has 18 vertices and 99 edges. Let  $V(\Gamma_2) = W_1 \cup W_2$ , where  $W_1 = \{1, 2, \dots, 9\}$ ,  $W_2 = \{1', 2', \dots, 9'\}$ , and  $E(\Gamma_2)$  is such that, for  $v, w \in W_1$ ,*

$$v' \sim w' \iff \begin{cases} \forall k \in \{0, 1, 2\}, v > 3k \iff w > 3k, & v \neq w, \quad \text{or} \\ w - v \equiv \pm 3 \pmod{9}, & \text{or} \\ [v, w] \in \{[1, 6], [2, 4], [3, 5]\}; \end{cases}$$

$$v \sim w \iff v' \approx w', \quad v \neq w;$$

$$v \approx w' \iff v, w \in \{1, 2, 3\}, \quad \text{or } v, w \in \{4, 5, 6\}.$$

**Construction 4.6** Let us also construct the graph  $\Gamma_3$  on 27 vertices and 171 edges. Let  $V(\Gamma_3) = \mathcal{O}_1 \cup \mathcal{O}_{1'} \cup \mathcal{O}_{1''}$ , where  $\mathcal{O}_1 = \{1, 2, \dots, 9\}$ ,  $\mathcal{O}_{1'} = \{1', 2', \dots, 9'\}$  and  $\mathcal{O}_{1''} = \{1'', 2'', \dots, 9''\}$ . The edge set of  $\Gamma_3$  is such that, for every  $v, w \in \mathcal{O}_1$ , we have

$$\begin{aligned} v' &\approx w'; v'' \approx w''; \\ v \sim w &\iff v \neq w; \\ v \sim w'' &\iff \begin{cases} w - v \equiv 1 \pmod{3}, & \text{or} \\ w - v \equiv 0, 3 \pmod{9}; \end{cases} \\ v \sim w' &\iff \begin{cases} w - v \equiv 0, 2k, 4k \pmod{9}, v \equiv k \pmod{3} \text{ for some } k \in \{1, 2\}, & \text{or} \\ w - v \equiv 0, \pm 1 \pmod{9}, v \equiv 0 \pmod{3}; \end{cases} \\ v' \sim w'' &\iff w \sim v'. \end{aligned}$$

**Construction 4.7** Let  $\Gamma$  be the graph on 10 vertices with automorphism group  $\text{Aut}(\Gamma) \cong C_4$  given in Example 2.7. Let  $\Gamma'$  be a graph on 16 vertices constructed according to Babai's Construction 2.6 for the group  $Q_8$ . We define  $\Gamma_4$  to be the graph  $\Gamma_4 = \Gamma \cup \Gamma'$ .

**Construction 4.8** Let  $r \geq 3$ . We let  $\Gamma$  be a graph on  $2^{r+1}$  vertices constructed according to Babai's Construction 2.6 for the generalised quaternion group  $Q_{2^r}$  and  $\Gamma'$  to be a graph on 6 vertices such that  $\text{Aut}(\Gamma') \cong C_2 \times C_2 \times C_2$ , which exists since  $\alpha(C_2 \times C_2 \times C_2) = 6$  (see Table 3 and [1, Theorem 8.1] for a proof). Then, the graph  $\Gamma_{r+2} = \Gamma \cup \Gamma'$  on  $2^{r+1} + 6$  vertices has automorphism group  $\text{Aut}(\Gamma_{r+2}) \cong G_{r+2}$ , since  $\Gamma$  is a connected component of  $\Gamma_{r+2}$  of size  $2^{r+1} > 6$  and  $\text{Aut}(\Gamma) \cong Q_{2^r}$ ,  $\text{Aut}(\Gamma') \cong C_2 \times C_2 \times C_2$ .

**Proposition 4.9** If  $G$  is the dihedral group  $D_{2n}$ , where  $3 \leq n \leq 5$ , or one of the groups  $G_1, G_2, G_3, G_4, G_{r+2}$  ( $r \geq 3$ ) in (7) then  $\alpha(G) \leq |G|$ .

**Proof** The  $n$ -cycle has full automorphism group  $D_{2n}$ .

The graphs  $\Gamma_i$  in Constructions 4.4–4.8 are designed to have at most  $|G_i|$  vertices and automorphism groups  $\text{Aut}(\Gamma_i) \cong G_i$ , for each  $i \geq 1$ . We omit the proof that  $\text{Aut}(\Gamma_i) \cong G_i$  for  $1 \leq i \leq 4$ , which we verified using the mathematical software GAP [5].  $\square$

We will now show that Theorem 4.1 also holds for the generalised dicyclic groups  $G$  that are different from  $Q_{2^r} \times C_2 \times C_2 \times C_2$ ,  $r \geq 3$ , completing the proof of Theorem 4.1.

**Proposition 4.10** Let  $G$  be a generalised dicyclic group such that

- (i)  $G$  is not a generalised quaternion group,
- (ii)  $G$  is not a generalised dicyclic group of the form  $Q_{2^r} \times C_2$  or  $Q_{2^r} \times C_2 \times C_2 \times C_2$ ,
- (iii)  $G$  is not one of the groups  $\text{Dic}_3, \text{Dic}_5, \text{Dic}_6, G_{16}$  that appear in Table 1.

Then  $\alpha(G) \leq |G|$ .

The rest of this section concerns the proof of Proposition 4.10.

Let  $G = \text{Dic}(A, b^2)$  be a generalised dicyclic group as in Proposition 4.10 and let

$$A = A_2 \oplus A_{2'}$$

where  $A_2$  is the Sylow 2-subgroup of  $A$  and  $A_{2'}$  is the Hall  $2'$ -subgroup of  $A$ . Then, by Lemma 4.3, there exist  $y \in A_2$  and  $B_2 < A_2$  such that

$$A_2 = \langle y \rangle \oplus B_2 \quad \text{and} \quad b^2 \in \langle y \rangle.$$

Setting  $X = B_2 \oplus A_{2'}$ , we get  $A = X \oplus \langle y \rangle$ .

Let  $r, k$  be such that  $\langle y \rangle \cong C_{2^r}$ ,  $|X| = k$ . We note that the quotient group  $G/X$  is isomorphic to the generalised quaternion group  $Q_{2^{r+1}}$ , for  $r > 1$ , and to the cyclic group  $C_4$ , for  $r = 1$ . Moreover, the quotient group  $G/\langle y \rangle$  is isomorphic to the generalised dihedral group  $\text{Dih}(X)$ .

**Construction 4.11** *We will construct a graph  $\Gamma$  such that  $\text{Aut}(\Gamma) \cong G$ . We start by defining two graphs,  $\Gamma_1, \Gamma_2$ , with the property that  $\text{Aut}(\Gamma_1) \cong G/X$ ,  $\text{Aut}(\Gamma_2) \cong G/\langle y \rangle$ .*

For  $r \geq 2$ , we let  $\Gamma_1$  be the graph with vertex set  $V(\Gamma_1) = G/X \cup (G/X)'$  that arises from Babai’s Construction 2.6 for the generalised quaternion group  $G/X$  with respect to the minimal generating set  $H = \{yX, bX\}$ . Furthermore, we partition the set of vertices  $G/X$  of  $\Gamma_1$  into the sets  $T_1, T_2$ , where  $T_i = \{y^n b^{(i-1)}X \mid n \in \mathbb{N}\}$ ,  $i = 1, 2$ . For  $r = 1$ , we define  $\Gamma_1$  to be the graph with automorphism group isomorphic to the cyclic group  $C_4$  that was presented in Example 2.7. Likewise, we partition its vertex set into the sets  $T_i$ , where  $T_i = \{2k + i \mid 0 \leq k \leq 4\}$ ,  $i = 1, 2$ .

Let us describe the graph  $\Gamma_2$  for all values of  $k$ . The conditions (i), (ii) in Proposition 4.10 ensure that  $k \geq 3$ . For  $3 \leq k \leq 5$ , we construct  $\Gamma_2$  with vertex set  $V(\Gamma_2) = G/\langle y \rangle \cup (G/\langle y \rangle)'$  according to Babai’s Construction 2.6 for the generalised dihedral group  $G/\langle y \rangle$ , with respect to some minimal generating set  $K = \{k_1, k_2, \dots, k_d\}$  of  $G/\langle y \rangle$  such that  $k_1 = b\langle y \rangle$ . For  $k \geq 6$ ,  $k \neq 9$ , we choose a GRR for  $G/\langle y \rangle$ , which exists by assumption (ii) in Proposition 4.10 and Lemma 4.2, and define the graph  $\Gamma_2$  to be either this GRR or its complement, with the additional property that the number  $k + \rho_{\Gamma_2}(v)$  is even, for  $v \in V(\Gamma_2)$ . Furthermore, for  $k \geq 3$ ,  $k \neq 9$ , we partition the set of vertices  $G/\langle y \rangle$  of  $\Gamma_2$  into  $S_i = \{xb^{(i-1)}\langle y \rangle \mid x \in X\}$ ,  $i = 1, 2$ . Finally, for  $k = 9$ , we let  $\Gamma_2$  be the graph on 18 vertices presented in Proposition 4.9, and  $S_i = W_i$ , where  $W_i$  is as in Construction 4.5, for  $i = 1, 2$ .

Let us now define the graph  $\Gamma$  such that

$$\begin{aligned} V(\Gamma) &= V(\Gamma_1) \cup V(\Gamma_2); \\ E(\Gamma) &= E(\Gamma_1) \cup E(\Gamma_2) \cup E, \quad \text{where} \\ E &= \left\{ [t_i, s_i] \mid t_i \in T_i, s_i \in S_i, i = 1, 2 \right\}. \end{aligned}$$

**Lemma 4.12** *Let the sets of vertices  $V(\Gamma_1), V(\Gamma_2)$  of  $\Gamma$  be fixed by  $\phi$ , for every  $\phi \in \text{Aut}(\Gamma)$ . Then  $\text{Aut}(\Gamma) \cong G$ .*

**Proof** By construction, the groups  $G/X$  and  $G/\langle y \rangle$  act on  $V(\Gamma_1)$  and  $V(\Gamma_2)$ , respectively. Using these actions, we associate a map  $g : V(\Gamma) \rightarrow V(\Gamma)$  to every  $g \in G$  by setting

$$g(v) = \begin{cases} gX(v), & \text{if } v \in V(\Gamma_1) \\ g\langle y \rangle(v), & \text{if } v \in V(\Gamma_2). \end{cases}$$

In other words, the map  $g$  is defined to satisfy  $g \upharpoonright_{V(\Gamma_1)} = gX$  and  $g \upharpoonright_{V(\Gamma_2)} = g\langle y \rangle$ .

Let  $g \in G$ . We will show that the map  $g$  is an automorphism of  $\Gamma$ . If  $v, w \in V(\Gamma_i)$  for some  $i \in \{1, 2\}$  then  $g$  is an automorphism of  $\Gamma$ , since

$$v \sim w \text{ in } \Gamma \iff v \sim w \text{ in } \Gamma_i \iff g \upharpoonright_{V(\Gamma_i)} v \sim g \upharpoonright_{V(\Gamma_i)} w \text{ in } \Gamma_i \iff gv \sim gw \text{ in } \Gamma.$$

Let  $v \in V(\Gamma_1), w \in V(\Gamma_2)$ . By construction of  $\Gamma$ ,  $g(T_i \times S_i) = T_j \times S_j$ , where  $i=j$  if and only if  $g \in \langle X, y \rangle$ ,  $i, j \in \{1, 2\}$ . Thus,

$$[v, w] \in E(\Gamma) \iff (v, w) \in \bigcup_{i=1}^2 (T_i \times S_i) \iff (gv, gw) \in \bigcup_{i=1}^2 (T_i \times S_i).$$

In other words,  $v \sim w \iff gv \sim gw$ . Since  $\langle y \rangle \cap X = \{1\}$ , we have  $G \leq \text{Aut}(\Gamma)$ .

For the opposite inclusion, let  $\phi \in \text{Aut}(\Gamma)$ . Since, by assumption,  $\phi$  fixes  $V(\Gamma_1)$ , the restriction  $\phi \upharpoonright_{V(\Gamma_1)}$  of  $\phi$  is an automorphism of  $\Gamma_1$ . Since  $\text{Aut}(\Gamma_1) \cong G/X$ , we have that  $\phi \upharpoonright_{V(\Gamma_1)} = g_1X$ , for some  $g_1 \in G$ . Then the automorphism  $g_1^{-1}\phi$  acts trivially on  $V(\Gamma_1)$ . As the set of vertices  $T_1$  is fixed by  $g_1^{-1}\phi$ , the set consisting of all neighbours of  $T_1$  is also fixed by the same automorphism. However,  $g_1^{-1}\phi$  fixes all neighbours of  $T_1$ , except, possibly, from elements of the set  $S_1$ . Therefore,  $S_1$  is fixed by  $g_1^{-1}\phi$ .

Likewise, we consider the restriction of the automorphism  $g_1^{-1}\phi$  on  $\Gamma_2$  to conclude that there exists some  $g_2 \in G$  such that the automorphism  $g_2^{-1}g_1^{-1}\phi$  acts trivially on  $V(\Gamma_2)$ ; without loss of generality, we let  $g_2 \in \langle X, b \rangle$ . Since the graph automorphisms  $g_1^{-1}\phi$  and  $g_2^{-1}g_1^{-1}\phi$  fix the set  $S_1$ , we conclude that  $g_2 \in X$ . However,  $X$  acts trivially on  $V(\Gamma_1)$ . Therefore, the graph automorphism  $g_2^{-1}g_1^{-1}\phi$  is the identity automorphism of  $\Gamma$ . Thus,

$$\phi = g_1g_2 \in G$$

and hence  $\text{Aut}(\Gamma) \leq G$ . □

**Lemma 4.13** *Let  $\phi \in \text{Aut}(\Gamma)$ . The sets of vertices  $V(\Gamma_1), V(\Gamma_2)$  are fixed by  $\phi$ .*

**Proof** First, we partition  $V(\Gamma_1), V(\Gamma_2)$  into the sets  $V(X_i), V(Y_i)$ , where  $i \in \{1, 2\}$ , and

$$V(X_1) = \begin{cases} G/X, & \text{if } r \geq 2 \\ V(\Gamma_1), & \text{if } r = 1, \end{cases} \quad V(X_2) = \begin{cases} (G/X)', & \text{if } r \geq 2 \\ \emptyset, & \text{if } r = 1, \end{cases}$$

$$V(Y_1) = \begin{cases} G/\langle y \rangle, & \text{if } k \neq 9 \\ V(\Gamma_2), & \text{if } k = 9, \end{cases} \quad V(Y_2) = \begin{cases} (G/\langle y \rangle)', & \text{if } 3 \leq k \leq 5 \\ \emptyset, & \text{if } k \geq 6. \end{cases}$$

We will examine all possible values of  $r, k$  in order to show that  $V(\Gamma_2)$  is fixed by  $\phi$ , using the automorphisms' property to preserve the valency of the vertices permuting.

The valency of a vertex  $v \in V(\Gamma_i)$  in  $\Gamma$  is

$$\rho(v) = \rho_{\Gamma_i}(v) + v_{\Gamma_j}(v), \tag{8}$$

where  $v_{\Gamma_j}(v)$  is the number of neighbours of  $v$  that lie in  $V(\Gamma_j)$  and  $i, j \in \{1, 2\}, i \neq j$ .

*Case 1* Assume that  $r \geq 2, k \geq 6$ . Let  $x_i \in V(X_i), y_i \in V(Y_i), i \in \{1, 2\}$ . By (8),

$$\rho(x_1) = 5 + k, \quad \rho(y_1) = \rho_{\Gamma_2}(y_1) + 2^r,$$

which, together with the assumption that the number  $k + \rho_{\Gamma_2}(y_1)$  is even, implies that  $\rho(x_1) \neq \rho(y_1)$ .

Suppose now that  $\phi(v) \in V(\Gamma_1)$ , for some  $v \in V(\Gamma_2)$ . Then  $\rho(v) = \rho(y_1) \neq \rho(x_1)$ , hence  $\phi(v) \in V(X_2)$ . Since  $\rho(\phi(v)) = \rho(x_2) \neq \rho(x_1)$ , the set  $V(X_1)$  is fixed by  $\phi$ . Therefore, the number of neighbours of  $v$  that lie in  $V(X_1)$ , which is  $2^r$ , is equal to the number of neighbours of  $\phi(v)$  in  $V(X_1)$ , which is 3; a contradiction. Hence  $V(\Gamma_2)$  is fixed by  $\phi$ .

*Case 2* Suppose that  $3 \leq k \leq 5$ ; we assume that  $G$  satisfies Proposition 4.10, (iii), thus  $r \geq 2$ . If  $x_i \in V(X_i), y_i \in V(Y_i), i \in \{1, 2\}$ , then, by (8),

$$\begin{aligned} \rho(x_1) &= 5 + k, & \rho(y_1) &= \rho_{\Gamma_2}(y_1) + 2^r, \\ \rho(x_2) &= 2^{r+1}, & \rho(y_2) &= d + 2. \end{aligned}$$

We will show that the sets of vertices  $V(Y_2)$  and  $V(Y_1)$  are fixed by  $\phi$ . Considering all abelian groups of order 3, 4 or 5, we conclude that the size of a minimal generating set of a generalised dihedral group of size  $2k$ , such that  $3 \leq k \leq 5$ , is between 2 and 3; hence  $4 \leq \rho(y_2) \leq 5$ . On the other hand, by construction,  $\min\{\rho(x_1), \rho(x_2), \rho(y_1)\} > 5$ . As graph automorphisms preserve the valency of the vertices they permute,  $V(Y_2)$  is fixed by  $\phi$ . It is implied that the set of neighbours of  $V(Y_2)$ , which is  $V(Y_1)$ , is also fixed by  $\phi$ .

*Case 3* Assume now that  $r = 1, k \geq 6, k \neq 9$ . First, we will compute the valency of each vertex of  $\Gamma$ . By (8), the valency of the vertices that lie in the sets of vertices  $\{1, 2\}, \{3, 4, 5, 6\}, \{7, 8, 9, 10\}$  is  $k + 4, k + 5, k + 3$ , respectively, and the valency of the vertices  $v \in V(\Gamma_2)$  is  $\rho_{\Gamma_2}(v) + 5$ .

Suppose that  $\phi(v) \in V(\Gamma_1)$  for some  $v \in V(\Gamma_2)$ . Without loss of generality, we assume that  $v \in S_1$ . Graph automorphisms preserve the valency of the vertices they permute so

$$\rho_{\Gamma_2}(v) + 2 = k + j, \quad \text{for some } j \in \{0, 1, 2\}.$$

The graph  $\Gamma$  was constructed so that the number  $\rho_{\Gamma_2}(y_1) + k$  is even, hence  $j$  is even. In other words, the set of vertices  $\{1, 2\}$  is fixed by  $\phi$ , as is either the set  $\{7, 8, 9, 10\}$

or the set  $\{3, 4, 5, 6\}$ . The vertex  $1 \in V(\Gamma_1)$  is connected to all other vertices in  $T_1 = \{1, 3, 5, 7, 9\}$  and no vertex in  $T_2$ . Furthermore, the set of neighbours of  $v$  that lie in  $V(\Gamma_1)$  is  $T_1$ . These properties combine to say that  $\phi(v)$  is adjacent to either the vertices  $1, 3 + 2j, 5 + 2j$  or the vertices  $2, 4 + 2j, 6 + 2j$ . However, there exists no vertex in  $V(\Gamma_1)$  that is adjacent to any of these triplets of vertices. Thus,  $\phi$  fixes  $V(\Gamma_2)$ .

*Case 4* Finally, let  $r = 1$ ,  $k = 9$ . We confirmed that the graph  $\Gamma$  on 28 vertices constructed has the desired property using the mathematical software GAP [5].  $\square$

The last step in the proof of Proposition 4.10 is to show that the order of the graph  $\Gamma$  constructed is bounded by the order of the group  $G$ .

**Lemma 4.14** *The graph  $\Gamma$ , defined in Construction 4.11, has at most  $|G|$  vertices.*

**Proof** The graph  $\Gamma$  was constructed so that the set  $V(\Gamma_1)$  has size 10, for  $r = 1$ , and  $2|G/X|$ , for  $r \geq 2$ . Moreover, the size of  $V(\Gamma_2)$  is  $2|G/\langle y \rangle|$ , for  $3 \leq k \leq 5$ , and  $|G/\langle y \rangle|$ , for  $k \geq 6$ . Thus,

$$|V(\Gamma)| = |V(\Gamma_1)| + |V(\Gamma_2)| = \max\left\{2 \frac{|G|}{|X|}, 10\right\} + \left(1 + \left\lfloor \frac{5}{k} \right\rfloor\right) \frac{|G|}{|\langle y \rangle|},$$

where  $\left\lfloor \frac{5}{k} \right\rfloor$  is the integer part of the real number  $\frac{5}{k}$ . Let us now explain why  $|V(\Gamma)| \leq |G|$ . If  $k = 3$  then the assumption that  $r \geq 3$  (Proposition 4.10, (iii)) implies that  $|V(\Gamma)| = \frac{2}{3}|G| + \frac{1}{2^{r-1}}|G| < |G|$ . Similarly, if  $4 \leq k \leq 5$  then  $r \geq 2$ , hence  $|V(\Gamma)| \leq \frac{1}{2}|G| + \frac{1}{2^{r-1}}|G| \leq |G|$ . Finally, if  $k \geq 6$  then  $|V(\Gamma)| \leq \frac{1}{3}|G| + \frac{1}{2^r}|G| < |G|$ .  $\square$

## 5 Proof of Theorem 1: the bound $\alpha(G) > |G|$

In this section we prove the bound  $\alpha(G) > |G|$  and compute  $\alpha(G)$  for groups  $G$  that are non-abelian and satisfy one of (iii), (iv) in Theorem 1.

**Theorem 5.1** *Let  $G$  be a group such that one of the following holds:*

- (i)  $G$  is a generalised quaternion group,
- (ii)  $G$  is a generalised dicyclic group of the form  $Q_{2r} \times C_2$ ,
- (iii)  $G$  is one of the non-abelian groups that appear in Table 1.

*Then  $\alpha(G) > |G|$ ; indeed, if (i) holds,  $\alpha(G) = 2|G|$ ; if (ii) holds,  $\alpha(G) = |G| + 2$ ; if (iii) holds,  $\alpha(G)$  is as shown in Table 1.*

The proof of Theorem 5.1 is the subject of this section. Specifically, we compute the value of  $\alpha(G)$  when  $G$  is contained in one of the families of groups mentioned in Theorem 5.1, (i), (ii), in Propositions 5.4 and 5.5. Then, we calculate  $\alpha(G)$  for the non-abelian groups  $G$  that are shown in Table 1 in Propositions 5.7, 5.10, 5.14 and 5.17.

Let us start by presenting two lemmas that will be used throughout the section.

**Lemma 5.2** *Let  $G$  be the dicyclic group of order  $2^{r+1}q$ , where  $q$  is an odd prime or  $q = 1$ . Let  $\Gamma$  be a graph such that  $G \cong \text{Aut}(\Gamma)$  and consider the action of  $G$  on the vertex set  $V(\Gamma)$ . If every orbit has size at most  $\max\{2^{r+1}, 2^r q\}$  then there exist at least 2 orbits of size  $2^{r+1}$ .*

**Proof** Let  $G = \langle y, x, b \mid y^{2^r} = x^q = 1, y^{2^{r-1}} = b^2, yx = xy, byb^{-1} = y^{-1}, bxb^{-1} = x^{-1} \rangle$ .

As the action of  $G$  on  $V(\Gamma)$  is faithful, there exists a vertex  $w$  of  $\Gamma$  such that  $b^2 \notin G_w$ . Then, since  $b^2$  is the only element of order 2, by Cauchy’s Theorem, 2 does not divide  $|G_w|$ . Therefore, by the orbit-stabilizer lemma,  $2^{r+1}$  divides  $|\mathcal{O}_w|$ . Thus  $|\mathcal{O}_w| = 2^{r+1}$ , since  $|\mathcal{O}_w| \leq \max\{2^{r+1}, 2^r q\}$ . Suppose that  $\mathcal{O}_w$  is the only orbit of size  $2^{r+1}$ .

Let  $B = \{y^k bw \mid k \in \mathbb{N}\}$ . We will show that the map  $\phi : V(\Gamma) \rightarrow V(\Gamma)$ , where

$$\phi(v) = \begin{cases} b^2v, & \text{if } v \in B, \\ v, & \text{if } v \notin B, \end{cases}$$

is an automorphism of  $\Gamma$ . Indeed, the property  $v_1 \sim v_2 \iff \phi(v_1) \sim \phi(v_2)$  holds for every  $v_1, v_2 \in V(\Gamma)$  as

- For  $v_1, v_2 \in B$ ,  $b^2 \in \text{Aut}(\Gamma)$  hence  $v_1 \sim v_2 \iff b^2v_1 \sim b^2v_2$ ,
- For  $v_1, v_2 \notin B$ , clearly  $v_1 \sim v_2 \iff \phi(v_1) \sim \phi(v_2)$ ,
- For  $v_1 = g_1w \notin B, v_2 = g_2w \in B, g_1 \in \langle y, x \rangle, g_2 \in \langle y, x \rangle b$ , we have  $g_1w \sim g_2w \iff g_1g_2^{-1}g_1w \sim g_1w \iff b^2g_2w \sim g_1w$ , since  $g_1g_2^{-1} \in \text{Aut}(\Gamma)$  and  $g_1g_2^{-1}g_1 = b^2g_2$ ,
- For  $v_1 \notin \mathcal{O}_w, v_2 \in B$ , we have that  $v_1 \sim v_2 \iff v_1 \sim b^2v_2$ , as  $b^2 \in \text{Aut}(\Gamma)$  and  $b^2 \in G_{v_1}$ , by the assumption that 2 divides  $|G_{v_1}|$ .

We have reached a contradiction since  $G_w = G_{bw} = \langle x \rangle$ . Hence there exists a second orbit of size  $2^{r+1}$ . □

**Lemma 5.3** *Let  $G = \text{Dic}_q, q \in \{3, 5\}$ , and let  $\Gamma$  be a graph on at most  $4q + 4$  vertices such that  $\text{Aut}(\Gamma) \cong G$ . Then, there is no orbit of size  $|G| = 4q$  in the action of  $G$  on  $V(\Gamma)$ .*

**Proof** Let  $G = \langle x, b \rangle$ , where  $x^q = b^4 = 1$ .

Suppose that there is a vertex  $v \in V(\Gamma)$  with stabilizer  $G_v = \{1\}$ . If there exists an orbit of size 4, let  $u \in V(\Gamma)$  be a vertex with stabilizer  $G_u = \langle x \rangle$ . By possibly replacing the graph  $\Gamma$  with its complement,  $\bar{\Gamma}$ , we assume that  $v$  is adjacent to up to two vertices in the orbit  $\mathcal{O}_u$ , if it exists. Without loss of generality, we also assume that if  $v$  is adjacent to a vertex in  $\mathcal{O}_u$  then  $v \sim u$ . Let  $B = \{x^k b^l v \mid k \in \mathbb{N}, l \in \{1, 3\}\}$  and let  $\phi : V(\Gamma) \rightarrow V(\Gamma)$ ,

$$\phi(w) = \begin{cases} b^2w, & \text{if } w \in B, \\ b^{(-1)^k l} w, & \text{if } w = b^k u \text{ and } v \sim b^l u, \text{ where } k \in \mathbb{N}, l \in \{1, 3\}, \\ b^{-k} w, & \text{if } w = b^k u \text{ and } v \not\sim b^l u, \forall l \in \{1, 3\}, \text{ where } k \in \mathbb{N}, \\ w, & \text{if 2 divides } |G_w| \text{ or } w \in \mathcal{O}_v \setminus B. \end{cases}$$

We will show that  $\phi \in \text{Aut}(\Gamma)$ . The property  $v_1 \sim v_2 \iff \phi(v_1) \sim \phi(v_2)$  holds for every  $v_1, v_2 \in V(\Gamma)$ , as

- If  $v_1, v_2 \in B$  then  $v_1 \sim v_2 \iff b^2v_1 \sim b^2v_2$ , since  $b^2 \in \text{Aut}(\Gamma)$ ,
- If  $v_1, v_2$  are fixed by  $\phi$  then clearly  $v_1 \sim v_2 \iff \phi(v_1) \sim \phi(v_2)$ ,
- If  $v_1 = g_1v \notin B, v_2 = g_2v \in B, g_1 \in \langle x, b^2 \rangle, g_2 \in \langle x, b^2 \rangle b$ , then  $g_1v \sim g_2v \iff g_1g_2^{-1}g_1v \sim g_1v \iff b^2g_2v \sim g_1v$ , since  $g_1g_2^{-1} \in \text{Aut}(\Gamma)$  and  $g_1g_2^{-1}g_1 = b^2g_2$ ,
- If  $v_1 \in \mathcal{O}_u$  then  $v_2 \in \mathcal{O}_v \cup \mathcal{O}_u$  ( $\mathcal{O}_v$  and  $\mathcal{O}_u$  are the only orbits, as  $|V(\Gamma)| \leq 4q+4$ );  $\phi$  was constructed to preserve adjacency and non-adjacency between  $v_1$  and  $v_2$ ,
- If  $v_1 \in B, 2$  divides  $|G_{v_2}|$  then  $b^2 \in G_{v_2}$ , hence  $v_1 \sim v_2 \iff b^2v_1 \sim v_2$ .

We have reached a contradiction, since  $\phi$  fixes  $v$  but not  $bv$  and  $G_v = G_{bv} = \{1\}$ .  $\square$

Using Lemma 5.2 we recover the following result, which was first proven in [8].

**Proposition 5.4** *The generalised quaternion group  $Q_{2^{r+1}}$  satisfies  $\alpha(Q_{2^{r+1}}) = 2^{r+2}$ .*

**Proof** By Babai’s Theorem 2.5,  $\alpha(Q_{2^{r+1}}) \leq 2^{r+2}$ . The inequality  $\alpha(Q_{2^{r+1}}) \geq 2^{r+2}$  follows from Lemma 5.2.  $\square$

**Proposition 5.5** *The generalised dicyclic group  $Q_{2^{r+1}} \times C_2$  satisfies  $\alpha(Q_{2^{r+1}} \times C_2) = 2^{r+2} + 2$ .*

**Proof** Let  $G = \langle y, x, b \mid y^{2^r} = x^2 = 1, y^{2^{r-1}} = b^2, yx = xy, bx = xb, byb^{-1} = y^{-1} \rangle$ .

Let  $\Gamma$  be a graph on at most  $2^{r+2} + 1$  vertices with automorphism group isomorphic to  $G$ . The faithfulness of the action of  $G$  on  $V(\Gamma)$  implies the existence of some  $w \in V(\Gamma)$  such that  $b^2 \notin G_w$ . Then  $G_w \in \{\{1\}, \langle x \rangle, \langle b^2x \rangle\}$ . Let  $u \in V(\Gamma)$  be such that  $G_u \in \{\langle x \rangle, \langle b^2x \rangle\}$  and  $u \notin \mathcal{O}_w$ ; if no such vertex exists, let  $u = w$ . Since  $|V(\Gamma)| \leq 2^{r+2} + 1$ , there exist at most two orbits of size  $2^{r+1}$  or one of size  $2^{r+2}$ . Therefore, if  $u \neq w$  then  $G_u \neq G_w$ , as the action of  $G$  on  $V(\Gamma)$  is faithful.

Let  $B = \{y^k x^l b z \mid z \in \{w, u\}, k, l \in \mathbb{N}\}$  and let  $\phi : V(\Gamma) \rightarrow V(\Gamma)$  be the map

$$\phi(v) = \begin{cases} b^2v, & \text{if } v \in B, \\ v, & \text{if } v \notin B. \end{cases}$$

We will show that  $\phi$  is an automorphism of  $\Gamma$  by proving that  $v_1 \sim v_2 \iff \phi(v_1) \sim \phi(v_2)$ , for every  $v_1, v_2 \in V(\Gamma)$ . Indeed,

- If  $v_1, v_2 \in B$ , then  $v_1 \sim v_2 \iff b^2v_1 \sim b^2v_2$ , since  $b^2 \in \text{Aut}(\Gamma)$ ,
- If  $v_1, v_2 \notin B$ , then clearly  $v_1 \sim v_2 \iff \phi(v_1) \sim \phi(v_2)$ ,
- If  $v_1 = g_1z \notin B, v_2 = g_2z \in B, z \in \{w, u\}, g_1 \in \langle y, x \rangle, g_2 \in \langle y, x \rangle b$ , then  $g_1z \sim g_2z \iff g_1g_2^{-1}g_1z \sim g_1z \iff b^2g_2z \sim g_1z$ , as  $g_1g_2^{-1} \in \text{Aut}(\Gamma), g_1g_2^{-1}g_1 = b^2g_2$ ,
- If  $v_1 \notin B, v_2 \in B$  and  $G_{v_1} = \langle b^{2^n}x \rangle, G_{v_2} = \langle b^{2^{(n+1)}}x \rangle$  for some  $n \in \{1, 2\}$ , then we have  $v_1 \sim v_2 \iff v_1 \sim (b^{2^n}x)(b^{2^{(n+1)}}x)v_2 \iff v_1 \sim b^2v_2$ ,
- If  $v_1 \notin \mathcal{O}_w \cup \mathcal{O}_u, v_2 \in B$ , then  $v_1 \sim v_2 \iff v_1 \sim b^2v_2$ , since  $b^2 \in G_{v_1}$ .

The map  $\phi$  fixes  $w$  but not  $bw$  and  $G_w = G_{bw}$ ; a contradiction. Thus,  $\alpha(G) \geq 2^{r+2} + 2$ .

Let us now construct a graph  $\Gamma$  on  $2^{r+2} + 2$  vertices such that  $\text{Aut}(\Gamma) \cong G$ .



**Construction 5.6** Let  $\Gamma_1$  be a graph on  $2^{r+2}$  vertices constructed according to Babai’s Construction 2.6 for the generalised quaternion group  $Q_{2^{r+1}}$  and  $\Gamma_2$  be the connected graph on 2 vertices. The graph  $\Gamma = \Gamma_1 \cup \Gamma_2$  has  $\text{Aut}(\Gamma) \cong G$ , since it consists of two connected components,  $\Gamma_1, \Gamma_2$ , of different size and  $\text{Aut}(\Gamma_1) \cong Q_{2^{r+1}}, \text{Aut}(\Gamma_2) \cong C_2$ .

□

**Proposition 5.7** The generalised dicyclic group  $G_{16} = \langle x, b \mid x^4 = b^4 = 1, bxb^{-1} = x^3 \rangle$  satisfies  $\alpha(G_{16}) = 18$ .

We will prove Proposition 5.7 using the following lemma.

**Lemma 5.8** Suppose that  $\Gamma$  is a graph on at most 17 vertices such that  $\text{Aut}(\Gamma) \cong G$ , where  $G = G_{16}$ , and consider the action of  $G$  on  $V(\Gamma)$ . Then, there is no vertex with stabilizer equal to  $\langle x^2b^2 \rangle$ . Moreover, there are two orbits,  $\mathcal{O}_{v_1}, \mathcal{O}_{v_2}$ , such that  $G_{v_1}, G_{v_2} \in \{ \langle x \rangle, \langle b^2x \rangle \}$ .

**Proof** The action of  $G$  on  $V(\Gamma)$  is faithful; thus, there exists some vertex  $v_1 \in V(\Gamma)$  such that  $b^2 \notin G_{v_1}$ . Let  $v_1, v_2, \dots, v_s \in V(\Gamma)$  form a maximal set of vertices such that  $b^2 \notin G_{v_i}$ , for  $i \in \{1, \dots, s\}$ , and the orbits  $\mathcal{O}_{v_1}, \dots, \mathcal{O}_{v_s}$  are distinct. Since  $|\mathcal{O}_{v_i}| \geq 4$ , for  $i \in \{1, \dots, s\}$ , the assumption  $|V(\Gamma)| \leq 17$  implies that  $s \leq 4$ .

Suppose that there do not exist distinct  $i, j \in \{1, \dots, s\}$  such that  $G_{v_i}, G_{v_j} \in \{ \langle x \rangle, \langle b^2x \rangle \}$  or there exists  $i \in \{1, \dots, s\}$  such that  $G_{v_i} = \langle x^2b^2 \rangle$ . Since  $|V(\Gamma)| \leq 17$  and the action of  $G$  on  $V(\Gamma)$  is faithful, if  $s \geq 3$  then there exists  $i \in \{1, \dots, s\}$  such that  $x^2 \notin G_{v_i}$ ; hence  $G_{v_i} = \langle x^2b^2 \rangle$ . Moreover, since the action is faithful, there is at most one  $i \in \{1, \dots, s\}$  such that  $G_{v_i} = \langle x^2b^2 \rangle$ . To sum up, we have  $s \leq 2$  or  $G_{v_i} = \langle x^2b^2 \rangle$  for a unique  $i \in \{1, \dots, s\}$ . If the latter is true, without loss of generality let  $v_1$  have stabilizer  $G_{v_1} = \langle x^2b^2 \rangle$ ; alternatively, let  $v_1$  be such that  $|\mathcal{O}_{v_1}| \geq |\mathcal{O}_{v_i}|$ , for  $i \in \{1, s\}$ . Moreover, if  $s = 2$ , without loss of generality we assume that if  $v_1$  is connected to  $\mathcal{O}_{v_2}$  then  $v_1 \sim v_2$ . Finally, by possibly replacing  $\Gamma$  with its complement, let  $v_1$  be adjacent to at most half the vertices of  $\mathcal{O}_{v_2}$ .

Let  $B = \{x^k b^l v_1 \mid k \in \mathbb{N}, l \in \{1, 3\}\}$ . We will show that the map  $\phi : V(\Gamma) \rightarrow V(\Gamma)$ ,

$$\phi(v) = \begin{cases} b^2v, & \text{if } v \in B, \\ v, & \text{if } v \in \mathcal{O}_{v_2} \text{ and } G_{v_1} = \langle x^2b^2 \rangle, \\ b^{(-1)^k l}v, & \text{if } v = b^k v_2, v_1 \sim b^l v_2, \text{ where } k \in \mathbb{N}, l \in \{1, 3\}, \text{ and } G_{v_1} \neq \langle x^2b^2 \rangle, \\ b^{-k}v, & \text{if } v = b^k v_2, v_1 \not\sim b^l v_2, \forall l \in \{1, 3\}, \text{ where } k \in \mathbb{N}, \text{ and } G_{v_1} \neq \langle x^2b^2 \rangle, \\ v, & \text{if } v \notin (B \cup \mathcal{O}_{v_2}), \end{cases}$$

is an automorphism of  $\Gamma$ . Indeed,  $u_1 \sim u_2 \iff \phi(u_1) \sim \phi(u_2)$ , for all  $u_1, u_2 \in V(\Gamma)$ , as

- If  $u_1, u_2 \in B$  then  $u_1 \sim u_2 \iff b^2u_1 \sim b^2u_2$ , since  $b^2 \in \text{Aut}(\Gamma)$ ,
- If  $u_1, u_2$  are fixed by  $\phi$  then clearly  $u_1 \sim u_2 \iff \phi(u_1) \sim \phi(u_2)$ ,
- If  $u_1 = g_1v_1 \notin B, u_2 = g_2v_1 \in B, g_1 \in \langle x, b^2 \rangle, g_2 \in \langle x, b^2 \rangle b$ , then  $g_1v_1 \sim g_2v_1 \iff g_1g_2^{-1}g_1v_1 \sim g_1v_1 \iff b^2g_2v_1 \sim g_1v_1$ , since  $g_1g_2^{-1} \in \text{Aut}(\Gamma)$  and  $g_1g_2^{-1}g_1 = b^2g_2$ ,

- If  $u_1 \in \mathcal{O}_{v_2}, u_2 \in \mathcal{O}_{v_1} \cup \mathcal{O}_{v_2}$  and  $G_{v_1} \neq \langle x^2 b^2 \rangle$ , then  $\phi$  was constructed to preserve adjacency and non-adjacency between  $u_1, u_2$ ,
- If  $u_1 \in B, u_2 \in \mathcal{O}_{v_2}$  and  $G_{v_1} = \langle x^2 b^2 \rangle$ , then  $u_1 \sim u_2 \iff x^2(x^2 b^2)u_1 \sim u_2 \iff b^2 u_1 \sim u_2$  since  $x^2 b^2 \in G_{v_1}, x^2 \in G_{v_2}$ ,
- If  $\phi(u_1) = b^2 u_1$  and  $u_2 \notin \mathcal{O}_{v_i}$  for all  $i \in \{1, \dots, s\}$ , then  $u_1 \sim u_2 \iff b^2 u_1 \sim u_2$ , as  $b^2 \in G_{u_2}$ ,
- If  $\phi(u_1) = b^l u_1$  for some  $l \in \{1, 3\}$ , and  $u_2 \notin \mathcal{O}_{v_i}$  for all  $i \in \{1, \dots, s\}$ , then, by assumption,  $G_{v_1} = \langle x^2 \rangle$  and  $G_{v_2} = G_{u_1} \in \{\langle x \rangle, \langle b^2 x \rangle\}$ . By faithfulness, there is  $w \in V(\Gamma), x^2 \notin G_w$ . Since  $|V(\Gamma)| \leq 17, G_w = \langle x^k b \rangle$ , for some  $k \in \mathbb{N}$ . In any case,  $G_{u_1} \in \{\langle x \rangle, \langle b^2 x \rangle\}, G_{u_2} \in \{\langle x^k b \rangle \mid k \in \mathbb{N}\} \cup G$  imply that  $u_1 \sim u_2 \iff b^l u_1 \sim u_2$ .

We have reached a contradiction since  $G_{v_1} = G_{bv_1}$  and  $\phi$  fixes  $v_1$  but not  $bv_1$ .  $\square$

**Proof of Proposition 5.7** Let  $G = G_{16}$ . Suppose that  $\Gamma$  is a graph on at most 17 vertices such that  $\text{Aut}(\Gamma) \cong G$ . As the action of  $G$  on  $V(\Gamma)$  is faithful, there exists  $w_1 \in V(\Gamma)$  such that  $x^2 \notin G_{w_1}$ ; by Lemma 5.8,  $G_{w_1} \neq \langle 1 \rangle, \langle x^2 b^2 \rangle$ , hence  $G_{w_1} = \langle b^2 \rangle$  or  $G_{w_1} = \langle x^k b \rangle$  for some  $k \in \mathbb{N}$ . Let  $\mathcal{O}_{v_1}, \mathcal{O}_{v_2}$  be two distinct orbits such that  $G_{v_1}, G_{v_2} \in \{\langle x \rangle, \langle b^2 x \rangle\}$ , which exist by Lemma 5.8. Since  $|V(\Gamma)| \leq 17$ , there are up to two orbits, of total size at most eight, containing vertices that are not fixed by  $x^2$ . If there are exactly eight such vertices, let  $w_2 \in V(\Gamma)$  be such that  $x^2 \notin G_{w_2}$  and the vertices  $\{w_1, xw_1, x^2 w_1, x^3 w_1, w_2, xw_2, x^2 w_2, x^3 w_2\}$  are distinct; if only four vertices of  $\Gamma$  are not fixed by  $x^2$ , let  $w_2 = w_1$ . By possibly replacing  $\Gamma$  with its complement, we assume that the vertex  $w_1$  is adjacent to up to two vertices of the set  $\{w_2, xw_2, x^2 w_2, x^3 w_2\}$ . Without loss of generality, if  $w_1 \neq w_2$ , let these vertices be  $w_2$  and  $x^\delta w_2, \delta \in \{0, 1, 2, 3\}$ ; if  $w_1$  is adjacent to exactly one vertex of the set  $\{x^k w_2 \mid k \in \mathbb{N}\}$  or  $w_1 = w_2$ , let  $\delta = 0$ . Then

$$w_1 \sim x^n w_2 \iff w_1 \sim x^{\delta-n} w_2, \quad \forall n \in \mathbb{N}. \quad (9)$$

We will show that the map  $\psi : V(\Gamma) \rightarrow V(\Gamma)$ , where

$$\psi(v) = \begin{cases} x^{-k} w_1, & \text{if } v = x^k w_1, \quad k \in \mathbb{N}, \\ x^{\delta-l} w_2, & \text{if } v = x^l w_2, \quad l \in \mathbb{N}, \\ v, & \text{if } x^2 \in G_v, \end{cases}$$

is an automorphism of  $\Gamma$ . Indeed,  $u_1 \sim u_2 \iff \psi(u_1) \sim \psi(u_2)$  for all  $u_1, u_2 \in V(\Gamma)$ , as

- if  $u_1 = x^k w_1, u_2 = x^l w_1, k, l \in \mathbb{N}, i \in \{1, 2\}$ , then  $x^k w_1 \sim x^l w_1 \iff x^{j-k} w_1 \sim x^{j-l} w_1$ , for  $j = 0, \delta$ , since  $x^{j-k-l} \in \text{Aut}(\Gamma)$ ,
- If  $u_1, u_2$  are fixed by  $\psi$  then clearly  $u_1 \sim u_2 \iff \psi(u_1) \sim \psi(u_2)$ ,
- If  $u_1 = x^k w_1, u_2 = x^l w_2$  for  $k, l \in \mathbb{N}$  and  $w_1 \neq w_2$  then, by (9),  $x^k w_1 \sim x^l w_2 \iff w_1 \sim x^{l-k} w_2 \iff w_1 \sim x^{\delta-l+k} w_2 \iff x^{-k} w_1 \sim x^{\delta-l} w_2$ ,
- If  $u_1 \in \mathcal{O}_{v_1} \cup \mathcal{O}_{v_2}$  or  $|\mathcal{O}_{u_1}| = 1$ , and  $u_2 \in \mathcal{O}_{w_1} \cup \mathcal{O}_{w_2}$ , then  $u_1 \sim u_2 \iff u_1 \sim x^k u_2$ , for all  $k \in \mathbb{N}$ , since  $G_{u_1} \in \{\langle x \rangle, \langle b^2 x \rangle, G\}$  and  $b^2 \in G_{u_2}$ ,
- If  $|\mathcal{O}_{u_1}| = 2$  and  $u_2 = x^k w_1, k \in \{1, 3\}$ , then  $x^2 \in G_{u_1}$  hence  $u_1 \sim x^k w_1 \iff u_1 \sim x^{k+2} w_1 \iff \psi(u_1) \sim \psi(u_2)$ ; note that  $w_1 = w_2$ , since  $|V(\Gamma)| \leq 17$ .

We have reached a contradiction:  $\psi$  fixes  $v_1$  and  $w_1$  but  $G_{v_1} \cap G_{w_1} = \{1\}$ . Thus  $\alpha(G) \geq 18$ .

We will complete the proof by constructing a graph  $\Gamma$  on 18 vertices having  $\text{Aut}(\Gamma) \cong G$ .

**Construction 5.9** Let  $\Gamma$  be a graph with vertex set  $V(\Gamma) = \mathcal{O}_1 \cup \mathcal{O}_{1'} \cup \mathcal{O}_{1''} \cup \mathcal{O}_{1'''}$ , where  $\mathcal{O}_1 = \{1, 2, \dots, 8\}$ ,  $\mathcal{O}_{1'} = \{1', 2'\}$ ,  $\mathcal{O}_{1''} = \{1'', 2'', 3'', 4''\}$  and  $\mathcal{O}_{1'''} = \{1''', 2''', 3''', 4'''\}$ . We define the edge set of  $\Gamma$  to be such that, for  $v, w \in \mathcal{O}_1$ ,

$$\begin{aligned} v' \approx w'; \quad v' \approx w''; \quad v' \approx w'''; \quad v''' \approx w'''; \\ v \sim w' &\iff v - w \equiv 0 \pmod{2}; \\ v'' \sim w''' &\iff w - v \equiv 0, 1 \pmod{4}; \\ v'' \sim w'' &\iff v - w \equiv 2 \pmod{4}; \\ v \sim w'' &\iff (v, w) \in \bigcup_{k \in \{0,1\}} (\{4k + i \mid 1 \leq i \leq 4\} \times \{k + 1, k + 3\}); \\ v \sim w''' &\iff v \sim w''; \\ v \sim w &\iff w - v \equiv 0, (-1)^k \pmod{4}, \quad v \neq w, \quad \text{and} \\ &k \in \{0, 1\} \text{ is such that } v \in \{4k + i \mid 1 \leq i \leq 4\}, w \notin \{4k + i \mid 1 \leq i \leq 4\}. \end{aligned}$$

Using the mathematical software GAP [5] we verified that  $\text{Aut}(\Gamma) \cong G$ . □

The proofs of Propositions 5.10, 5.14, and 5.17 that follow are similar in nature to the proof of Proposition 5.7. Therefore, some of the technical details are omitted.

**Proposition 5.10** For the groups  $\text{Dic}_3, \text{Dic}_5, \text{Dic}_6, Q_8 \times C_3$  we have that  $\alpha(\text{Dic}_3) = 17, \alpha(\text{Dic}_5) = 23$ , and  $\alpha(\text{Dic}_6) = \alpha(Q_8 \times C_3) = 25$ .

**Proof** Let  $G = \langle y, x, b \rangle$  be a generating set for  $G$  such that  $y^{2^r} = x^q = 1, y^{2^{r-1}} = b^2, yx = xy$ , where  $r \in \{1, 2\}, q \in \{3, 5\}$ . Suppose that there exists a graph  $\Gamma$  such that  $\text{Aut}(\Gamma) \cong G$  and  $|V(\Gamma)| < 3q + 2^{r+2}$ .

Suppose, in addition, that less than  $3q$  vertices are not fixed by  $x$ . The faithfulness of the action of  $G$  on  $V(\Gamma)$  implies the existence of  $v \in V(\Gamma)$  such that  $x \notin G_v$ , hence  $q$  divides  $|\mathcal{O}_v|$ . Let  $V = \{x^k z \mid z \in \{v, u\}, k \in \mathbb{N}\}$  be the set of vertices of  $\Gamma$  that are not fixed by  $x$ , where  $u \notin \{x^k v \mid k \in \mathbb{N}\}$ , if there exist two orbits of size  $q$  or one of size  $2q$ , and  $u = v$ , if there is exactly one orbit of size  $q$ .

We may assume that  $v$  is adjacent to up to two vertices of the set  $\{x^k u \mid k \in \mathbb{N}\}$ . If  $v \neq u$ , let us consider these vertices to be  $u$  and  $x^\delta u$ , where  $\delta \in \{0, 1, \dots, q - 1\}$ ; if  $v$  is adjacent to exactly one vertex in  $\{x^k u \mid k \in \mathbb{N}\}$  or  $v = u$ , let  $\delta = 0$ . In any case, we have

$$v \sim x^n u \iff v \sim x^{\delta-n} u, \quad \forall n \in \mathbb{N}. \tag{10}$$

Similar to the map  $\psi$  in Proposition 5.7, the map  $\psi : V(\Gamma) \rightarrow V(\Gamma)$ , where

$$\psi(z) = \begin{cases} x^{-k} v, & \text{if } z = x^k v, \quad k \in \mathbb{N}, \\ x^{\delta-l} u, & \text{if } z = x^l u, \quad l \in \mathbb{N}, \\ z, & \text{if } q \text{ divides } |G_z|, \end{cases}$$

is an automorphism of  $\Gamma$ . The faithfulness of the action implies the existence of  $w \in V(\Gamma)$  such that  $b^2 \notin G_w$ , hence  $G_w = \langle x \rangle$  ( $G_w \neq \{1\}$ , since there exist less than  $3q$  vertices that are not fixed by  $x$ ). Then,  $\psi \in G_w \cap G_v = \{1\}$  and  $\psi(xv) = x^{-1}v$ ; a contradiction.

We will show that there is no orbit of size  $|G|$ . Indeed, if  $|G| = 24$  then we assumed that  $|V(\Gamma)| \leq 24$  hence, by the GRR theorem, there exists no orbit of size 24. If  $|G| = 12, 20$  then, by Lemma 5.3, every orbit has size at most  $2^r q$ .

By Lemma 5.2, there exist at least two orbits of size  $2^{r+1}$  (a similar statement to Lemma 5.2 holds for the group  $G = Q_8 \times C_3$  and the proof is analogous). Considering the number of vertices that are fixed or not fixed by  $x$  we conclude that  $|V(\Gamma)| \geq 3q + 2^{r+2}$ ; a contradiction. Hence,  $\alpha(G) \geq 3q + 2^{r+2}$ .

Let us now consider each case for  $G \in \{\text{Dic}_3, \text{Dic}_5, \text{Dic}_6, Q_8 \times C_3\}$  and construct a graph  $\Gamma$  such that  $\text{Aut}(\Gamma) \cong G$ , completing the proof that  $\alpha(G) = 3q + 2^{r+2}$ .

**Construction 5.11** Assume that  $G = \text{Dic}_q$  for some  $q \in \{3, 5\}$ . Let  $\Gamma$  be a graph with vertex set  $V(\Gamma) = \mathcal{O}_1 \cup \mathcal{O}_{1'} \cup \mathcal{O}_{1''} \cup \mathcal{O}_{1'''}$ , where  $\mathcal{O}_1 = \{1, 2, \dots, 2q\}$ ,  $\mathcal{O}_{1'} = \{1', 2', \dots, q'\}$ ,  $\mathcal{O}_{1''} = \{1'', 2'', 3'', 4''\}$ ,  $\mathcal{O}_{1'''} = \{1''', 2''', 3''', 4'''\}$ , and edge set such that, given  $v, w \in \mathcal{O}_1$ ,

$$\begin{aligned} &v' \approx w'; v''' \approx w'''; v' \approx w''; v' \approx w'''; \\ &v \sim w \iff w - v \equiv 0 \pmod{q}, \quad v \neq w; \\ &v'' \sim w'' \iff w - v \equiv 1 \pmod{4}; \\ &v \sim w' \iff w - v \equiv 0 \pmod{q}, v \leq q, \quad \text{or } w - v \equiv 1 \pmod{q}, v > q; \\ &v \sim w'' \iff w - v \equiv 0 \pmod{2}, v > q, \quad \text{or } w - v \equiv 1 \pmod{2}, v \leq q; \\ &v \sim w''' \iff v \sim w''; \\ &v'' \sim w''' \iff w - v \equiv 0, 1 \pmod{4}. \end{aligned}$$

**Construction 5.12** For  $G = \text{Dic}_6$ , we let  $\Gamma$  be the graph with vertex set  $V(\Gamma) = \mathcal{O}_1 \cup \mathcal{O}_{1'} \cup \mathcal{O}_{1''} \cup \mathcal{O}_{1'''}$ , where  $\mathcal{O}_1 = \{1, 2, \dots, 8\}$ ,  $\mathcal{O}_{1'} = \{1', 2', \dots, 8'\}$ ,  $\mathcal{O}_{1''} = \{1'', 2'', \dots, 6''\}$ ,  $\mathcal{O}_{1'''} = \{1''', 2''', 3'''\}$ , and edge set such that, for  $v, w \in \mathcal{O}_1$ ,

$$\begin{aligned} &v \approx w; v''' \approx w'''; v \approx w''; v' \approx w''; v' \approx w'''; \\ &v' \sim w' \iff w - v \equiv 4 \pmod{8}; \\ &v'' \sim w'' \iff w - v \equiv 1 \pmod{3}, v \leq 3, w > 3, \quad \text{or } w - v \equiv 2 \pmod{3}, v > 3, w \leq 3; \\ &v \sim w' \iff w - v \equiv 0 \pmod{8}, \quad \text{or } w - v \equiv 3 \pmod{4} \text{ and } (v \leq 4 \iff w \leq 4); \\ &v \sim w'' \iff v \leq 4, w \leq 3, \quad \text{or } v > 4, w > 3; \\ &v'' \sim w''' \iff w - v \equiv 0 \pmod{3}. \end{aligned}$$

**Construction 5.13** Finally, for  $G = Q_8 \times C_3$ , let  $\Gamma_1$  be a graph on 16 vertices constructed according to Babai’s Construction 2.6 for the group  $Q_8$  and let  $\Gamma_2$  be a graph on 9 vertices such that  $\text{Aut}(\Gamma_2) \cong C_3$ , which exists by Proposition 3.3. We let  $\Gamma = \Gamma_1 \cup \Gamma_2$ .

Using GAP [5], we confirmed that each graph has the desired automorphism group.  $\square$

**Proposition 5.14** *Let  $G = \langle a, b \mid a^8 = b^2 = 1, bab^{-1} = a^5 \rangle$ . Then  $\alpha(G) = 18$ .*

Suppose that there exists graph  $\Gamma$  with  $V(\Gamma) \leq 17$  and  $\text{Aut}(\Gamma) \cong G$ . As  $G$  acts faithfully on  $V(\Gamma)$ , there exists  $w \in V(\Gamma)$  such that  $a^4 \notin G_w$ , hence  $G_w \in \{\langle 1 \rangle, \langle a^4b \rangle, \langle b \rangle\}$ . If  $G_w = \langle 1 \rangle$  then the subgraph  $\Gamma_1$  of  $\Gamma$  induced by  $\mathcal{O}_w$  has order 16 and  $\text{Aut}(\Gamma_1) \cong G$ , contradicting the non-existence of a GRR for  $G$ . Since  $\langle a^4b \rangle, \langle b \rangle$  are conjugate, we may assume that  $G_w = \langle b \rangle$ . Let  $u \in V(\Gamma)$  such that  $u \notin \mathcal{O}_w$  and  $G_u = \langle b \rangle$ , if there exists a second orbit with elements not fixed by  $a^4$ ; if no such orbit exists, let  $u = w$ .

**Lemma 5.15** *The vertex  $w$  is adjacent to exactly one of the vertices  $a^2u, a^6u$  in  $\Gamma$ , with  $\Gamma$  as above.*

**Proof** Suppose, conversely, that

$$w \sim a^2u \iff w \sim a^6u. \tag{11}$$

Let  $B = \{w, a^4w, u, a^4u\}$ . Then, the map  $\phi : V(\Gamma) \rightarrow V(\Gamma)$ , where

$$\phi(v) = \begin{cases} a^4v, & \text{if } v \in B, \\ v, & \text{if } v \notin B, \end{cases}$$

is an automorphism of  $\Gamma$ . The proof is similar to that of the other automorphisms defined in this section. However,  $\phi$  fixes  $a^2w$  but not  $w$ , contradicting the equality  $G_w = G_{a^2w}$ .  $\square$

**Proof of Proposition 5.14** If  $\Gamma$  is a graph on at most 17 vertices having  $\text{Aut}(\Gamma) \cong G$  and  $\mathcal{O}_w, \mathcal{O}_u$  are the orbits of size 8 listed above then by Lemma 5.15 either  $w \sim a^2u$  or  $w \sim a^6u$  (hence  $w \neq u$ ). Arguing analogously, we can show that  $w \sim u \iff w \sim a^4u$ . Without loss of generality, we assume that  $w \sim u, w \sim a^2u$ . Moreover, since  $b \in G_w, b \in G_u$ , we have that  $w \sim a^3u \iff w \sim ba^3bu \iff w \sim a^7u$ . Hence, it holds for every  $n \in \mathbb{N}$  that

$$w \sim a^nu \iff w \sim a^{2-n}u. \tag{12}$$

Since  $|V(\Gamma)| \leq 17$  and  $|\mathcal{O}_w \cup \mathcal{O}_u| = 16$ , there is at most one additional orbit, which has size 1. Similar to the map  $\psi$  in Proposition 5.7, the map  $\psi : V(\Gamma) \rightarrow V(\Gamma)$ , where

$$\psi(v) = \begin{cases} a^{-k}u, & \text{if } v = a^ku, k \in \mathbb{N}, \\ a^{6-l}w, & \text{if } v = a^lw, l \in \mathbb{N}, \\ v, & \text{if } |\mathcal{O}_v| = 1. \end{cases}$$

is an automorphism. This is a contradiction as  $G_u = G_{a^2u}$  and  $\psi$  fixes  $u$  but not  $a^2u$ .

We complete the proof by constructing a graph  $\Gamma$  with  $V(\Gamma) = 18$  and  $\text{Aut}(\Gamma) \cong G$ .

**Construction 5.16** Let  $\Gamma$  be a graph with  $V(\Gamma) = V_1 \cup V_2 \cup V_3$ , where  $V_1 = \{1, 2, \dots, 8\}$ ,  $V_2 = \{1', 2', \dots, 8'\}$  and  $V_3 = \{1'', 2''\}$ . We define the edge set of  $\Gamma$  to be such that, for  $v, w \in V_1$ ,

$$\begin{aligned} v'' &\approx w''; \\ v \sim w &\iff w - v \equiv 1, 7 \pmod{8}; \\ v' \sim w' &\iff w - v \equiv 3, 5 \pmod{8}; \\ v \sim w' &\iff w - v \equiv 0, 1, 3 \pmod{8}; \\ v \sim w'' &\iff w = 1; \\ v' \sim w'' &\iff w = 2. \end{aligned}$$

Using GAP [5] we computed that  $\text{Aut}(\Gamma) \cong G$ . □

**Proposition 5.17** The alternating group  $A_4$  satisfies  $\alpha(A_4) = 16$ .

Let  $G = \langle a, b \rangle$ , where  $a = (1\ 2\ 3)$  and  $b = (1\ 2)(3\ 4)$ . Suppose that there exists a graph  $\Gamma$  on at most 15 vertices with  $\text{Aut}(\Gamma) \cong G$ .

Since  $G$  has no subgroup of order 6, there is no orbit of size 2. Furthermore, there exists  $z \in V(\Gamma)$  such that  $b \notin G_z$ . Then  $|G_z| \in \{1, 2, 3\}$ , hence the orbit  $\mathcal{O}_z$  has size 4, 6 or 12. We examine each of these cases.

**Lemma 5.18** Let  $G = A_4$  and let  $\Gamma$  be as in the previous paragraph and consider the action of  $G$  on  $V(\Gamma)$ . Then, there exists no orbit of size 4.

**Proof** Suppose, in contrast, that there exists some orbit of size 4. If there also exists an orbit of size 6 as well as an orbit of size 3, then without loss of generality we let  $w \in V(\Gamma)$  be such that  $|\mathcal{O}_w| = 3$  and  $u \sim aw \iff u \sim a^2w$ , for every  $u \in V(\Gamma)$  having  $G_u = \langle b \rangle$ ; this is possible since the bound  $|V(\Gamma)| \leq 15$  ensures that there is at most one orbit of size 6. Otherwise, let  $w$  be such that  $G_w = \langle aba \rangle$ . Then the map  $\phi : V(\Gamma) \rightarrow V(\Gamma)$ , where

$$\phi(v) = \begin{cases} a^2v, & \text{if } G_v = \langle b \rangle, G_v = \langle aba \rangle \text{ or } v = w, \\ av, & \text{if } G_v = \langle a^2ba \rangle, G_v = \langle ab \rangle \text{ or } v = a^2w, \\ v, & \text{otherwise,} \end{cases}$$

is an automorphism. The proof is more technical but similar to others in this section. For a detailed justification, we refer the reader to the arXiv version of this article [3].

Practically,  $\phi$  interchanges two pairs of vertices in the orbit of size 6, if it exists, one pair in the orbit of size 3, if both an orbit of size 3 and 6 exist, and one pair in every orbit of size 4. However,  $\phi$  fixes two vertices,  $v_1, v_2$ , such that  $G_{v_1} = \langle a \rangle$ ,  $G_{v_2} = \langle a^2b \rangle$ , but  $G_{v_1} \cap G_{v_2} = \{1\}$ ; a contradiction. □

**Lemma 5.19** Let  $G = A_4$  and let  $\Gamma$  be as in Proposition 5.17 and consider the action of  $G$  on  $V(\Gamma)$ . Then, there exists no orbit of size 12.

**Proof** Suppose that there exists an orbit of size 12. In [21, Proposition 3.7], Watkins proved that  $G$  has no GRR. The arguments in [21, Proposition 3.7] extend to the case

that  $\Gamma$  contains an additional orbit of size 3, or up to three additional orbits of size 1. For details on the extension we refer the reader to the arXiv version of this article [3].

□

**Proof of Proposition 5.17** We assumed that  $\Gamma$  is a graph on at most 15 vertices having  $\text{Aut}(\Gamma) \cong G$ ; then  $b \notin G_z$  for some  $z \in V(\Gamma)$ . By Lemmas 5.18 and 5.19, there is no orbit of size 4 or 12, and  $|G_z| = 6$ . Using the group structure of  $G$  we can prove that  $\chi : V(\Gamma) \rightarrow V(\Gamma)$ ,

$$\chi(v) = \begin{cases} aba^2v, & \text{if } G_v = \langle b \rangle, \\ v, & \text{if } G_v \neq \langle b \rangle, \end{cases}$$

is an automorphism of  $\Gamma$ .

However,  $\chi$  fixes two vertices,  $v_1, v_2 \in \mathcal{O}_z$  such that  $G_{v_1} = \langle aba^2 \rangle, G_{v_2} = \langle a^2ba \rangle$ , contradicting the property  $G_{v_1} \cap G_{v_2} = \{1\}$ . Therefore,  $\alpha(G) \geq 16$ .

We will show that  $\alpha(G) = 16$  by constructing a graph  $\Gamma$  on 16 vertices with  $\text{Aut}(\Gamma) \cong G$ .

**Construction 5.20** Let  $\Gamma$  be a graph with vertex set  $V(\Gamma) = \mathcal{O}_1 \cup \mathcal{O}_{1'} \cup \mathcal{O}_{1''}$ , where  $\mathcal{O}_1 = \{1, 2, \dots, 6\}, \mathcal{O}_{1'} = \{1', 2', \dots, 6'\}$  and  $\mathcal{O}_{1''} = \{1'', 2'', 3'', 4''\}$ . We define the edge set of  $\Gamma$  to be such that, for  $v, w \in \mathcal{O}_1$ ,

$$\begin{aligned} v' \approx w'; \quad v' \approx w''; \quad v' \approx w'''; \quad v''' \approx w'''; \\ v \sim w' &\iff v - w \equiv 0 \pmod{2}; \\ v'' \sim w''' &\iff w - v \equiv 0, 1 \pmod{4}; \\ v'' \sim w'' &\iff v - w \equiv 2 \pmod{4}; \\ v \sim w'' &\iff (v, w) \in \bigcup_{k=0}^1 (\{4k + i \mid 1 \leq i \leq 4\} \times \{k + 1, k + 3\}); \\ v \sim w''' &\iff v \sim w''; \\ v \sim w &\iff w - v \equiv 0, (-1)^k \pmod{4}, \quad v \neq w, \quad \text{and} \\ &k \in \{0, 1\} \text{ is such that } v \in \{4k + i \mid 1 \leq i \leq 4\}, w \notin \{4k + i \mid 1 \leq i \leq 4\}. \end{aligned}$$

Using the mathematical software GAP [5] we computed that  $\text{Aut}(\Gamma) \cong G$ . □

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