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Tight chiral polytopes

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Abstract

A chiral polytope with Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$ has at least $2p_1 \cdots p_{n-1}$ flags, and it is called *tight* if the number of flags meets this lower bound. The Schläfli symbols of tight chiral polyhedra were classified in an earlier paper, and another paper proved that there are no tight chiral n-polytopes with $n \geq 6$. Here we prove that there are no tight chiral 5-polytopes, describe 11 families of tight chiral 4-polytopes, and show that every tight chiral 4-polytope covers a polytope from one of those families.

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1 Introduction

An *abstract n-polytope* is a partially ordered set that satisfies many of the properties of the face lattices of convex n-polytopes. The maximal chains (called *flags*) are analogous to the simplices in the barycentric subdivision of a convex polytope. Automorphisms are order-preserving bijections and are the combinatorial analogue of symmetries of convex polytopes.

The group of automorphisms of an abstract polytope acts semiregularly on the set of flags, and if the action is transitive (and thus regular), then the polytope is said to be regular. These polytopes are regarded as the most symmetric and have been extensively studied. The automorphism group of a regular polytope has a standard generating set, and it is possible to recover the polytope from a group in this form, making it possible to study regular polytopes completely in terms of their groups.

An abstract polytope is *chiral* whenever the automorphism group has two orbits on the flags such that flags that differ in only one element are in opposite orbits. This is the combinatorial analogue to having all symmetry by rotations but none by

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reflections. As with regular polytopes, the automorphism group of a chiral polytope has a standard form, and we can build a chiral polytope out of such a group. The study of chiral polytopes grew out from the study of chiral maps and twisted honeycombs (see [7,8]), and while chiral 3-polytopes and chiral 4-polytopes are nowadays plentiful, constructing chiral n-polytopes with $n \geq 5$ seems to be much harder. To date, there is no known natural family of chiral n-polytopes with one polytope for each n (whereas there are many examples of families of regular n-polytopes, such as n-cubes). There is a construction, described in [21], that takes a chiral n-polytope as input and produces a chiral $(n+1)$-polytope, but the polytopes constructed this way are so large that their individual study is out of reach with the current computational means available.

How can we find small examples of chiral polytopes? One strategy is to specify part of the local structure (such as what kind of sub-units the polytope is built from) and then use that local structure to put a lower bound on the number of flags. This idea was used in [3] to find the smallest regular polytopes of each rank and in [11] to explore bounds in the size of chiral polytopes. A polytope is called *tight* if its number of flags is equal to some lower bound. For example, a chiral polyhedron (3-polytope) with p-gonal faces and q edges at each vertex must have at least $2pq$ flags, and so a tight chiral polyhedron has exactly $2pq$ flags (see [10]).

In [12], the first author determined the pairs (p, q) such that there is a tight chiral polyhedron with p-gonal faces and q edges at each vertex. Furthermore, the first author showed in [11] that there are no tight chiral n-polytopes with $n \geq 6$. In this work, we exhibit 11 families of tight chiral 4-polytopes (see Table 4) and show that every tight chiral 4-polytope covers one of the polytopes in these families. Furthermore, we prove the following theorem.

Theorem 1 *There are no tight chiral* 5*-polytopes.*

2 Background

In this section, we summarize relevant definitions and results.

2.1 Abstract polytopes

Regular abstract polytopes are a combinatorial generalization of the notion of (geometric) polyhedra explored by Petrie, Coxeter, Grünbaum and Dress in the 20th Century (see [6,15,16,18]). In what follows, we recall the basic definitions. For further details see [19].

An *abstract polytope* $(\mathcal{P}, \leq)$ of rank n is a partially ordered set satisfying the following four axioms.

(I) It has a unique minimal element F_{-1} and a unique maximal element F_n.
(II) All maximal chains have precisely $n + 2$ faces, including F_{-1} and F_n. This induces a strictly increasing *rank function* $\text{rank} : \mathcal{P} \to \{-1, \ldots, n\}$ where $\text{rank}(F_{-1}) = -1$ and $\text{rank}(F_n) = n$.

(III) *Diamond condition*: Given two elements F, G with $\mathrm{rank}(G) = \mathrm{rank}(F) + 2$, there exist precisely two elements H_1 and H_2 with $\mathrm{rank}(H_1) = \mathrm{rank}(H_2) = \mathrm{rank}(F) + 1$ such that $F \le H_i \le G$ for $i \in \{1, 2\}$.

(IV) *Strong connectivity*: For any pair of incident elements $\{F, G\} \subseteq \mathcal{P}$ with $\mathrm{rank}(G) - \mathrm{rank}(F) \ge 3$, the incidence graph of the open interval (F, G) is connected. (The *incidence graph* of a partially ordered set has the elements as vertices, and two are adjacent if and only if the corresponding elements are incident.)

Throughout this paper, we will encounter only abstract polytopes and we shall refer to them simply as 'polytopes.' Rank 2 and 3 polytopes are also called *poylgons* and *polyhedra*, respectively. For convenience, we refer to the polytope $(\mathcal{P}, \le)$ simply as $\mathcal{P}$. Two elements F, G of $\mathcal{P}$ are said to be *incident* if either $F \le G$ or $G \le F$.

The elements of $\mathcal{P}$ are called *faces*. Those of rank i are called i*-faces*. Following the tradition, the 0- 1- and $(n-1)$-faces are called *vertices*, *edges* and *facets*, respectively. For $i \in \{1, \dots, n-2\}$, we define the i*-skeleton* of $\mathcal{P}$ as the partially ordered set consisting of all the j-faces for $j \le i$. If F_0 is a vertex and F_{n-1} is an incident facet, we say that the closed interval $[F_0, F_{n-1}]$ is a *medial section* of $\mathcal{P}$.

The closed intervals of a polytope (also called *sections*) satisfy the axioms of abstract polytopes. In particular, any medial section of a polytope is a polytope. The section $[F_0, F_n]$, where F_0 is a vertex, is called the *vertex-figure* at F_0. Every face F may be identified with the section $[F_{-1}, F]$, and in this way it may be considered as an abstract polytope.

The maximal chains of $\mathcal{P}$ are called *flags*. Due to the diamond condition, for any flag Φ and any rank $i \in \{0, \dots, n-1\}$ there exists a unique flag Φ^i that differs from Φ precisely in the element of rank i. The flag Φ^i is called the i*-adjacent flag* of Φ. We extend this notation recursively in such a way that if w is a word on the alphabet $\{0, \dots, n-1\}$ and $i \in \{0, \dots, n-1\}$ then $(\Phi^w)^i = \Phi^{wi}$.

The *dual* $\mathcal{P}^\delta$ of a polytope $\mathcal{P}$ consists of the same elements as $\mathcal{P}$ with the partial order reversed. In this way, if F is an i-face of an n-polytope $\mathcal{P}$, then it is an $(n-i-1)$-face of $\mathcal{P}^\delta$.

An n-polytope is said to be *flat* whenever every vertex is incident to every facet. Given $0 \le k < m \le n$, we say that it is (k, m)-flat if every k-face is incident to every m-face.

There is a unique polytope of rank 0 and a unique polytope of rank 1. They correspond to the face lattices of a single point and of a line segment (with its two endpoints). For each integer $k \ge 2$, there is a unique polygon with k vertices, that corresponds to the face lattice of a convex k-gon. There is also a unique *apeirogon* with infinitely many vertices, corresponding to the face lattice of the tiling of the real line by unit intervals. Therefore the rank 2 sections of a polytope are all isomorphic to k-gons for some k or to apeirogons.

We say that a polytope is *equivelar* if, for every $i \in \{1, \dots, n-1\}$, all sections between an $(i-2)$-face and an incident $(i+1)$-face are p_i-gons for some numbers p_i, regardless of the choice of $(i-2)$-face and $(i+1)$-face. Regular and chiral polytopes defined below are examples of equivelar polytopes. The *Schläfli type* (or *type* for short) of an equivelar polytope is $\{p_1, \dots, p_{n-1}\}$.

We say that an n-polytope $\mathcal{Q}$ is a *quotient* of a polytope $\mathcal{P}$ whenever there exists a rank and adjacency preserving mapping from the faces of $\mathcal{P}$ to the faces of $\mathcal{Q}$. (We say that two i-faces are *adjacent* if they are incident to a common $(i-1)$-face and $(i+1)$-face.) In such cases, we say that $\mathcal{P}$ *covers* $\mathcal{Q}$.

An *automorphism* of $\mathcal{P}$ is an order-preserving bijection of its faces. The automorphism group is denoted by $\Gamma(\mathcal{P})$ and acts freely on the set of flags. It follows from the strong connectivity of $\mathcal{P}$ that all orbits of flags have the same size $|\Gamma(\mathcal{P})|$.

2.2 Regularity and chirality

In this subsection, we provide a general background on regular and chiral polytopes.

Our main interest in this paper is on chiral polytopes; hence, we shall follow the approach given in [22] to the study of the automorphism groups of these two classes of objects, and not the one in [19] for regular polytopes.

We say that an n-polytope $\mathcal{P}$ is *regular* whenever $\Gamma(\mathcal{P})$ acts transitively on the set of flags, and it is *chiral* whenever $\Gamma(\mathcal{P})$ induces two orbits on the flags in such a way that adjacent flags belong to distinct orbits. If $\mathcal{P}$ is regular or chiral we say that it is *rotary*.

For every $i \in \{0, \dots, n-1\}$ the automorphism group of a rotary polytope acts transitively on the i-faces. As a consequence, rotary polytopes are equivelar.

It is well-known that for every integers $p_1, \dots, p_{n-1} \geq 2$ there is a regular polytope with type $\{p_1, \dots, p_{n-1}\}$ (see [19, Chapter 3]. This is not the case for chiral polytopes, as shown by the following lemma.

Lemma 1 *If the last entry of the type of a polytope $\mathcal{P}$ is 2 then $\mathcal{P}$ is not chiral.*

Proof If $\mathcal{P}$ is an n-polytope with a 2 as the last entry of its type then all $(n-3)$-faces belong to precisely two facets. By the diamond condition, also the $(n-2)$-faces belong to two facets. The connectivity of the $(n-2)$-skeleton shows that $\mathcal{P}$ has precisely two facets and all i-faces are incident to them for $i \leq n-2$.

The function that fixes every i-face for $i \leq n-2$ and interchanges the two $(n-1)$-faces is then an automorphism, and it maps every flag to its $(n-1)$-adjacent. Hence $\mathcal{P}$ is not chiral. □

Every finite polygon is isomorphic to the face lattice of some convex regular polygon, and hence it is regular. Also the unique infinite 2-polytope is regular. Hence the rank of a non-regular polytope must be at least 3. Chiral polytopes exist in ranks 3 and higher (see [21]).

All sections of regular polytopes are regular. The facets and vertex-figures of a chiral n-polytope may be either regular or chiral; however, the $(n-2)$-faces must be regular (see [22, Proposition 9]). Note that chiral polytopes with chiral facets must have rank at least 4.

Much of the work on chiral polytopes has been done through a particular presentation of their automorphism groups that we explain next. For another useful presentation see for example [5].

Given a fixed *base flag* Φ of a rotary n-polytope $\mathcal{P}$ there exist $\sigma_i \in \Gamma(\mathcal{P})$ for $i \in \{1, \dots, n-1\}$ such that $\Phi\sigma_i = \Phi^{i(i-1)}$. We shall denote the group $\langle \sigma_1, \dots, \sigma_{n-1} \rangle$

by $\Gamma^+(\mathcal{P})$ and call it the *rotation group* of $\mathcal{P}$. The automorphisms σ_i are called *standard generators* of $\Gamma^+(\mathcal{P})$. If $\mathcal{P}$ has type $\{p_1, \dots, p_{n-1}\}$, then the order of σ_i is p_i and therefore $\Gamma^+(\mathcal{P})$ is a suitable quotient of the even subgroup $[p_1, \dots, p_{n-1}]^+$ of the Coxeter group $[p_1, \dots, p_{n-1}]$ (see for example [19, Chapter 3]).

If $\mathcal{P}$ is chiral then $\Gamma(\mathcal{P}) = \Gamma^+(\mathcal{P})$. Whenever $\mathcal{P}$ is regular, $\Gamma^+(\mathcal{P})$ has index at most 2 in $\mathcal{P}$; if the index is 2 we say that $\mathcal{P}$ is *orientably regular*, and it is *non-orientably regular* if $\Gamma(\mathcal{P}) = \Gamma^+(\mathcal{P})$. In any of these cases, if F is an i-face and G is a j-face such that $F \leq G$ and their ranks differ in at least 3 then $\Gamma^+([F, G]) = \langle \sigma_{i+2}, \dots, \sigma_{j-1} \rangle$.

For a rotary polytope $\mathcal{P}$, the standard generators of $\Gamma^+(\mathcal{P})$ satisfy

$$(\sigma_i \dots \sigma_j)^2 = id \quad \text{for every } 1 \leq i < j \leq n-1, \tag{1}$$

as well as the intersection condition

$$A_I \cap A_J = A_{I \cup J} \quad \text{for every } I, J \subseteq \{0, \dots, n-1\}, \tag{2}$$

where for $I \subseteq \{0, \dots, n-1\}$ the set A_I denotes the stabilizer in $\Gamma^+(\mathcal{P})$ of those faces F_i of the base flag with ranks $i \in I$. If $I = \{1, \dots, n-1\} \setminus \{i, i+1, \dots, j\}$ with $i < j$ then $A_I = \langle \sigma_{i+1}, \dots, \sigma_j \rangle$, which allows us to state the following lemma. For other sets, I the generating sets $\mathcal{X}_I$ of these stabilizers are more complicated (see [22, Sect. 3]).

Lemma 2 *Let $\mathcal{P}$ be a rotary polytope with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$. If $j \leq i+1 \leq k$ then*

$$\langle \sigma_1, \dots, \sigma_i \rangle \cap \langle \sigma_j, \dots, \sigma_k \rangle = \langle \sigma_j, \dots, \sigma_i \rangle. \tag{3}$$

If $\mathcal{P}$ is chiral, we may choose the base flag in one or in the other flag orbit. These two choices produce non-equivalent sets of standard generators σ_i, in the sense that the defining relations for $\Gamma^+(\mathcal{P})$ will not be the same for the two sets. One may think of these two ways of looking at $\mathcal{P}$ as a *left* and *right form* of the same object; we can go from one to the other just by 'reflecting' our setting from the base flag into any of its adjacent flags. When doing this, we may take $\{\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \sigma_4, \dots, \sigma_{n-1}\}$ as the new set of standard generators for $\Gamma(\mathcal{P})$. For a chiral polyhedron, another convenient new set of generators is $\{\sigma_1^{-1}, \sigma_2^{-1}\}$. The *enantiomorph* of a chiral polytope $\mathcal{P}$ (with an implicit base flag chosen) consists of the same polytope but where we change the base flag to any of its adjacent flags. We denote the enantiomorph of $\mathcal{P}$ by $\mathcal{P}^*$. For more details about these forms see [23].

We mentioned that the rotation group of a rotary polytope is a group with a generating set satisfying (1) and the intersection condition (2). Conversely, a group with a generating set satisfying (1) and a suitable version of (2) is the rotation group of an orientable rotary polytope (that is, orientably regular or chiral).

The construction of the polytope from a group $\Gamma = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ is detailed in [22, Sect. 5]. It defines the i-face of the base flag as the subgroup of Γ generated by the elements $\mathcal{X}_{\{i\}}$ of $A_{\{i\}}$ mentioned before Lemma 2. The remaining i-faces are the cosets of the base i-face under the right action of Γ. It also establishes that two faces are incident if they have non-empty intersection. In particular, the sets of facets may be identified with the right cosets of $\langle \sigma_1, \ldots, \sigma_{n-2} \rangle$ under Γ. Note that this construction can be performed even if the group does not satisfy the intersection condition. The output will still have well-defined flags, and it is possible to talk about regularity through the action of its automorphism group.

If $\mathcal{P}$ is non-orientably regular then that construction will produce the orientable double cover of $\mathcal{P}$. It follows that there is a one-to-one correspondence between orientable rotary polytopes and groups satisfying (1) together with some version of (2). For our purposes, we find convenient the following version of (2) that can be easily deduced from [22, Lemma 10].

Lemma 3 *Let $\Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ be a group where each σ_i is nontrivial and the order of $\sigma_i \ldots \sigma_j$ is 2, for every $1 \le i < j \le n-1$. Then Γ satisfies the intersection condition (2) if and only if*

$$\langle \sigma_1, \ldots, \sigma_i \rangle \cap \langle \sigma_j, \ldots, \sigma_{i+1} \rangle = \langle \sigma_j, \ldots, \sigma_i \rangle, \tag{4}$$

for every $2 \le j \le i+1 \le n-1$, where if $j = i+1$ then we interpret the right-hand side as being the trivial group.

If $\mathcal{P}$ is orientably regular (resp. chiral) with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ then $\mathcal{P}^\delta$ is also orientably regular (resp. chiral) and, with respect to some flag, the i-th standard generator of $\Gamma^+(\mathcal{P}^\delta)$ is σ_{n-1-i}^{-1}, for $i \in \{1, \ldots, n-1\}$.

In upcoming sections, we will be interested in normal subgroups contained in $\langle \sigma_i \rangle$ for some i. In those situations the following result will prove useful.

Lemma 4 *Let $\mathcal{P}$ be a rotary 4-polytope, and let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$.*

(a) For every k, $\sigma_3 \sigma_1^k \sigma_3^{-1} = \sigma_2^{-1} \sigma_1^{-k} \sigma_2$.
(b) If K is a subgroup of $\langle \sigma_1 \rangle$, then $\sigma_2^{-1} K \sigma_2 = K$ if and only if $\sigma_3^{-1} K \sigma_3 = K$.

Proof We start with

$$\sigma_3 \sigma_1 = (\sigma_1 \sigma_2)^2 \sigma_3 \sigma_1 (\sigma_2 \sigma_3)^2 = \sigma_1 \sigma_2 (\sigma_1 \sigma_2 \sigma_3)^2 \sigma_2 \sigma_3 = \sigma_1 \sigma_2^2 \sigma_3. \tag{5}$$

It follows that

$$\sigma_3 \sigma_1^k = (\sigma_1 \sigma_2^2)^k \sigma_3.$$

Then

$$\sigma_3 \sigma_1^k \sigma_3^{-1} = (\sigma_1 \sigma_2^2)^k = (\sigma_2^{-1} \sigma_1^{-1} \sigma_2)^k = \sigma_2^{-1} \sigma_1^{-k} \sigma_2.$$

That proves part (a). Part (b) follows since $K = \langle \sigma_1^k \rangle$ for some k. $\square$

2.3 Covers and quotients

From the definition of cover, we know that if $\mathcal{P}$ and $\mathcal{Q}$ are orientable rotary n-polytopes such that $\mathcal{P}$ covers $\mathcal{Q}$, then the flags of $\mathcal{P}$ in one orbit under $\Gamma^+(\mathcal{P})$ are mapped by the covering to the flags of $\mathcal{Q}$ in one orbit under $\Gamma^+(\mathcal{Q})$. As a consequence, there exists $N \triangleleft \Gamma^+(\mathcal{P})$ such that $\mathcal{Q} \cong \mathcal{P}/N$. In other words, the faces of $\mathcal{Q}$ can be taken as the orbits of faces of $\mathcal{P}$ under the action of N, and two of them are incident whenever an element in the orbit of one face is incident to some element in the orbit of the other face. (See [20, Example 2.15] for an example of a cover of polytopes that is not induced by the action of a group of automorphisms.)

Conversely, given $N \triangleleft \Gamma^+(\mathcal{P})$, the quotient $\mathcal{P}/N$ is a polytope if and only if $\Gamma^+(\mathcal{P}/N)$ satisfies (1) and the intersection condition (4) with respect to the generators $\{\sigma_i N\}_{i\in\{1,\dots,n-1\}}$.

Whenever $\mathcal{P}$ is chiral, there exists a normal subgroup $X(\mathcal{P})$ of $\Gamma(\mathcal{P})$ satisfying that $\mathcal{P}/X(\mathcal{P})$ is a regular structure (in the sense that all flags belong to the same orbit under $\Gamma(\mathcal{P}/X(\mathcal{P}))$ and that if $N \triangleleft \Gamma(\mathcal{P})$ is such that $\mathcal{P}/N$ is a regular structure then $N \geq X(\mathcal{P})$. The group $X(\mathcal{P})$ is called the *chirality group* of $\mathcal{P}$. Note that $\mathcal{P}$ is regular if and only if $X(\mathcal{P})$ is trivial.

Elsewhere the chirality group has been introduced in other terms (see for example [1, 2] and [9]), but for our purposes the universal property of the chirality group mentioned here is more convenient.

The *mix* of two polytopes $\mathcal{P}$ and $\mathcal{Q}$ with base flags $\Phi_{\mathcal{P}}$ and $\Phi_{\mathcal{Q}}$, respectively, is the smallest structure $\mathcal{P}\diamondsuit\mathcal{Q}$ (which itself may or may not be a polytope) with well-defined ranks and adjacencies that covers simultaneously $\mathcal{P}$ and $\mathcal{Q}$, while mapping the base flag of $\mathcal{P}\diamondsuit\mathcal{Q}$ to $\Phi_{\mathcal{P}}$ and $\Phi_{\mathcal{Q}}$, respectively. As noted in [14, Section 3], the choice of base flags may be relevant when performing the mix of two chiral polytopes. This is often taken into account by choosing a base flag from which to construct the standard generators of the automorphism group.

If $\mathcal{P}$ and $\mathcal{Q}$ are orientable rotary polytopes with $\Gamma^+(\mathcal{P}) = \langle\sigma_1, \dots, \sigma_{n-1}\rangle$ and $\Gamma^+(\mathcal{Q}) = \langle\sigma'_1, \dots, \sigma'_{n-1}\rangle$ then $\Gamma^+(\mathcal{P}\diamondsuit\mathcal{Q}) = \langle\tau_1, \dots, \tau_{n-1}\rangle \leq \Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$, where $\tau_i = (\sigma_i, \sigma'_i)$. For convenience, we also denote $\Gamma^+(\mathcal{P}\diamondsuit\mathcal{Q})$ by $\Gamma^+(\mathcal{P})\diamondsuit\Gamma^+(\mathcal{Q})$.

The mix of two orientably regular polytopes is orientably regular. However, the mix of an orientable rotary polytope with a chiral polytope may be either orientably regular or chiral.

The next lemma relates the notions of quotient and mix of orientable rotary polytopes.

Lemma 5 *Let $\mathcal{P}$ be an orientable rotary polytope with base flag Φ_0 and let K, N be normal subgroups of $\Gamma^+(\mathcal{P})$. Then*

$$\mathcal{P}/(K \cap N) \cong (\mathcal{P}/K)\diamondsuit(\mathcal{P}/N),$$

where the base flags of $\mathcal{P}/K$ and $\mathcal{P}/N$ are taken as $\Phi_0 \cdot K$ and $\Phi_0 \cdot N$, respectively.

Proof The regular structure $\mathcal{P}/(K \cap N)$ (which may or may not be a polytope) covers $\mathcal{P}/K$ mapping a face $F \cdot (K \cap N)$ to the face $F \cdot K$. Similarly, it covers $\mathcal{P}/N$. Hence $\mathcal{P}/(K \cap N)$ covers the mix $(\mathcal{P}/K)\diamondsuit(\mathcal{P}/N)$.

Let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$. Then there is a group epimorphism from $\Gamma^+(\mathcal{P}/(K \cap N))$ to $\Gamma^+((\mathcal{P}/K)\Diamond(\mathcal{P}/N))$ mapping $\sigma_i \cdot (K \cap N)$ to $(\sigma_i \cdot K, \sigma_i \cdot N)$ for $i \in \{1, \ldots, n-1\}$. This epimorphism sends the element $\sigma_{i_1} \cdots \sigma_{i_k} \cdot (K \cap N)$ to $(\sigma_{i_1} \cdots \sigma_{i_k} \cdot K, \sigma_{i_1} \cdots \sigma_{i_k} \cdot N)$. The latter is trivial if and only if $\sigma_{i_1} \cdots \sigma_{i_k} \in K \cap N$. Since the kernel of the epimorphism is trivial, the isomorphism holds. □

Given a chiral polytope $\mathcal{P}$, there exists a smallest regular structure $\mathcal{R}$ with well-defined ranks and adjacencies of flags that covers $\mathcal{P}$ (even if this structure is not a polytope itself), in the sense that every regular polytope that covers $\mathcal{P}$ also covers $\mathcal{R}$. We shall call this structure the *smallest regular cover* of $\mathcal{P}$.

Sometimes the smallest regular cover of $\mathcal{P}$ is a polytope itself; for example, when the facets or the vertex-figures are regular (see [20, Corollary 7.5]). If the smallest regular cover of $\mathcal{P}$ is a polytope then it is elsewhere also called the *minimal regular cover* of $\mathcal{P}$; otherwise, $\mathcal{P}$ may have multiple polytopal regular covers that are minimal in the partial order given by the covering relation.

The smallest regular cover $\mathcal{R}$ of a chiral polytope $\mathcal{P}$ is the regular structure constructed (in the sense of [22]) from the group $\Gamma(\mathcal{P})\Diamond\Gamma(\mathcal{P}^*)$, where $\mathcal{P}^*$ is the enantiomorph of $\mathcal{P}$ (see [20, Sect. 7]). We may assume that if $\Gamma(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ then

$$\Gamma^+(\mathcal{R}) = \langle (\sigma_1, \sigma_1^{-1}), (\sigma_2, \sigma_1^2\sigma_2), (\sigma_3, \sigma_3), \ldots, (\sigma_{n-1}, \sigma_{n-1}) \rangle.$$

We next relate the chirality group of a chiral polytope with its smallest regular cover. This is a direct consequence of [20, Remark 7.3].

Lemma 6 *Let $\mathcal{P}$ be a chiral polytope and $\mathcal{R}$ its smallest regular cover. Then $X(\mathcal{P})$ is isomorphic to the kernel of the quotient from $\Gamma^+(\mathcal{R})$ to $\Gamma(\mathcal{P})$.*

The following result relates the smallest regular covers of chiral polytopes with that of one of its facets.

Lemma 7 *Let $\mathcal{P}$ be a chiral polytope with chiral facets isomorphic to $\mathcal{Q}$. Then the facets of the smallest regular cover of $\mathcal{P}$ are isomorphic to the smallest regular cover of $\mathcal{Q}$.*

Proof Since the facets of $\mathcal{P}$ are chiral, $\mathcal{P}$ has rank $n \geq 4$.

Let $\Gamma(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$, let $\mathcal{R}_\mathcal{P}$ be the smallest regular cover of $\mathcal{P}$ and let $\mathcal{R}_\mathcal{Q}$ be the smallest regular cover of $\mathcal{Q}$. Then $\Gamma^+(\mathcal{R}_\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle \Diamond \langle \sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \ldots, \sigma_{n-1} \rangle$ and $\Gamma^+(\mathcal{R}_\mathcal{Q}) = \langle \sigma_1, \ldots, \sigma_{n-2} \rangle \Diamond \langle \sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \ldots, \sigma_{n-2} \rangle$. Since the orientation preserving automorphism group of the facet of $\mathcal{R}_P$ is $\Gamma^+(\mathcal{R}_\mathcal{Q})$, the lemma holds. □

We conclude this section with a result that relates the chirality group of a chiral polytope $\mathcal{P}$ with that of its facets.

Lemma 8 *Let $\mathcal{P}$ be a chiral polytope with chiral facets isomorphic to $\mathcal{Q}$. Then $X(\mathcal{Q}) \leq X(\mathcal{P})$.*

Proof Let $\mathcal{R}_\mathcal{P}$ be the smallest regular cover of $\mathcal{P}$, and let $\mathcal{R}_\mathcal{Q}$ be the smallest regular cover of $\mathcal{Q}$. Then, by Lemma 7, the facets of $\mathcal{R}_\mathcal{P}$ are isomorphic to $\mathcal{R}_\mathcal{Q}$. By Lemma 6, $X(\mathcal{Q})$ is the kernel of the natural covering $\eta_\mathcal{Q}$ from $\Gamma^+(\mathcal{R}_\mathcal{Q})$ to $\Gamma^+(\mathcal{Q})$, whereas $X(\mathcal{P})$ is the kernel of the natural covering $\eta_\mathcal{P}$ from $\Gamma^+(\mathcal{R}_\mathcal{P})$ to $\Gamma^+(\mathcal{P})$. Since the kernel of $\eta_\mathcal{Q}$ is contained in the kernel of $\eta_\mathcal{P}$, the result follows. □

2.4 Tight polytopes

For the rest of the paper, all polytopes we deal with will be finite. A polytope of type $\{p_1, p_2, \ldots, p_{n-1}\}$ has at least $2p_1p_2\cdots p_{n-1}$ flags, and if it has exactly that many flags, we say it is *tight* [10, Prop. 3.3].

The first mention of the property of tightness occured in [3], while searching for the smallest regular polytopes of each rank. There it was proven that for $n \geq 4$, the regular n-polytopes with fewest flags are always tight. Their study was extended in [10] to equivelar polytopes that may not be regular. In particular, it was proven there that an equivelar polytope is tight if and only if every section of rank 3 is flat. It follows that every section of a tight polytope is itself tight. The following lemma is a natural consequence of this fact.

Lemma 9 *Let $\mathcal{P}$ and $\mathcal{Q}$ be tight rotary polytopes with types $\{p, q\}$ and $\{q, r\}$, respectively. Suppose that $\Gamma^+(\mathcal{P}) = [p, q]^+/N_1$ and $\Gamma^+(\mathcal{Q}) = [q, r]^+/N_2$ where $N_1 \lhd [p, q]^+$ and $N_2 \lhd [q, r]^+$ are subgroups induced by the sets of relations R_1 and R_2, respectively. Then a rotary 4-polytope with facets isomorphic to $\mathcal{P}$ and vertex-figures isomorphic to $\mathcal{Q}$ exists if and only if the group $[p, q, r]^+/N_3$ has order pqr and satisfies the intersection condition (4), where N_3 is the subgroup induced by the relations in R_1 in the first two generators and the relations R_2 in the last two generators. Moreover, such a 4-polytope must be unique.*

Tight regular and chiral polyhedra were studied more deeply in [4,12] and [13]. We summarize relevant results on these polyhedra in Sect. 3. Some results on regular polytopes of higher ranks can be found in [4].

The next proposition summarizes Corollary 3.4 and Theorem 3.5 of [11].

Proposition 1 (a) *If $\mathcal{P}$ is a tight chiral 4-polytope then it has chiral facets or chiral vertex-figures (or both).*
(b) *If $\mathcal{P}$ is a tight chiral 5-polytope then it has chiral facets, vertex-figures, and medial sections.*
(c) *There are no tight chiral n-polytopes for $n \geq 6$.*

Since we shall work with the automorphism groups of chiral polytopes in place of the polytopes themselves, it is useful to have a characterization of tightness that is entirely group-theoretic.

Proposition 2 *Suppose that $\mathcal{P}$ is an orientable rotary n-polytope of type $\{p_1, \ldots, p_{n-1}\}$, with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$. Then the following are equivalent:*

(a) *$\mathcal{P}$ is tight.*
(b) $|\Gamma^+(\mathcal{P})| = p_1 \cdots p_{n-1}$.

(c) $\Gamma^+(\mathcal{P}) = \langle\sigma_1\rangle \cdots \langle\sigma_{n-1}\rangle$.

Proof The equivalence of (a) and (b) follows from the fact that $|\Gamma^+(\mathcal{P})|$ is equal to half the number of flags.

Next we show that (b) and (c) are equivalent. For each $1 \leq i \leq n-1$, let

$$S_i = \langle\sigma_i\rangle \cdots \langle\sigma_{n-1}\rangle.$$

Then $|S_{n-1}| = p_{n-1}$, and for $i < n-1$,

$$S_i = \langle\sigma_i\rangle S_{i+1}.$$

Therefore,

$$|S_i| = \frac{|\langle\sigma_i\rangle| \cdot |S_{i+1}|}{|\langle\sigma_i\rangle \cap S_{i+1}|},$$

and since $\Gamma^+(\mathcal{P})$ satisfies the intersection condition (4), the intersection on bottom is trivial, and so

$$|S_i| = p_i \cdot |S_{i+1}|.$$

It follows that $|S_1| = p_1 \cdots p_{n-1}$. This shows that (c) implies (b).

Conversely, if $|\Gamma^+(\mathcal{P})| = p_1 \cdots p_{n-1}$, then $\Gamma^+(\mathcal{P})$ has the same order as its subset S_1, which implies that $\Gamma^+(\mathcal{P}) = S_1$. □

Note that (b) and (c) are equivalent only in the presence of the intersection condition.

In light of Proposition 2, we will say that the group $\Gamma = \langle\sigma_1, \ldots, \sigma_{n-1}\rangle$ is tight provided that $\Gamma = \langle\sigma_1\rangle \cdots \langle\sigma_{n-1}\rangle$. Then Γ is the rotation group of a tight orientable rotary polytope if and only if Γ is tight, and it satisfies the intersection condition (4). The following result is immediate:

Proposition 3 *If Γ is tight, then any quotient of Γ is tight. If $\mathcal{P}$ is a tight orientable rotary polytope then any quotient of $\mathcal{P}$ is tight.*

Proposition 3 imposes a restriction on the quotients of tight orientable rotary polytopes. The contrapositive of the next proposition imposes another restriction to quotients of tight orientably regular polytopes, namely that tight regular polytopes do not have chiral quotients.

Proposition 4 *If $\mathcal{P}$ is a tight orientable rotary n-polytope that covers a chiral n-polytope then $\mathcal{P}$ itself is chiral.*

Proof Let $\mathcal{Q}$ be a chiral quotient of $\mathcal{P}$. We proceed by induction over n. By Proposition 1 (c), it is only necessary to show the statement for $n \in \{3, 4, 5\}$.

The case when $n = 3$ was proven in [12, Prop. 2.5]. If $n \in 4, 5$, then by Proposition 1 either the facets or the vertex-figures of $\mathcal{Q}$ are chiral $(n-1)$-polytopes. Since the facets and vertex-figures of $\mathcal{Q}$ are quotients of the facets and vertex-figures of $\mathcal{P}$, the inductive hypothesis implies that the facets or vertex-figures of $\mathcal{P}$ must be chiral. Hence $\mathcal{P}$ is chiral. □

Propositions 3 and 4 have the following consequence. When taking polytopal quotients of a tight chiral polytope $\mathcal{P}$ by normal subgroups of $\Gamma^+(\mathcal{P})$, we obtain tight orientably regular or chiral polytopes, and if $\mathcal{P}$ is orientably regular then the quotients are tight and regular. This suggests to try to find successive proper quotients of tight chiral polytopes until we obtain tight regular polytopes. As we shall see, this is always possible. Proposition 6 gives a condition for such quotients to exist. Other conditions will be given in Sects. 4 and 5.

The chiral polytopes we will be interested in typically have a cyclic chirality group, generated by a power of some σ_i. The following result describes circumstances where this property is preserved when taking quotients.

Lemma 10 *Let $\mathcal{P}$ be a tight chiral polytope with $\Gamma(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_n \rangle$ and $\mathcal{Q}$ a chiral quotient of $\mathcal{P}$ with $\Gamma(\mathcal{Q}) = \langle \sigma_1', \ldots, \sigma_n' \rangle$. If $X(\mathcal{P}) \leq \langle \sigma_2 \rangle$ then $X(\mathcal{Q}) \leq \langle \sigma_2' \rangle$.*

Proof Let $K \lhd \Gamma(\mathcal{P})$ such that $\mathcal{Q} = \mathcal{P}/K$, and let $\mathcal{R} = \mathcal{P}/KX(P)$. Then $\mathcal{R}$ is a quotient of $\mathcal{P}/X(\mathcal{P})$, and since the latter is regular, Propositions 3 and 4 imply that $\mathcal{R}$ is regular as well. Now, $\mathcal{R}$ is the quotient of $\mathcal{Q}$ by $KX(P)/K$, and since $\mathcal{R}$ is regular, that implies that $X(\mathcal{Q})$ is contained in $KX(\mathcal{P})/K$, which is the image of $X(\mathcal{P})$ in $\Gamma(\mathcal{Q})$, and thus contained in $\langle \sigma_2' \rangle$. □

Next, we describe useful structural properties of the normal subgroups of the rotation group of tight orientable rotary polytopes.

Lemma 11 *Let $\mathcal{P}$ be a tight orientable rotary n-polytope with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and let $K \lhd \Gamma^+(\mathcal{P})$ such that $\mathcal{P}/K$ is a tight orientable rotary n-polytope. Then there exist nonnegative integers $\alpha_1, \ldots, \alpha_{n-1}$ such that*

$$K = \langle \sigma_1^{\alpha_1} \rangle \langle \sigma_2^{\alpha_2} \rangle \cdots \langle \sigma_{n-1}^{\alpha_{n-1}} \rangle.$$

Moreover, $\mathcal{P}/K$ has type $\{\alpha_1, \ldots, \alpha_{n-1}\}$.

Proof For $1 \leq i \leq n-1$, let α_i be the smallest positive integer such that $\sigma_i^{\alpha_i} \in K$, and let $H = \langle \sigma_1^{\alpha_1} \rangle \cdots \langle \sigma_{n-1}^{\alpha_{n-1}} \rangle$. Then clearly $H \subseteq K$. To show the reverse inclusion, let $\gamma \in K$. By Proposition 2, we may write γ as $\sigma_1^{\beta_1} \cdots \sigma_{n-1}^{\beta_{n-1}}$ for some exponents β_i. Since $\gamma \in K$, we have that for every i,

$$K\sigma_1^{\beta_1} \cdots \sigma_i^{\beta_i} = K(\sigma_{i+1}^{\beta_{i+1}} \cdots \sigma_{n-1}^{\beta_{n-1}})^{-1}.$$

Then, writing $\overline{\sigma_i}$ for the image of σ_i in $\Gamma^+(\mathcal{P})/K$, we get that

$$\overline{\sigma_1^{\beta_1} \cdots \sigma_i^{\beta_i}} = \overline{(\sigma_{i+1}^{\beta_{i+1}} \cdots \sigma_{n-1}^{\beta_{n-1}})^{-1}}.$$

Since $\Gamma^+(\mathcal{P})/K$ is the rotation group of a rotary polytope, Equation (3) implies that $\overline{\sigma_1^{\beta_1} \cdots \sigma_i^{\beta_i}} = 1$, which means that $\sigma_1^{\beta_1} \cdots \sigma_i^{\beta_i} \in K$ for every i. In particular, $\sigma_1^{\beta_1} \in K$, from which it follows that $\sigma_2^{\beta_2} \in K$ (since $\sigma_1^{\beta_1}\sigma_2^{\beta_2} \in K$), and continuing in this way it follows that each $\sigma_i^{\beta_i} \in K$. By our choice of exponents α_i, that means that each β_i is divisible by α_i, and so $\gamma \in H$.

The type of $\mathcal{P}/K$ follows from Proposition 3 since K has order $p_1 \cdots p_{n-1}/\alpha_1 \cdots \alpha_{n-1}$. □

Proposition 5 *Suppose that $\mathcal{P}$ is a tight orientable rotary n-polytope with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$, and let $N = \langle \sigma_1^{a_1} \rangle \langle \sigma_2^{a_2} \rangle \cdots \langle \sigma_{n-1}^{a_{n-1}} \rangle$ be a normal subgroup of $\Gamma^+(\mathcal{P})$. If N does not contain any generator σ_i, then $\Gamma^+(\mathcal{P})/N$ is the rotation group of a tight orientable rotary polytope.*

Proof Let $\Gamma^+(\mathcal{P})/N = \langle \overline{\sigma_1}, \ldots, \overline{\sigma_{n-1}} \rangle$. Since no generator σ_i is in N, it follows that each $\overline{\sigma_i}$ has order at least 2. Then to prove that $\Gamma^+(\mathcal{P})/N$ is the rotation group of an orientable rotary polytope, by Lemma 3 it suffices to show that

$$\langle \overline{\sigma_1}, \ldots, \overline{\sigma_i} \rangle \cap \langle \overline{\sigma_j}, \ldots, \overline{\sigma_{i+1}} \rangle = \langle \overline{\sigma_j}, \ldots, \overline{\sigma_i} \rangle,$$

for all i and j such that $2 \leq j \leq i+1 \leq n-1$. (In fact, it suffices to show that the subgroup on the left is included in the subgroup on the right, since the reverse inclusion is obvious.) Tightness will then follow from Proposition 3.

Consider an element of $\Gamma^+(\mathcal{P})/N$ that lies in

$$\langle \overline{\sigma_1}, \ldots, \overline{\sigma_i} \rangle \cap \langle \overline{\sigma_j}, \ldots, \overline{\sigma_{i+1}} \rangle.$$

We may write this element as $\overline{\varphi_1} = \overline{\varphi_2}$, where

$$\varphi_1 \in \langle \sigma_1, \ldots, \sigma_i \rangle$$

and

$$\varphi_2 \in \langle \sigma_j, \ldots, \sigma_{i+1} \rangle.$$

Then $\varphi_1 = \gamma \varphi_2$ for some $\gamma \in N$. Since $\mathcal{P}$ is tight, Proposition 2(c) says that we may write $\gamma = \sigma_1^{b_1} \cdots \sigma_{n-1}^{b_{n-1}}$. Setting $\gamma_1 = \sigma_1^{b_1} \cdots \sigma_{j-1}^{b_{j-1}}$ and $\gamma_2 = \sigma_j^{b_j} \cdots \sigma_{n-1}^{b_{n-1}}$, we have that by definition γ_1 and γ_2 both lie in N. Now,

$$\gamma_1^{-1} \varphi_1 = \gamma_2 \varphi_2,$$

and it follows that

$$\gamma_1^{-1} \varphi_1 \in \langle \sigma_1, \ldots, \sigma_i \rangle \cap \langle \sigma_j, \ldots, \sigma_{n-1} \rangle.$$

Then since $\Gamma^+(\mathcal{P})$ satisfies the intersection condition, it follows from Lemma 2 that $\gamma_1^{-1} \varphi_1 \in \langle \sigma_j, \ldots, \sigma_i \rangle$. And since $\gamma_1 \in N$, this implies that $\overline{\varphi_1} \in \langle \overline{\sigma_j}, \ldots, \overline{\sigma_i} \rangle$, which is what we wanted to show. □

When considering $\Gamma^+(\mathcal{P})$ as a group acting on the set of i-faces of $\mathcal{P}$ for some i, the kernel of this action is a natural normal subgroup of $\mathcal{P}$ to consider. (Recall that the *kernel* of the action of a group Γ on a set X is the subgroup of Γ fixing X pointwise.) The next results give sufficient conditions for the kernel of the action on the vertex set to be nontrivial.

Lemma 12 *Let $\mathcal{P}$ be a tight orientable rotary polyhedron. If $\gamma \in \Gamma^+(\mathcal{P})$ fixes a vertex and one of its neighbors then it fixes all vertices of $\mathcal{P}$.*

Proof Let u_0 be the base vertex of $\mathcal{P}$. Let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \sigma_2 \rangle$, and let $\gamma \in \Gamma^+(\mathcal{P})$ such that it fixes u_0 and one of its neighbors v_0.

Since the stabilizer of u_0 is $\langle \sigma_2 \rangle$ then $\gamma = \sigma_2^a$ for some a. Now, if σ_2^a fixes v_0 then it must fix all neighbors of u_0, since all of them are images of v_0 under $\langle \sigma_2 \rangle$. Since the choice of base vertex is arbitrary, we have proven that if γ fixes a vertex u and one of its neighbors then it fixes all neighbors of u.

The result then follows from the connectivity of the 1-skeleton of $\mathcal{P}$. □

The fact that the base facet of a tight polytope $\mathcal{P}$ contains all vertices of $\mathcal{P}$ implies the following corollary.

Corollary 1 *Let $\mathcal{P}$ be a tight orientable rotary n-polytope with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$. If σ_2^a fixes a neighbor of the base vertex, then it fixes all vertices of $\mathcal{P}$.*

Corollary 2 *Let $\mathcal{P}$ be a tight orientable rotary n-polytope with type $\{p_1, \ldots, p_{n-1}\}$ with $p_1 \leq p_2$. Then the kernel of the action of $\Gamma^+(\mathcal{P})$ on the vertex set is nontrivial.*

Proof If $\mathcal{P}$ is a tight polytope of type $\{p_1, \ldots, p_{n-1}\}$, then it has p_1 vertices. The automorphism σ_2 fixes the base vertex while permuting the remaining $p_1 - 1$. If $p_1 \leq p_2$, then each neighbor of the base vertex must have a nontrivial stabilizer under $\langle \sigma_2 \rangle$, since the group has order p_2, which is larger than the largest possible orbit. □

Now we are ready to exhibit a proper normal subgroup N of $\Gamma^+(\mathcal{P})$ that is a key element in discussions in Sects. 4 and 5.

Proposition 6 *Let $\mathcal{P}$ be a tight orientable rotary n-polytope with $n \geq 3$ with type $\{p_1, \ldots, p_{n-1}\}$ satisfying that $p_1 \geq p_2$, and rotation group $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$. Then there exists an integer k such that $\langle \sigma_1^k \rangle$ is a nontrivial normal subgroup of $\Gamma^+(\mathcal{P})$.*

Proof By the dual version of Corollary 2 the group $\langle \sigma_1, \sigma_2 \rangle$ has a nontrivial kernel when acting on the 2-faces of the base 3-face of $\mathcal{P}$. These 2-faces correspond to cosets of $\langle \sigma_1 \rangle$ in $\langle \sigma_1, \sigma_2 \rangle$. Then there exists $k \in \{1, \ldots, p_1 - 1\}$ such that $\langle \sigma_1 \rangle \sigma_2^\ell \sigma_1^k = \langle \sigma_1 \rangle \sigma_2^\ell$ for every ℓ. In particular, when $\ell = -1$ this implies that $\sigma_2^{-1} \sigma_1^k \sigma_2 \in \langle \sigma_1 \rangle$. Since the latter group is cyclic, we have that $\langle \sigma_1^k \rangle$ is normal in $\langle \sigma_1, \sigma_2 \rangle$. The result follows from Lemma 4 and commutativity of σ_1^k with σ_i for every $i \geq 4$. □

3 Tight orientable rotary polyhedra and 4-polytopes

Much of the discussion on tight chiral n-polytopes for $n \geq 4$ in Sects. 4, 5 and 6 is based on what we know about tight orientable rotary polyhedra. In this section, we summarize some important facts about them.

We start with a simple result related to Lemma 1, and one of its consequences for tight orientable rotary polyhedra.

Lemma 13 *For every $p \geq 2$, there is a unique polyhedron of type $\{p, 2\}$ and it is regular.*

Proof Let $\mathcal{P}$ be a polyhedron with type $\{p, 2\}$. Then every vertex of $\mathcal{P}$ is incident with precisely two edges and precisely two facets. Since adjacent vertices belong to the same two facets, the connectivity of $\mathcal{P}$ forces $\mathcal{P}$ itself to have only two facets. It follows that $\mathcal{P}$ is isomorphic to the face lattice of the map on the sphere whose 1-skeleton is an equatorial p-gon, and its two facets are the northern and southern hemispheres. Clearly $\mathcal{P}$ is regular. □

Lemma 14 *If $\mathcal{P}$ is an orientable rotary polyhedron with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \sigma_2 \rangle$ and $\langle \sigma_1 \rangle \lhd \Gamma^+(\mathcal{P})$, then $\mathcal{P}$ has type $\{p, 2\}$ for some p. In particular, $\mathcal{P}$ is regular.*

Proof If $\sigma_2^{-1} \langle \sigma_1 \rangle \sigma_2 = \langle \sigma_1 \rangle$ then $\sigma_2^{-1} \sigma_1 \sigma_2 = \sigma_1^k$ for some k. Now, $\sigma_2^{-1} \sigma_1 \sigma_2 = \sigma_2^{-1} \sigma_2^{-1} \sigma_1^{-1}$, implying that $\sigma_2^{-2} = \sigma_1^{k+1}$. The intersection condition (4) tells us that σ_2 has order 2, and hence, the type of $\mathcal{P}$ is $\{p, 2\}$ for some p. Lemma 13 implies the regularity of $\mathcal{P}$. □

The rotation groups of tight orientable rotary polyhedra have many normal subgroups contained in the vertex or facet stabilizer. In the next result, we describe some of these normal subgroups.

Proposition 7 *Suppose $\mathcal{P}$ is a chiral or orientable rotary polyhedron of type $\{p, q\}$, with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \sigma_2 \rangle$. If $\langle \sigma_2^a \rangle \lhd \Gamma^+(\mathcal{P})$, then $\sigma_2^a \sigma_1 = \sigma_1 \sigma_2^{sa}$ for some s such that $s^2 \equiv 1 \pmod{q/a}$. In particular, σ_1^2 commutes with σ_2^a, and if p is odd, then σ_2^a is central.*

Proof Without loss of generality, we may assume that a is a positive divisor of q. The subgroup $\langle \sigma_2^a \rangle$ is normal if and only if $\sigma_1^{-1} \sigma_2^a \sigma_1 = \sigma_2^{sa}$ for some s. Furthermore, we note that

$$\begin{aligned}
\sigma_2^a &= (\sigma_1 \sigma_2)^{-2} \sigma_2^a (\sigma_1 \sigma_2)^2 \\
&= (\sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1}) \sigma_2^{sa} (\sigma_2 \sigma_1 \sigma_2) \\
&= \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{sa} \sigma_1 \sigma_2 \\
&= \sigma_2^{-1} \sigma_2^{s^2 a} \sigma_2 \\
&= \sigma_2^{s^2 a},
\end{aligned}$$

so that $a \equiv s^2 a \pmod{q}$, and thus $s^2 \equiv 1 \pmod{q/a}$. It is now clear then that σ_1^2 commutes with σ_2^a, and if p is odd, then $\langle \sigma_1^2 \rangle = \langle \sigma_1 \rangle$ so that σ_1 commutes with σ_2^a as well. □

Lemma 15 *Let $\mathcal{P}$ be a tight regular polyhedron with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \sigma_2 \rangle$ and $\langle \sigma_2 \rangle$ core-free. Then $\langle \sigma_1^2 \rangle \lhd \Gamma^+(\mathcal{P})$.*

Proof First, [13, Theorem 3.3] says that $\Gamma^+(\mathcal{P})$ is the quotient of $[p,q]^+$ by the extra relation $\sigma_2^{-1}\sigma_1 = \sigma_1^i\sigma_2^j$ for some i and j. By [13, Proposition 3.2(a)], the subgroup $\langle\sigma_2^{j-1}\rangle$ is normal, so since we are supposing that $\langle\sigma_2\rangle$ is core-free, we need $j = 1$. Then taking the relation $\sigma_2^{-1}\sigma_1 = \sigma_1^i\sigma_2$ and multiplying on the left by σ_1^{-1} and then rewriting $\sigma_1^{-1}\sigma_2^{-1}$ as $\sigma_2\sigma_1$ gives us $\sigma_2\sigma_1^2 = \sigma_1^{i-1}\sigma_2$. Furthermore, i must be odd; Sect. 4 of [13] uses a parameter k which is shown at the end of the section to satisfy $i = 1-k$, and k and p are both even by [13, Lemma 4.8]. (Note that Lemma 4.8 requires the polyhedron to have no multiple edges; this is equivalent to asking for $\langle\sigma_2\rangle$ to be core-free, by [13, Proposition 4.6].) Thus σ_2 normalizes $\langle\sigma_1^2\rangle$ and thus this subgroup is normal. □

Tight orientably regular polyhedra with no multiple edges were classified in [13, Theorem 4.13]. The next theorem is a direct consequence.

Theorem 2 *The types of the tight orientably regular polyhedra with no multiple edges are:*

(a) $\{p,2\}$ *for some* $p \geq 2$,
(b) $\{2q,q\}$ *for some odd integer* $q \geq 3$,
(c) $\{p,q\}$ *with* $p = 2^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$ *for some* $\alpha_1 > 0$, *some distinct odd primes* $P_2,\ldots,P_k$, *and* q *a proper even divisor of* p *satisfying that*
   – *the maximal power of* 2 *dividing* q *is either* 2, 4 *or* 2^{α_1-1}, *and if it is* 4 *then* $\alpha_1 \geq 3$,
   – *for* $i \in \{2,\ldots,k\}$, *either* $P_i^{\alpha_i}$ *divides* q *or* P_i *is coprime with* q.

In [12], an *atomic* chiral polyhedron was defined as a tight chiral polyhedron with type $\{p,q\}$ that covers no chiral polyhedron of type $\{p',q\}$ or of type $\{p,q'\}$ for p' a proper divisor of p and q' a proper divisor of q. It is easy to see that every tight chiral polyhedron covers an atomic chiral polyhedron. Furthermore, we will see in Corollary 4 that a stronger condition is true: atomic chiral polyhedra do not cover *any* chiral polyhedra.

The atomic chiral polyhedra were classified in [12, Lemma 4.10, Theorem 4.11, Theorem 4.14]. Here we summarize and slightly simplify this classification (see [12, Theorem 4.15]).

Theorem 3 *Every atomic chiral polyhedron* $\mathcal{P}$ *is one of the polyhedra in Table 1, with chirality group* $X(\mathcal{P})$ *and enantiomorph* $\mathcal{P}^*$ *as described in the table.*

Proof First we will prove the claim for atomic chiral polyhedra of type $\{2m, m^\alpha\}$ and $\{m^\alpha, 2m\}$. We start by noting that for any rotation group $\langle\sigma_1,\sigma_2\rangle$ and for all t, the relation $\sigma_2^{-1}\sigma_1 = \sigma_1^3\sigma_2^t$ is equivalent to $\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^t$, since:

$$\begin{aligned} \sigma_2^{-1}\sigma_1 &= \sigma_1^3\sigma_2^t && \text{Multiply both sides by } \sigma_1^{-1} \text{ on the left} \\ \sigma_1^{-1}\sigma_2^{-1}\sigma_1 &= \sigma_1^2\sigma_2^t && \text{Use } \sigma_1^{-1}\sigma_2^{-1} = \sigma_2\sigma_1 \\ \sigma_2\sigma_1^2 &= \sigma_1^2\sigma_2^t. \end{aligned}$$

Table 1 The atomic chiral polyhedra. An atomic chiral polyhedron with name $\mathcal{P}(p,q)_t$ has an automorphism group that is a quotient of $[p,q]^+$ by the relation(s) given in the “Extra relations” column, and the subscript indicates the name of an additional parameter

$\mathcal{P}$	Extra relations	$X(\mathcal{P})$	$\mathcal{P}^*$	Notes
$\mathcal{P}(2m, m^\alpha)_k$	$\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^{1+km^{\alpha-1}}$	$\langle\sigma_2^{m^{\alpha-1}}\rangle$	$\mathcal{P}(2m, m^\alpha)_{m-k}$	m odd prime, $\alpha \geq 2$
				$1 \leq k \leq m-1$
$\mathcal{P}(m^\alpha, 2m)_k$	$\sigma_2^2\sigma_1 = \sigma_1^{1+km^{\alpha-1}}\sigma_2^2$	$\langle\sigma_1^{m^{\alpha-1}}\rangle$	$\mathcal{P}(m^\alpha, 2m)_{m-k}$	m odd prime, $\alpha \geq 2$
				$1 \leq k \leq m-1$
$\mathcal{P}(8, 2^\beta)_\epsilon$	$\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^{1+\epsilon 2^{\beta-2}}$	$\langle\sigma_2^{2^{\beta-1}}\rangle$	$\mathcal{P}(8, 2^\beta)_{-\epsilon}$	$\beta \geq 5, \epsilon = \pm 1$
$\mathcal{P}(2^\beta, 8)_\epsilon$	$\sigma_1\sigma_2^2 = \sigma_2^2\sigma_1^{1+\epsilon 2^{\beta-2}}$	$\langle\sigma_1^{2^{\beta-1}}\rangle$	$\mathcal{P}(2^\beta, 8)_{-\epsilon}$	$\beta \geq 5, \epsilon = \pm 1$
$\mathcal{P}(2^{\beta-1}, 2^\beta)_\epsilon$	$\sigma_2^{-1}\sigma_1 = \sigma_1^{-1+2^{\beta-2}}\sigma_2^{-3+\epsilon 2^{\beta-2}}$	$\langle\sigma_2^{2^{\beta-1}}\rangle$	$\mathcal{P}(2^{\beta-1}, 2^\beta)_{-\epsilon}$	$\beta \geq 5, \epsilon = \pm 1$
	$\sigma_2\sigma_1^{-1} = \sigma_1^{1+2^{\beta-2}}\sigma_2^{3+\epsilon 2^{\beta-2}}$			
$\mathcal{P}(2^\beta, 2^{\beta-1})_\epsilon$	$\sigma_1^{-1}\sigma_2 = \sigma_2^{-1+2^{\beta-2}}\sigma_1^{-3+\epsilon 2^{\beta-2}}$	$\langle\sigma_1^{2^{\beta-1}}\rangle$	$\mathcal{P}(2^\beta, 2^{\beta-1})_{-\epsilon}$	$\beta \geq 5, \epsilon = \pm 1$
	$\sigma_1\sigma_2^{-1} = \sigma_2^{1+2^{\beta-2}}\sigma_1^{3+\epsilon 2^{\beta-2}}$			

Similarly, for all t, the relation $\sigma_2\sigma_1^{-1} = \sigma_1^{-3}\sigma_2^t$ is equivalent to $\sigma_1^2\sigma_2 = \sigma_2^{-t}\sigma_1^2$, since:

$$\begin{aligned}
\sigma_2\sigma_1^{-1} &= \sigma_1^{-3}\sigma_2^t && \text{Multiply both sides by } \sigma_1 \text{ on the left}\\
\sigma_1\sigma_2\sigma_1^{-1} &= \sigma_1^{-2}\sigma_2^t && \text{Use } \sigma_1\sigma_2 = \sigma_2^{-1}\sigma_1^{-1}\\
\sigma_2^{-1}\sigma_1^{-2} &= \sigma_1^{-2}\sigma_2^t && \text{Invert both sides}\\
\sigma_1^2\sigma_2 &= \sigma_2^{-t}\sigma_1^2.
\end{aligned}$$

Now, suppose that $\mathcal{P}$ is the atomic chiral polyhedron of type $\{2m, m^\alpha\}$ whose group is the quotient of $[2m, m^\alpha]^+$ by the relations $\sigma_2^{-1}\sigma_1 = \sigma_1^3\sigma_2^{1+km^{\alpha-1}}$ and $\sigma_2\sigma_1^{-1} = \sigma_1^{-3}\sigma_2^{-1+km^{\alpha-1}}$. (See [12, Theorem 4.11].) Then the above discussion shows that this group is equivalent to the quotient of $[2m, m^\alpha]^+$ by the relations $\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^{1+km^{\alpha-1}}$ and $\sigma_1^2\sigma_2 = \sigma_2^{1-km^{\alpha-1}}\sigma_1^2$. Furthermore, the second of those relations is superfluous, since if $\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^{1+km^{\alpha-1}}$ then

$$\sigma_2^{1-km^{\alpha-1}}\sigma_1^2 = \sigma_1^2\sigma_2^{(1-km^{\alpha-1})(1+km^{\alpha-1})} = \sigma_1^2\sigma_2.$$

So $\Gamma(\mathcal{P})$ may be written in the form as it appears in Table 1.

Next, the proof of [12, Theorem 3.6] shows that $\langle\sigma_2^{m^{\alpha-1}}\rangle$ is normal and that the quotient of $\mathcal{P}$ by this normal subgroup is regular. Thus $X(\mathcal{P})$ is a nontrivial subgroup of $\langle\sigma_2^{m^{\alpha-1}}\rangle$, and since the latter has prime order m, this implies that $X(\mathcal{P}) = \langle\sigma_2^{m^{\alpha-1}}\rangle$.

To find a presentation for $\Gamma(\mathcal{P}^*)$, we may change the defining relations of $\Gamma(\mathcal{P})$ by replacing σ_1 with σ_1^{-1} and replacing σ_2 with σ_2^{-1}. This yields:

$$\begin{aligned}
\sigma_2^{-1}\sigma_1^{-2} &= \sigma_1^{-2}\sigma_2^{-1-km^{\alpha-1}} && \text{Invert both sides}\\
\sigma_1^2\sigma_2 &= \sigma_2^{1+km^{\alpha-1}}\sigma_1^2.
\end{aligned}$$

From this, we obtain $\sigma_1^2\sigma_2^{1-km^{\alpha-1}} = \sigma_2^{(1+km^{\alpha-1})(1-km^{\alpha-1})}\sigma_1^2 = \sigma_2\sigma_1^2$. Thus, the enantiomorph replaces the parameter k with $-k$ (or equivalently, $m-k$).

A presentation for the dual of $\mathcal{P}$ (with respect to the same base flag as $\mathcal{P}$) is obtained by changing each defining relation, replacing σ_1 with σ_2^{-1} and σ_2 with σ_1^{-1}. Applying this to the relation $\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^{1+km^{\alpha-1}}$ and then inverting both sides yields $\sigma_2^2\sigma_1 = \sigma_1^{1+km^{\alpha-1}}\sigma_2^2$, matching the second row of Table 1.

This finishes the proof for atomic chiral polyhedra of type $\{2m, m^\alpha\}$ and their duals. The proof for the remaining polyhedra is analogous (referencing [12, Theorems 3.7 and 3.8]), except that for type $\{2^{\beta-1}, 2^\beta\}$ and its dual, it is not possible to simplify the presentation in the same way that we can for the other two cases. □

Corollary 3 *Let $\mathcal{P}$ be an atomic chiral polyhedron with type $\{p, q\}$ and $p \geq q$. Then p is a prime power.*

It turns out that the atomic chiral polyhedra satisfy a stronger condition than their definition would seem to imply.

Corollary 4 *If $\mathcal{P}$ is an atomic chiral polyhedron, then it does not properly cover any chiral polyhedron.*

Proof Suppose that $\mathcal{P}$ is an atomic chiral polyhedron of type $\{p, q\}$, and without loss of generality, assume that $p \geq q$ so that p is a prime power m^α (where we could have $m = 2$). By the definition of atomic, $\mathcal{P}$ does not properly cover any chiral polyhedra of type $\{p, q'\}$ or $\{p', q\}$. Furthermore, if $\mathcal{Q}$ is an orientable rotary polyhedron of type $\{p', q'\}$ where p' is a proper divisor of p, then the kernel of the natural map from $\Gamma(\mathcal{P})$ to $\Gamma(\mathcal{Q})$ contains $\langle \sigma_1^{m^{\alpha-1}} \rangle$ (see Table 1), and since that is the chirality group of $\mathcal{P}$, it follows that $\mathcal{Q}$ is regular. □

In light of Corollary 4, let us now make a (harmless) redefinition of what it means to be atomic, while simultaneously generalizing the definition to higher rank.

Definition 1 A chiral polytope is *atomic* if it is tight and it does not properly cover any chiral polytopes.

The following result is an immediate consequence of the definition of atomicity and [12, Corollary 4.3], which states that every tight chiral polyhedron of type $\{p, q\}$ covers a tight orientable rotary polyhedron of type $\{p', q\}$ or $\{p, q'\}$.

Proposition 8 *If $\mathcal{P}$ is a tight chiral polyhedron of type $\{p, q\}$ that is not atomic, then it covers a tight chiral polyhedron of type $\{p', q\}$ or $\{p, q'\}$.*

When mixing tight orientable rotary polyhedra we may not get a tight structure, as shown next.

Proposition 9 *Let $\mathcal{P}$ and $\mathcal{Q}$ be distinct atomic chiral polyhedra of types $\{p, q\}$ and $\{p, q'\}$, respectively, with q' a divisor of q (not necessarily proper). Then $\mathcal{P} \diamondsuit \mathcal{Q}$ is not tight, regardless of the choice of base flags.*

Proof The mix of $\mathcal{P}$ and $\mathcal{Q}$ with respect to any choice of base flags must have type $\{p, q\}$, and if it were tight then it should be isomorphic to $\mathcal{P}$ and have $\mathcal{Q}$ as a proper quotient. This is not possible since $\mathcal{P}$ is atomic. □

We conclude this section with some technical lemmas that allow us to find polytopal quotients of tight orientable rotary 4-polytopes.

Lemma 16 *Let $\mathcal{P}$ be a tight chiral polyhedron with type $\{p, q\}$ and $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2 \rangle$. Then $\langle \sigma_i^2 \rangle$ is not normal in $\Gamma(\mathcal{P})$.*

Proof Let $\mathcal{Q}$ be an atomic chiral polyhedron covered by $\mathcal{P}$ with automorphism group $\langle \tau_1, \tau_2 \rangle$. If we assume that $\langle \sigma_i^2 \rangle \triangleleft \Gamma(\mathcal{P})$, then by the correspondence theorem in group theory we must also have that $\langle \tau_i^2 \rangle \triangleleft \Gamma(\mathcal{Q})$.

It was proven in [12, Proposition 4.1] that if $\mathcal{Q}$ has type $\{p', q'\}$ and $p' > q'$ then $\langle \tau_1 \rangle$ has a proper subgroup normal in $\Gamma(\mathcal{Q})$. On the other hand, it is shown in [12, Proposition 4.5] that either $\langle \tau_1 \rangle$ or $\langle \tau_2 \rangle$ is core-free in $\Gamma(\mathcal{Q})$. Up to duality, we may assume that $p' > q'$, and hence, we only need to show that $\langle \tau_1^2 \rangle$ is not normal in $\Gamma(\mathcal{Q})$.

Now, using the classification of atomic chiral polyhedra we see that if $\{p, q\} = \{m^\alpha, 2m\}$ then $\langle \tau_1^2 \rangle = \langle \tau_1 \rangle$ and this is not normal in $\Gamma(\mathcal{Q})$ (see Lemma 14). On the other hand, if p and q are powers of 2, the dual version of [12, Lemma 4.13] tells us that the core of $\langle \tau_1 \rangle$ is $\langle \tau_1^4 \rangle$. Hence, $\langle \tau_1^2 \rangle$ is not normal in $\Gamma(\mathcal{Q})$. □

Lemma 17 *Let $\mathcal{P}$ be a chiral 4-polytope with chiral facets and let K be the kernel of the action of $\Gamma(\mathcal{P})$ on the vertex set. Then $\sigma_i \notin K$ for $i \in \{1, 2, 3\}$.*

Proof The group K is a normal subgroup of $\Gamma(\mathcal{P})$ that is contained in $\langle\sigma_2, \sigma_3\rangle$ since it fixes the base vertex. The intersection condition (3) implies that $\sigma_1 \notin K$.

If $\sigma_2 \in K$, then also $\sigma_1\sigma_2\sigma_1^{-1} \in K$, which implies that $\sigma_2^{-1}\sigma_1^{-2} \in K$ and so $\sigma_1^2 \in K$. Since $\langle\sigma_1\rangle$ has trivial intersection with K (again by the intersection condition), this implies that $\sigma_1^2 = id$, which contradicts Lemma 1.

Similarly, if $\sigma_3 \in K$, then also $\sigma_2\sigma_3\sigma_2^{-1} \in K$, which implies that $\sigma_3^{-1}\sigma_2^{-2} \in K$ and so $\sigma_2^2 \in K$. It follows that $\sigma_1^{-1}\sigma_2^2\sigma_1 \in K$. Then $\sigma_1^{-1}\sigma_2^2\sigma_1$ lies in the intersection of $\langle\sigma_1, \sigma_2\rangle$ with $\langle\sigma_2, \sigma_3\rangle$, and so it must lie in $\langle\sigma_2\rangle$. This implies that $\langle\sigma_2^2\rangle$ is normal in $\langle\sigma_1, \sigma_2\rangle$, contradicting Lemma 16. □

Lemma 18 *Let $\mathcal{P}$ be a tight orientable rotary 4-polytope, with $\Gamma^+(\mathcal{P}) = \langle\sigma_1, \sigma_2, \sigma_3\rangle$. Let K be the kernel of the action of $\Gamma^+(\mathcal{P})$ on the vertex set. Then:*

(a) There are integers a and b such that $K = \langle\sigma_2^a\rangle\langle\sigma_3^b\rangle$.
(b) $\mathcal{P}/K$ is a tight orientable rotary 4-polytope.

Proof Let a be the smallest positive integer such that $\sigma_2^a \in K$, and let b be the smallest positive integer such that $\sigma_3^b \in K$. (We allow the possibility that $\sigma_2^a = id$ or $\sigma_3^b = id$.) Let $N = \langle\sigma_2^a\rangle\langle\sigma_3^b\rangle$. Then clearly N is contained in K. To prove the first part, it remains to show that K is contained in N.

Let $H = \langle\sigma_2, \sigma_3\rangle$, and suppose that the order of σ_1 is p. Since $\mathcal{P}$ is tight, it has p vertices, which we can identify with the cosets $H, H\sigma_1, \ldots, H\sigma_1^{p-1}$. The action of each automorphism on the vertices is by multiplication on the right. Now, suppose that $\varphi \in K$, which in particular implies that $\varphi \in \langle\sigma_2, \sigma_3\rangle$. Since $\mathcal{P}$ is tight, Proposition 2 implies that we may write $\varphi = \sigma_2^c\sigma_3^d$. Since $\sigma_2^c\sigma_3^d$ fixes all vertices, it follows that the action of σ_2^c on vertices is the same as the action of σ_3^{-d} on vertices. Note that σ_3^{-1} fixes the neighbor of the base vertex in the base edge, namely,

$$H\sigma_1^{-1}\sigma_3^{-1} = H(\sigma_3\sigma_1)^{-1} = H(\sigma_1\sigma_2^2\sigma_3)^{-1} = H\sigma_1^{-1}.$$

It follows that σ_3^{-d} fixes that vertex, and thus so does σ_2^c. However, by Corollary 1, if a power of σ_2 fixes a neighbor of the base vertex, then it fixes all vertices. Therefore, $\sigma_2^c \in K$, from which it follows that $\sigma_3^d \in K$. Then by our choice of a and b, it follows that $\varphi \in N$.

The second part follows from the first along with Lemma 17. □

4 Atomic chiral 4-polytopes with chiral facets and vertex-figures

To understand the structure of tight chiral 4-polytopes, we use a strategy similar to what was done with tight chiral polyhedra. Recall that a tight chiral 4-polytope is *atomic* if it does not properly cover any chiral polytopes. It is clear that every tight chiral polytope covers an atomic chiral polytope. Our goal will be to classify the atomic chiral 4-polytopes.

By Proposition 1 (a), the facets or the vertex-figures of an atomic chiral 4-polytope must be chiral. In this section, we classify all atomic chiral 4-polytopes that have chiral facets and chiral vertex-figures, leaving the case when one of them is regular for Sect. 5. We will show in Theorem 4 that an atomic chiral 4-polytope with chiral facets and chiral vertex-figures must have atomic chiral facets and atomic chiral vertex-figures. The classification of atomic chiral polyhedra will be then used to find all atomic chiral 4-polytopes with chiral facets and vertex-figures.

4.1 The structure of atomic chiral 4-polytopes with chiral facets and vertex-figures

Now we study atomic chiral 4-polytopes with chiral facets and chiral vertex-figures. We find several restrictions on atomic chiral 4-polytopes, culminating in Theorem 4.

Proposition 10 *Let $\mathcal{P}$ be an atomic chiral 4-polytope of type $\{p, q, r\}$, with chiral facets and vertex-figures, and with $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$. Then:*

(a) $\langle \sigma_1 \rangle$ and $\langle \sigma_3 \rangle$ are core-free in $\Gamma(\mathcal{P})$.
(b) $q > p$ and $q > r$.

Proof By duality, for the first part it suffices to prove that $\langle \sigma_1 \rangle$ is core-free. Suppose that $\mathcal{P}$ is a tight chiral 4-polytope with chiral facets and vertex-figures and suppose that $\langle \sigma_1 \rangle$ is not core-free. In other words, there is a nontrivial normal subgroup $N = \langle \sigma_1^a \rangle$ of $\Gamma(\mathcal{P})$. If $\sigma_1 \in N$, then $\langle \sigma_1 \rangle$ is normal in $\langle \sigma_1, \sigma_2 \rangle$, and by Lemma 14, this implies that the facets are regular, contradicting our assumptions. So $\sigma_1 \notin N$. Then the dual of Proposition 5 shows that $\Gamma(\mathcal{P})/N$ is the rotation group of a tight rotary polytope $\mathcal{Q}$. Since $\langle \sigma_2, \sigma_3 \rangle$ has trivial intersection with N, the vertex-figures of $\mathcal{Q}$ must be isomorphic to the vertex-figures of $\mathcal{P}$, which are chiral. Thus $\mathcal{Q}$ is chiral, which means that $\mathcal{P}$ is not atomic. This proves part (a).

By Proposition 6, if $p \geq q$ then there exists a proper divisor k of p such that $\langle \sigma_1^k \rangle \lhd \Gamma(\mathcal{P})$ contradicting part (a). A dual argument follows if $r \geq q$. ◻

Proposition 11 *Let $\mathcal{P}$ be an atomic chiral 4-polytope of type $\{p, q, r\}$, with chiral facets and vertex-figures, and with $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$. Then:*

(a) The chirality group $X(\mathcal{P})$ is $\langle \sigma_2^{q'} \rangle$ for some q' with q/q' prime,
(b) The chirality groups of the base facet and vertex-figure are isomorphic to $X(\mathcal{P})$.

Proof Let H and K be the kernels of the actions of $\Gamma(\mathcal{P})$ on the vertices and on the facets of $\mathcal{P}$, respectively. By Proposition 10(b) together with Corollary 2 and its dual form, $H \leq \langle \sigma_2, \sigma_3 \rangle$ and $K \leq \langle \sigma_1, \sigma_2 \rangle$ are nontrivial normal subgroups of $\Gamma(\mathcal{P})$. Therefore $H \cap K$ is a normal subgroup of $\Gamma(\mathcal{P})$ that by the intersection condition is contained in $\langle \sigma_2 \rangle$.

Now, Lemma 18 and its dual show that $\mathcal{P}/H$ and $\mathcal{P}/K$ are polytopes, and since H and K are nontrivial and $\mathcal{P}$ is atomic, $\mathcal{P}/H$ and $\mathcal{P}/K$ are regular. Moreover, by Lemma 5, $\mathcal{P}/(H \cap K) \cong \mathcal{P}/H \Diamond \mathcal{P}/K$ is also regular, implying that $H \cap K$ is nontrivial.

Since $\mathcal{P}/(H \cap K)$ is regular, $X(\mathcal{P}) \leq H \cap K = \langle \sigma_2^m \rangle$ for some m. If q/m is not prime, then $\langle \sigma_2^{mk} \rangle \lhd \Gamma(\mathcal{P})$ for any k, in particular, for some k such that q/mk is prime.

By atomicity of $\mathcal{P}$, its quotient by $\langle\sigma_2^{mk}\rangle$ is regular and, since it is a maximal quotient, $X(\mathcal{P}) = \langle\sigma_2^{mk}\rangle$. This concludes part (a).

Part (b) follows from Part (a) and Lemma 8. □

Proposition 12 *Let $\mathcal{P}$ be an atomic chiral 4-polytope of type $\{p, q, r\}$ with chiral facets and vertex-figures. If q is a prime power then the facets and vertex-figures of $\mathcal{P}$ are atomic chiral polyhedra.*

Proof Suppose $\mathcal{P}$ has facets isomorphic to $\mathcal{Q}_1$ and vertex-figures isomorphic to $\mathcal{Q}_2$. By Proposition 11, $X(\mathcal{P}) = X(\mathcal{Q}_1) = X(\mathcal{Q}_2)$, and these groups are cyclic of prime order. If q is a prime power then $X(\mathcal{P})$ is contained in all proper subgroups of $\langle\sigma_2\rangle$, and so $\mathcal{Q}_1$ does not cover any tight chiral polyhedra of type $\{p, q'\}$ with q' a proper divisor of q. Proposition 10 says that $\langle\sigma_1\rangle$ is core-free, and so also $\mathcal{Q}_1$ does not cover any tight chiral polyhedra of type $\{p', q\}$ with p' a proper divisor of p. It follows that $\mathcal{Q}_1$ is atomic, and a dual argument proves that $\mathcal{Q}_2$ is atomic as well. □

We are now ready to prove the main necessary condition for a tight chiral 4-polytope with chiral facets and chiral vertex-figures to be atomic.

Theorem 4 *If $\mathcal{P}$ is an atomic chiral 4-polytope with chiral facets and chiral vertex-figures, then the facets and vertex-figures are atomic chiral polyhedra.*

Proof Assume that $\mathcal{P}$ has type $\{p, q, r\}$. The facets and vertex-figures of $\mathcal{P}$ are isomorphic to some chiral polyhedra $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively. Proposition 11 (b) tells us that $X(\mathcal{P}) = \langle\sigma_2^{q'}\rangle$ with q/q' prime and that $X(\mathcal{P}) = X(\mathcal{Q}_1) = X(\mathcal{Q}_2)$.

Assume to the contrary that $\mathcal{Q}_i$ is not atomic for some $i \in \{1, 2\}$. Then, by Proposition 12, q must have at least two distinct prime factors, which by Corollary 3 implies that neither $\mathcal{Q}_1$ nor $\mathcal{Q}_2$ is atomic. Let $m = q/q'$, which is prime (but not necessarily odd). Then $q = m^\alpha t$ for some α and some t not divisible by m.

Since $\mathcal{Q}_1$ is not atomic there exists $N_1 \triangleleft \langle\sigma_1, \sigma_2\rangle$ such that $\mathcal{Q}_1/N_1$ is an atomic chiral polyhedron. By Lemma 11, there exist a and b such that $N_1 = \langle\sigma_1^a\rangle\langle\sigma_2^b\rangle$ and $\mathcal{Q}_1/N_1$ has type $\{a, b\}$.

Now $X(\mathcal{Q}_1) = X(\mathcal{P})$, and therefore, $\sigma_2^{q'} \notin N_1$. It follows that q/b divides t and m^α divides b. Lemma 10 implies that $X(\mathcal{Q}_1/N_1)$ is contained in the subgroup generated by the second standard generator of $\Gamma(\mathcal{Q}_1/N_1)$. Since $\mathcal{Q}_1/N_1$ is atomic, we can conclude that $a < b$ by the classification of atomic chiral polyhedra. Corollary 3 now tells us that $b = m^\alpha$.

We proceed in a dual manner to observe that there exists $N_2 = \langle\sigma_2^{b'}\rangle\langle\sigma_3^c\rangle \leq \langle\sigma_2, \sigma_3\rangle$ such that $\mathcal{Q}_2/N_2$ is an atomic chiral polyhedron with type $\{b', c\} = \{m^\alpha, c\}$. In particular, $b = b'$.

Let $K = \langle\sigma_1^a\rangle\langle\sigma_2^b\rangle\langle\sigma_3^c\rangle$. We claim that $K \triangleleft \Gamma(\mathcal{P})$. To see this, note that

$$\sigma_2^{-1}(\sigma_1^{k_1 a}\sigma_2^{k_2 b}\sigma_3^{k_3 c})\sigma_2 = (\sigma_2^{-1}\sigma_1^{k_1 a}\sigma_2^{k_2 b}\sigma_2)(\sigma_2^{-1}\sigma_3^{k_3 c}\sigma_2) \in (\langle\sigma_1^a\rangle\langle\sigma_2^b\rangle)(\langle\sigma_2^b\rangle\langle\sigma_3^c\rangle),$$

and as noted in the proof of Lemma 4,

$$\sigma_3(\sigma_1^{k_1 a}\sigma_2^{k_2 b}\sigma_3^{k_3 c})\sigma_3^{-1} = (\sigma_2^{-1}(\sigma_1^{-k_1 a})\sigma_2)(\sigma_3\sigma_2^{k_2 b}\sigma_3^{k_3 c}\sigma_3^{-1}) \in (\langle\sigma_1^a\rangle\langle\sigma_2^b\rangle)(\langle\sigma_2^b\rangle\langle\sigma_3^c\rangle).$$

A dual argument shows that K is invariant under conjugation by σ_1.

Now, Lemma 17 and Proposition 5 imply that $\mathcal{P}/K$ is a polytope, and since $\mathcal{P}$ is atomic this polytope must be regular of type $\{a, b, c\}$. In particular, this implies that the facets are regular polyhedra of type $\{a, b\}$. On the other hand, the facets must be a quotient of $\mathcal{Q}_1/N_1$, which is a tight chiral polyhedron of type $\{a, b\}$. But no tight polyhedron properly covers another polyhedron of the same type, and so we have a contradiction. □

Now let us show that the conditions in Proposition 10 and Theorem 4 suffice if we want to build an atomic chiral 4-polytope.

Corollary 5 *A tight chiral* 4*-polytope* $\mathcal{P}$ *with chiral facets and vertex-figures is atomic if and only if*

(a) The facets and vertex-figures are atomic, and
(b) $\langle\sigma_1\rangle$ *and* $\langle\sigma_3\rangle$ *are core-free in* $\Gamma(\mathcal{P})$.

Proof Theorem 4 and Proposition 10 prove that the conditions are necessary. Now, suppose that $\mathcal{P}$ satisfies the conditions. If $\mathcal{Q}$ is a proper chiral quotient of $\mathcal{P}$, then $\mathcal{Q}$ is still tight, and so Proposition 1 says that either the facets or vertex-figures are chiral. Without loss of generality, suppose that the facets of $\mathcal{Q}$ are chiral. The facets of $\mathcal{P}$ cover the facets of $\mathcal{Q}$, and since the facets of $\mathcal{P}$ are atomic, this implies that $\mathcal{Q}$ has the same facets. In particular, if $\mathcal{P}$ has type $\{p, q, r\}$, then $\mathcal{Q}$ has type $\{p, q, r'\}$ for some r' dividing r. By tightness, $|\Gamma(\mathcal{P})| = pqr$ and $|\Gamma(\mathcal{Q})| = pqr'$, and so since $\mathcal{Q}$ is a proper quotient of $\mathcal{P}$, we have $r' \neq r$. Furthermore, $\Gamma(\mathcal{Q}) = \Gamma(\mathcal{P})/\langle\sigma_3^{r'}\rangle$. But this contradicts that $\langle\sigma_3\rangle$ is core-free in $\Gamma(\mathcal{P})$. □

4.2 Classification of atomic chiral 4-polytopes with chiral facets and vertex-figures

In light of Lemma 9, once we know the possible types of facets and vertex-figures of an atomic chiral 4-polytope, all we need to do is try amalgamating the compatible pairs and see which ones give us a group of the proper size that satisfies the intersection condition. Theorem 4 implies that the facets and vertex-figures must appear on Table 1. Combined with Proposition 10, we find that the automorphism group of an atomic chiral 4-polytope with chiral facets and vertex-figures must be one of the groups in Table 2. For simplicity, we avoid including the various parameters (such as m, α, and k_1) in the names of the groups. The "extra relations" show how to define the group as a quotient of the given parent group.

Using GAP [17], we verified that Γ_2, Γ_3, and Γ_4 have the correct order and satisfy the intersection condition for $\beta = 5$ and $\beta = 6$, and for all four choices of (ϵ_1, ϵ_2). Thus, for these parameter values, the group is the automorphism group of a tight chiral polytope. We similarly verified that Γ_1 is the automorphism group of a tight chiral polytope for $m = 3, \alpha \in \{2, 3\}, k_1 = k_2 \in \{1, 2\}$ and for $m = 5, \alpha = 2, k_1 = k_2 \in \{1, \ldots, 4\}$. Furthermore, for these values of m and α, we verified that Γ_1 does not have the proper order when $k_1 \neq k_2$, and so does not define the automorphism group of a tight chiral polytope.

For the group Γ_1, we will show that we do in fact need $k_1 = k_2$. Then, for each group we will describe a permutation representation of the group. There is a standard strategy

Table 2 The candidate groups of atomic chiral 4-polytopes with chiral facets and vertex-figures

Group name	Parent group	Extra relations	Notes
Γ_1	$[2m, m^{\alpha}, 2m]^{+}$	$\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^{1+k_1m^{\alpha-1}}$	m odd prime, $\alpha \geq 2$,
		$\sigma_3^2\sigma_2 = \sigma_2^{1+k_2m^{\alpha-1}}\sigma_3^2$	$k_1, k_2 \in \{1, \ldots, m-1\}$
Γ_2	$[8, 2^{\beta}, 8]^{+}$	$\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^{1+\epsilon_1 2^{\beta-2}}$	$\beta \geq 5, \epsilon_1, \epsilon_2 \in \{-1, 1\}$
		$\sigma_3^2\sigma_2 = \sigma_2^{1+\epsilon_2 2^{\beta-2}}\sigma_3^2$	
Γ_3	$[2^{\beta-1}, 2^{\beta}, 2^{\beta-1}]^{+}$	$\sigma_2^{-1}\sigma_1 = \sigma_1^{-1+2^{\beta-2}}\sigma_2^{-3+\epsilon_1 2^{\beta-2}}$	$\beta \geq 5, \epsilon_1, \epsilon_2 \in \{-1, 1\}$
		$\sigma_2\sigma_1^{-1} = \sigma_1^{1+2^{\beta-2}}\sigma_2^{3+\epsilon_1 2^{\beta-2}}$	
		$\sigma_3^{-1}\sigma_2 = \sigma_2^{3+\epsilon_2 2^{\beta-2}}\sigma_3^{1+2^{\beta-2}}$	
		$\sigma_3\sigma_2^{-1} = \sigma_2^{-3+\epsilon_2 2^{\beta-2}}\sigma_3^{-1+2^{\beta-2}}$	
Γ_4	$[8, 2^{\beta}, 2^{\beta-1}]^{+}$	$\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^{1+\epsilon_1 2^{\beta-2}}$	$\beta \geq 5, \epsilon_1, \epsilon_2 \in \{-1, 1\}$
		$\sigma_3^{-1}\sigma_2 = \sigma_2^{3+\epsilon_2 2^{\beta-2}}\sigma_3^{1+2^{\beta-2}}$	
		$\sigma_3\sigma_2^{-1} = \sigma_2^{-3+\epsilon_2 2^{\beta-2}}\sigma_3^{-1+2^{\beta-2}}$	

that we used to determine the permutation representation, based on the following facts. If $\mathcal{P}$ is a tight chiral 4-polytope of type $\{p, q, r\}$, then the cosets of $\langle\sigma_1\rangle$ are of the form $\langle\sigma_1\rangle\sigma_2^b\sigma_3^c$, and $\Gamma(\mathcal{P})$ acts on the set of cosets by right multiplication. Furthermore, since $\Gamma(\mathcal{P})$ is tight, then for every i we can rewrite $\langle\sigma_1\rangle\sigma_2^b\sigma_3^c\sigma_i$ as $\langle\sigma_1\rangle\sigma_2^{b'}\sigma_3^{c'}$ for some b' and c'. So for each i, we determined how b' and c' depend on b and c. We then encode the coset $\langle\sigma_1\rangle\sigma_2^b\sigma_3^c$ as the pair $(b, c) \in \mathbb{Z}_q \times \mathbb{Z}_r$ and write down a description of the multiplication.

Once we have a permutation representation, the following lemma will show that we indeed have found the group of a tight chiral polytope.

Lemma 19 *Suppose that $\mathcal{P}$ is a tight orientable rotary polyhedron and that $\mathcal{Q}$ is a tight chiral polyhedron. Let $\Gamma = \langle\sigma_1, \sigma_2, \sigma_3\rangle = [p, q, r]^+/N_3$, the amalgamation of $\Gamma^+(\mathcal{P})$ with $\Gamma^+(\mathcal{Q})$ as defined in Lemma 9. Suppose that there is a permutation group $G = \langle\pi_1, \pi_2, \pi_3\rangle$ on $\mathbb{Z}_q \times \mathbb{Z}_r$ such that the function that sends each σ_i to π_i determines a group epimorphism. Further, suppose that:*

(a) π_1 *fixes* $(0, 0)$.
(b) *There is some point* (b, c) *such that the smallest power of* π_1 *that fixes* (b, c) *is* π_1^p.
(c) $(b, 0)\pi_2 = (b + 1, 0)$ *for all* b.
(d) $(b, c)\pi_3 = (b, c + 1)$ *for all* b *and* c.

Then $\Gamma \cong G$, and Γ is the rotation group of a tight chiral polytope of type $\{p, q, r\}$.

Proof First, note that since G is a quotient of Γ, then $\pi_1^p = \pi_2^q = \pi_3^r = id$. The given conditions then imply that no smaller powers of any π_i will equal the identity. Now, since Γ is a tight quotient of $[p, q, r]^+$, it follows that $|G| \leq |\Gamma| \leq pqr$. If we can show that G satisfies the intersection condition, then Proposition 2 will imply that $|G| = pqr$ and thus that $G \cong \Gamma$ and that Γ is the rotation group of a tight orientable rotary polytope of type $\{p, q, r\}$. Furthermore, by Lemma 9, such a polytope will have chiral vertex-figures isomorphic to $\mathcal{Q}$ and thus it will be chiral itself.

To show that G satisfies the intersection condition, we first need to show that

$$\langle\pi_1\rangle \cap \langle\pi_2\rangle = \{id\} = \langle\pi_2\rangle \cap \langle\pi_3\rangle.$$

If $\varphi = \pi_1^a = \pi_2^b$, then

$$(0, 0) = (0, 0)\pi_1^a = (0, 0)\pi_2^b = (b, 0),$$

and so $b \equiv 0 \pmod q$, which implies that φ is trivial. Similarly, if $\varphi = \pi_2^b = \pi_3^c$, then

$$(b, 0) = (0, 0)\pi_2^b = (0, 0)\pi_3^c = (0, c),$$

which implies that φ is trivial. Finally, we need to show that

$$\langle\pi_1, \pi_2\rangle \cap \langle\pi_2, \pi_3\rangle.$$

Consider φ in this intersection. Since G is a quotient of the tight group Γ, we may write $\varphi = \pi_1^a \pi_2^b = \pi_2^{b'} \pi_3^c$ for some a, b, b', c. We have

$$(0,0)\pi_1^a \pi_2^b = (0,0)\pi_2^b = (b,0)$$

and

$$(0,0)\pi_2^{b'} \pi_3^c = (b',0)\pi_3^c = (b',c).$$

It follows that $c \equiv 0 \pmod{r}$ and thus that $\pi_3^c = id$. So $\varphi = \pi_2^{b'} \in \langle \pi_2 \rangle$, as desired. □

Theorem 5 *The group Γ_1 is the automorphism group of an atomic chiral 4-polytope of type $\{2m, m^\alpha, 2m\}$ if and only if $k_1 = k_2$.*

Proof First let us show that $k_1 = k_2$. Note that

$$\begin{aligned}
\sigma_1 \sigma_2^{k_1 m^{\alpha-1}} &= \underline{\sigma_1 \sigma_2^{-1}} \sigma_2^{1+k_1 m^{\alpha-1}} \\
&= \sigma_2^{1-k_1 m^{\alpha-1}} \underline{\sigma_1^3 \sigma_2^{1+k_1 m^{\alpha-1}}} \\
&= \sigma_2^{1-k_1 m^{\alpha-1}} \sigma_2^{-1} \sigma_1 \\
&= \sigma_2^{-k_1 m^{\alpha-1}} \sigma_1
\end{aligned}$$

Thus, conjugation by σ_1 inverts $\sigma_2^{k_1 m^{\alpha-1}}$, and since $1 \leq k_1 \leq m-1$ and m is prime, this implies that conjugation by σ_1 inverts $\sigma_2^{m^{\alpha-1}}$. A similar argument shows that conjugation by σ_3 inverts $\sigma_2^{m^{\alpha-1}}$. Then, using Lemma 4(a) we see that

$$\begin{aligned}
\sigma_3 \underline{\sigma_2^{-1} \sigma_1} &= \underline{\sigma_3 \sigma_1^3} \sigma_2^{1+k_1 m^{\alpha-1}} \\
&= \sigma_2^{-1} \sigma_1^{-3} \underline{\sigma_2 \sigma_3 \sigma_2} \sigma_2^{k_1 m^{\alpha-1}} \\
&= \sigma_2^{-1} \sigma_1^{-3} \underline{\sigma_3^{-1} \sigma_2^{k_1 m^{\alpha-1}}} \\
&= \underline{\sigma_2^{-1} \sigma_1^{-3}} \sigma_2^{-k_1 m^{\alpha-1}} \sigma_3^{-1} \\
&= \sigma_1^{-1} \sigma_2^{1-2k_1 m^{\alpha-1}} \sigma_3^{-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\underline{\sigma_3 \sigma_2^{-1}} \sigma_1 &= \sigma_2^{1+k_2 m^{\alpha-1}} \underline{\sigma_3^3 \sigma_1} \\
&= \sigma_2^{k_2 m^{\alpha-1}} \underline{\sigma_2 \sigma_1 \sigma_2} \sigma_3^{-3} \sigma_2^{-1} \\
&= \underline{\sigma_2^{k_2 m^{\alpha-1}} \sigma_1^{-1}} \sigma_3^{-3} \sigma_2^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sigma_1^{-1}\sigma_2^{-k_2m^{\alpha-1}}\underline{\sigma_3^{-3}\sigma_2^{-1}}\\
&= \sigma_1^{-1}\sigma_2^{1-2k_2m^{\alpha-1}}\sigma_3^{-1}.
\end{aligned}
$$

Thus $\sigma_2^{1-2k_1m^{\alpha-1}} = \sigma_2^{1-2k_2m^{\alpha-1}}$, and since k_1 and k_2 are defined modulo m (which is an odd prime), it follows that $k_1 = k_2$.

Now, fix $k_1 = k_2 = k$. Let $D = km^{\alpha-1}$. For $b \in \mathbb{Z}_{m^\alpha}$, we define $\overline{b} = -b + \dfrac{b(b-1)}{2}D$. Then we define permutations of $\mathbb{Z}_{m^\alpha} \times \mathbb{Z}_{2m}$ as follows:

$$
\begin{aligned}
(b,c)\pi_1 &= \begin{cases} (\overline{b} + \frac{c}{2}D, -c), & \text{if } c \text{ is even},\\ (\overline{b} + 2 - \frac{c-1}{2}D, 2-c), & \text{if } c \text{ is odd}, \end{cases}\\
(b,c)\pi_2 &= \begin{cases} (b + 1 + \frac{c}{2}D, c), & \text{if } c \text{ is even},\\ (b - 1 - \frac{c-1}{2}D, c-2), & \text{if } c \text{ is odd}, \end{cases}\\
(b,c)\pi_3 &= (b, c+1).
\end{aligned}
$$

We want to show that $\langle \pi_1, \pi_2, \pi_3\rangle$ satisfies the defining relations of Γ_1. Here are several intermediate calculations; the first three formulas help verify the fourth and fifth.

(a) $\overline{b}D = -bD$
(b) $\overline{b + tD} = \overline{b} - tD$
(c) $\overline{\overline{b}} = b(1+D)$
(d) $(b,c)\pi_1\pi_2 = (\overline{b}+1, -c)$
(e) $(b,c)\pi_1^2 = \begin{cases} (b(1+D) - cD, c) & \text{if } c \text{ is even},\\ (b(1-D) + cD, c) & \text{if } c \text{ is odd}. \end{cases}$
(f) $(b,c)\pi_2^m = \begin{cases} (b+m, c), & \text{if } c \text{ is even},\\ (b-m, c), & \text{if } c \text{ is odd}. \end{cases}$

From the above, it is straightforward to show that

$$
(b,c)\pi_1^{2t} = \begin{cases} (b(1+tD) - tcD, c) & \text{if } c \text{ is even},\\ (b(1-tD) + tcD, c) & \text{if } c \text{ is odd}. \end{cases}
$$

Then $(b,c)\pi_1^{2m} = (b,c)$ since $mD \equiv 0 \pmod{m^\alpha}$. We note that the action of π_1 on the second coordinate makes it clear that π_1 has even order, and for $1 \le t \le m-1$ we have $(1,0)\pi_1^{2t} = (1+tD, 0) \ne (1,0)$. So π_1 has order $2m$ (and not a proper divisor).

From the sixth calculation above, it is clear that $\pi_2^{m^\alpha} = id$. It's also clear that π_3 has order $2m$.

Next, we want to show that $(\pi_1\pi_2)^2 = (\pi_1\pi_2\pi_3)^2 = id$. Since $(b,c)\pi_1\pi_2 = (\overline{b}+1, -c)$, we have:

$$\begin{aligned}(b,c)(\pi_1\pi_2)^2 &= (\overline{b}+1,-c)\pi_1\pi_2\\ &= (\overline{\overline{b}+1}+1,c),\end{aligned}$$

and

$$\begin{aligned}\overline{\overline{b}+1}+1 &= -(\overline{b}+1)+\frac{(\overline{b}+1)\overline{b}}{2}D+1\\ &= -\overline{b}+\frac{(-b+1)(-b)}{2}D\\ &= b-\frac{b(b-1)}{2}D+\frac{(-b+1)(-b)}{2}D\\ &= b.\end{aligned}$$

So $(b,c)(\pi_1\pi_2)^2 = (b,c)$. Essentially the same proof shows that $(b,c)(\pi_1\pi_2\pi_3)^2 = (b,c)$. Verifying that $(b,c)(\pi_2\pi_3)^2 = (b,c)$ is straightforward.

Finally, verifying that $\pi_2\pi_1^2 = \pi_1^2\pi_2^{1+D}$ and $\pi_3^2\pi_2 = \pi_2^{1+D}\pi_3^2$ is relatively straightforward with the hints above. Lemma 19 and Corollary 5 then finish the proof. □

Theorem 6 *The group Γ_2 is the automorphism group of an atomic chiral 4-polytope of type $\{8,2^\beta,8\}$ for all four choices of (ϵ_1,ϵ_2) and for every $\beta \geq 5$.*

Proof Let $D = 2^{\beta-3}$. For $b \in \mathbb{Z}_{2^\beta}$, we define $\overline{b} = -b+b(b-1)D\epsilon_1$. Then we define permutations of $\mathbb{Z}_{2^\beta}\times\mathbb{Z}_8$ as follows:

$$\begin{aligned}(b,c)\pi_1 &= \begin{cases}(\overline{b}+D\epsilon_2 c,-c), & \text{if } c \text{ is even},\\ (\overline{b}+2-D\epsilon_2(c-1),2-c), & \text{if } c \text{ is odd},\end{cases}\\ (b,c)\pi_2 &= \begin{cases}(b+1+D\epsilon_2 c,c), & \text{if } c \text{ is even},\\ (b-1-D\epsilon_2(c-1),c-2), & \text{if } c \text{ is odd},\end{cases}\\ (b,c)\pi_3 &= (b,c+1).\end{aligned}$$

The following intermediate calculations can be used to verify that there is a well-defined epimorphism from Γ_2 to $\langle\pi_1,\pi_2,\pi_3\rangle$ sending each σ_i to π_i.

(a) $4D \equiv 2^{\beta-1}$ and $8D \equiv 0 \pmod{2^\beta}$
(b) If $\beta = 5$, then $D^2 \equiv 2^{\beta-1} \pmod{2^\beta}$, and if $\beta \geq 6$ then $D^2 \equiv 0 \pmod{2^\beta}$.
(c) $\overline{bD} = -bD$
(d) $\overline{b+2tD} = \overline{b}-2tD$ for all t
(e) $\overline{\overline{b}} = b(1+2D\epsilon_1)$
(f) $(b,c)\pi_1\pi_2 = (\overline{b}+1,-c)$
(g) $(b,c)\pi_1^2 = \begin{cases}(b(1+2D\epsilon_1)-2D\epsilon_2 c,c) & \text{if } c \text{ is even},\\ (b(1-2D\epsilon_1)+2D\epsilon_1+2D\epsilon_2(c-1),c) & \text{if } c \text{ is odd}.\end{cases}$
(h) $(b,c)\pi_2^8 = \begin{cases}(b+8,c), & \text{if } c \text{ is even},\\ (b-8,c), & \text{if } c \text{ is odd}.\end{cases}$

We omit the details of showing that $\langle \pi_1, \pi_2, \pi_3 \rangle$ satisfies the defining relations of Γ_2. Lemma 19 and Corollary 5 then finish the proof. □

Theorem 7 *The group Γ_3 is the automorphism group of an atomic chiral 4-polytope of type $\{2^{\beta-1}, 2^{\beta}, 2^{\beta-1}\}$ for all four choices of (ϵ_1, ϵ_2) and for every $\beta \geq 5$.*

Proof Let $D = 2^{\beta-3}$. For $b \in \mathbb{Z}_{2^\beta}$, we define

$$\overline{b} = \begin{cases} b(1 + D\epsilon_1) & \text{if } b \text{ is even,} \\ (b-1)(1 - D\epsilon_1) - 1 & \text{if } b \text{ is odd.} \end{cases}$$

Then we define permutations of $\mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^{\beta-1}}$ as follows:

$$(b, c)\pi_1 = \begin{cases} (\overline{b} + 2c + D\epsilon_2 c, c(D+1)) & \text{if } c \text{ is even,} \\ (\overline{b} + 2c - D\epsilon_2(c-1), c(D+1) - D) & \text{if } c \text{ is odd,} \end{cases}$$
$$(b, c)\pi_2 = \begin{cases} (b + 1 - 2c + D\epsilon_2 c, c(D-1)) & \text{if } c \text{ is even,} \\ (b + 1 - 2c - D\epsilon_2(c-1), c(D-1) - D) & \text{if } c \text{ is odd,} \end{cases}$$
$$(b, c)\pi_3 = (b, c+1).$$

The following intermediate calculations can be used to verify that there is a well-defined epimorphism from Γ_3 to $\langle \pi_1, \pi_2, \pi_3 \rangle$ sending each σ_i to π_i.

(a) $4D \equiv 2^{\beta-1}$ and $8D \equiv 0 \pmod{2^\beta}$
(b) If $\beta = 5$, then $D^2 \equiv 2^{\beta-1} \pmod{2^\beta}$, and if $\beta \geq 6$ then $D^2 \equiv 0 \pmod{2^\beta}$.
(c) $(b, c)\pi_1\pi_2 = (\overline{b} + 1, -c)$
(d) $\overline{\overline{b} + 1} = b - 1$
(e) $(b, c)\pi_2^4 = \begin{cases} (b+4, c) & \text{if } c \text{ is even} \\ (b + 4 + 4D, c) & \text{if } c \text{ is odd.} \end{cases}$
(f) $(b, c)\pi_2^8 = (b + 8, c)$.
(g) $(b, c)\pi_1^2 = \begin{cases} (b(1 + 2D\epsilon_1) + 2c(2 + D\epsilon_1), c) & \text{if } b \text{ is even,} \\ (b(1 - 2D\epsilon_1) + 2c(2 - D\epsilon_1) + 4D - 4, c) & \text{if } b \text{ is odd.} \end{cases}$
(h) $(b, c)\pi_1^4 = \begin{cases} (b + c(4D+8), c) & \text{if } b \text{ is even,} \\ (b + (c-1)(4D+8), c) & \text{if } b \text{ is odd.} \end{cases}$
(i) $(b, c)\pi_1^{2^{\beta-2}} = (b + 4D(b+c), c) =$
$\begin{cases} (b, c) & \text{if } b \text{ and } c \text{ have the same parity,} \\ (b + 4D, c) & \text{if } b \text{ and } c \text{ have opposite parity.} \end{cases}$

Here we give more details on how to verify that $\langle \pi_1, \pi_2, \pi_3 \rangle$ satisfies the extra relations from Table 2. To verify the relation $\pi_2^{-1}\pi_1 = \pi_1^{-1+2^{\beta-2}}\pi_2^{-3+\epsilon_1 2^{\beta-2}}$, we rewrite it:

$$\pi_2^{-1}\pi_1 = \pi_1^{-1+2^{\beta-2}}\pi_2^{-3+\epsilon_1 2^{\beta-2}} \quad \text{Multiply by } \pi_1^{-1} \text{ on the left}$$

$$\begin{aligned}
\pi_1^{-1}\pi_2^{-1}\pi_1 &= \pi_1^{-2+2^{\beta-2}}\pi_2^{-3+\epsilon_1 2^{\beta-2}} && \pi_1^{-1}\pi_2^{-1} = \pi_2\pi_1 \\
\pi_2\pi_1^2 &= \pi_1^{-2+2^{\beta-2}}\pi_2^{-3+\epsilon_1 2^{\beta-2}} && \text{Multiply by } \pi_1^2 \text{ on the left and } \pi_2^4 \text{ on the right} \\
\pi_1^2\pi_2\pi_1^2\pi_2^4 &= \pi_1^{2^{\beta-2}}\pi_2^{1+\epsilon_1 2^{\beta-2}}
\end{aligned}$$

Then we can show that both sides send (b, c) to

$$\begin{cases} (b + 4Db + 2D\epsilon_1 + 1 - 2c + D\epsilon_2 c, c(D-1)) & \text{if } c \text{ is even,} \\ (b + 4Db - 2D\epsilon_1 + 1 - 2c - D\epsilon_2(c-1), c(D-1) - D) & \text{if } c \text{ is odd.} \end{cases}$$

After showing that that relation holds, we can use it to rewrite the second relation into a form that is easier to verify:

$$\begin{aligned}
\pi_2\pi_1^{-1} &= \pi_1^{1+2^{\beta-2}}\pi_2^{3+\epsilon_1 2^{\beta-2}} && \text{Multiply by } \pi_1^{-1} \text{ on the left} \\
\pi_1^{-1}\pi_2\pi_1^{-1} &= \pi_1^{2^{\beta-2}}\pi_2^{3+\epsilon_1 2^{\beta-2}} && \text{Rewrite using first relation} \\
\pi_2^{3-\epsilon_1 2^{\beta-2}}\pi_1^{-2^{\beta-2}} &= \pi_1^{2^{\beta-2}}\pi_2^{3+\epsilon_1 2^{\beta-2}} && \text{Multiply by } \pi_2 \text{ on the left and right} \\
\pi_2^{4-\epsilon_1 2^{\beta-2}}\pi_1^{-2^{\beta-2}}\pi_2 &= \pi_2\pi_1^{2^{\beta-2}}\pi_2^{4+\epsilon_1 2^{\beta-2}}
\end{aligned}$$

Then we can show that both sides send (b, c) to

$$\begin{cases} (b + 4Db - 2D\epsilon_1 + 5 - 2c + D\epsilon_2 c, c(D-1)) & \text{if } c \text{ is even,} \\ (b + 4Db - 2D\epsilon_1 + 5 - 2c - D\epsilon_2(c-1), c(D-1) - D) & \text{if } c \text{ is odd.} \end{cases}$$

To verify the third relation, we rewrite it as $\pi_2\pi_3^{-1}\pi_2 = \pi_2^{4+2\epsilon_2 D}\pi_3^{1+2D}$. Then we can show that both sides send (b, c) to

$$\begin{cases} (b + 4 + 2\epsilon_2 D, c + 1 + 2D) & \text{if } c \text{ is even,} \\ (b + 4 - 2\epsilon_2 D, c + 1 + 2D) & \text{if } c \text{ is odd.} \end{cases}$$

To verify the fourth relation, we multiply both sides by π_2^4 on the left and π_2 on the right to obtain $\pi_2^4\pi_3 = \pi_2^{1+2\epsilon_2 D}\pi_3^{-1+2D}\pi_2$. Then we can show that both sides send (b, c) to

$$\begin{cases} (b + 4, c + 1) & \text{if } c \text{ is even,} \\ (b + 4 + 4D, c + 1) & \text{if } c \text{ is odd.} \end{cases}$$

Lemma 19 and Corollary 5 then finish the proof. □

Theorem 8 *The group Γ_4 is the automorphism group of an atomic chiral 4-polytope of type $\{8, 2^\beta, 2^{\beta-1}\}$ for all four choices of (ϵ_1, ϵ_2) and for every $\beta \geq 5$.*

Proof We use the same permutation representation as Theorem 7, except that we now define $\overline{b} = -b + b(b-1)D\epsilon_1$ as in Theorem 6. Note that since the relations of Γ_4 that involve only σ_2 and σ_3 are the same as the relations in Γ_3, and the permutation representation for those two elements is the same, the only relations that need to be verified are those that include σ_1. Here are some intermediate calculations:

(a) $(b, c)\pi_1^2 = \begin{cases} (b(1+2D\epsilon_1), c) & \text{if } c \text{ is even,} \\ (b(1-2D\epsilon_1) + 2D\epsilon_1, c) & \text{if } c \text{ is odd.} \end{cases}$

(b) $(b, c)\pi_1^4 = \begin{cases} (b(1+4D\epsilon_1), c) & \text{if } c \text{ is even,} \\ (b(1-4D\epsilon_1) - 4D, c) & \text{if } c \text{ is odd.} \end{cases}$

(c) $(b, c)\pi_1\pi_2 = (\overline{b} + 1, -c)$. (Note that since this calculation and the definition of $\overline{b}$ is the same as in Theorem 6, it follows at once that $\pi_1\pi_2$ and $\pi_1\pi_2\pi_3$ have order 2.)

(d) $(b, c)\pi_2^8 = (b + 8, c)$. (This follows from the same calculation in Theorem 7.)

Lemma 19 and Corollary 5 then finish the proof. □

Table 4 includes information on all of the atomic chiral 4-polytopes with chiral facets and vertex-figures.

5 Atomic chiral 4-polytopes with regular facets and chiral vertex-figures

Now we switch our attention to atomic chiral 4-polytopes with regular facets and chiral vertex-figures. The goal is to show that the vertex-figures are atomic chiral polyhedra and then use the classifications in Sect. 3 to find all atomic chiral 4-polytopes with regular facets.

5.1 The structure of atomic chiral 4-polytopes with regular facets and chiral vertex-figures

As in the previous section, we start by studying normal subgroups of the rotation group of atomic chiral 4-polytopes, in this case with regular facets.

Lemma 20 *Let $\mathcal{P}$ be an atomic chiral 4-polytope with regular facets, chiral vertex-figures and type $\{p, q, r\}$. If $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ then*

(a) *$\langle \sigma_1 \rangle$ is core-free,*
(b) *q is even,*
(c) *$p < q$.*

Proof If $\langle \sigma_1^k \rangle \triangleleft \Gamma(\mathcal{P})$, then by Proposition 5, $\mathcal{P}/\langle \sigma_1^k \rangle$ is a tight polytope with vertex-figures isomorphic to those of $\mathcal{P}$. The chirality of the vertex-figures of $\mathcal{P}$ contradicts atomicity, proving part (a).

To prove part (b), assume to the contrary that q is odd. Since $\langle \sigma_1 \rangle$ is core-free, the type of the facets of $\mathcal{P}$ must be the dual of one of the types listed in Theorem 2. The only possibility for q being odd is if the facets of $\mathcal{P}$ have type $\{2, q\}$. This contradicts Lemma 1.

Part (c) follows from part (1) and Proposition 6. □

Lemma 21 *Let $\mathcal{P}$ be an atomic chiral 4-polytope with regular facets and chiral vertex-figures. If $\mathcal{P}$ has type $\{p, q, r\}$, then the vertex-figures of $\mathcal{P}$ do not cover a chiral polyhedron with type $\{q, r'\}$ for $r' < r$.*

Proof Assume to the contrary that the vertex-figures of $\mathcal{P}$ cover a chiral polyhedron $\mathcal{Q}$ with type $\{q, r'\}$ with $r' < r$. Then $\langle \sigma_3^{r'} \rangle \triangleleft \langle \sigma_2, \sigma_3 \rangle$, and $\Gamma(\mathcal{Q}) = \langle \sigma_2, \sigma_3 \rangle / \langle \sigma_3^{r'} \rangle$. In particular, $\langle \sigma_3^{r'} \rangle$ is normalized by conjugation by σ_2. The dual version of Lemma 4 implies that it is also normalized by conjugation by σ_1 and hence $\langle \sigma_3^{r'} \rangle \triangleleft \Gamma(\mathcal{P})$.

By Proposition 5, $\mathcal{P}/\langle \sigma_3^{r'} \rangle$ is a 4-polytope. Furthermore, its vertex-figures are isomorphic to $\mathcal{Q}$, which is chiral. This contradicts the atomicity of $\mathcal{P}$. □

Lemma 22 *Let $\mathcal{P}$ be an atomic chiral 4-polytope with type $\{p, q, r\}$, regular facets and chiral vertex-figures. Then the vertex-figures of $\mathcal{P}$ do not cover a chiral polyhedron with type $\{q', r\}$ with either q' an even divisor of q or $q' < q/2$.*

Proof Let $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$.

Assume first that the vertex-figures of $\mathcal{P}$ cover a chiral polyhedron $\mathcal{Q}$ with type $\{q', r\}$ with q' even. Then $\langle \sigma_2^{q'} \rangle \triangleleft \langle \sigma_2, \sigma_3 \rangle$. By the dual version of Lemma 15, $\langle \sigma_2^2 \rangle \triangleleft \langle \sigma_1, \sigma_2 \rangle$. Since $\langle \sigma_2^{q'} \rangle \leq \langle \sigma_2^2 \rangle$ and the latter is cyclic, we have that $\langle \sigma_2^{q'} \rangle \triangleleft \Gamma(\mathcal{P})$. Then Proposition 5 shows that $\mathcal{P}/\langle \sigma_2^{q'} \rangle$ is a 4-polytope whose vertex-figures are isomorphic to $\mathcal{Q}$, contradicting atomicity of $\mathcal{P}$.

Now, if q' is odd and $q' < q/2$ then $\langle \sigma_2^{2q'} \rangle$ is invariant under conjugation by all generators σ_i and hence it is a proper normal subgroup of $\Gamma(\mathcal{P})$. It follows that $\mathcal{P}/\langle \sigma_2^{2q'} \rangle$ is a proper quotient of $\mathcal{P}$ whose vertex-figures cover $\mathcal{Q}$. Proposition 4 implies that $\mathcal{P}/\langle \sigma_2^{2q'} \rangle$ is a chiral quotient of $\mathcal{P}$, again contradicting atomicity of $\mathcal{P}$. □

We are now ready to prove the main necessary condition for a tight chiral 4-polytope with regular facets and chiral vertex-figures to be atomic.

Theorem 9 *If $\mathcal{P}$ is an atomic chiral 4-polytope with regular facets and chiral vertex-figures then the vertex-figures are atomic chiral polyhedra.*

Proof Let $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ and assume that the facets and vertex-figures of $\mathcal{P}$ are isomorphic to $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively. We shall abuse notation and write $\Gamma^+(\mathcal{Q}_1) = \langle \sigma_1, \sigma_2 \rangle$ and $\Gamma(\mathcal{Q}_2) = \langle \sigma_2, \sigma_3 \rangle$.

Assume to the contrary that $\mathcal{Q}_2$ is not atomic. Lemmas 21 and 22 imply that $\mathcal{Q}_2$ covers no chiral polyhedron with type $\{q, r'\}$ with $r' < r$ and that the only chiral polyhedron covered by $\mathcal{Q}_2$ with type $\{q', r\}$ for $q' < q$ is such that $q' = q/2$. Furthermore, $q/2$ must be odd.

First, we show that $\mathcal{Q}_2/\langle \sigma_2^{q/2} \rangle$ is atomic. Note that since $\sigma_2^{q/2}$ has order 2 and $\langle \sigma_2^{q/2} \rangle \triangleleft \Gamma(\mathcal{Q}_2)$, $\sigma_2^{q/2}$ is central in $\Gamma(\mathcal{Q}_2)$. Let $\Gamma(\mathcal{Q}_2/\langle \sigma_2^{q/2} \rangle) = \langle \hat{\sigma}_2, \hat{\sigma}_3 \rangle$. By Lemma 22, $\mathcal{Q}_2/\langle \sigma_2^{q/2} \rangle$ does not cover a chiral polyhedron with type $\{q'', r\}$ for

$q'' < q/2$. On the other hand, if $\mathcal{Q}_2/\langle\sigma_2^{q/2}\rangle$ covers a chiral polyhedron with type $\{q/2, r'\}$ for $r' < r$ then $\hat{\sigma}_2^{-1}\hat{\sigma}_3^{r'}\hat{\sigma}_2 = \hat{\sigma}_3^{ar'}$ for some integer a. Lifting this relation to $\Gamma(\mathcal{Q}_2)$, we have that $\sigma_2^{-1}\sigma_3^{r'}\sigma_2 = \sigma_2^{\epsilon q/2}\sigma_3^{ar'}$ for some $\epsilon \in \{0, 1\}$; however, by Lemma 21 $\langle\sigma_3^{r'}\rangle$ is not normal in $\Gamma(\mathcal{Q}_2)$ and hence $\epsilon = 1$. Then, conjugation by σ_2 interchanges the subgroups $\langle\sigma_3^{r'}\rangle$ and $\langle\sigma_2^{q/2}\sigma_3^{r'}\rangle$, implying that $\sigma_2^{-\ell}\sigma_3^{r'}\sigma_2^{\ell} \in \langle\sigma_3^{r'}\rangle$ if and only if ℓ is even. It follows that $\sigma_3^{r'} = \sigma_2^{q/2}\sigma_3^{r'}\sigma_2^{q/2} \notin \langle\sigma_3^{r'}\rangle$, a contradiction. Therefore $\mathcal{Q}_2/\langle\sigma_2^{q/2}\rangle$ is atomic.

Since $\mathcal{Q}_2/\langle\sigma_2^{q/2}\rangle$ is atomic and has type $\{q/2, r\}$ with $q/2$ odd, we have that $q/2 = m^\beta$ and $r = 2m$ for some odd prime m and positive integer β. In particular $q = 2m^\beta$ and, by Theorem 2, p must be the odd prime power m^β. Furthermore, by [12, Prop. 3.2 and Thm. 3.6], the atomic chiral polyhedron of type $\{m^\beta, 2m\}$ covers a tight regular polyhedron of type $\{m, 2m\}$, and so $\langle\sigma_2^m\rangle$ is normal in $\langle\sigma_2, \sigma_3\rangle/\langle\sigma_2^{q/2}\rangle$ and indeed in $\langle\sigma_2, \sigma_3\rangle$ itself. Then, since the dual version of Lemma 15 tells us that $\langle\sigma_2^2\rangle$ is normal in $\langle\sigma_1, \sigma_2\rangle$, it follows that $\langle\sigma_2^{2m}\rangle$ is normal in $\Gamma(\mathcal{P})$.

Abusing notation let $\Gamma^+(\mathcal{Q}_2/\langle\sigma_2^m\rangle) = \langle\sigma_2', \sigma_3\rangle$. Since m is odd and $\mathcal{Q}/\langle\sigma_2^m\rangle$ is regular, Proposition 7 and the dual version of Lemma 15 imply that σ_3^2 is central in $\Gamma^+(\mathcal{Q}_2/\langle\sigma_2^m\rangle)$. If $\Gamma^+(\mathcal{P}/\langle\sigma_2^{2m}\rangle) = \langle\sigma_1, \sigma_2'', \sigma_3\rangle$ then

$$\sigma_2''\sigma_3^2(\sigma_2'')^{-1} = \sigma_3^2(\sigma_2'')^{\varepsilon m}$$

for some $\varepsilon \in \{0, 1\}$. Now, $(\sigma_2'')^m$ generates a normal subgroup of order 2, and is thus central. Then

$$id = \sigma_2''\sigma_3^{2m}(\sigma_2'')^{-1} = (\sigma_3^2(\sigma_2'')^{\varepsilon m})^m = \sigma_3^{2m}(\sigma_2'')^{\varepsilon m^2} = (\sigma_2'')^{\varepsilon m^2}.$$

Since σ_2'' has order $2m$ and m^2 is odd, it follows that $\varepsilon = 0$, so in fact, σ_3^2 commutes with σ_2''. Then, by (5) we have that

$$\sigma_1^{-1}\sigma_3^2\sigma_1 = ((\sigma_2'')^2\sigma_3)^2 = \sigma_2''\sigma_3^{-2}(\sigma_2'')^{-1} = \sigma_3^{-2},$$

and so conjugation by σ_1 inverts σ_3^2. Since $p = m^\beta$ is odd, this implies that $\sigma_3^2 = \sigma_3^{-2}$, and so $2m$ (the order of σ_3) divides 4, which is impossible. □

Corollary 6 *A tight chiral 4-polytope $\mathcal{P}$ of type $\{p, q, r\}$ with regular facets and vertex-figures is atomic if and only if*

(a) The vertex-figures are atomic,
(b) q is even, and
(c) $\langle\sigma_1\rangle$ is core-free in $\Gamma(\mathcal{P})$.

Proof Theorem 9 and Lemma 20 prove that the conditions are necessary. To prove that they suffice, suppose that $\mathcal{P}$ satisfies the three conditions and suppose that $\mathcal{P}$ properly covers a chiral 4-polytope $\mathcal{Q}$. Then the facets of $\mathcal{Q}$ are covered by the regular facets of $\mathcal{P}$, and by Proposition 4, the facets of $\mathcal{Q}$ are regular. Then by Proposition 1, the vertex-figures of $\mathcal{Q}$ are chiral. These vertex-figures are covered by the vertex-figures of $\mathcal{P}$, which are atomic, and so $\mathcal{Q}$ has the same vertex-figures as $\mathcal{P}$. In particular, $\mathcal{Q}$ has type $\{p', q, r\}$ for some p' dividing p. By tightness, $|\Gamma(\mathcal{Q})| = p'qr$ and $|\Gamma(\mathcal{P})| = pqr$, and since $\mathcal{Q}$ is a proper quotient of $\mathcal{P}$, we have $p' < p$. Now, the kernel of the natural epimorphism from $\Gamma(\mathcal{P})$ to $\Gamma(\mathcal{Q})$ includes $\sigma_1^{p'}$. On the other hand, $|\langle\sigma_1^{p'}\rangle| = p/p'$ so that $|\Gamma(\mathcal{P})| = |\Gamma(\mathcal{Q})| \cdot |\langle\sigma_1^{p'}\rangle|$. It follows that $\sigma_1^{p'}$ generates a nontrivial normal subgroup of $\Gamma(\mathcal{P})$, contradicting that $\langle\sigma_1\rangle$ is core-free. So $\mathcal{P}$ must be atomic. □

5.2 Classification of atomic chiral 4-polytopes with regular facets and chiral vertex-figures

If $\mathcal{P}$ is an atomic chiral 4-polytope with regular facets and chiral vertex-figures, then Lemma 20 implies that the facets must be the dual of one of the polyhedra in Theorem 2, and Theorem 9 implies that the vertex-figures must be one of the polyhedra in Theorem 3 or its dual. The dual of Lemma 1 implies that the facets cannot be type $\{2, q\}$. Then, after some manipulation of the relations in [13, Sect. 4] we have the following lemma:

Lemma 23 *The facets of an atomic chiral 4-polytope with regular facets must be one of the following:*

(a) Type $\{m, 2m\}$ for an odd prime m, with rotation group $[m, 2m]^+/(\sigma_2^2\sigma_1 = \sigma_1\sigma_2^2)$.
(b) Type $\{4, 8\}$, with rotation group $[4, 8]^+/(\sigma_2^2\sigma_1 = \sigma_1\sigma_2^2)$.
(c) Type $\{4, 2^\beta\}$ for some $\beta \geq 5$, with rotation group $[4, 2^\beta]^+/(\sigma_2^{-1}\sigma_1 = \sigma_1^{-1}\sigma_2^{1+2^{\beta-1}})$.
(d) Type $\{2^{\beta-1}, 2^\beta\}$ for some $\beta \geq 5$, with rotation group $[2^{\beta-1}, 2^\beta]^+/(\sigma_2\sigma_1^{-1} = \sigma_1\sigma_2^{3-\epsilon 2^{\beta-1}})$, with $\epsilon \in \{0, 1\}$.

Now there are eight possibilities for the automorphism group of an atomic chiral 4-polytope with regular facets and chiral vertex-figures; see Table 3.

We will show that the first three groups do correspond to atomic chiral 4-polytopes, whereas the remaining groups do not.

Theorem 10 *The group Λ_1 is the automorphism group of an atomic chiral polytope of type $\{m, 2m, m^\alpha\}$, for each k satisfying $1 \leq k \leq m - 1$.*

Proof Let $D = km^{\alpha-1}$. Then we define permutations of $\mathbb{Z}_{2m} \times \mathbb{Z}_{m^\alpha}$ as follows:

$$(b, c)\pi_1 = \begin{cases} (b + 2c, c + \frac{c(c-1)}{2}D) & \text{if } b \text{ is even,} \\ (b + 2c - 2, c + \frac{c(c-1)}{2}D) & \text{if } b \text{ is odd,} \end{cases}$$
$$(b, c)\pi_2 = (b + 1 - 2c, -c + \frac{c(c-1)}{2}D),$$
$$(b, c)\pi_3 = (b, c + 1).$$

Here are a few intermediate calculations.

Table 3 The candidate groups of atomic chiral 4-polytopes with regular facets and chiral vertex-figures

Group name	Parent group	Extra relations	Notes
Λ_1	$[m, 2m, m^{\alpha}]^+$,	$\sigma_2^2\sigma_1 = \sigma_1\sigma_2^2$	m odd prime, $\alpha \geq 2$,
		$\sigma_3\sigma_2^2 = \sigma_2^2\sigma_3^{1+km^{\alpha-1}}$	$1 \leq k \leq m-1$
Λ_2	$[4, 8, 2^{\beta}]^+$	$\sigma_2^2\sigma_1 = \sigma_1\sigma_2^2$	$\beta \geq 5, \epsilon = \pm 1$
		$\sigma_3\sigma_2^2 = \sigma_2^2\sigma_3^{1+\epsilon 2^{\beta-2}}$	
Λ_3	$[4, 2^{\beta-1}, 2^{\beta}]^+$	$\sigma_2^{-1}\sigma_1 = \sigma_1^{-1}\sigma_2^{1+2^{\beta-2}}$	$\beta \geq 5, \epsilon = \pm 1$
		$\sigma_3^{-1}\sigma_2 = \sigma_2^{-1+2^{\beta-2}}\sigma_3^{-3+\epsilon 2^{\beta-2}}$	
		$\sigma_3\sigma_2^{-1} = \sigma_2^{1+2^{\beta-2}}\sigma_3^{3+\epsilon 2^{\beta-2}}$	
Λ_4	$[4, 2^{\beta}, 8]^+$	$\sigma_2^{-1}\sigma_1 = \sigma_1^{-1}\sigma_2^{1+2^{\beta-1}}$	$\beta \geq 5, \epsilon = \pm 1$
		$\sigma_2\sigma_3^2 = \sigma_3^2\sigma_2^{1+\epsilon 2^{\beta-2}}$	
Λ_5	$[4, 2^{\beta}, 2^{\beta-1}]^+$	$\sigma_2^{-1}\sigma_1 = \sigma_1^{-1}\sigma_2^{1+2^{\beta-1}}$	$\beta \geq 5, \epsilon = \pm 1$
		$\sigma_3^{-1}\sigma_2 = \sigma_2^{3+\epsilon 2^{\beta-2}}\sigma_3^{1-2^{\beta-2}}$	
		$\sigma_3\sigma_2^{-1} = \sigma_2^{-3+\epsilon 2^{\beta-2}}\sigma_3^{-1+2^{\beta-2}}$	
Λ_6	$[2^{\beta-1}, 2^{\beta}, 8]^+$	$\sigma_2\sigma_1^{-1} = \sigma_1\sigma_2^{3-\epsilon_1 2^{\beta-1}}$	$\beta \geq 5$,
		$\sigma_2\sigma_3^2 = \sigma_3^2\sigma_2^{1+\epsilon_2 2^{\beta-2}}$	$\epsilon_1 \in \{0, 1\}, \epsilon_2 = \pm 1$
Λ_7	$[2^{\beta-1}, 2^{\beta}, 2^{\beta-1}]^+$	$\sigma_2\sigma_1^{-1} = \sigma_1\sigma_2^{3-\epsilon_1 2^{\beta-1}}$	$\beta \geq 5$,
		$\sigma_3^{-1}\sigma_2 = \sigma_2^{3+\epsilon_2 2^{\beta-2}}\sigma_3^{1-2^{\beta-2}}$	$\epsilon_1 \in \{0, 1\}, \epsilon_2 = \pm 1$
		$\sigma_3\sigma_2^{-1} = \sigma_2^{-3+\epsilon_2 2^{\beta-2}}\sigma_3^{-1+2^{\beta-2}}$	
Λ_8	$[2^{\beta-2}, 2^{\beta-1}, 2^{\beta}]^+$	$\sigma_2\sigma_1^{-1} = \sigma_1\sigma_2^{3-\epsilon_1 2^{\beta-2}}$	$\beta \geq 5$,
		$\sigma_3^{-1}\sigma_2 = \sigma_2^{-1+2^{\beta-2}}\sigma_3^{-3+\epsilon_2 2^{\beta-2}}$	$\epsilon_1 \in \{0, 1\}, \epsilon_2 = \pm 1$
		$\sigma_3\sigma_2^{-1} = \sigma_2^{1+2^{\beta-2}}\sigma_3^{3+\epsilon_2 2^{\beta-2}}$	

(a) For all n, $(b, c)\pi_1^n = \begin{cases} (b + 2nc, c + n\frac{c(c-1)}{2}D) & \text{if } b \text{ is even,} \\ (b + 2nc - 2n, c + n\frac{c(c-1)}{2}D) & \text{if } b \text{ is odd.} \end{cases}$

(b) $(b, c)\pi_2^2 = (b + 2, c(1 + D))$

(c) $(b, c)\pi_1\pi_2 = \begin{cases} (b + 1, -c) & \text{if } b \text{ is even ,} \\ (b - 1, -c) & \text{if } b \text{ is odd.} \end{cases}$.

From these, it is routine to show that there is a well-defined epimorphism from Λ_1 to $\langle \pi_1, \pi_2, \pi_3 \rangle$ sending each σ_i to π_i. Lemma 19 and Corollary 6 then finish the proof. □

Theorem 11 *The group Λ_2 is the automorphism group of an atomic chiral polytope of type* $\{4, 8, 2^{\beta}\}$.

Proof Let $D = 2^{\beta-3}$. For $b \in \mathbb{Z}_8$, we define

$$\overline{b} = \begin{cases} b & \text{if } b \text{ is even,} \\ b-2 & \text{if } b \text{ is odd.} \end{cases}$$

Then we define permutations of $\mathbb{Z}_8 \times \mathbb{Z}_{2^\beta}$ as follows:

$$(b,c)\pi_1 = \begin{cases} (\overline{b} - 2c, c(1 + D\epsilon)) & \text{if } c \text{ is even,} \\ (\overline{b} - 2c + 4, c(1 - D\epsilon) + D\epsilon) & \text{if } c \text{ is odd,} \end{cases}$$
$$(b,c)\pi_2 = \begin{cases} (b + 1 - 2c, c(-1 + D\epsilon)) & \text{if } c \text{ is even,} \\ (b + 1 - 2c, c(-1 - D\epsilon) + D\epsilon) & \text{if } c \text{ is odd,} \end{cases}$$
$$(b,c)\pi_3 = (b, c+1).$$

We note that

$$(b,c)\pi_2^2 = \begin{cases} (b+2, c(1 - 2D\epsilon) & \text{if } c \text{ is even,} \\ (b+2, c(1 + 2D\epsilon) & \text{if } c \text{ is odd.} \end{cases}$$

Then it is routine to show that there is a well-defined epimorphism from Λ_2 to $\langle \pi_1, \pi_2, \pi_3 \rangle$ sending each σ_i to π_i. Lemma 19 and Corollary 6 then finish the proof. □

Theorem 12 *The group Λ_3 is the automorphism group of an atomic chiral polytope of type* $\{4, 2^{\beta-1}, 2^\beta\}$.

Proof Let $D = 2^{\beta-3}$. For $b \in \mathbb{Z}_{2^{\beta-1}}$, we define

$$\overline{b} = \begin{cases} b(-1 - D) & \text{if } b \text{ is even,} \\ b(-1 - D) + D & \text{if } b \text{ is odd.} \end{cases}$$

Then we define permutations of $\mathbb{Z}_{2^{\beta-1}} \times \mathbb{Z}_{2^\beta}$ as follows:

$$(b,c)\pi_1 = \begin{cases} (\overline{b} - cD, c(-1 + D\epsilon)) & \text{if } c \text{ is even,} \\ (\overline{b} - (c-1)D + 2, (1 - c)(1 + D\epsilon) + 1) & \text{if } c \text{ is odd,} \end{cases}$$
$$(b,c)\pi_2 = \begin{cases} (b + 1 + cD, c(1 + D\epsilon)) & \text{if } c \text{ is even,} \\ (b - 1 + (c-1)D, (c-1)(1 - D\epsilon) - 1 & \text{if } c \text{ is odd,} \end{cases}$$
$$(b,c)\pi_3 = (b, c+1).$$

Here are some intermediate calculations:

(a) If a is even, then $\overline{a+b} = \overline{a} + \overline{b}$.
(b) $\overline{\overline{b}} \equiv b(1 + 2D) \pmod{2^{\beta-1}}$.
(c) $(b,c)\pi_1^2 = \begin{cases} (b(1+2D), c(1 - 2D\epsilon)) & \text{if } c \text{ is even,} \\ (b(1+2D) - 2D, (c-1)(1 + 2D\epsilon) + 1) & \text{if } c \text{ is odd,} \end{cases}$

(d) $(b,c)\pi_2^8 = \begin{cases} (b+8,c), & \text{if } c \text{ is even,} \\ (b-8,c-16), & \text{if } c \text{ is odd.} \end{cases}$

(e) $(b,c)\pi_1\pi_2 = (\overline{b}+1,-c)$

Let us rewrite the first extra relation of Λ_3 as $\sigma_2\sigma_1^2 = \sigma_1^2\sigma_2^{1+2D}$ (see the proof of Theorem 3, noting that $\sigma_1^{-1} = \sigma_1^3$). Similarly, we rewrite the second extra relation by multiplying both sides on the left by σ_2, and the third relation by multiplying both sides on the right by σ_2. Then one can check that there is a well-defined epimorphism from Λ_3 to $\langle\pi_1,\pi_2,\pi_3\rangle$ sending each σ_i to π_i. Lemma 19 and Corollary 6 then finish the proof. □

In order to rule out the remaining cases, we will use the following lemma.

Lemma 24 *Suppose that Λ is a quotient of $[p,q,r]^+$ satisfying*

$$\begin{aligned} \sigma_2^2\sigma_1 &= \sigma_1\sigma_2^{2t}, \\ \sigma_3^{-1}\sigma_2 &= \sigma_2^a\sigma_3^c, \\ \sigma_3\sigma_2^{-1} &= \sigma_2^b\sigma_3^{-c}, \end{aligned}$$

and suppose that b is odd. Then

$$\sigma_2^{2t+2}\sigma_3 = \sigma_3\sigma_2^{-t(b+1)-1+a}.$$

Proof First, we note that

$$\sigma_1^{-1}\sigma_2^2\underline{\sigma_3\sigma_1} = \sigma_1^{-1}\underline{\sigma_2^2\sigma_1}\sigma_2^2\sigma_3 = \sigma_2^{2t+2}\sigma_3.$$

On the other hand,

$$\begin{aligned} \sigma_1^{-1}\underline{\sigma_2^2\sigma_3}\sigma_1 &= \sigma_1^{-1}\underline{\sigma_2\sigma_3^{-1}}\sigma_2^{-1}\sigma_1 \\ &= \sigma_1^{-1}\sigma_3^c\underline{\sigma_2^{-b-1}\sigma_1} \\ &= \underline{\sigma_1^{-1}\sigma_3^c\sigma_1}\sigma_2^{-t(b+1)} \\ &= \underline{\sigma_2\sigma_3^{-c}}\sigma_2^{-t(b+1)-1} \\ &= \sigma_3\sigma_2^{-t(b+1)-1+a}. \end{aligned}$$

□

Proposition 13 *The groups Λ_4 and Λ_6 satisfy*

$$\sigma_3^{-1}\sigma_2 = \sigma_2^{-1-\epsilon 2^{\beta-2}}\sigma_3^{-3}$$

and

$$\sigma_3\sigma_2^{-1} = \sigma_2^{1-\epsilon 2^{\beta-2}}\sigma_3^3.$$

Proof First:

$$\begin{aligned}
\sigma_2\sigma_3^2 &= \sigma_3^2\sigma_2^{1+\epsilon 2^{\beta-2}} && \text{Rewrite } \sigma_2\sigma_3 \text{ as } \sigma_3^{-1}\sigma_2^{-1}\\
\sigma_3^{-1}\sigma_2^{-1}\sigma_3 &= \sigma_3^2\sigma_2^{1+\epsilon 2^{\beta-2}} && \text{Multiply on the left by } \sigma_3\\
\sigma_2^{-1}\sigma_3 &= \sigma_3^3\sigma_2^{1+\epsilon 2^{\beta-2}} && \text{Invert both sides}\\
\sigma_3^{-1}\sigma_2 &= \sigma_2^{-1-\epsilon 2^{\beta-2}}\sigma_3^{-3}
\end{aligned}$$

Next, we note that since $\sigma_2\sigma_3^2 = \sigma_3^2\sigma_2^{1+\epsilon 2^{\beta-2}}$, it follows that

$$\sigma_2^{1-\epsilon 2^{\beta-2}}\sigma_3^2 = \sigma_3^2\sigma_2^{(1-\epsilon 2^{\beta-2})(1+\epsilon 2^{\beta-2})} = \sigma_3^2\sigma_2.$$

Finally:

$$\begin{aligned}
\sigma_3^2\sigma_2 &= \sigma_2^{1-\epsilon 2^{\beta-2}}\sigma_3^2 && \text{Rewrite } \sigma_3\sigma_2 \text{ as } \sigma_2^{-1}\sigma_3^{-1}\\
\sigma_3\sigma_2^{-1}\sigma_3^{-1} &= \sigma_2^{1-\epsilon 2^{\beta-2}}\sigma_3^2 && \text{Multiply on the right by } \sigma_3\\
\sigma_3\sigma_2^{-1} &= \sigma_2^{1-\epsilon 2^{\beta-2}}\sigma_3^3.
\end{aligned}$$

□

Theorem 13 *The groups Λ_4 and Λ_5 are not the automorphism groups of tight chiral 4-polytopes.*

Proof In Λ_4 and Λ_5, the relation $\sigma_2^{-1}\sigma_1 = \sigma_1^{-1}\sigma_2^{1+2^{\beta-1}}$ is equivalent to $\sigma_2\sigma_1^{-1} = \sigma_1\sigma_2^{-1+2^{\beta-1}}$ (see [13, Proposition 3.1]), and this implies that $\sigma_2^2\sigma_1 = \sigma_2\sigma_1^{-1}\sigma_2^{-1} = \sigma_1\sigma_2^{2(-1+2^{\beta-2})}$. Then Proposition 13 and Lemma 24 prove that both groups satisfy $\sigma_2^{2^{\beta-1}}\sigma_3 = \sigma_3$, and so σ_2 does not have order 2^β as required. □

Theorem 14 *The groups Λ_6 and Λ_7 are not the automorphism groups of tight chiral 4-polytopes.*

Proof If either group is the automorphism group of a tight chiral 4-polytope, then Proposition 6 implies that $\langle\sigma_1, \sigma_2\rangle$ and $\langle\sigma_2, \sigma_3\rangle$ both have a normal subgroup of the form $\langle\sigma_2^k\rangle$. It follows that $\langle\sigma_2^{2^{\beta-1}}\rangle$ is normal in $\langle\sigma_1, \sigma_2, \sigma_3\rangle$, which means that $\sigma_2^{2^{\beta-1}}$ is central.

Now, the relation $\sigma_2\sigma_1^{-1} = \sigma_1\sigma_2^{3-\epsilon_1 2^{\beta-1}}$ implies that $\sigma_2^2\sigma_1 = \sigma_1\sigma_2^{2(1+\epsilon_1 2^{\beta-2})}$. Proposition 13 implies that we may use Lemma 24, which then implies that in Λ_6, $\sigma_2^{4+\epsilon_1 2^{\beta-1}}\sigma_3 = \sigma_3\sigma_2^{-4-\epsilon_1 2^{\beta-1}}$. So conjugation by σ_3 inverts $\sigma_2^{4+\epsilon_1 2^{\beta-1}}$, and since $\sigma_2^{2^{\beta-1}}$ is central, this implies that conjugation by σ_3 inverts σ_2^4. Now, $\sigma_3^2\sigma_2 = \sigma_3\sigma_2^{-1}\sigma_3^{-1} = \sigma_2^{1+\epsilon_2 2^{\beta-2}}\sigma_3^2$, and it follows that $\sigma_3^4\sigma_2 = \sigma_2^{1+\epsilon_2 2^{\beta-1}}\sigma_3^4$. Then $\sigma_3^4\sigma_2^2 = \sigma_2^{2(1+\epsilon_2 2^{\beta-1})}\sigma_3^4 = \sigma_2^2\sigma_3^4$, and since σ_3 has order 8, this implies that $\sigma_2^2 = \sigma_3^4\sigma_2^2\sigma_3^4$. So:

$$\sigma_3\sigma_2^4 = \sigma_2^{-1}\underline{\sigma_3^{-1}\sigma_2}\sigma_2^2$$

$$
\begin{aligned}
&= \sigma_2^{-2+\epsilon_2 2^{\beta-2}} \underline{\sigma_3^{-3}\sigma_2^2} \\
&= \sigma_2^{-2+\epsilon_2 2^{\beta-2}} \underline{\sigma_3\sigma_2^2}\sigma_3^4 \\
&= \sigma_2^{-2+\epsilon_2 2^{\beta-2}} \sigma_2^{-2+\epsilon_2 2^{\beta-2}} \sigma_3^{-3}\sigma_3^4 \\
&= \sigma_2^{-4+\epsilon_2 2^{\beta-1}} \sigma_3.
\end{aligned}
$$

Since we also have that conjugation by σ_3 inverts σ_2^4, this implies that $\sigma_2^{2^{\beta-1}} = id$, and so σ_2 does not have the desired order.

In Λ_7, Lemma 24 implies that $\sigma_2^{4+\epsilon_1 2^{\beta-1}}\sigma_3 = \sigma_3\sigma_2^{4+\epsilon_1 2^{\beta-1}}$. Since $\sigma_2^{2^{\beta-1}}$ is central, this implies that σ_3 commutes with σ_2^4. However,

$$
\begin{aligned}
\sigma_3\sigma_2^{-4-\epsilon_2 2^{\beta-2}} &= \underline{\sigma_3\sigma_2^{-1}}\sigma_2^{-3-\epsilon_2 2^{\beta-2}} \\
&= \sigma_2^{-3+\epsilon_2 2^{\beta-2}} \underline{\sigma_3^{-1+2^{\beta-2}}\sigma_2^{-3-\epsilon_2 2^{\beta-2}}} \\
&= \sigma_2^{-4+\epsilon_2 2^{\beta-2}}\sigma_3.
\end{aligned}
$$

Then $\sigma_2^{-2^{\beta-2}} = \sigma_2^{2^{\beta-2}}$, which implies that $\sigma_2^{2^{\beta-1}} = id$, and again σ_2 does not have the desired order. □

Theorem 15 *The group Λ_8 is not the automorphism group of tight chiral* 4*-polytope.*

Proof If Λ_8 were the automorphism group of a tight chiral 4-polytope, then $\langle\sigma_2\rangle$ would be core-free in $\langle\sigma_2, \sigma_3\rangle$ (see Proposition 6 and [12, Proposition 4.5]). We will show that in fact, σ_3 normalizes a nontrivial subgroup of $\langle\sigma_2\rangle$.

From the relation $\sigma_2\sigma_1^{-1} = \sigma_1\sigma_2^{3-\epsilon_1 2^{\beta-2}}$, it follows that $\sigma_2^2\sigma_1 = \sigma_1\sigma_2^{2-\epsilon_1 2^{\beta-2}}$, and thus, for each k, $\sigma_2^{2k}\sigma_1 = \sigma_1\sigma_2^{(1-\epsilon_1 2^{\beta-3})2k}$. Then

$$
\begin{aligned}
\sigma_1^{-1}\sigma_3\underline{\sigma_2^2\sigma_1} &= \underline{\sigma_1^{-1}\sigma_3\sigma_1}\sigma_2^{2-\epsilon_1 2^{\beta-2}} \\
&= \sigma_2^2\sigma_3\sigma_2^{2-\epsilon_1 2^{\beta-2}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sigma_1^{-1}\underline{\sigma_3\sigma_2^2}\sigma_1 &= \sigma_1^{-1}\sigma_2^{-2+2^{\beta-2}}\underline{\sigma_3^{-3+\epsilon_2 2^{\beta-2}}\sigma_1} \\
&= \underline{\sigma_1^{-1}\sigma_2^{-2+2^{\beta-2}}\sigma_1}\sigma_2\sigma_3^{3-\epsilon_2 2^{\beta-2}}\sigma_2^{-1} \\
&= \sigma_2^{-1+2^{\beta-2}+\epsilon_1 2^{\beta-2}}\sigma_3^{3-\epsilon_2 2^{\beta-2}}\sigma_2^{-1} \\
&= \sigma_2^{-1+2^{\beta-2}+\epsilon_1 2^{\beta-2}}\underline{\sigma_3^{3-\epsilon_2 2^{\beta-2}}\sigma_2^{1-2^{\beta-2}}}\sigma_2^{-2+2^{\beta-2}} \\
&= \sigma_2^{-1+2^{\beta-2}+\epsilon_1 2^{\beta-2}}\sigma_2^{-1}\sigma_3\sigma_2^{-2+2^{\beta-2}}.
\end{aligned}
$$

Putting these together, we find that $\sigma_2^{-4+2^{\beta-2}+\epsilon_1 2^{\beta-2}}\sigma_3 = \sigma_3\sigma_2^{4-2^{\beta-2}-\epsilon_1 2^{\beta-2}}$, and so σ_3 normalizes a nontrivial subgroup of $\langle\sigma_2\rangle$. □

Table 4 summarizes the atomic chiral 4-polytopes with chiral vertex figures. The duals of the first three rows yield atomic chiral 4-polytopes with regular vertex-figures, and the last two rows correspond to a dual pair of chiral 4-polytopes. In total, there are 11 families of atomic chiral 4-polytopes. Thus we have shown:

Theorem 16 *Every tight chiral 4-polytope covers one of the polytopes in Table 4 or its dual.*

Proposition 14 *If $\mathcal{P}$ is an atomic chiral 4-polytope with regular facets, then $X(\mathcal{P})$ is contained in $\langle \sigma_3 \rangle$.*

Proof An atomic chiral 4-polytope $\mathcal{P}$ with regular facets has automorphism group Λ_1, Λ_2, or Λ_3. In each case, the chirality group of the vertex-figures is a cyclic group of prime order of the form $\langle \sigma_3^c \rangle$ that is normal in $\langle \sigma_2, \sigma_3 \rangle$ (see Table 1), and thus in $\Gamma(\mathcal{P})$ (by the dual of Lemma 4). The quotient of $\Gamma(\mathcal{P})$ by this normal subgroup is a polytope, by Proposition 5, and thus, it is regular (by atomicity). Therefore, the chirality group of the vertex-figures contains the chirality group of $\mathcal{P}$, and since the former has prime order and the latter is nontrivial, it follows that the two coincide, proving the claim. □

6 Tight chiral 5-polytopes

Recall that a tight chiral 5-polytope must have chiral facets and chiral vertex-figures (see Proposition 1 (c)). In this section, we prove Theorem 1, that is, that no such polytope exists.

Recall that a tight chiral 5-polytope $\mathcal{P}$ with $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$ is *atomic* if it does not properly cover any tight chiral polytope. Clearly, every tight chiral 5-polytope covers an atomic chiral 5-polytope.

We start by giving properties that atomic chiral 5-polytopes must satisfy, should they exist.

Lemma 25 *Let $\mathcal{P}$ be a tight chiral 5-polytope with type $\{p, q, r, s\}$ where $q \geq r$. Then the kernel of the action of $\Gamma(\mathcal{P})$ on the chains containing a 3-face and a facet is nontrivial.*

Proof The stabilizer of the chain containing the base 3-face and the base facet is $\Delta = \langle \sigma_1, \sigma_2 \rangle$. The remaining chains can be associated to right cosets of Δ. Proposition 6 implies that there is a nontrivial subgroup $\langle \sigma_2^k \rangle$ that is normal in $\langle \sigma_2, \sigma_3, \sigma_4 \rangle$. Then it follows that for all a and b we have $(\langle \sigma_1, \sigma_2 \rangle \sigma_3^a \sigma_4^b) \sigma_2^k = \langle \sigma_1, \sigma_2 \rangle \sigma_3^a \sigma_4^b$, and so σ_2^k fixes all chains containing a 3-face and a facet. □

Lemma 26 *Let $\mathcal{P}$ be an atomic chiral 5-polytope with $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$ and type $\{p, q, r, s\}$. If $q \geq r$ then*

(a) $X(\mathcal{P})$ is $\langle \sigma_2^{q'} \rangle$ for some q' satisfying that q/q' is prime,

(b) The chirality groups of the base facet and the base vertex-figure are also $\langle \sigma_2^{q'} \rangle$.

Table 4 The atomic chiral 4-polytopes with chiral vertex-figures

$\{p, q, r\}$	Facets	$\sigma_2^{-1}\sigma_1$	$\sigma_2\sigma_1^{-1}$	$\sigma_3^{-1}\sigma_2$	$\sigma_3\sigma_2^{-1}$	Notes
$\{m, 2m, m^\alpha\}$	Regular	$\sigma_1^{-1}\sigma_2^{-3}$	$\sigma_1\sigma_2^3$	$\sigma_2^3\sigma_3^{1+km^{\alpha-1}}$	$\sigma_2^{-3}\sigma_3^{-1+km^{\alpha-1}}$	m odd prime, $\alpha \geq 2$, $1 \leq k \leq m-1$
$\{4, 8, 2^\beta\}$	Regular	$\sigma_1^{-1}\sigma_2^{-3}$	$\sigma_1\sigma_2^3$	$\sigma_2^3\sigma_3^{1+\epsilon 2^{\beta-2}}$	$\sigma_2^{-3}\sigma_3^{-1+\epsilon 2^{\beta-2}}$	$\beta \geq 5, \epsilon = \pm 1$
$\{4, 2^{\beta-1}, 2^\beta\}$	Regular	$\sigma_1^{-1}\sigma_2^{1+2^{\beta-2}}$	$\sigma_1\sigma_2^{-1+2^{\beta-2}}$	$\sigma_2^{-1+2^{\beta-2}}\sigma_3^{-3+\epsilon 2^{\beta-2}}$	$\sigma_2^{1+2^{\beta-2}}\sigma_3^{3+\epsilon 2^{\beta-2}}$	$\beta \geq 5, \epsilon = \pm 1$
$\{2m, m^\alpha, 2m\}$	Chiral	$\sigma_1^3\sigma_2^{1+km^{\alpha-1}}$	$\sigma_1^{-3}\sigma_2^{-1+km^{\alpha-1}}$	$\sigma_2^{-1+km^{\alpha-1}}\sigma_3^{-3}$	$\sigma_2^{1+km^{\alpha-1}}\sigma_3^3$	m odd prime, $\alpha \geq 2$, $1 \leq k \leq m-1$
$\{8, 2^\beta, 8\}$	Chiral	$\sigma_1^3\sigma_2^{1+\epsilon_1 2^{\beta-2}}$	$\sigma_1^{-3}\sigma_2^{-1+\epsilon_1 2^{\beta-2}}$	$\sigma_2^{-1+\epsilon_2 2^{\beta-2}}\sigma_3^{-3}$	$\sigma_2^{1+\epsilon_2 2^{\beta-2}}\sigma_3^3$	$\beta \geq 5, \epsilon_1, \epsilon_2 = \pm 1$
$\{2^{\beta-1}, 2^\beta, 2^{\beta-1}\}$	Chiral	$\sigma_1^{-1+2^{\beta-2}}\sigma_2^{-3+\epsilon_1 2^{\beta-2}}$	$\sigma_1^{1+2^{\beta-2}}\sigma_2^{3+\epsilon_1 2^{\beta-2}}$	$\sigma_2^{3+\epsilon_2 2^{\beta-2}}\sigma_3^{1+2^{\beta-2}}$	$\sigma_2^{-3+\epsilon_2 2^{\beta-2}}\sigma_3^{-1+2^{\beta-2}}$	$\beta \geq 5, \epsilon_1, \epsilon_2 = \pm 1$
$\{8, 2^\beta, 2^{\beta-1}\}$	Chiral	$\sigma_1^3\sigma_2^{1+\epsilon_1 2^{\beta-2}}$	$\sigma_1^{-3}\sigma_2^{-1+\epsilon_1 2^{\beta-2}}$	$\sigma_2^{3+\epsilon_2 2^{\beta-2}}\sigma_3^{1+2^{\beta-2}}$	$\sigma_2^{-3+\epsilon_2 2^{\beta-2}}\sigma_3^{-1+2^{\beta-2}}$	$\beta \geq 5, \epsilon_1, \epsilon_2 = \pm 1$
$\{2^{\beta-1}, 2^\beta, 8\}$	Chiral	$\sigma_1^{-1+2^{\beta-2}}\sigma_2^{-3+\epsilon_1 2^{\beta-2}}$	$\sigma_1^{1+2^{\beta-2}}\sigma_2^{3+\epsilon_1 2^{\beta-2}}$	$\sigma_2^{-1+\epsilon_2 2^{\beta-2}}\sigma_3^{-3}$	$\sigma_2^{1+\epsilon_2 2^{\beta-2}}\sigma_3^3$	$\beta \geq 5, \epsilon_1, \epsilon_2 = \pm 1$

Proof Let H and K be the kernels of the actions of $\Gamma(\mathcal{P})$ on the vertices and on the chains consisting of a 3-face and a 4-face, respectively. By Corollary 2 and Lemma 25, H and K are nontrivial. Therefore $H \cap K$ is a normal subgroup of $\Gamma(\mathcal{P})$ that by the intersection condition is contained in $\langle \sigma_2 \rangle$. The rest of the proof is as in Proposition 11. □

Now we can prove Theorem 1.

Proof of Theorem 1 It suffices to show that there are no atomic chiral 5 polytopes. Suppose to the contrary that $\mathcal{P}$ is an atomic chiral 5-polytope. Up to duality, we may assume that $q \geq r$. Let $\mathcal{K}$ be the base facet. It must be a tight chiral 4-polytope, by Proposition 1(b), and since the facets of the facets of a chiral polytope are always regular, $\mathcal{K}$ has regular facets. Now, Lemma 26 says that $\mathcal{K}$ has chirality group contained in $\langle \sigma_2 \rangle$. Let $\mathcal{K}'$ be an atomic chiral 4-polytope that is covered by $\mathcal{K}$, with $\Gamma(\mathcal{K}') = \langle \sigma_1', \sigma_2', \sigma_3' \rangle$. Then Lemma 10 says that $X(\mathcal{K}')$ is contained in $\langle \sigma_2' \rangle$, which contradicts Proposition 14. □

7 Concluding remarks

The study of tight chiral polytopes was originated in the search for chiral polytopes with a small number of flags. In ranks 3 and 4, the atomic chiral polytopes are now classified; this constitutes the first step for a full classification of tight chiral 3- and 4-polytopes. However, the techniques used to classify tight regular polyhedra fail in the chiral setting, and the full classification seems to require several more steps.

The nonexistence of tight chiral n-polytopes for $n \geq 5$ strengthens the general belief that for each $n \geq 5$ the chiral n-polytopes with the fewest flags have considerably more flags than the regular n-polytopes with the fewest flags. See also [11, Theorem 5.5].

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