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Tetravalent edge-transitive Cayley graphs of Frobenius groups

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Abstract
In this paper, we give a characterisation for a class of edge-transitive Cayley graphs and provide a method for constructing edge-transitive graphs of valency 4 with arbitrarily large vertex stabiliser. In particular, in the last section, we obtain certain extensions of the results of Li et al. (Tetravalent edge-transitive Cayley graphs with odd number of vertices, J Comb Theory Ser B 96:164–181, 2006) on half-transitive graphs.

Keywords Frobenius group · Edge-transitive graph · Coset graph · Cayley graph

Mathematics Subject Classification 05C25 · 05E18

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1 Introduction

Graphs considered in this paper are assumed to be finite, simple, connected and undirected. For a graph Γ, let $V\Gamma$, $E\Gamma$ and $\mathsf{Aut}\Gamma$ denote its vertex set, edge set and the full automorphism group, respectively. If a subgroup $X \leq \mathsf{Aut}\Gamma$ acts transitively on $V\Gamma$ or $E\Gamma$, then the graph Γ is said to be X*-vertex-transitive* or X*-edge-transitive*, respectively. A sequence $v_0, v_1, \ldots, v_s$ of vertices of Γ is called an s-arc if $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$, and $\{v_i, v_{i+1}\}$ is an edge for $0 \leq i \leq s-1$. The graph Γ is called (X, s)*-arc-transitive* if X is transitive on the s*-arcs* of Γ; if in addition X is not transitive on the $(s+1)$-arcs, then Γ is said to be (X, s)*-transitive*. In particular, a 1-arc is simply called an *arc*, and Γ is simply called X*-arc-transitive* if it is $(X, 1)$-arc-transitive.

A graph Γ is called a *Cayley graph* if there exists a group G and a subset $S \subset G\backslash\{1\}$ with $S = S^{-1} := \{g^{-1} \mid g \in S\}$ such that the vertices of Γ may be identified with the elements of G in such a way that x is adjacent to y if and only if $yx^{-1} \in S$. The Cayley graph Γ is denoted by $\mathsf{Cay}(G, S)$. Throughout this paper, denote by $\mathbf{1}$ the vertex of $\mathsf{Cay}(G, S)$ corresponding to the identity of G.

It is well known that a graph Γ is a Cayley graph of a group G if and only if the full automorphism group $\mathsf{Aut}\Gamma$ contains a subgroup which is regular on vertices and isomorphic to G. In particular, a Cayley graph $\mathsf{Cay}(G, S)$ is vertex-transitive, but of course not necessarily edge-transitive. In the literature, the Cayley graphs which are edge-transitive have received much attention, and special classes of edge-transitive Cayley graphs have been well investigated. For instance, see [10,22,35,37] for those with valency 4; see [23,32,33] for characterisations of edge-transitive Cayley graphs of metacyclic Frobenius groups; see [5] for a classification of normal edge-transitive Cayley graphs of Frobenius groups of order a product of two primes; see [26] for a classification of cubic arc-transitive Cayley graphs on Frobenius groups. In this paper, we investigate tetravalent edge-transitive Cayley graphs of Frobenius groups.

An edge-transitive graph Γ is called *half-transitive* if $\mathsf{Aut}\Gamma$ is transitive on the vertices but not on the arcs of Γ. In view of the fact that 4 is the smallest admissible valency for a half-transitive graph, special attention has been given to the study of tetravalent half-transitive graphs (for example, see [10–12,27,35,40]). In fact, many of the interesting families of half-transitive graphs are constructed as metacirculants, see [28,40] for reference. Kutnar et al. [18] gave one family of half-transitive graphs that are not metacirculant. It is therefore worth mentioning some families of tetravalent half-transitive graphs of non-metacirculants. The main results (Theorems 1.1, 1.2) provide a generic construction of half-transitive graphs of valency 4, which are not metacirculants. To state our results, we need more definitions.

A typical method for studying vertex-transitive graphs is taking certain quotients. For an X-vertex-transitive graph Γ and a normal subgroup $N \lhd X$, the *normal quotient graph* Γ_N induced by N is the graph that has vertex set $V\Gamma_N = \{u^N \bigm| u \in V\Gamma\}$ such that u^N and v^N are adjacent if and only if u is adjacent in Γ to some vertex in v^N. If the valency of Γ_N equals the valency of Γ, then Γ is called a *normal cover* of Γ_N.

For an integer $m \geq 3$, we denote by $\mathbf{C}_{m[2]}$ the *lexicographic product* of the empty graph $2\mathbf{K_1}$ of order 2 by a cycle $\mathbf{C}_m$ of size m, which has vertex set $\{(i, j) \bigm| 1 \leq$

$i \leq m, 1 \leq j \leq 2\}$ such that (i, j) and (i', j') are adjacent if and only if $i - i' \equiv \pm 1 \pmod m$.

A Frobenius group G is a semidirect product of a normal subgroup W by a subgroup H such that none of the non-identity elements of H centralises a non-identity element of W, refer to Dixon and Mortimer [7].

Let $\mathbb{F}$ be a field, G a group and V a vector space over $\mathbb{F}$ such that $G \leq \mathrm{GL}(V)$. Suppose that $V = V_1 \oplus \cdots \oplus V_r$ $(r > 1)$, where V_i are subspaces of V which are transitively permuted by the action of G. We call G *imprimitive* on V if there exists such a decomposition. Otherwise, G is called *primitive* on V. For positive integers p and n, we call d the order of p modulo n if n divides $p^d - 1$ but n does not divide $p^i - 1$ for $i < d$, and denote d by $\mathrm{ord}_n(p)$.

Theorem 1.1 *Let $G = W{:}H \cong \mathbb{Z}_p^d{:}\mathbb{Z}_n$ be a Frobenius group, where $d = \mathrm{ord}_n(p)$ for a prime p and a positive integer n. Assume that Γ is a connected tetravalent X-edge-transitive Cayley graph of G, where $G \leq X \leq \mathsf{Aut}\Gamma$. If X is soluble, then one of the following statements holds:*

(1) *G is normal in X, and $X_{\mathbf{1}} \leq \mathrm{D}_8$;*
(2) *$G \cong \mathrm{D}_{2p}$, $\Gamma \cong \mathbf{C}_{p[2]}$ and $\mathsf{Aut}\Gamma \cong \mathbb{Z}_2^p{:}\mathrm{D}_{2p}$;*
(3) *$X = W{:}((N{:}H).\mathcal{O})$ with $\mathsf{soc}(X) = W \times L$, and $X_{\mathbf{1}} = N.\mathcal{O}$, where $N \cong \mathbb{Z}_2^\ell$ with $2 \leq \ell \leq d$, $L \cong 1$ or $\mathbb{Z}_2$, and $\mathcal{O} \cong 1$ or $\mathbb{Z}_2$, satisfying the following statements:*
   (i) *there exist $x_1, \ldots, x_d \in W$ and $\tau_1, \ldots, \tau_d \in N$ such that $W = \langle x_1, \ldots, x_d\rangle$, $\langle x_i, \tau_i\rangle \cong \mathrm{D}_{2p}$ and $N = \langle \tau_i\rangle \times \mathbf{C}_N(x_i)$ for $1 \leq i \leq d$;*
   (ii) *H does not centralise N, and H is imprimitive on W;*
   (iii) *$X/(WN) \cong \mathbb{Z}_n$ or D_{2n}, and Γ is X-arc-transitive if and only if $X/(WN) \cong \mathrm{D}_{2n}$;*
(4) *$\Gamma_W \cong \mathbf{C}_{\frac{n}{2}[2]}$, Γ is a normal cover of Γ_W, and $X = W{:}((NH).\mathcal{O})$ such that*
   (i) *$X_{\mathbf{1}} \leq N.\mathcal{O}$, $N \cap H \cong \mathbb{Z}_2$, and H normalises N, but H does not centralise N, where $N \cong \mathbb{Z}_2^\ell$ with $2 \leq \ell \leq n/2$, and $\mathcal{O} \cong 1$ or $\mathbb{Z}_2$;*
   (ii) *W is the unique minimal normal subgroup of X, and H is imprimitive on W;*
   (iii) *$X/(WN) \cong \mathbb{Z}_{\frac{n}{2}}$ or D_n, and Γ is X-arc-transitive if and only if $X/(WN) \cong \mathrm{D}_n$;*
(5) *$X = ((WN){:}H).\mathcal{O}$, $X_{\mathbf{1}} = N.\mathcal{O}$, and Γ is X-arc-transitive if and only if $X/(WN) \cong \mathrm{D}_{2n}$, where W and N are 2-groups, and $\mathcal{O} \cong 1$ or $\mathbb{Z}_2$.*

Remarks on Theorem 1.1.

(a) The Cayley graph Γ in part (1), called a *normal edge-transitive graph*, is studied in [31]. Furthermore, if $X = \mathsf{Aut}\Gamma$, then Γ is called a *normal Cayley graph*, introduced in [38].
(b) Note that $\mathrm{ord}_n(p) = d$ if and only if H acts irreducibly on W (such n is called a *primitive divisor* of $p^d - 1$), refer to [6, Proposition 2.3].
(c) The group X satisfies part (3) or part (4) if and only if H is imprimitive on W, see Lemmas 4.4 and 4.5. In addition, H is imprimitive on W if and only if there exists some prime r dividing d such that n divides $r(p^{d/r} - 1)$, see [6, Proposition 2.8].

Table 1 Insoluble automorphism groups with metacyclic Frobenius subgroups

X	G	X_1
$\mathrm{PSL}(3,3){:}\mathbb{Z}_2$	D_{26}	$\mathbb{Z}_3^2{:}\mathrm{GL}(2,3)$
$\mathrm{PGL}(2,7)$	$\mathrm{D}_{14}, \mathbb{Z}_7{:}\mathbb{Z}_3, \mathbb{Z}_7{:}\mathbb{Z}_6$	$\mathrm{S}_4, \mathrm{D}_{16}, \mathrm{D}_8$
$\mathrm{PSL}(2,23)$	$\mathbb{Z}_{23}{:}\mathbb{Z}_{11}$	S_4
$\mathrm{PSL}(2,11)$	$\mathbb{Z}_{11}{:}\mathbb{Z}_5$	A_4
$\mathrm{PGL}(2,11)$	$\mathbb{Z}_{11}{:}\mathbb{Z}_5, \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	$\mathrm{S}_4, \mathrm{A}_4$
$\mathrm{PGL}(2,11)\times\mathbb{Z}_2$	$\mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	S_4

(d) Constructions 3.3 and 3.5 show that the graph Γ indeed exists when the group X satisfies part (3) or part (4) with $\mathcal{O}=1$.

Theorem 1.2 *Using the notation defined in Theorem 1.1, if X is insoluble, then one of the following holds:*

(1) $G\cong\mathbb{Z}_p^4{:}\mathbb{Z}_5$, *$X=W.\overline{X}$ and $\Gamma_W\cong\mathbf{K}_5$, where* $\mathsf{soc}(\overline{X})\cong\mathrm{A}_5$*;*
(2) $G\cong\mathbb{Z}_p^4{:}\mathbb{Z}_{10}$, *$X=W.(\overline{X}\times\mathbb{Z}_2)$ and $\Gamma_W\cong\mathbf{K}_{5,5}-5\mathbf{K}_2$, where* $\mathsf{soc}(\overline{X})\cong\mathrm{A}_5$*;*
(3) *X is almost simple with one exception, and the triple (X,G,X_1) lies in Table 1.*

Remarks on Theorem 1.2.

(a) Constructions 3.7 and 3.9 show that the graph Γ indeed exists when the group X satisfies part (1) or part (2).
(b) Kuzman [19] classified all arc-transitive elementary abelian covers of the complete graph $\mathbf{K}_5$, and in [9,39], Du et al. classified all regular covers of the graph $\mathbf{K}_{n,n}-n\mathbf{K}_2$ with the covering transformation group $\mathbb{Z}_p^2$ or $\mathbb{Z}_p^3$. However, it seems difficult at the moment to classify such graph $\mathbf{K}_{5,5}-5\mathbf{K}_2$ with the covering transformation group $\mathbb{Z}_p^4$.

Theorems 1.1 and 1.2 provide a method for characterising some classes of half-transitive graphs of valency 4. The following theorem is such an example and generalises some of the results in [22].

Theorem 1.3 *Let $G=W{:}\langle h\rangle\cong\mathbb{Z}_p^d{:}\mathbb{Z}_n$ be a Frobenius group, where $\mathrm{ord}_n(p)=d>1$ is an odd integer, p is an odd prime, and $n\nmid r(p^{d/r}-1)$ for any prime r dividing d. Let Γ be a connected tetravalent edge-transitive Cayley graph of G. Then, $\mathsf{Aut}\Gamma=G{:}\mathbb{Z}_2$, Γ is half-transitive, and $\Gamma\cong\Gamma_i=\mathsf{Cay}(G,S_i)$, where $1\le i\le\lfloor\frac{n-1}{2}\rfloor$, $(n,i)=1$, and*

$$S_i=\{ah^i, a^{-1}h^i, (ah^i)^{-1}, (a^{-1}h^i)^{-1}\},\ \text{where}\ a\in W\setminus\{1\}.$$

Moreover, if $p^k i\equiv\pm j \pmod n$ for some $k\ge0$, then $\Gamma_i\cong\Gamma_j$.

Remark If $n\mid r(p^{d/r}-1)$ for some prime r dividing d, then $\langle h\rangle$ is imprimitive on W, refer to Detinko and Flannery [6, Proposition 2.8]. Construction 3.3 shows that there exist infinite such groups G, such that the Cayley graphs Γ of G are not half-transitive. Therefore, the condition that $n\nmid r(p^{d/r}-1)$ for any prime r dividing d is needed.

2 Preliminary results

In this section, we collect the notation and elementary facts as well as some technical lemmas. Some basic facts will be used in the sequel without further reference.

For a core-free subgroup H of group X and an element $a \in X \setminus H$, let $[X{:}H] := \{Hx \mid x \in X\}$, and define the coset graph

$$\Gamma := \mathsf{Cos}(X, H, H\{a, a^{-1}\}H)$$

with vertex set $[X{:}H]$ such that Hx and Hy are adjacent if and only if $yx^{-1} \in H\{a, a^{-1}\}H$. The properties stated in the following lemma are well known.

Lemma 2.1 *For a coset graph $\Gamma = \mathsf{Cos}(X, H, H\{a, a^{-1}\}H)$, the following hold:*

(i) *Γ is X-edge-transitive;*
(ii) *Γ is X-arc-transitive if and only if $HaH = Ha^{-1}H$, or equivalently, $HaH = HbH$ for some $b \in X \setminus H$ such that $b^2 \in H \cap H^b$;*
(iii) *Γ is connected if and only if $X = \langle H, a \rangle$;*
(iv) *the valency of Γ equals*

$$val(\Gamma) = \begin{cases} |H : H \cap H^a| & \text{if } HaH = Ha^{-1}H, \\ 2|H : H \cap H^a| & \text{otherwise.} \end{cases}$$

Lemma 2.2 *Let $\sigma \in \mathsf{Aut}(X)$. Then, σ induces an automorphism from $\mathsf{Cos}(X, H, H\{a, a^{-1}\}H)$ to $\mathsf{Cos}(X, H^\sigma, H^\sigma\{a^\sigma, (a^\sigma)^{-1}\}H^\sigma)$. In particular, if $\sigma \in \mathbf{N}_{\mathsf{Aut}(X)}(H)$, then*

$$\mathsf{Cos}(X, H, H\{a, a^{-1}\}H) \cong \mathsf{Cos}(X, H, H\{a^\sigma, (a^\sigma)^{-1}\}H).$$

Proof Let $\Gamma = \mathsf{Cos}(X, H, H\{a, a^{-1}\}H)$ and $\Gamma' = \mathsf{Cos}(X, H^\sigma, H^\sigma\{a^\sigma, (a^\sigma)^{-1}\}H^\sigma)$. For any $x, y \in X$, we have $xy^{-1} \in H\{a, a^{-1}\}H$ if and only if $x^\sigma(y^\sigma)^{-1} = (xy^{-1})^\sigma \in H^\sigma\{a^\sigma, (a^\sigma)^{-1}\}H^\sigma$, and so $\{Hx, Hy\} \in E\Gamma$ if and only if $\{H^\sigma x^\sigma, H^\sigma y^\sigma\} \in E\Gamma'$. □

The vertex stabiliser for s-arc-transitive graphs of valency 4 is known, refer to [36].

Lemma 2.3 *Let Γ be a connected (X, s)-transitive graph of valency 4. Then, s and the stabiliser X_u are listed in the following table,*

s	1	2	3	4	7
X_u	2-group	A_4, S_4	$\mathbb{Z}_3 \times \mathrm{A}_4$, $(\mathbb{Z}_3 \times \mathrm{A}_4).\mathbb{Z}_2$, $\mathrm{S}_3 \times \mathrm{S}_4$	$\mathbb{Z}_3^2{:}\mathrm{GL}(2,3)$	$[3^5]{:}\mathrm{GL}(2,3)$

where $[3^5]$ is a 3-group of order 3^5. In particular, $|X_u|$ divides $2^4 \cdot 3^6$ if $s \geq 2$.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph. Assume that $X \leq \mathsf{Aut}\Gamma$ is transitive on both $V\Gamma$ and $E\Gamma$. By [1, Proposition 3.1] along with [21, Lemma 2.9], we have the following conclusion.

Lemma 2.4 *Let $N \lhd X$. Then, the valency of Γ_N is a divisor of the valency of Γ. In particular, if Γ is of valency 4 and X/N is insoluble, then Γ is a normal cover of Γ_N.*

For a Cayley graph $\Gamma = \mathsf{Cay}(G, S)$, let $\mathsf{Aut}(G, S) = \{\alpha \in \mathsf{Aut}(G) \mid S^\alpha = S\}$. It is easily shown that $\mathsf{Aut}(G, S)$ is a subgroup of $\mathsf{Aut}\Gamma$ that fixes the vertex $\mathbf{1}$ and normalises the regular subgroup of $\mathsf{Aut}\Gamma$. Then, we have the following property.

Lemma 2.5 *Let $G = W{:}H = \mathbb{Z}_p^d{:}\mathbb{Z}_n$ be a Frobenius group, where $\mathrm{ord}_n(p) = d$ for a prime p and an integer n. Let $\Gamma = \mathsf{Cay}(G, S)$ be connected of valency 4. Assume that $\mathsf{Aut}\Gamma$ has a subgroup X such that Γ is X-edge-transitive and $G \unlhd X$. Then, $X_{\mathbf{1}} \le \mathrm{D}_8$.*

Proof Since Γ is connected, we have $\langle S\rangle = G$, and so $\mathsf{Aut}(G, S)$ is faithful on S. Hence, $\mathsf{Aut}(G, S) \le \mathrm{S}_4$. By [13, Lemma 2.1], we obtain $X \le \mathbf{N}_{\mathsf{Aut}\Gamma}(G) = G{:}\mathsf{Aut}(G, S)$, and so $X_{\mathbf{1}} \le \mathsf{Aut}(G, S) \le \mathrm{S}_4$. Suppose that 3 divides $|X_{\mathbf{1}}|$. Then, $X_{\mathbf{1}}$ is 2-transitive on S. Hence, Γ is $(X, 2)$-arc-transitive, and all elements in S are involutions, see for example [20]. In particular, $|G|$ is even. If $p = 2$, then $\langle S\rangle \le W < G$, which is a contradiction.

Thus, $p > 2$, and so $|H|$ is even. For this case, G has a cyclic Sylow 2-subgroup, and so all involutions of G are conjugate. Consequently, $\langle S\rangle = G \cong \mathbb{Z}_p^d{:}\mathbb{Z}_2$. As W is minimal normal in G, one has $d = 1$, namely, $\langle S\rangle = G \cong \mathrm{D}_{2p}$. Thus, $\mathsf{Aut}(G) \cong \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$. However, since $X_{\mathbf{1}}$ is 2-transitive on S, we have $X_{\mathbf{1}} \cong \mathrm{A}_4$ or S_4, which is impossible. Therefore, $X_{\mathbf{1}} \le \mathrm{D}_8$. □

We will also need to know about the order of the *maximal p-elements* in $\mathrm{GL}(d, p)$.

Lemma 2.6 [24, Lemma 2.5] *Let p be a prime and d a positive integer. If $d \ge 2$, then the largest order p^e of p-elements of $\mathrm{GL}(d, p)$ satisfies $p^e \ge d > p^{e-1}$.*

Finally, we quote a result about simple groups, which will be used later.

Lemma 2.7 (Kazarin [17]) *Let T be a non-abelian simple group which has a $2'$-Hall subgroup. Then, $T = \mathrm{PSL}(2, p)$, where $p = 2^e - 1$ is a prime. Furthermore, $T = GH$, where $G = \mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}$ and $H = \mathrm{D}_{p+1} = \mathrm{D}_{2^e}$.*

3 Existence of graphs satisfying Theorem 1.1 and Theorem 1.2

In this section, we first construct some examples of graphs satisfying Theorem 1.1.

The following construction produces edge-transitive graphs admitting a group X satisfying part (3) of Theorem 1.1 with $L \cong \mathbb{Z}_2$, and $\mathcal{O} = 1$.

Hypothesis 3.1 Let $p = 2^\ell m + 1$ be a prime, where m is an odd number and ℓ is a positive integer. Let $d > 1$ be an integer.

(i) For $1 \le i \le d$, let $G_i = \langle x_i\rangle{:}\langle \tau_i\rangle \cong \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ be a Frobenius group, and $C_i = \langle c_i\rangle \cong \mathbb{Z}_2$.

(ii) Let $Y = ((G_1 \times C_1) \times \cdots \times (G_d \times C_d)){:}\langle \pi\rangle \cong ((\mathbb{Z}_p{:}\mathbb{Z}_{p-1}) \times \mathbb{Z}_2) \wr \mathbb{Z}_d = ((\mathbb{Z}_p{:}\mathbb{Z}_{p-1}) \times \mathbb{Z}_2)^d{:}\mathbb{Z}_d$, where $(x_i, \tau_i, c_i)^\pi = (x_{i+1}, \tau_{i+1}, c_{i+1})$ (reading the subscripts modulo d).

Lemma 3.2 *Assume Hypothesis 3.1. If d divides m, then $\mathrm{ord}_{2^i md}(p) = d$ for $1 \le i \le \ell$.*

Proof Let $\mathrm{ord}_{2^i md}(p) = k$ for some k. Then, $2^i md \mid p^k - 1$. As d is odd, we deduce that

$$d \,\big|\, (p^{k-1} + p^{k-2} + \cdots + p + 1). \tag{1}$$

As d divides m, d divides $p-1$, and so d divides $p^j - 1$ for $j \ge 1$. We derive from (1) that d divides k. On the other hand, as $2^i md$ divides $p^d - 1$, one has $k \le d$, and so $d = k$. □

By Lemma 3.2, if d divides m, then $\mathrm{ord}_{2md}(p) = d$. Therefore, for convenience, let

$$n = 2md, \text{ where } d \text{ divides } m. \tag{2}$$

Construction 3.3 Assume Hypothesis 3.1 and (2). Let

$$X = W{:}(N{:}\langle h\rangle) \cong \mathbb{Z}_p^d{:}(\mathbb{Z}_2^d : \mathbb{Z}_n)$$

be a subgroup of Y, such that $W \cong \mathbb{Z}_p^d$, $N \cong \mathbb{Z}_2^d$ and $\langle h\rangle \cong \mathbb{Z}_n$ satisfy

$$W = \prod_{i=1}^{d} \langle x_i\rangle,\ N = \prod_{i=1}^{d} \langle \tau_i^{\frac{p-1}{2}}\rangle \quad \text{and} \quad h = c_1 \tau_1^{\frac{p-1}{2m}} \pi.$$

Let $y = x_1 h$. Set

$$\Gamma = \mathsf{Cos}(X, N, N\{y, y^{-1}\}N).$$

Lemma 3.4 *Let Γ be the graph constructed in Construction 3.3, and let $G = W{:}\langle h\rangle$. Then, Γ is a connected tetravalent X-edge-transitive Cayley graph of Frobenius group G, and G is not normal in X.*

Proof By the definition, N is core-free in X, and hence, $X \le \mathsf{Aut}\Gamma$. Now, $X = GN$ and $G \cap N = 1$, and hence, G acts regularly on the vertex set $[X{:}N]$. Thus, Γ is a Cayley graph of G. By Hypothesis 3.1, h does not centralise N, and so G is not normal in X.

Let $H = \langle h\rangle$. It is easily shown that H is faithful on W. We claim that H acts fixed-point-freely on W. Assume otherwise. Let $U = \langle w \,\big|\, w^h = w, w \in W\rangle$. Then, U is a proper subgroup of W. By Maschke's Theorem, W can be decomposed as $W = U \times V$ such that H normalises both U and V. Note that H is fixed-point-free on V. Let $k = \dim_{\mathbb{F}_p} V$. Then, $k < d$. Since $o(h) = n = 2md$, we deduce that $2md$ divides $p^k - 1$, contradicting Lemma 3.2. This establishes the claim. Thus, G is a Frobenius group.

Let $\tau = \tau_1^{(p-1)/2} \cdots \tau_d^{(p-1)/2}$. Note that y is defined in Construction 3.3. Then, as $[\tau, c_1] = [\tau, \pi] = [\tau, \tau_1^{\frac{p-1}{2m}}] = 1$ and $x_1^\tau = x_1^{-1}$, we have

$$y^\tau y^{-1} = (x_1 c_1 \tau_1^{\frac{p-1}{2m}} \pi)^\tau (x_1 c_1 \tau_1^{\frac{p-1}{2m}} \pi)^{-1} = x_1^{-2} \in \langle N, y \rangle.$$

Thus, as $o(x_1) = p$ with p odd, we have $x_1 \in \langle N, y \rangle$. Since W is minimal normal in G, all the x_i belong to $\langle N, y \rangle$, and hence, $\langle N, y \rangle = X$. So Γ is connected. Let $c = c_1 \cdots c_d$. Then, we calculate that $c = h^{dm}\tau$, and so $c \in X$. Therefore, $\mathsf{soc}(X) = W \times \langle c \rangle$.

Finally, let $\sigma_i = \tau_i^{\frac{p-1}{2}}$ where $1 \le i \le d$. Then, calculations show $\sigma_i^y = \sigma_{i+1}$ for $2 \le i \le d-1$, and $\sigma_d^y = \sigma_1$. Since $\sigma_1^{x_1} = x_1^{-2}\sigma_1$, we have

$$\sigma_1^y = \sigma_1^{x_1 c_1 \tau_1^{\frac{p-1}{2m}} \pi} = (x_1^{-2}\sigma_1)^{\tau_1^{\frac{p-1}{2m}} \pi} = ((x_1^{-2})^{\tau_1^{\frac{p-1}{2m}}} \sigma_1)^\pi = (x_2^{-2})^{\tau_2^{\frac{p-1}{2m}}} \sigma_2 \notin N.$$

Thus, $N \cap N^y = \langle \sigma_1, \sigma_3, \ldots, \sigma_d \rangle \cong \mathbb{Z}_2^{d-1}$, and so $|N : N \cap N^y| = 2$. Since $X \le \mathsf{Aut}\Gamma$, Γ is not a cycle. By Lemma 2.1, Γ is connected, X-edge-transitive and of valency 4. □

Remark The normal quotient Γ_W induced by W is a cycle, see Lemmas 4.4 and 4.5.

As a matter of fact, there are many Frobenius groups which satisfy Construction 3.3. For example, $G = \mathbb{Z}_7^3{:}\mathbb{Z}_{18}$, $\mathbb{Z}_{13}^3{:}\mathbb{Z}_{18}$, $\mathbb{Z}_{41}^5{:}\mathbb{Z}_{50}$ and so on.

The following construction produces edge-transitive graphs admitting a group X satisfying part (4) of Theorem 1.1 with $\mathcal{O} = 1$.

Construction 3.5 Assume Hypothesis 3.1 and (2). Let $X = W{:}\langle N, h \rangle$ be a subgroup of Y, such that $W \cong \mathbb{Z}_p^d$, $N \cong \mathbb{Z}_2^{d-1}$ and $\langle h \rangle \cong \mathbb{Z}_n$ satisfy

$$W = \prod_{i=1}^{d} \langle x_i \rangle, N = \prod_{i \ne 1} \langle \tau_i^{\frac{p-1}{2}} \rangle \text{ and } h = \tau_1^{\frac{p-1}{2m}} \pi.$$

Let $y = x_1 h$. Set

$$\Gamma = \mathsf{Cos}(X, N, N\{y, y^{-1}\}N).$$

Lemma 3.6 *Let Γ be the graph constructed in Construction 3.5, and $G = W{:}\langle h \rangle$. Then, Γ is a connected tetravalent X-edge-transitive Cayley graph of Frobenius group G, and G is not normal in X.*

Proof Arguing similarly as in Lemma 3.4, G is a Frobenius group, and Γ is an X-edge-transitive Cayley graph of G. By the definition, G is not normal in X.

Let $\sigma_i = \tau_i^{\frac{p-1}{2}}$ where $1 \le i \le d$. Then, calculations show $\sigma_i^y = \sigma_{i+1}$ for $2 \le i \le d-1$, and $\sigma_d^y = \sigma_1$. It follows that $N \cap N^y = \langle \sigma_3, \sigma_4, \ldots, \sigma_d \rangle \cong \mathbb{Z}_2^{d-2}$, and so

$|N : N \cap N^y| = 2$. Thus, Γ is of valency 4. Now, $y^{-1} = \pi^{-1}\tau_1^{-\frac{p-1}{2m}} x_1^{-1}$, $\sigma_2^{\pi^{-1}} = \sigma_1$ and $\sigma_1^{x_1^{-1}} = x_1^2\sigma_1$, and we have

$$\sigma_2^{y^{-1}} = \sigma_2^{\pi^{-1}\tau_1^{-\frac{p-1}{2m}} x_1^{-1}} = \sigma_1^{x_1^{-1}} = x_1^2\sigma_1 \in \langle N, y\rangle.$$

Since $\sigma_d^y = \sigma_1 \in \langle N, y\rangle$, we obtain $x_1 \in \langle N, y\rangle$. Since W is minimal normal in G, all the x_i belong to $\langle N, y\rangle$, and so $\langle N, y\rangle = X$. Consequently, Γ is connected. Thus, the statement follows. $\square$

Remark By the definition, h does not normalise N, and thus, X cannot satisfy the properties in part (ii) of Lemma 4.4. However, h normalises $\langle N, h^{\frac{n}{2}}\rangle$; namely, X satisfies the properties in part (ii) of Lemma 4.5. Thus, $\Gamma_W \cong \mathbf{C}_{\frac{n}{2}[2]}$, where N, W and Γ appear in Construction 3.5.

We now construct an example of graph satisfying part (1) of Theorem 1.2.

Although arc-transitive elementary abelian covers of the complete graph $\mathbf{K}_5$ were classified by Kuzman [19], we present here a distinct and independent construction by using the techniques of groups, and building upon coset graphs.

Let p be a prime, such that 5 is a primitive divisor of $p^4 - 1$. Set

$$V = \langle a_1\rangle \times \cdots \times \langle a_5\rangle \cong \mathbb{Z}_p^5.$$

Let $N = \mathsf{Alt}(\{1, \ldots, 5\})$. Then, for $n \in N$, n acts on V as follows:

$$(a_1^{\lambda_1} \cdots a_5^{\lambda_5})^{n^{-1}} = a_{1^n}^{\lambda_1} \cdots a_{5^n}^{\lambda_5}, \quad \text{where } 1 \le \lambda_i \le p.$$

Let $\overline{a}_i = a_5 a_i^{-1}$ for $1 \le i \le 4$. Set

$$W = \langle \overline{a}_1\rangle \times \langle \overline{a}_2\rangle \times \langle \overline{a}_3\rangle \times \langle \overline{a}_4\rangle.$$

Then, N is faithful on W, and so N can be embedded into $\mathrm{GL}(W)$.

Construction 3.7 Let $X = W{:}N \cong \mathbb{Z}_p^4{:}\mathrm{A}_5$, and $G = W{:}\langle h\rangle$ with $h = (12345)$. Let $R = \mathsf{Alt}(\{2, 3, 4, 5\}) \cong \mathrm{A}_4$, and let $g = \overline{a}_1(15)(24)$. Set

$$\Gamma = \mathsf{Cos}(X, R, RgR).$$

Lemma 3.8 *Let Γ be the graph constructed in Construction 3.7. Then, Γ is a connected tetravalent $(X, 2)$-arc-transitive Cayley graph of Frobenius group G, and G is not normal in X. In particular, Γ is a normal cover of Γ_W and $\Gamma_W \cong \mathbf{K}_5$.*

Proof Let $H = \langle h\rangle$. By the definition, H is fixed-point-free on W, and so G is a Frobenius group. Noting that N has a decomposition HR, it implies that R is core-free in X, and hence, $X \le \mathsf{Aut}\Gamma$. Now, $X = GR$ and $G \cap R = 1$, and so G is regular on the vertex set $[X{:}R]$. Thus, Γ is a Cayley graph of G. Obviously, G is not normal in X.

Denote by u and v the vertices R and Rg, respectively. Then, $X_u = R$ and $X_v = R^g$. Since $X_{uv} = X_u \cap X_v$, we have $X_{uv} = \langle (234) \rangle$, and so $|R : R \cap R^g| = 4$. By Lemma 2.1, Γ is of valency 4. Let $x = (25)(34)$. Then, $x \in R$. Noting that $g = \overline{a}_1 h x$, we deduce that $\overline{a}_1 h \in \langle R, g \rangle$. Now, $(\overline{a}_1 h)^x = (a_5 a_1^{-1} h)^x = a_{5^x} a_{1^x}^{-1} h^x = a_2 a_1^{-1} h^{-1}$, so

$$\begin{aligned}\overline{a}_1 h (\overline{a}_1 h)^x &= (a_5 a_1^{-1} h)(a_2 a_1^{-1} h^{-1}) = a_5 a_1^{-1} a_{2^h} a_{1^h}^{-1} = a_5 a_1^{-1} a_3 a_2^{-1} \\ &= \overline{a}_1 \overline{a}_2 \overline{a}_3^{-1} \in \langle R, g \rangle.\end{aligned}$$

Let $y = (23)(45)$. Then, $y \in \langle R, g \rangle$. Similarly, we calculate that

$$(\overline{a}_1 \overline{a}_2 \overline{a}_3^{-1})^y = \overline{a}_1 \overline{a}_2^{-1} \overline{a}_3 \overline{a}_4^{-1} \quad \text{and} \quad (\overline{a}_1 \overline{a}_2^{-1} \overline{a}_3 \overline{a}_4^{-1})^{xy} = \overline{a}_1 \overline{a}_2^{-1} \overline{a}_4^{-1}.$$

The two equations above yield $\overline{a}_3 \in \langle R, g \rangle$, and so $\overline{a}_i \in \langle R, g \rangle$ for $i = 1, 2, 4$. Thus, $W \leq \langle R, g \rangle$. Since $\overline{a}_1 h \in \langle R, g \rangle$, we have $h \in \langle R, g \rangle$, forcing $X = \langle R, g \rangle$. Thus, Γ is connected. Notice that X/W is insoluble. By Lemma 2.4, Γ is a normal cover of Γ_W, and thus, $\Gamma_W \cong \mathbf{K}_5$ by [2, Theorem 1.2]. □

We end this section by presenting an example satisfying part (2) of Theorem 1.2. Let p be a prime for which $p^2 \equiv -1 \pmod{10}$. Set

$$V = \langle a_1, \ldots, a_5, a_{1'}, \ldots, a_{5'} \rangle \cong \mathbb{Z}_p^{10}.$$

Let $S = \mathrm{Sym}(\{1, 1', \ldots, 5, 5'\}) \cong \mathrm{S}_{10}$. Then, for $t \in S$, t acts on V as follows:

$$(a_1^{\lambda_1} \cdots a_5^{\lambda_5} a_{1'}^{\lambda_{1'}} \cdots a_{5'}^{\lambda_{5'}})^{t^{-1}} = a_{1^t}^{\lambda_1} \cdots a_{5^t}^{\lambda_5} a_{(1')^t}^{\lambda_{1'}} \cdots a_{(5')^t}^{\lambda_{5'}} \quad \text{where } 1 \leq \lambda_i, \lambda_{i'} \leq p.$$

Let $T = \langle (12345)(1'2'3'4'5'), (12)(1'2') \rangle$ and $g = (11') \cdots (55')$. Then, $T \leq S$ and $g \in S$. Since g centralises T, we may set

$$N = T \times \langle g \rangle \cong \mathrm{S}_5 \times \mathbb{Z}_2.$$

Let $u_i = a_i a_{i'}^{-1}$ where $1 \leq i \leq 5$. Set $U = \langle u_1, \ldots, u_5 \rangle$. It is straightforward to verify that N is faithful on U, and so N can be embedded into $\mathrm{GL}(U)$. Let $a = a_1 \cdots a_5$ and $a' = a_{1'} \cdots a_{5'}$. Then, $\langle a(a')^{-1} \rangle \leq U$. Let $w_i = u_i \langle a(a')^{-1} \rangle$ where $1 \leq i \leq 5$. Set

$$W = U / \langle a(a')^{-1} \rangle = \langle w_1, \ldots, w_4 \rangle.$$

Then, $W \cong \mathbb{Z}_p^4$. Now, N normalises $\langle a(a')^{-1} \rangle$, and so N induces a faithful action on W. Therefore, N can be embedded into $\mathrm{GL}(W)$.

Construction 3.9 Let $X = W{:}N \cong \mathbb{Z}_p^4{:}(\mathrm{S}_5 \times \mathbb{Z}_2)$. Let $R = \langle (1234)(1'2'3'4'), (12)(1'2') \rangle \cong \mathrm{S}_4$ and $y = w_1 w_5 (15)(1'5')g$. Set

$$\Gamma = \mathsf{Cos}(X, R, RyR).$$

Let $G = W{:}H$ where $H = \langle h, g\rangle$ with $h = (12345)(1'2'3'4'5')$.

Arguing similarly as in Lemma 3.8, we have the following conclusion.

Lemma 3.10 *Let Γ be the graph constructed in Construction 3.9. Then, Γ is a connected tetravalent $(X, 2)$-arc-transitive Cayley graph of Frobenius group G, and G is not normal in X. In particular, Γ is a normal cover of Γ_W and $\Gamma_W \cong \mathbf{K}_{5,5} - 5\mathbf{K}_2$.*

4 Soluble automorphism groups

In this section, let $G = W{:}H = \mathbb{Z}_p^d{:}\mathbb{Z}_n$ be a Frobenius group, where $\mathrm{ord}_n(p) = d$ for a prime p and a positive integer n. Let $\Gamma = \mathsf{Cay}(G, S)$ be a connected tetravalent X-edge-transitive Cayley graph, where $G \le X \le \mathsf{Aut}\Gamma$. We first handle the case where X is soluble. Let $G \cong \mathrm{D}_{2p}$ with p an odd prime. By virtue of Li et al. [21, Theorem 1.1], we have the following conclusions.

Lemma 4.1 *Let Γ be a connected edge-transitive tetravalent Cayley graph of G, where $G \cong \mathrm{D}_{2p}$ with p an odd prime. Then, either*

(i) *Γ is arc-regular and $\mathsf{Aut}\Gamma \cong \mathrm{D}_{2p}{:}\mathbb{Z}_4$, or*
(ii) *$\Gamma \cong \mathbf{C}_{p[2]}$, and $\mathsf{Aut}\Gamma \cong \mathbb{Z}_2^p{:}\mathrm{D}_{2p}$.*

In the remainder of this section, assume that $G \ncong \mathrm{D}_{2p}$, unless specified otherwise.

Let F be the *Fitting subgroup* of X. If X is soluble, then $\mathbf{C}_X(F) \le F$ and $F \ne 1$, see [14]. For a prime q, by F_q we mean a Sylow q-subgroup of F.

Lemma 4.2 *Use the notation defined above, then $G \cap F = W$.*

Proof By the definition, $F \lhd X$ and $\mathbf{C}_X(F) \le F$; namely, F is self-centralising.

We claim that $W \le F$. Suppose for a contradiction that $W \nleq F$. Then, $G \cap F = 1$. Since $X = GX_\mathbf{1}$, it follows that $|F|$ divides $|X_\mathbf{1}|$. By Lemma 2.3, we deduce that F is either a q-group with $q = 2$ or 3, or a $\{2, 3\}$-group. For convenience, let $\overline{X} = X/F$, $\overline{G} = GF/F = \overline{W}{:}\overline{H}$, and $F_{\overline{X}}$ the Fitting subgroup of $\overline{X}$.

Case 1: F is a q-group. Assume that Γ_F is a cycle. By [30, Theorem 4.1], Γ is not $(X, 2)$-arc-transitive. By Lemma 2.3, $X_\mathbf{1}$ is a 2-group. Noting that $|F|$ divides $|X_\mathbf{1}|$, F is a 2-group. Since $K = FK_\mathbf{1}$, K is also a 2-group, and hence, $K = F$. Noting that W induces the identity on $V\Gamma_F$, we have $W \le F$, which is a contradiction.

Therefore, Γ is a normal cover of Γ_F. Then, $|F|$ divides $|G|$, and so $|F|$ divides p^d or n. Suppose that $|F|$ divides n. Let $\overline{F} = F/\Phi(F)$. Then, $\overline{F}$ is an elementary abelian group of order q^ℓ, where ℓ is a positive integer. By Gorenstein [14, p.174, Theorem 1.4], W induces a faithful action on $\overline{F}$, and so H can be embedded into $\mathsf{Aut}(\overline{F})$. Thus, $\mathsf{Aut}(\overline{F})$ contains an element of order q^ℓ. By Lemma 2.6, we deduce that $q^{\ell-1} < \ell$. This is not possible.

Thus, $|F|$ divides p^d. Then, $F_{\overline{X}}$ is a p'-group, and so $F_{\overline{X}} \cap \overline{G} = 1$. Therefore, $|F_{\overline{X}}|$ divides $|X_\mathbf{1}|$. It implies that $|X_\mathbf{1}|$ is divisible by two distinct primes, and $F_{\overline{X}}$ is a r-group, where $r \ne p$ is a prime. Via Lemma 2.3, Γ is $(X, 2)$-arc-transitive. Let $\Sigma = \Gamma_F$. Then, Σ is $(\overline{X}, 2)$-arc-transitive. Since $\overline{G}$ is transitive and faithful on $V\Sigma$, we deduce that $|F| < p^d$. Note that $\overline{W}$ is the unique minimal normal subgroup of $\overline{G}$. By Praeger [30, Theorem 4.1], we derive that Σ is a normal cover of $\Sigma_{F_{\overline{X}}}$, and so $|F_{\overline{X}}|$ divides $|\overline{H}|$. Let $\overline{F}_{\overline{X}} = F_{\overline{X}}/\Phi(F_{\overline{X}})$. Arguing as in the above paragraph, $\mathsf{Aut}(\overline{F}_{\overline{X}})$ contains an element of order $|\overline{F}_{\overline{X}}|$, which is impossible.

Case 2: F is a $\{2, 3\}$-group. Then, Γ is $(X, 2)$-arc-transitive, and so $|F|$ divides $2^4 \cdot 3^6$. By Praeger [30, Theorem 4.1], Γ is a normal cover of Γ_F, or $\Gamma_F = \mathbf{K_2}$, or F is transitive on $V\Gamma$.

Assume that Γ is a normal cover of Γ_F. Pick $u \in V\Gamma_F$. Since G is regular on $V\Gamma$, we deduce that $|\overline{G}_u| = |F|$, and hence, Γ_F is $(\overline{G}, 2)$-arc-transitive. Note that $\overline{G}_u$ is a Frobenius group. By Lemma 2.3, we conclude $\overline{G}_u \cong \mathrm{A}_4$, and so $|F| = 12$. Thus, $\mathsf{Aut}(F) \le \mathrm{S}_3 \times \mathbb{Z}_2$, and so W centralises F, contradicting the fact that F is self-centralising.

Assume that $\Gamma_F = \mathbf{K_2}$. Then, $|G|$ divides $2^5 \cdot 3^6$, and hence, $p = 3$. Since G is a Frobenius group, we deduce that $G = \mathbb{Z}_3^2{:}\mathbb{Z}_4$, $\mathbb{Z}_3^2{:}\mathbb{Z}_8$ or $\mathbb{Z}_3^4{:}\mathbb{Z}_{16}$. Noting that F is a $\{2, 3\}$-group, we have $F_2 \ne 1$ and $F_3 \ne 1$. For this case, Γ is a normal cover of Γ_{F_2} or Γ_{F_3}, and so $|F|$ divides $2^2 \cdot 3^2$, $2^3 \cdot 3^2$ or $2^4 \cdot 3^4$ according to whether $G = \mathbb{Z}_3^2{:}\mathbb{Z}_4$, $\mathbb{Z}_3^2{:}\mathbb{Z}_8$ or $\mathbb{Z}_3^4{:}\mathbb{Z}_{16}$. However, one can quickly verifies by Magma [3] that W centralises F, which is a contradiction. In a similar fashion, we exclude the case that F is transitive on $V\Gamma$.

Consequently, $W \le F$, and so $G \cap F = W$, completing the proof. □

Lemma 4.3 *With the notation before Lemma 4.1, the following statements hold:*

(i) *if p is an odd prime, then $W \trianglelefteq X$;*
(ii) *if $p = 2$, then $F = \mathrm{O}_2(X)$, and either*

  (a) *$W < F$, Γ_F is a cycle, and $X = (F{:}H).\mathcal{O}$, where $\mathcal{O} = 1$ or $\mathbb{Z}_2$, or*
  (b) *$W = F$ and further, W is characteristic in X.*

Proof Let p be an odd prime. By Lemma 4.2, $W \le F$, and so $W \le F_p$. Let $p > 3$. Then, as $|F_pG : G| = |F_p : W|$ divides $|X_\mathbf{1}|$, we conclude that $W = F_p$, and so $W \trianglelefteq X$. Let $p = 3$. If $W < F_3$, then Γ is $(X, 2)$-arc-transitive, and so $\Gamma_{F_3} = \mathbf{K_2}$. Thus, $|G|$ divides $2|F_3|$, and so $|H| = 2$. Since W is minimal normal in G, we deduce that $G \cong \mathrm{D}_6$, contradicting our assumption. Thus, $W = F_3$. So part (i) holds.

Let $p = 2$. By Lemma 4.2, $W \le F_2$, and so either $W = F_2 \trianglelefteq X$ or $W < F_2$. Assume that the latter case occurs. If $F_3 \ne 1$, then 3 divides $|X_\mathbf{1}|$. By Lemma 2.3, Γ is $(X, 2)$-arc-transitive. By Praeger [30, Theorem 4.1], Γ is a normal cover of Γ_{F_2}, and so $F_2 = W$, which is a contradiction. Thus, F is a 2-group, and Γ_F is a cycle. It is easily shown that F is the kernel of X acting on $V\Gamma_F$. Consequently, $X = (F{:}H).\mathcal{O}$ where $\mathcal{O} \cong 1$ or $\mathbb{Z}_2$. Thus, either Γ_F is a cycle or $W \trianglelefteq X$, as in part (ii). □

We now assume that $W \trianglelefteq X$. By virtue of Lemma 2.4, either the normal quotient Γ_W is a cycle, or Γ is a normal cover of Γ_W. We first handle the case where Γ_W is a cycle.

Lemma 4.4 *Let K be the kernel of X acting on $V\Gamma_W$. Then, the following hold.*

(i) *$X = ((WK_\mathbf{1}){:}H).\mathcal{O}$, $X_\mathbf{1} = K_\mathbf{1}.\mathcal{O}$ and $W \cong \mathbb{Z}_2^d$, where $\mathcal{O} \cong 1$ or $\mathbb{Z}_2$;*

(ii) *Assume p is an odd prime. Then, either*

(1) *G is normal in X, or*

(2) *G is not normal in X, and*

(a) *$X = W{:}((K_{\mathbf{1}}{:}H).\mathcal{O})$, and H does not centralise $K_{\mathbf{1}}$ where $K_{\mathbf{1}} \cong \mathbb{Z}_2^{\ell}$ with $2 \le \ell \le d$, and $\mathcal{O} \cong 1$ or $\mathbb{Z}_2$;*

(b) *there exist $x_1, \ldots, x_d \in W$ and $\tau_1, \ldots, \tau_d \in K_{\mathbf{1}}$ such that $W = \langle x_1, \ldots, x_d \rangle$, $\langle x_i, \tau_i \rangle \cong \mathrm{D}_{2p}$, and $K_{\mathbf{1}} = \langle \tau_i \rangle \times \mathbf{C}_{K_{\mathbf{1}}}(x_i)$ for $1 \le i \le d$;*

(c) *$\mathsf{soc}(X) = W \times L$, where $L \cong 1$ or $\mathbb{Z}_2$;*

(d) *H is imprimitive on W.*

Proof Let B be a vertex of Γ_W. Then, W acts regularly on B. Thus, $K = WK_{\mathbf{1}}$ and $K \cap H = 1$, where $K_{\mathbf{1}}$ is a 2-group. For this case, Γ_W is a connected Cayley graph of G/W. Since H is of order n, Γ_W is a cycle of size n. It follows that $X/K \cong \mathbb{Z}_n$ or D_{2n}. Further, Γ is X-arc-transitive if and only if $X/K \cong \mathrm{D}_{2n}$.

Assume first that $p = 2$. Since $(|K|, |H|) = 1$, we conclude that $K{:}H \le X$. Noting that X/K is isomorphic to a subgroup of D_{2n}, it follows that $X = (K{:}H).\mathcal{O}$ and $X_{\mathbf{1}} = K_{\mathbf{1}}.\mathcal{O}$ where $\mathcal{O} \cong 1$ or $\mathbb{Z}_2$, as in part (i).

Assume now that p is an odd prime. Furthermore, we assume that G is not normal in X. If $K_{\mathbf{1}} = 1$, then $K = W$, and hence, $G \lhd X$, which is a contradiction. Thus, $K_{\mathbf{1}} \ne 1$.

Let $U = \mathbf{N}_X(K_{\mathbf{1}})$. Since $K_{\mathbf{1}} \ntrianglelefteq X$, it implies that $U \ne X$. Noting that $(|W|, |K_{\mathbf{1}}|) = 1$, we obtain that $\mathbf{N}_{X/W}(K/W) = \mathbf{N}_{X/W}(WK_{\mathbf{1}}/W) = \mathbf{N}_X(K_{\mathbf{1}})W/W = UW/W$. It follows from $K/W \trianglelefteq X/W$ that $X = WU$. Since $W \lhd X$, $W \cap U \lhd U$. Furthermore, $W \cap U \lhd W$ since W is abelian. Thus, $W \cap U \lhd \langle U, W \rangle = UW = X$. If $W \le U$, then $K = WK_{\mathbf{1}} = W \times K_{\mathbf{1}}$, and hence, $K_{\mathbf{1}} \lhd X$, which is a contradiction. Thus, $W \cap U < W$. If $W \cap U \ne 1$, then $\mathbf{C}_W(K_{\mathbf{1}}) \ne 1$, and so $\mathbf{C}_W(K) \ne 1$. Since W is minimal in G, we deduce that $\mathbf{C}_W(K) = W$, and so $K_{\mathbf{1}} \lhd X$, again a contradiction. Thus, $W \cap U = 1$, and so $K \cap U = WK_{\mathbf{1}} \cap U = (W \cap U)K_{\mathbf{1}} = K_{\mathbf{1}}$. Now, $X/K = UW/K = UK/K \cong U/(K \cap U) = U/K_{\mathbf{1}}$, and hence, $U = (K_{\mathbf{1}}.\hat{H}).\mathcal{O}$, where $\hat{H} \cong \mathbb{Z}_n$ and $\mathcal{O} \cong 1$ or $\mathbb{Z}_2$. Since $X = WU$ and $(|U|, |W|) = 1$, we conclude that $H^g \le U$ for some $g \in W$. For convenience, we may assume that $H \le U$, and so $U = (K_{\mathbf{1}}{:}H).\mathcal{O}$. Thus, $X_{\mathbf{1}} = K_{\mathbf{1}}.\mathcal{O}$. Furthermore, since G is not normal in X, it follows that H does not centralise $K_{\mathbf{1}}$.

Set $Y = W{:}(K_{\mathbf{1}}{:}H)$. Then, Y has index at most 2 in X, and Γ is Y-edge-transitive. Clearly, Γ is not Y-arc-transitive. Hence, $\Gamma = \mathsf{Cos}(Y, K_{\mathbf{1}}, K_{\mathbf{1}}\{y, y^{-1}\}K_{\mathbf{1}})$, where $y \in Y$ is such that $\langle K_{\mathbf{1}}, y \rangle = Y$ and $K_{\mathbf{1}} \cap K_{\mathbf{1}}^y$ has index 2 in $K_{\mathbf{1}}$. We may choose $y \in G$ such that $y = hx$ where $x \in W$ and $\langle h \rangle = H$. Then, $K_{\mathbf{1}} \cap K_{\mathbf{1}}^y = K_{\mathbf{1}} \cap K_{\mathbf{1}}^x$.

We claim that $K_{\mathbf{1}} \cap K_{\mathbf{1}}^x = \mathbf{C}_{K_{\mathbf{1}}}(x)$. For any $\sigma \in K_{\mathbf{1}} \cap K_{\mathbf{1}}^x$, we have $\sigma^{x^{-1}} \in K_{\mathbf{1}}$, and so $\sigma^{-1}\sigma^{x^{-1}} \in K_{\mathbf{1}}$. Since $x \in W$ and $W \lhd WK_{\mathbf{1}}$, we obtain that $\sigma^{-1}\sigma^{x^{-1}} = (\sigma^{-1}x\sigma)x^{-1} \in W$. So $\sigma^{-1}\sigma^{x^{-1}} \in W \cap K_{\mathbf{1}} = 1$, and $\sigma^{x^{-1}} = \sigma$. Thus, σ centralises x. It follows that $K_{\mathbf{1}} \cap K_{\mathbf{1}}^x \le \mathbf{C}_{K_{\mathbf{1}}}(x)$. Clearly, $\mathbf{C}_{K_{\mathbf{1}}}(x) \le K_{\mathbf{1}} \cap K_{\mathbf{1}}^x$. So $\mathbf{C}_{K_{\mathbf{1}}}(x) = K_{\mathbf{1}} \cap K_{\mathbf{1}}^x$, as required.

Noting that W is a minimal normal subgroup of X and $X = WU$, we obtain that $W = \langle x^{\sigma_1} \rangle \times \langle x^{\sigma_2} \rangle \times \cdots \times \langle x^{\sigma_d} \rangle$ where $\sigma_1 = 1, \sigma_2, \ldots, \sigma_d \in U$. Then, $\mathbf{C}_{K_{\mathbf{1}}}(x^{\sigma_i}) = (\mathbf{C}_{K_{\mathbf{1}}}(x))^{\sigma_i} < K_{\mathbf{1}}^{\sigma_i} = K_{\mathbf{1}}$. The intersection $\cap_{i=1}^{d} \mathbf{C}_{K_{\mathbf{1}}}(x^{\sigma_i}) \le \mathbf{C}_K(W) = W$, and hence, $\cap_{i=1}^{d}\mathbf{C}_{K_\mathbf{1}}(x^{\sigma_i})=1$. Since $\mathbf{C}_{K_\mathbf{1}}(x^{\sigma_i})$ is a maximal subgroup of $K_\mathbf{1}$, the Frattini subgroup $\Phi(K_\mathbf{1})\le\cap_{i=1}^{d}\mathbf{C}_{K_\mathbf{1}}(x^{\sigma_i})=1$. Thus, $K_\mathbf{1}$ is an elementary abelian 2-group, that is, $K_\mathbf{1}\cong\mathbb{Z}_2^{\ell}$ for some $\ell\ge 1$. Since $\cap_{i=1}^{d}\mathbf{C}_{K_\mathbf{1}}(x^{\sigma_i})=1$, it follows that $\ell\le d$. Assume $\ell=1$. Then, $K_\mathbf{1}\cong\mathbb{Z}_2$ and so $K_\mathbf{1}\le\mathbf{C}_X(H)$. Thus, $G\lhd X$, which contradicts the fact that G is not normal in X. Therefore, $\ell>1$, as in part (a).

Since $\mathbf{C}_{K_\mathbf{1}}(x)$ has index 2 in $K_\mathbf{1}$, there exists some τ_1 belonging to $K_\mathbf{1}$ such that $K_\mathbf{1}=\langle\tau_1\rangle\times\mathbf{C}_{K_\mathbf{1}}(x)$. Set $x_1=x^{-1}x^{\tau_1}$. Then, $x_1\ne 1$, $x_1^{\tau_1}=x_1^{-1}$ and $\mathbf{C}_{K_\mathbf{1}}(x)=\mathbf{C}_{K_\mathbf{1}}(x_1)$, and hence, $K_\mathbf{1}=\langle\tau_1\rangle\times\mathbf{C}_{K_\mathbf{1}}(x_1)$. Noticing that W is a minimal normal subgroup of $X=WU$, there exist $\mu_1=1,\mu_2,\ldots,\mu_d\in U$ such that $W=\langle x_1^{\mu_1}\rangle\times\langle x_1^{\mu_2}\rangle\times\cdots\times\langle x_1^{\mu_d}\rangle$. Let $x_i=x_1^{\mu_i}$ and $\tau_i=\tau_1^{\mu_i}$, where $i=1,\ldots,d$. Then, $\mathbb{Z}_2^{\ell-1}\cong(\mathbf{C}_{K_\mathbf{1}}(x_1))^{\mu_i}=\mathbf{C}_{K_\mathbf{1}^{\mu_i}}(x_1^{\mu_i})=\mathbf{C}_{K_\mathbf{1}}(x_i)$, and $K_\mathbf{1}=K_\mathbf{1}^{\mu_i}=\langle\tau_i\rangle\times\mathbf{C}_{K_\mathbf{1}}(x_i)$. Furthermore, $x_i^{\tau_i}=x_1^{\tau_1\mu_i}=(x_1^{-1})^{\mu_i}=x_i^{-1}$, and thereby, $\langle x_i,\tau_i\rangle\cong\mathrm{D}_{2p}$, as in part (b).

Recall that $W\cong\mathbb{Z}_p^d$ for an odd prime p. Since G is not normal in X, we conclude that $d>1$. Assume that X has a minimal normal subgroup $L\ne W$. Then, $W\cap L=1$, and so $LK/K\lhd X/K\le\mathrm{D}_{2n}$. It follows that either $L\le K$ or $L\cap K=1$. If $L\le K$, then L is a 2-group. Since $K_\mathbf{1}$ is a Sylow 2-subgroup of K, we conclude that $L\unlhd K_\mathbf{1}$, and thereby, $L=1$, which is impossible. Thus, $L\cap K=1$, and so $|L|$ divides $|X_\mathbf{1}|$. Consequently, $L\cong\mathbb{Z}_2$. So $\mathsf{soc}(X)=W\times L$, as in part (c).

By the previous paragraph, we obtain $\mathbf{C}_X(W)=W\times L$ where $L\cong 1$ or $\mathbb{Z}_2$. Let $\overline{X}=X/L$, $\overline{G}=GL/L$ and $\overline{K_\mathbf{1}}=K_\mathbf{1}L/L$. Then, $\overline{G}\cong G$ and $\overline{K_\mathbf{1}}\cong K_\mathbf{1}$. Write $\overline{G}=\overline{W}{:}\overline{H}$. Then, $\overline{H}$ normalises $\overline{K_\mathbf{1}}$, and $\overline{H}\,\overline{K_\mathbf{1}}$ is faithful and irreducible on $\overline{W}$. It is well known that each irreducible representation of $\overline{K_\mathbf{1}}$ over $\mathbb{F}_p$ is of dimension 1. Via Clifford's Theorem, $\overline{W}$ can be decomposed as

$$\overline{W}=W_1\times\cdots\times W_t \text{ with } W_i=U_i^e(1\le i\le t),$$

such that $\overline{K_\mathbf{1}}$ normalises each U_i, and all the U_i are pairwise non-isomorphic irreducible with respect to the action of $\overline{K_\mathbf{1}}$. Assume that $t=1$. Since $\overline{K_\mathbf{1}}$ is faithful on $\overline{W}$, we deduce that $\overline{K_\mathbf{1}}$ is faithful and irreducible on U_1. By Gorenstein [14, Theorem 2.3, p.65], $\overline{K_\mathbf{1}}$ is cyclic, which is a contradiction. Thus, $t\ge 2$. Now, $\overline{H}$ normalises $\overline{K_\mathbf{1}}$, and we conclude that $\overline{H}$ preserves such decomposition. Consequently, $\overline{H}$ is imprimitive on $\overline{W}$, and so H is imprimitive on W, as in part (d).

This completes the proof of Lemma 4.4. □

We now handle the case where Γ is a normal cover of Γ_W.

Lemma 4.5 *Assume that Γ is a normal cover of Γ_W. Then, either*

(i) *G is normal in X, or*
(ii) *G is not normal in X, and*

   (a) *$\Gamma_W\cong\mathbf{C}_{\frac{n}{2}[2]}$;*
   (b) *$X=W{:}((NH).\mathcal{O})$, $X_\mathbf{1}\le N.\mathcal{O}$, $N\cap H\cong\mathbb{Z}_2$, and H normalises N, but H does not centralise N, where $N\cong\mathbb{Z}_2^{\ell}$ with $2\le\ell\le n/2$, and $\mathcal{O}\cong 1$ or $\mathbb{Z}_2$;*
   (c) *W is unique minimal normal in X, and H is imprimitive on W;*
   (d) *$X/(WN)\cong\mathbb{Z}_{\frac{n}{2}}$ or D_n, and Γ is X-arc-transitive if and only if $X/(WN)\cong\mathrm{D}_n$.*

Proof Let $\overline{H} = G/W$ and $\overline{X} = X/W$. We first note that Γ_W is a Cayley graph of $\overline{H}$. By Baik et al. [2, Theorem 1.2], we deduce that $\overline{H} \trianglelefteq \mathsf{Aut}\Gamma_W$, or $\Gamma_W = \mathbf{K_5}$ with $\overline{H} = \mathbb{Z}_5$, or $\Gamma_W = \mathbf{K}_{5,5} - 5\mathbf{K}_2$ with $\overline{H} = \mathbb{Z}_{10}$, or $\Gamma_W = \mathbf{C}_{\frac{n}{2}[2]}$.

Let $\overline{H} \trianglelefteq \mathsf{Aut}\Gamma_W$. Then, $\overline{H} \trianglelefteq \overline{X}$, and so $G \trianglelefteq X$. Let $\Gamma_W = \mathbf{K_5}$ with $\overline{H} = \mathbb{Z}_5$. Then, $\mathsf{Aut}\Gamma_W = \mathrm{S}_5$. Since X is soluble, $\overline{X}$ is also soluble. Also, since $\overline{H} \leq \overline{X}$, we deduce that $\overline{X} \leq \mathbb{Z}_5{:}\mathbb{Z}_4$, and so $\overline{H} \trianglelefteq \overline{X}$. Thus, $G \trianglelefteq X$. Let $\Gamma_W = \mathbf{K}_{5,5} - 5\mathbf{K}_2$ with $\overline{H} = \mathbb{Z}_{10}$. Then, $\mathsf{Aut}\Gamma_W = \mathrm{S}_5 \times \mathbb{Z}_2$. As above, we obtain that $\overline{H} \trianglelefteq \overline{X}$, and so $G \trianglelefteq X$.

Let $\Gamma_W = \mathbf{C}_{\frac{n}{2}[2]}$. Write $n = 2m$. Then, $\mathsf{Aut}\Gamma_W \cong \mathbb{Z}_2^m{:}\mathrm{D}_n$. Let $\overline{K} \cong \mathbb{Z}_2^m$ be such that $\overline{K} \trianglelefteq \mathsf{Aut}\Gamma_W$. Then, $\mathsf{Aut}\Gamma_W = (\overline{K}\,\overline{H}){:}\overline{\mathrm{O}}$, where $\overline{\mathrm{O}} \cong \mathbb{Z}_2$. Let u be a vertex of Γ_W for which $\mathbf{1} \in u$. Choose $\overline{M} \leq \overline{K}$ such that $|\overline{M}| = 2^{m-1}$ and $(\mathsf{Aut}\Gamma_W)_u = \overline{M}{:}\overline{\mathrm{O}}$.

Noting that $\overline{X}\,\overline{K}/\overline{K} \cong \overline{H}\,\overline{\mathcal{O}}/((\overline{H}\,\overline{\mathcal{O}}) \cap \overline{K})$ where $\overline{\mathcal{O}} = 1$ or $\overline{\mathrm{O}}$, we conclude that $\overline{X} = (\overline{X} \cap \overline{K})\overline{H}\,\overline{\mathcal{O}}$, and Γ is X-arc-transitive if and only if $\overline{\mathcal{O}} = \overline{\mathrm{O}}$. Let $\widehat{K} = \overline{X} \cap \overline{K}$. Then, $\widehat{K} \trianglelefteq \overline{X}$ and $\widehat{K} \cap \overline{H} \cong \mathbb{Z}_2$. Thus, $X = W.((\widehat{K}\overline{H}).\mathcal{O})$, where $\mathcal{O} \cong \overline{\mathcal{O}}$. Let K be the preimage of $\widehat{K}$ under $X \to X/W$. Note that W is of odd order. By Schur–Zassenhaus's Theorem, $K = W{:}N$, where $N \cong \widehat{K}$. It further implies that $N \cong \mathbb{Z}_2^k$, where $1 \leq k \leq m$.

Now, $(|N|, |W|) = 1$, we obtain $X/W = \mathbf{N}_{X/W}(NW/W) = \mathbf{N}_X(N)W/W$, and hence, $X = W\mathbf{N}_X(N)$. Since $H \leq X$, we may assume without loss of generality that H is a subgroup of $\mathbf{N}_X(N)$. Thus, $X = W{:}((NH).\mathcal{O})$. By the previous paragraph, we conclude that $N \cap H \cong \mathbb{Z}_2$. If $k = 1$, then $NH = H$, and so $G \trianglelefteq X$. In what follows, we assume that G is not normal in X. Thus, $k \geq 2$, and so $2 \leq k \leq m$.

Set $Y = W{:}(NH)$. Since $G \leq Y$, we have $Y = GY_{\mathbf{1}}$. Noting that $|Y| = \frac{|W||H||N|}{|H \cap N|} = |G||Y_{\mathbf{1}}|$, we have $|Y_{\mathbf{1}}| = \frac{|N|}{|N \cap H|} = \frac{|N|}{2}$. Let $\overline{Y} = Y/W$. By the above paragraph, we deduce that $|\overline{Y}_u| = \frac{|N|}{2}$, and thus, $Y_{\mathbf{1}}W/W = \overline{Y}_u$. Since $\overline{Y}_u = \overline{Y} \cap \overline{M} \leq \widehat{K}$ and $\widehat{K} = NW/W$, it follows that $Y_{\mathbf{1}}W/W \leq NW/W$. Consequently, $Y_{\mathbf{1}} \leq NW$. Via Sylow's Theorem, we may assume that $Y_{\mathbf{1}}$ is a subgroup of N, and so $Y_{\mathbf{1}}$ has index 2 in N. Thus, $X_{\mathbf{1}} \leq N.\mathcal{O}$, and $X_{\mathbf{1}}$ has index 2 in $N.\mathcal{O}$, as in part (b).

Let $C{:} = \mathbf{C}_{NH}(W)$. Then, C is normal in Y. In what follows, we prove that $C = 1$. Suppose for a contradiction that $C \neq 1$. Without loss of generality, we assume that C is minimal in Y. Since H acts fixed-point-freely on W, we have $C \cap H = 1$. Let $\overline{C}$ be the image of C under $X \to X/W$. Then, $\overline{C}$ is minimal normal in $\overline{Y}$, and hence, $\overline{C}$ is a subgroup of $\widehat{K}$. It implies that $C \cong \mathbb{Z}_2^\ell$ for some ℓ.

Let $\overline{K} = \langle \sigma_1, \ldots, \sigma_m \rangle$. Note that $\overline{H}$ acts by conjugation transitively on $\{\sigma_1, \ldots, \sigma_m\}$. Write $\overline{H} = \langle \overline{h} \rangle$. Then, $\overline{h} = \sigma\pi$, where $\sigma \in \overline{K}$, and π is a m-cycle. Relabeling if necessary, we may take $\pi^{-1} = (12 \cdots m)$. Let $\overline{K}_u = \prod\limits_{i \neq 1} \langle \sigma_i \rangle$. Choose $v, w \in \Gamma_W(u)$ for which $\overline{K}_v = \prod\limits_{i \neq 2} \langle \sigma_i \rangle$ and $\overline{K}_w = \prod\limits_{i \neq m} \langle \sigma_i \rangle$. Pick $x \in \overline{C}$ such that $x \in \overline{K}_u$, but $x \notin \overline{K}_v$. Then,

$$x = \sigma_2 \cdots \sigma_i \cdots \sigma_r, \quad \text{where } 2 < \cdots < i < \cdots < r \leq m.$$

Let $t = m - r$. We then calculate that

$$x^{\overline{h}^t} = \sigma_{2^{\pi^t}} \cdots \sigma_{r^{\pi^t}} = \sigma_{(2+t)} \cdots \sigma_m.$$

It follows that $\langle x, x^{\overline{h}^t}\rangle \le \overline{K}_u \cap \overline{C}$, and hence Γ_W is $\overline{C}{:}\overline{H}$-edge-transitive.

Let $Z = (W \times C){:}H$. By the previous paragraph, Γ is Z-edge-transitive. However, Γ is not Z-arc-transitive. By Lemma 2.1, $\Gamma = \mathsf{Cos}(Z, Z_{\mathbf{1}}, Z_{\mathbf{1}}\{g, g^{-1}\}Z_{\mathbf{1}})$, where $Z = \langle Z_{\mathbf{1}}, g\rangle$ and $|Z_{\mathbf{1}} : (Z_{\mathbf{1}} \cap Z_{\mathbf{1}}^g)| = 2$. By Lemma 2.2 along with Sylow's Theorem, we may assume that $Z_{\mathbf{1}} \le HC$. Write $H = \langle h\rangle$. Since $Z_{\mathbf{1}} \cong \mathbb{Z}_2^{\ell}$, we deduce that $Z_{\mathbf{1}} \le R{:} = \langle C, h^m\rangle$, and hence, $|Z_{\mathbf{1}} \cap C| = 2^{\ell-1}$. Let $L{:} = Z_{\mathbf{1}} \cap C = \langle \tau_1, \ldots, \tau_{\ell-1}\rangle$. Then, $C = \langle \tau_1, \ldots, \tau_{\ell-1}, \tau_\ell\rangle$ for $\tau_\ell \in C\backslash Z_{\mathbf{1}}$, and thereby,

$$Z_{\mathbf{1}} = \langle \tau_1, \ldots, \tau_{\ell-1}, \tau_\ell h^m\rangle.$$

Since C is minimal in Z, there exists $\tau \in L$ such that $\tau^g = x_\ell \tau_\ell$, where $x_\ell \in L$. If $(h^m)^g \in R$, then $Z \le R{:}\langle g\rangle$, which contradicts the fact that G is a Frobenius group. Thus, $(h^m)^g \notin R$, and so $(\tau_\ell h^m)^g \notin R$. Let $T = \langle \tau^g, (\tau_\ell h^m)^g\rangle$. Then, $T \cap Z_{\mathbf{1}} = 1$, and thereby, $Z_{\mathbf{1}} \cap Z_{\mathbf{1}}^g$ has index at least 4 in $Z_{\mathbf{1}}$, which is a contradiction. Thus, $C = 1$, and so W is the unique minimal normal subgroup of X. Arguing similarly as in Lemma 4.4, we obtain that H is imprimitive on W, as in part (c).

Let $M = WN$. Then, Γ_M is a cycle. Since X/M is transitive on $V\Gamma_M$, one has $X/M \cong \mathbb{Z}_m$ or D_n. Further, Γ is arc-transitive if and only if $X/M \cong \mathrm{D}_n$, as in part (d). □

With the above preparations, we are ready to embark on the proof of Theorem 1.1.

Proof of Theorem 1.1 If $G \lhd X$, then by Lemma 2.5, we have $X_{\mathbf{1}} \le \mathrm{D}_8$, as in Theorem 1.1 (1). In what follows, we assume that G is not normal in X.

Suppose that $p > 2$. By Lemmas 4.1-4.5, if W is not normal in X, then $G \cong \mathrm{D}_{2p}$, $\Gamma \cong \mathbf{C}_{p[2]}$ and $\mathsf{Aut}\Gamma \cong \mathbb{Z}_2^p{:}\mathrm{D}_{2p}$, as in Theorem 1.1 (2). Now, we may assume that W is normal in X. If Γ_W is a cycle, then part (3) of Theorem 1.1 occurs by Lemma 4.4. If Γ is a normal cover of Γ_W, it follows from Lemma 4.5 that part (4) of Theorem 1.1 follows.

Suppose that $p = 2$. By Lemmas 4.3 and 4.4, Theorem 1.1 (5) occurs. □

5 Insoluble automorphism groups

Let $G = W{:}H = \mathbb{Z}_p^d{:}\mathbb{Z}_n$ be a Frobenius group where $\mathrm{ord}_n(p) = d$ for a prime p and an integer n. Assume that $\Gamma = (V\Gamma, E\Gamma)$ is a connected X-edge-transitive tetravalent Cayley graph of G, where $G \le X \le \mathsf{Aut}\Gamma$. In this section, we study the case where the automorphism group X is insoluble.

Remark For any triple $(X, G, X_{\mathbf{1}})$ listed in Table 2, the corresponding graph Γ does exist, refer to Li et al. [21, Theorem 1.1].

We now determine the structure of insoluble group X. Denote by $R(X)$ the maximal soluble normal subgroup of X. We first treat the case where $R(X) = 1$.

Lemma 5.1 *Let N be minimal normal in X. If $R(X) = 1$, then $\mathbf{C}_X(N) = 1$.*

Table 2 Almost simple automorphism groups

X	G	X_1
$\mathrm{PSL}(3,3){:}\mathbb{Z}_2$	D_{26}	$\mathbb{Z}_3^2{:}\mathrm{GL}(2,3)$
$\mathrm{PGL}(2,7)$	$\mathrm{D}_{14}, \mathbb{Z}_7{:}\mathbb{Z}_3, \mathbb{Z}_7{:}\mathbb{Z}_6$	$\mathrm{S}_4, \mathrm{D}_{16}, \mathrm{D}_8$
$\mathrm{PSL}(2,23)$	$\mathbb{Z}_{23}{:}\mathbb{Z}_{11}$	S_4
$\mathrm{PSL}(2,11)$	$\mathbb{Z}_{11}{:}\mathbb{Z}_5$	A_4
$\mathrm{PGL}(2,11)$	$\mathbb{Z}_{11}{:}\mathbb{Z}_5, \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	$\mathrm{S}_4, \mathrm{A}_4$

Proof Note that N is minimal in X. Since $R(X) = 1$, we have $N \cong T^k$, where T is a non-abelian simple group, and k is an integer. Clearly, $\mathbf{Z}(N) = 1$. Let $C{:}= \mathbf{C}_X(N)$. Since $N \trianglelefteq X$, we have $C \trianglelefteq X$. Suppose that $C \neq 1$. By our assumption, C is insoluble. Noticing that $N \cap G \trianglelefteq G$, we conclude that $N \cap G = 1$ or $W \le N \cap G$. For the former, $|N|$ divides $|X_1|$, and so N is soluble, contrary to our assumption. Thus, $W \le N \cap G$. Similarly, $W \le C \cap G$. It follows that $W \le N \cap C$, a contradiction. Thus, $C = 1$. □

Lemma 5.2 *If $R(X) = 1$, then X is almost simple.*

Proof Suppose for a contradiction that X is not almost simple. Then, by our assumption, there exists a minimal normal subgroup N of X, such that $N = T_1 \times \cdots \times T_k$, where $T_i \cong T$ is non-abelian simple and $k \ge 2$. By [17], we obtain that T is one of the following:

$$\begin{aligned} &\mathrm{PSL}(2,q)(q>3),\ \mathrm{PSL}(3,q)(q<9),\ \mathrm{PSL}(4,2),\ \mathrm{PSp}(4,3),\\ &\quad \mathrm{PSU}(3,8) \quad \text{and} \quad \mathrm{M}_{11}. \end{aligned} \tag{3}$$

By Frattini argument, one has $X = GX_u$, where $u \in V\Gamma$. By Lemma 2.3, either X_u is a 2-group or $|X_u|$ divides $2^4 \cdot 3^6$. Let $r > 3$ be a prime divisor of $|T|$. Noting that r divides $|X|$ and $(|W|, |H|) = 1$, we conclude that r divides either $|W|$ or $|H|$.

Case 1: Suppose that r divides $|W|$. Let $W_i = T_i \cap W$ where $1 \le i \le k$. Then, $W_i \neq 1$ for all i. Assume that $N \cap H = 1$. Then, $G \cap N = W$, and hence, $|N : W|$ divides $|X_u|$. So does $\prod_{i=1}^{k} |T_i : W_i|$. This implies that $|T|$ has exactly three prime divisors. By (3), together with [15, p. 12-14, 135-136], the only possibility is that T is one of the following groups:

$$\mathrm{A}_5,\ \mathrm{A}_6,\ \mathrm{PSL}(2,7),\ \mathrm{PSL}(2,8),\ \mathrm{PSL}(2,17) \quad \text{and} \quad \mathrm{PSL}(3,3).$$

Let $T = \mathrm{A}_5$. Then, $N \cap G = \mathbb{Z}_5^k$, and so $2^{2k} \cdot 3^k$ divides $|X_u|$. Thus, $k = 2$. Since X can be embedded into $(\mathrm{S}_5 \times \mathrm{S}_5){:}\mathbb{Z}_2$, we deduce that $G = \mathbb{Z}_5^2{:}\mathbb{Z}_8$. However, it is easily shown that $N \cap G = \mathbb{Z}_5^2{:}\mathbb{Z}_2$, contradicting our assumption. Let $T = \mathrm{A}_6$. Then, $N \cap G = \mathbb{Z}_5^k$, and so $2^{3k} \cdot 3^{2k}$ divides $|X_u|$, which is impossible. In a similar fashion, we can exclude the remaining cases.

Thus, $N \cap H \neq 1$. Let $\widehat{H} = N \cap H$. Then, $G \cap N = W{:}\widehat{H}$. Let H_i be the projection of $\widehat{H}$ on T_i where $1 \le i \le k$. Let $G_i = W_i{:}H_i$. Since G is a Frobenius group, G_i is a

Frobenius group, and so $\widehat{H}$ is a diagonal subgroup of $H_1 \times \cdots \times H_k$. Hence, $\widehat{H} \cong H_i$ for each i. Since $X = GX_u$, we deduce that $|N : (G \cap N)|$ divides $|X_u|$.

Let $T = \mathrm{PSL}(2, q)$ with $q > 3$. Write $q = s^e$ for a prime s and $e \geq 1$. Let $f = (2, s - 1)$.

Assume that $p = s$. Then, $|W_i| = p^\ell$ where $\ell \leq e$, and so $|H_i|$ divides $p^\ell - 1$. If $\ell < e$, then $|H_i| < \frac{q-1}{f}$. For this case, since $(q - 1, q + 1) = f$, we deduce that $|T_i : G_i|$ is divisible by three distinct primes. So is $|X_u|$ for $|T_i : G_i|$ divides $|X_u|$, which is a contradiction. Thus, $\ell = e$. By Suzuki [34, Theorem 6.17], we conclude that $\mathbf{N}_{T_i}(W_i) \cong \mathbb{Z}_p^e{:}\mathbb{Z}_{\frac{q-1}{f}}$, and so $G_i \lesssim \mathbb{Z}_p^e{:}\mathbb{Z}_{\frac{q-1}{f}}$. Let $M \cong \mathbb{Z}_p^{ke}{:}\mathbb{Z}_{\frac{q-1}{f}}$ be a subgroup of N. Then, $G \cap N$ can be embedded into M. Since $|N : M|$ divides $|X_u|$, we deduce that

$$(q + 1)^k [f^{-1}(q - 1)]^{k-1} \text{ divides } 2^4 \cdot 3^6.$$

A straightforward calculation shows $q = s = 5$ and $k = 2$. Then, $G \cap N \cong \mathbb{Z}_5^2{:}\mathbb{Z}_2$. Since X can be embedded into $(\mathrm{S}_5 \times \mathrm{S}_5){:}\mathbb{Z}_2$, we deduce that $G = \mathbb{Z}_5^2{:}\mathbb{Z}_8$. By the above discussion, $2^3 \cdot 3^2$ divides $|X_u|$, and so by Lemma 2.3, we deduce that either

$$X = (\mathrm{S}_5 \times \mathrm{S}_5){:}\mathbb{Z}_2 \quad \text{and} \quad X_u = \mathrm{S}_3 \times \mathrm{S}_4 \qquad \text{or} \tag{4}$$

$$X = ((\mathrm{A}_5 \times \mathrm{A}_5){:}\mathbb{Z}_2){:}\mathbb{Z}_2 \quad \text{and} \quad X_u = (\mathbb{Z}_3 \times \mathrm{A}_4).\mathbb{Z}_2. \tag{5}$$

Suppose that part (4) follows. Write $X = (G_1 \times G_2){:}\langle \pi \rangle$, where $G_1 \cong G_2 \cong \mathrm{S}_5$ and π interchanges G_1 and G_2. By Magma [3], there is just one conjugacy class of G and two conjugacy classes of X_u in X, such that $X = GX_u$ and $G \cap X_u = 1$. Choose $v \in \Gamma(u)$. By Lemma 2.1, write $\Gamma = \mathrm{Cos}(X, X_u, X_u g X_u)$, where $g \in \mathbf{N}_X(X_{uv}) \setminus X_u$ and $g^2 \in X_{uv}$. Since Γ is X-arc-transitive, one has $|X_u : X_{uv}| = 4$, and so $|X_{uv}| = 36$. However, one can quickly verifies by Magma [3] that there is no $g \in \mathbf{N}_X(X_{uv})$ such that $\langle X_u, g \rangle = X$; namely, Γ is not connected. Similarly, part (5) does not occur.

Assume that $p \neq s$. For $1 \leq i \leq k$, let $L_i \cong \mathbb{Z}_{\frac{q+\epsilon}{f}}$ be a subgroup of T_i, where $\epsilon = \pm 1$. By Sylow's Theorem, W_i can be embedded into L_i. By Suzuki [34, Theorem 6.23], we deduce that $\mathbf{N}_{T_i}(W_i) \cong \mathrm{D}_{\frac{2(q+\epsilon)}{f}}$, and so $G_i \lesssim \mathrm{D}_{\frac{2(q+\epsilon)}{f}}$. Let $M \cong \mathbb{Z}_{\frac{q+\epsilon}{f}}^k{:}\mathbb{Z}_2$ be a subgroup of N. Then, $G \cap N$ can be embedded into M. Since $|N : M|$ divides $|X_u|$, and hence, $|N : M|$ divides $2^4 \cdot 3^6$, it follows that

$$q^k (q - \epsilon)^k \text{ divides } 2^5 \cdot 3^6.$$

Calculations show that $q = 4$, $k = 2$ and $\epsilon = 1$. Then, $f = 1$, and so $G \cap N = M = \mathbb{Z}_5^2{:}\mathbb{Z}_2$. Note that X can be embedded into $(\mathrm{S}_5 \times \mathrm{S}_5){:}\mathbb{Z}_2$. By the definition, we deduce that $G = \mathbb{Z}_5^2{:}\mathbb{Z}_8$. Arguing as above, one can prove that this case does not occur.

Let $T = \mathrm{PSL}(3, q)$ with $q < 9$. Assume that $q = 2$. By the Atlas [4], we have $G_i \cong \mathbb{Z}_7{:}\mathbb{Z}_3$ where $1 \leq i \leq k$. Then, $G \cap N = W{:}\widehat{H} \cong \mathbb{Z}_7^k{:}\mathbb{Z}_3$, and so $|N : (G \cap N)| = 2^{3k} \cdot 3^{k-1}$ dividing $|X_u|$, a contradiction. In a similar fashion, one can prove that $T \neq \mathrm{PSL}(3, q)$ with $q \geq 3$. Let $T = \mathrm{PSL}(4, 2)$. By the Atlas [4], we derive that $35 \nmid |G_i|$ where $i \geq 1$. It implies that 5 or 7 divides $|X_u|$, which is impossible. Let $T = \mathrm{PSp}(4, 3)$. By the Atlas [4], one has $G_i \cong \mathbb{Z}_2^4{:}\mathbb{Z}_5$, and so $G \cap N \cong \mathbb{Z}_2^{4k}{:}\mathbb{Z}_5$. Thus, $|N : (G \cap N)|$ is divisible by 5, so is $|X_u|$, a contradiction. Similarly, T can neither equal to $\mathrm{PSU}(3, 8)$ nor M_{11}.

Case 2: Suppose that r divides $|H|$. If $r \nmid |H_i|$, then r divides $|X_u|$, a contradiction. Thus, r divides $|H_i|$ for each i. Since $G \cap N = W{:}\widehat{H}$ with $\widehat{H} \cong H_i$, r divides $|N : (G \cap N)|$, and so $|X_u|$ is divisible by r, again a contradiction.

Therefore, X is almost simple. □

Lemma 5.2 tells us that if X is insoluble and $R(X) = 1$, then X is almost simple. The next two lemmas determine the graph Γ for the case where X is almost simple.

Lemma 5.3 *Let X be an almost simple group with $\mathsf{soc}(X) = \mathrm{PSL}(2, 7)$. If Γ is not $(X, 2)$-arc-transitive, then $X = \mathrm{PGL}(2, 7)$ and $(X_{\mathbf{1}}, G) = (\mathrm{D}_8, \mathbb{Z}_7{:}\mathbb{Z}_6)$ or $(\mathrm{D}_{16}, \mathbb{Z}_7{:}\mathbb{Z}_3)$.*

Proof Denote by u the vertex $\mathbf{1}$. By Frattini argument, we have $X = GX_u$. Since Γ is not $(X, 2)$-arc-transitive, X_u is a 2-group. Note that G is a Frobenius group. Checking the subgroups of $\mathrm{PGL}(2, 7)$ in the Atlas [4], we obtain $G = \mathbb{Z}_7{:}\mathbb{Z}_6$ or $\mathbb{Z}_7{:}\mathbb{Z}_3$.

Assume first that $G = \mathbb{Z}_7{:}\mathbb{Z}_6$. Since $\mathbb{Z}_7{:}\mathbb{Z}_3$ is maximal in $\mathsf{soc}(X)$, we have $X = \mathrm{PGL}(2, 7)$. It follows that $X_u = \mathrm{D}_8$. Assume now that $G = \mathbb{Z}_7{:}\mathbb{Z}_3$. Furthermore, assume that $X = \mathrm{PSL}(2, 7)$. Then, Γ is a connected tetravalent X-edge-transitive Cayley graph, and $X_u = \mathrm{D}_8$ is a Sylow 2-subgroup of X. Choose $v \in \Gamma(u)$. Then, $|X_u : X_{uv}| = 2$ or 4. Since Γ is X-vertex-transitive, we write Γ as a coset graph $\mathsf{Cos}(X, H, H\{x, x^{-1}\}H)$, where $H = X_u = \mathrm{D}_8$ and $x \in X$ is such that $\langle H, x\rangle = X$; in particular, $x \notin H$.

Suppose that $|X_u : X_{uv}| = 4$. Then, Γ is X-arc-transitive. By Lemma 2.1, we choose x such that $(u, v)^x = (v, u)$, yielding $x \in \mathbf{N}_X(X_{uv}) \cong \mathrm{D}_8$. In particular, $\mathbf{N}_X(X_{uv}) \neq X_u$. Then, $|\mathbf{N}_{X_u}(X_{uv})| = 4$. Hence, $\mathbf{N}_{X_u}(X_{uv})$ is normal in both $\mathbf{N}_X(X_{uv})$ and X_u, and so $\mathbf{N}_{X_u}(X_{uv}) \trianglelefteq \langle X_u, \mathbf{N}_X(X_{uv})\rangle$. Checking the subgroups of $\mathrm{PSL}(2, 7)$ in the Atlas [4], we obtain that $\langle X_u, \mathbf{N}_X(X_{uv})\rangle \cong \mathrm{S}_4$, which contradicts the fact that $\langle X_u, x\rangle = X$.

Suppose that $|X_u : X_{uv}| = 2$. Then, $|X_{uv}| = 4$, and so $X_{uv} \trianglelefteq M := \langle X_u, X_v\rangle$. Thus, $M \cong \mathrm{S}_4$. By Lemma 2.1, we may choose x such that $u^x = v$. Noting that X_u and X_v are two Sylow 2-subgroups of M, there exists some $y \in M$ such that $X_u^y = X_v = X_u^x$. Hence, $xy^{-1} \in \mathbf{N}_X(X_u) = X_u$, so $\langle X_u, x\rangle \leq \langle X_u, xy^{-1}, y\rangle \leq M$, again a contradiction. Thus, $X = \mathrm{PGL}(2, 7)$. □

For a graph Γ, and $X \leq \mathsf{Aut}\Gamma$, the permutation group induced by X_u on $\Gamma(u)$ is denoted by $X_u^{\Gamma(u)}$, and the kernel (of X_u acting on $\Gamma(u)$) is denoted by $X_u^{[1]}$. Then, $X_u^{\Gamma(u)} \cong X_u/X_u^{[1]}$. For a positive integer n and a prime divisor p, denote by n_p the p-part of n. That is to say, n/n_p is indivisible by p.

Lemma 5.4 *If X is almost simple, then $(X, G, X_{\mathbf{1}})$ is one of the triples listed in Table 2.*

Proof Let $T = \mathsf{soc}(X)$. By Kazarin [17], we conclude that T is one of the following groups:

$$\mathrm{PSL}(2, q)(q > 3),\ \ \mathrm{PSL}(3, q)(q < 9),\ \ \mathrm{PSL}(4, 2),\ \ \mathrm{PSp}(4, 3),\ \ \mathrm{PSU}(3, 8) \text{ and } \mathrm{M}_{11}.$$

For convenience, denote by u the vertex $\mathbf{1}$ of Γ. Let $\widehat{G} = G \cap T$. Since X_u is soluble, we deduce that $\widehat{G} \neq 1$, and so $W \leq \widehat{G}$. By Lemma 2.3, X_u is either a 2-group or a $\{2,3\}$-group. Noting that $|T : \widehat{G}| = |TG : G|$ divides $|X_u|$, it follows that $|\widehat{G}|$ is divisible by $\frac{|T|}{|T|_2|T|_3}$. Since $\widehat{G}$ is soluble, $\widehat{G}$ contains a $\{2,3\}'$-Hall subgroup R of T.

Let $T = \mathrm{PSL}(3,3)$. Then, $X = \mathrm{PSL}(3,3){:}\mathbb{Z}_2$, $G = \mathrm{D}_{26}$, $X_u = \mathbb{Z}_3^2{:}\mathrm{GL}(2,3)$, and the corresponding graph Γ does exist, refer to Li et al. [21, Theorem 1.1].

Let $T = \mathrm{PSL}(3,4)$. Then, $|R| = 35$. However, T does not contain such R, a contradiction occurs. Similarly, T is neither $\mathrm{PSL}(3,q)$ with $5 \leq q \leq 8$ nor $\mathrm{PSU}(3,8)$. Let $T = \mathrm{PSL}(4,2)$. Then, R is a subgroup of T of order 35. By the Atlas [4], R is a cyclic subgroup of A_7, a contradiction. Let $T = \mathrm{PSp}(4,3)$. Note that 5 divides $|\widehat{G}|$ and $\widehat{G}$ is a Frobenius group. By the Atlas [4], $|\widehat{G}|$ divides $2^4 \cdot 5$, and so $|T_u|_3 = 3^4$. Thus, $|X_u|_3 = 3^4$, contradicting Lemma 2.3. Let $T = \mathrm{M}_{11}$. Then, $X = \mathrm{M}_{11}$, and so 55 divides $|G|$. By the Atlas [4], we deduce that $G = \mathbb{Z}_{11}{:}\mathbb{Z}_5$, and so $X_u = \mathbb{Z}_3^2{:}\mathrm{Q}_8.2$, contrary to Lemma 2.3.

Let $T = \mathrm{PSL}(2,q)$ with $q > 3$. If $q = 4$ or 5, then as 5 divides $|G|$, we have $X = \mathrm{S}_5$, $G = \mathrm{D}_{10}$ and $X_u = \mathrm{A}_4$. However, one can quickly verifies by Magma [3] that there is no factorisation $X = GX_u$. If $q = 7$, then 7 divides $|G|$. We check using Magma [3] that $X = \mathrm{PGL}(2,7)$, $G = \mathrm{D}_{14}$ and $X_u = \mathrm{S}_4$. If $q = 11$, then 55 divides $|G|$. By the Atlas [4], we deduce that $\widehat{G} = \mathbb{Z}_{11}{:}\mathbb{Z}_5$. By Magma [3], $X = \mathrm{PSL}(2,11).\mathcal{O}$, $G = \mathbb{Z}_{11}{:}(\mathbb{Z}_5 \times \mathcal{O}_1)$ and $X_u = \mathrm{A}_4.\mathcal{O}_2$, where $\mathcal{O}_1\mathcal{O}_2 = \mathcal{O}$ with $\mathcal{O} = 1$ or 2. If $q = 23$, then $11 \cdot 23$ divides $|G|$. By the Atlas [4], $\widehat{G} = \mathbb{Z}_{23}{:}\mathbb{Z}_{11}$. By Magma [3], $X = \mathrm{PSL}(2,23)$, $G = \mathbb{Z}_{23}{:}\mathbb{Z}_{11}$ and $X_u = \mathrm{S}_4$.

In what follows, we assume that $q \neq 4, 5, 7, 11$ or 23. Write $q = r^e$ for a prime r and $e \geq 1$. Let $f = (2, q-1)$. By [25, Proposition 4.1], either

$$\widehat{G} \leq \mathrm{D}_{\frac{2(r^e+1)}{f}} \quad \text{and} \quad \mathbb{Z}_r^e \trianglelefteq T_u \leq \mathbb{Z}_r^e{:}\mathbb{Z}_{\frac{r^e-1}{f}}, \quad \text{or} \tag{6}$$

$$\mathbb{Z}_r^e \trianglelefteq \widehat{G} \leq \mathbb{Z}_r^e{:}\mathbb{Z}_{\frac{r^e-1}{f}} \quad \text{and} \quad T_u \leq \mathrm{D}_{\frac{2(r^e+1)}{f}}. \tag{7}$$

Assume that (6) follows. Then, $\frac{r^e(r^e-1)}{2}$ divides $|T : \widehat{G}|$. Since $|T : \widehat{G}|$ divides $|X_u|$, it follows that $\frac{r^e(r^e-1)}{2} \mid 2^4 \cdot 3^6$. We calculate $r^e = 9$, and so $r = 3$ and $e = 2$. Then, $\mathbb{Z}_3^2 \trianglelefteq T_u \leq \mathbb{Z}_3^2{:}\mathbb{Z}_4$, and hence, $T_u^{\Gamma(u)} = \mathbb{Z}_3$ or S_3. Now $T_u^{\Gamma(u)} \trianglelefteq X_u^{\Gamma(u)} \leq \mathrm{S}_4$, and so $X_u^{\Gamma(u)} = \mathrm{S}_3$. By Lemma 2.3, this is a contradiction. Thus, (7) follows.

Case 1: Suppose that T_u is a $\{2,3\}$-group. So does $T_u^{\Gamma(u)}$. Since $T_u \leq \mathrm{D}_{\frac{2(r^e+1)}{f}}$ and $T_u^{\Gamma(u)} \leq \mathrm{S}_4$, we deduce that $T_u^{\Gamma(u)} = \mathrm{S}_3$. Consequently, $X_u^{\Gamma(u)} = \mathrm{S}_3$, a contradiction.

Case 2: Suppose that T_u is a 2-group. Then, $|T : \widehat{G}|$ is a power of 2. By Guralnick [16, Theorem 1], $|T : \widehat{G}| = q + 1 = 2^\ell$ for $\ell \geq 3$. Thus, $|\widehat{G}| = \frac{|T|}{q+1} = \frac{q(q-1)}{2}$, and so $\widehat{G}$ contains a $2'$-Hall subgroup of T. Then, $G \cap T$ contains a $2'$-Hall subgroup of T. By Lemma 2.7, we have that $q = r = 2^\ell - 1$, $\widehat{G} = \mathbb{Z}_r{:}\mathbb{Z}_{\frac{r-1}{2}}$ and $T_u = \mathrm{D}_{r+1}$. Suppose that $\ell = 3$. Then, $q = 7$. By Lemma 5.3, we are done. In what follows, we assume that $\ell \geq 5$.

Suppose that $G = \widehat{G}$. Then, as $|\widehat{G}|$ is odd, by Li et al. [22, Theorem 1.1], $X = \mathrm{PGL}(2,7)$ and $X_u = \mathrm{D}_{16}$, contradicting our assumption. Thus, $\widehat{G} < G$. Noting that $\widehat{G}$

is maximal in T, we deduce that $X = \mathrm{PGL}(2, r)$, $G = \mathbb{Z}_r{:}\mathbb{Z}_{r-1}$, and $X_u = T_u = \mathrm{D}_{r+1}$. Let $v \in \Gamma(u)$. By Lemma 2.1, X_{uv} has index 2 or 4 in both X_u and X_v. Since $\ell \geq 5$, X_{uv} contains a subgroup $C \cong \mathbb{Z}_4$. It is easily shown that C is normal in both X_u and X_v, and so $C \lhd L := \langle X_u, X_v\rangle$. By Suzuki [34, p.417], both X_u and X_v are maximal in T, and hence $L = X_u = X_v$. By the connectedness of Γ, L fixes each vertex of Γ, which is impossible.

This completes the proof of Lemma 5.4. □

We now handle the case where $R(X) \neq 1$.

Lemma 5.5 *If $R(X) \cap G = 1$, then $G = \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$ and $X = \mathrm{PGL}(2, 11) \times \mathbb{Z}_2$.*

Proof Write $\overline{X} = X/R(X)$, $\overline{G} = GR(X)/R(X)$ and $\overline{X_{\mathbf{1}}} = X_{\mathbf{1}}R(X)/R(X)$. Then, $\overline{G} \cong G$ and $\overline{X_{\mathbf{1}}} \cong X_{\mathbf{1}}$. Since $X = GX_{\mathbf{1}}$, we have $\overline{X} = \overline{G}\,\overline{X_{\mathbf{1}}}$. Let $\Sigma = \Gamma_{R(X)}$. Noting that $\overline{X}$ is insoluble, it follows from Lemma 2.4 that Γ is a normal cover of Σ. Pick $u \in V\Sigma$ such that $\mathbf{1} \in u$, so that $\overline{X_{\mathbf{1}}} \leq \overline{X}_u$, and $\overline{X} = \overline{G}\,\overline{X}_u$ because $\overline{G}$ is transitive on $V\Sigma$. Further, since $\overline{G}$ is not regular on $V\Sigma$, it follows that $\overline{G} \cap \overline{X}_u \neq 1$.

Assume that $\overline{X}$ is not almost simple. By Lemma 5.1, $\overline{X}$ has a unique minimal normal subgroup $\overline{N}$. Arguing as in the proof of Lemma 5.2, we only need to deal with the case where $\overline{N} = \mathrm{A}_5 \times \mathrm{A}_5$. For this case, $\overline{X}$ can be embedded into $(\mathrm{S}_5 \times \mathrm{S}_5){:}\mathbb{Z}_2$. By the definition, $\overline{G} = \mathbb{Z}_5^2{:}\mathbb{Z}_8$, and so $\overline{N} \cap \overline{G} = \mathbb{Z}_5^2{:}\mathbb{Z}_2$. Thus, as $|\overline{N} : \overline{N} \cap \overline{G}|$ divides $\frac{|\overline{X}_u|}{|\overline{G}\cap\overline{X}_u|}$, $2^3 \cdot 3^2$ divides $\frac{|\overline{X}_u|}{|\overline{G}\cap\overline{X}_u|}$. Noting that $\overline{G} \cap \overline{X}_u \neq 1$, it follows that $2^4 \cdot 3^2$ divides $|\overline{X}_u|$, and so by Lemma 2.3, $\overline{X}_u = \mathrm{S}_3 \times \mathrm{S}_4$. For this case, $\overline{X} = ((\mathrm{A}_5 \times \mathrm{A}_5){:}\mathbb{Z}_2){:}\mathbb{Z}_2$, and so $\overline{G} \cap \overline{X}_u \cong \mathbb{Z}_2$. By Magma [3], there is just one conjugacy class of $\overline{X}_u$ and two conjugacy classes of $\overline{G}$ in $\overline{X}$, such that $\overline{X} = \overline{G}\,\overline{X}_u$ and $\overline{G} \cap \overline{X}_u \cong \mathbb{Z}_2$. Choose $v \in \Sigma(u)$. By Lemma 2.1, write $\Sigma = \mathsf{Cos}(\overline{X}, \overline{X}_u, \overline{X}_u g\overline{X}_u)$, where $g \in \mathbf{N}_{\overline{X}}(\overline{X}_{uv})\backslash\overline{X}_u$ and $g^2 \in \overline{X}_{uv}$. Since Σ is $\overline{X}$-arc-transitive, we deduce that $|\overline{X}_u : \overline{X}_{uv}| = 4$, and so $|\overline{X}_{uv}| = 36$. However, one can quickly verifies by Magma [3] that there is no $g \in \mathbf{N}_{\overline{X}}(\overline{X}_{uv})$ such that $\langle\overline{X}_u, g\rangle = \overline{X}$; namely, Σ is not connected. Thus, $\overline{X}$ is almost simple.

Let $\overline{T} = \mathsf{soc}(\overline{X})$. By Kazarin [17], we obtain that $\overline{T}$ is one of the following groups:

$$\mathrm{PSL}(2, q)(q > 3),\ \mathrm{PSL}(3, q)(q < 9),\ \mathrm{PSL}(4, 2),\ \mathrm{PSp}(4, 3),\ \mathrm{PSU}(3, 8) \text{ and } \mathrm{M}_{11}.$$

Let $\overline{T} = \mathrm{PSL}(2, q)$ where $q = 4, 5, 7, 11$ or 23. If $q = 4$ or 5, then the only possibility is that $\overline{G} \cong \mathbb{Z}_5{:}\mathbb{Z}_4$ by Li et al. [21, Theorem 1.1]. For this case, Γ is a Cayley graph of order 20, and so by Pan et al. [29, Theorem 5.3], G is normal in X, which is a contradiction. If $q = 7$, 11 or 23, then $|\overline{G}|$ is square-free. The same is true for G. By Li et al. [21, Theorem 1.1], if $q = 7$ or 23, then X is almost simple, contradicting our assumption, and if $q = 11$, then $X = \mathrm{PGL}(2, 11) \times \mathbb{Z}_2$, $G = \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$, $X_{\mathbf{1}} = \mathrm{S}_4$, and the corresponding graph Γ does exist.

Let $\overline{T}$ be one of the remaining groups. Arguing as in the proof of Lemma 5.4 with $\overline{X} = \overline{G}\,\overline{X}_u$ in the place $X = GX_{\mathbf{1}}$, we rule out these possibilities.

This completes the proof of Lemma 5.5. □

Lemma 5.6 *If $R(X) \cap G \neq 1$, then the following statements hold:*

(a) $G \cong \mathbb{Z}_p^4{:}\mathbb{Z}_5$, $X = W.\overline{X}$ *and* $\Gamma_W \cong \mathbf{K}_5$, *where* $\mathsf{soc}(\overline{X}) \cong \mathrm{A}_5$*;*
(b) $G \cong \mathbb{Z}_p^4{:}\mathbb{Z}_{10}$, $X = W.(\overline{X} \times \mathbb{Z}_2)$ *and* $\Gamma_W \cong \mathbf{K}_{5,5} - 5\mathbf{K}_2$, *where* $\mathsf{soc}(\overline{X}) \cong \mathrm{A}_5$.

Proof Let $R = R(X) \cap G$. Then, $R \lhd G$. By our assumption, $R(X) \cap G \neq 1$, and thereby, $R \geq W$ because W is minimal normal in G. Since $X/R(X)$ is insoluble, it follows from Lemma 2.4 that Γ is a normal cover of $\Gamma_{R(X)}$, and so $GR(X)/R(X) \leq \mathsf{Aut}\Gamma_{R(X)}$.

Let $\widehat{H} = HR(X)/R(X)$. Note that Γ is a Cayley graph of G, so $\Gamma_{R(X)}$ is a Cayley graph of $\widehat{H}$. Thus, $|R(X)||\widehat{H}| = |G|$. Since $|G| = |W||H|$, we calculate that

$$|R(X)| = |W||R(X) \cap H|.$$

It follows that $R(X) \leq G$, and so $R = R(X)$. This implies that $W \unlhd X$. Let $\overline{H} = G/W$. By [2, Theorem 1.2], either $\Gamma_W \cong \mathbf{K}_5$ and $\overline{H} \cong \mathbb{Z}_5$, or $\Gamma_W \cong \mathbf{K}_{5,5} - 5\mathbf{K}_2$ and $\overline{H} \cong \mathbb{Z}_{10}$. For the former, we have $\mathsf{Aut}\Gamma_W \cong \mathrm{S}_5$, and for the latter, $\mathsf{Aut}\Gamma_W \cong \mathrm{S}_5 \times \mathbb{Z}_2$.

Let $\overline{X} = X/W$. Since Γ is a normal cover of Γ_W, we have $\overline{X} \leq \mathsf{Aut}\Gamma_W$. Let $\Gamma_W \cong \mathbf{K}_5$. Noting that $\overline{X}$ is insoluble, we conclude that $\mathsf{soc}(\overline{X}) \cong \mathrm{A}_5$. Let $\Gamma_W \cong \mathbf{K}_{5,5} - 5\mathbf{K}_2$. Since $\overline{H} \cong \mathbb{Z}_{10}$, we obtain $\overline{X} = L \times \mathbb{Z}_2$ where $\mathsf{soc}(L) \cong \mathrm{A}_5$. Note that $H \cong \mathbb{Z}_5$ or $\mathbb{Z}_{10}$. By the definition, we deduce that $d = 2$ or 4. However, since $\mathrm{GL}(2, p)$ does not contain A_5, it follows that $d = 4$. Therefore, $G \cong \mathbb{Z}_p^4{:}\mathbb{Z}_5$ or $\mathbb{Z}_p^4{:}\mathbb{Z}_{10}$. This completes the proof. □

The assertion of Theorem 1.2 follows from Lemmas 5.2–5.6.

6 Half-transitive graphs

In the last section, we apply Theorems 1.1 and 1.2 to prove Theorem 1.3.

Let p be an odd prime and $d > 1$ an odd integer. Let n be a primitive divisor of $p^d - 1$, such that n does not divide $r(p^{d/r} - 1)$ for any prime r dividing d. Set

$$G = W{:}\langle h\rangle = \mathbb{Z}_p^d{:}\mathbb{Z}_n < \mathrm{AGL}(1, p^d).$$

Construction 6.1 Let i be coprime to n for $1 \leq i \leq n-1$, and let $a \in W\backslash\{1\}$. Set

$$\begin{cases} S_i = \{ah^i, a^{-1}h^i, (ah^i)^{-1}, (a^{-1}h^i)^{-1}\}, \\ \Gamma_i = \mathsf{Cay}(G, S_i). \end{cases}$$

With this preparation, we are ready to embark on the proof of Theorem 1.3.

Proof of Theorem 1.3 Let $X = \mathsf{Aut}\Gamma$. Let $\Gamma = \mathsf{Cay}(G, S)$ be connected, edge-transitive and of valency 4. By our assumption, $\langle h\rangle$ is primitive on W, $d > 1$ is odd, and p is an odd prime. By Theorems 1.1–1.2, we obtain that G is normal in X. By virtue of Godsil [13, Lemma 2.1], we have $X = G{:}\mathsf{Aut}(G, S)$.

By Lemma 2.5, one has $X_1 = \mathsf{Aut}(G, S) \leq \mathrm{D}_8$. By Doerk [8, Proposition 12.10], $\mathsf{Aut}(G) = A\Gamma\mathrm{L}(1, p^d) \cong \mathbb{Z}_p^d{:}(\mathbb{Z}_{p^d-1}{:}\mathbb{Z}_d)$, and so $\mathsf{Aut}(G)$ has a cyclic Sylow 2-subgroup. It follows that $X_1 = \langle\sigma\rangle \cong \mathbb{Z}_4$ or $\mathbb{Z}_2$. Thus, σ fixes an element of G of order n, say $f \in G$ such that $o(f) = n$ and $f^\sigma = f$. Then, $G = W{:}\langle f\rangle$, and $X = G{:}\langle\sigma\rangle = (W{:}\langle f\rangle){:}\langle\sigma\rangle$. Moreover, since a Sylow 2-subgroup of $\mathsf{Aut}(G)$ is cyclic, all involutions of $\mathsf{Aut}(G)$ are conjugate. It is easy to verify that every involution of $\mathsf{Aut}(G)$ inverts all non-identity elements of W.

Since Γ is connected, $\langle S\rangle = G$ and $\mathsf{Aut}(G, S)$ is faithful on S. Assume that S contains an involution. Noting that Γ is X-edge-transitive, S consists of involutions. By the proof of Lemma 2.5, $G \cong \mathrm{D}_{2p}$, against our assumption. Hence, S does not contain an involution. For this case, we may write $S = \{x, x^{-1}, y, y^{-1}\}$ such that either $o(\sigma) = 2$ and $(x, y)^\sigma = (y, x)$, or $o(\sigma) = 4$ and $(x, y)^\sigma = (y, x^{-1})$, refer to Praeger [31, Proposition 1]. Now, $x = af^i$, where $a \in W$ and $i \geq 0$. Suppose that $o(\sigma) = 4$. Then, $y = x^\sigma = (af^i)^\sigma = a^\sigma f^i$, and $a' f^{-i} = f^{-i}a^{-1} = (af^i)^{-1} = x^{-1} = x^{\sigma^2} = a^{\sigma^2} f^i = a^{-1} f^i$. It follows that $f^{2i} = 1$, and hence, f^i has order 1 or 2. If $f^i = 1$, then $x = a$, and $y = x^\sigma = a^\sigma$, belonging to W, and so $\langle S\rangle \leq W < G$, which is a contradiction. Thus, f^i has order 2. Noting that f^i inverts each element of W, we conclude that x has order 2, again a contradiction. Thus, σ is an involution, and so $(x, y)^\sigma = (y, x)$, $x = af^i$, and $y = x^\sigma = a^\sigma f^i = a^{-1} f^i$. In particular, Γ is not arc-transitive, and $S = \{af^i, a^{-1} f^i, (af^i)^{-1}, (a^{-1} f^i)^{-1}\}$.

Notice that $f, h \in G$ with $o(f) = o(h) = n$. Since $(|W|, n) = 1$, it follows from Schur–Zassenhaus's Theorem that there exists $b \in W$ such that $h^b \in \langle f\rangle$. So $f^{b^{-1}} = h^r$ for some r coprime to n. Let $\tau = \sigma^{b^{-1}}$. Then, as $f^\sigma = f$, we have $h^\tau = h$, and so $X = G{:}\langle\tau\rangle$. Moreover, $S^{b^{-1}} = \{ah^{ir}, a^{-1}h^{ir}, (ah^{ir})^{-1}, (a^{-1}h^{ir})^{-1}\}$. Let $ir \equiv j \pmod{n}$ and $1 \leq j \leq n - 1$. Then, $S_j := \{ah^j, a^{-1}h^j, (ah^j)^{-1}, (a^{-1}h^j)^{-1}\}$. Since $\Gamma \cong \mathsf{Cay}(G, S_j)$ is connected, it follows from Li et al. [22, Lemma 6.2(ii)] that $(j, n) = 1$. Let Γ_i and Γ_j be as in Construction 6.1 with $(i, n) = (j, n) = 1$. Furthermore, if $p^k i \equiv j$ or $-j \pmod{n}$ for some $k \geq 0$, then $\Gamma_i \cong \Gamma_j$, refer to Li et al. [22, Lemma 6.2(iii)].

This completes the proof of Theorem 1.3. □

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