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Integral and distance integral Cayley graphs over generalized dihedral groups

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Abstract

A graph is said to be integral (resp. distance integral) if all the eigenvalues of its adjacency matrix (resp. distance matrix) are integers. Let H be a finite abelian group, and let $\mathscr{H} = \langle H, b \mid b^2 = 1, bhb = h^{-1}, h \in H\rangle$ be the generalized dihedral group of H. The contribution of this paper is threefold. Firstly, based on the representation theory of finite groups, we obtain a necessary and sufficient condition for a Cayley graph over $\mathscr{H}$ to be integral, which naturally contains the main results obtained in Lu et al. (J Algebr Comb 47:585–601, 2018). Secondly, a closed-form decomposition formula for the distance matrix of Cayley graphs over any finite groups is derived. As applications, a necessary and sufficient condition for the distance integrality of Cayley graphs over $\mathscr{H}$ is displayed. Some simple sufficient (or necessary) conditions for the integrality and distance integrality of Cayley graph are exhibited, respectively, from which several infinite families of integral and distance integral Cayley graphs over $\mathscr{H}$ are constructed. And lastly, some necessary and sufficient conditions for the equivalence of integrity and distance integrity of Cayley graphs over generalized dihedral groups are obtained.

Keywords Integral Cayley graph · Generalized dihedral group · Character · Irreducible representation

Mathematics Subject Classification 05C50

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1 Introduction

Throughout this paper, we only consider simple graphs $\Gamma = (V_\Gamma, E_\Gamma)$ with vertex set V_Γ and edge set E_Γ. The *distance* between two vertices $x, y \in V_\Gamma$, denoted by $d_\Gamma(x, y)$, is the length of a shortest path connecting them.

The *adjacency matrix* $A(\Gamma)$ of Γ is a 0–1 $n \times n$ matrix whose (x, y)-entry is equal to 1 if and only if vertices x and y are adjacent, whereas the *distance matrix* $D(\Gamma)$ of Γ is an $n \times n$ square matrix whose (x, y)-entry is equal to $d_\Gamma(x, y)$, where $n := |V_\Gamma|$. The set of all eigenvalues of $A(\Gamma)$ and $D(\Gamma)$ are called the *spectrum* and *distance spectrum* of Γ, respectively. Note that both $A(\Gamma)$ and $D(\Gamma)$ are real symmetric matrices. Hence, all the eigenvalues of $A(\Gamma)$ [resp. $D(\Gamma)$] are real. Γ is said to be *integral* (resp. *distance integral*) if all the eigenvalues of $A(\Gamma)$ [resp, $D(\Gamma)$] are integers.

The notion of integral graphs was originally defined by Harary and Schwenk [12]; in the same paper, they proposed the following interesting problem: "Which graphs have integral spectra in Graphs and Combinatorics?" Since then, classifying and constructing integral graphs have become important research topics in algebraic graph theory. However, for a general graph, giving a systemic and complete solution to the aforementioned problem turns out to be extremely difficult, and the problem is yet far from being solved. Many researchers then tried to make some progress in solving this problem by studying the integrity of some special classes of graphs, for example, see [6,24]. One of the most popular among them is the study on the integrity of Cayley graphs.

Let G be a finite group, and let S be a subset of G such that $1_G \notin S$ and $S^{-1} = S$, where 1_G denotes the identity element of G. The *Cayley graph* $Cay(G, S)$ over G with respect to S is the graph with vertices given by the elements of G, and two vertices $g, h \in G$ are adjacent if and only if $gh^{-1} \in S$. A remarkable achievement for the spectrum of Cayley graphs is due to Babai [4] who gave an expression for the spectrum of a Cayley graph $Cay(G, S)$ in terms of irreducible characters of the finite group G in 1979. Bridges and Mena [5] derived a complete characterization of integral Cayley graphs over abelian groups. So [22] characterized integral graphs among circulant graphs by using a different approach. Klotz and Sander [14,15] classified finite abelian Cayley integral groups as finite abelian groups of exponent dividing 4 or 6; they also proposed the determination of all non-abelian Cayley integral groups. (A group G is said to be *integral* if the Cayley graph $Cay(G, S)$ is integral for any $S \subseteq G$ satisfying $1_G \in S$ and $S^{-1} = S$.) Alperin and Peterson [3] presented a necessary and sufficient condition for the integrity of Cayley graphs $Cay(G, S)$ on abelian groups G by describing the structure of S. Ahmady, Bell and Mohar [2] classified all finite groups that have a non-trivial integral Cayley graph. Recently, Lu et al. [17] have obtained a necessary and sufficient condition for the integrality of Cayley graphs over dihedral groups D_n by analyzing the irreducible characters of D_n. For more results on integral Cayley graphs, one may be referred to [1,8,10,18,19].

In distinction from the extensive studies on integral Cayley graphs, not much work was done on the distance integrity of Cayley graphs, which is also a very important research object in algebraic graph theory. One possible reason is that it is not easy to find the distance spectrum of graphs. Along with these directions, some special classes of distance integrity of Cayley graphs were studied. Renteln [20] showed that the distance spectrum of a Cayley graph over a real reflection group with respect to the set of all reflections is integral and provided a combinatorial formula for such spectrum. Foster-Greenwood and Kriloff [9] proved that the eigenvalues and distance eigenvalues of Cayley graphs over complex reflection groups with connection sets consisting of all reflections are integers.

The relationships between the integrity and distance integrity of Cayley graphs have also been considered in the literature. Ilić [13] proved that all the distance eigenvalues of integral Cayley graphs over cyclic groups are integers. Two years later, Klotz and Sander [16] extended the above result from cyclic groups to abelian groups. All these conclusions suggest that there are some intrinsic connections between the integrity and distance integrity of Cayley graphs, although it maybe not clear to find these connections just by their definitions. A very natural problem is that whether or not the integrity and the distance integrity of Cayley graphs are equivalent; if possible, under what conditions they may be equivalent.

The aforementioned works [3,9,13,16,17,20] lead us to the study on the integrity and distance integrity of Cayley graphs over generalized dihedral groups in this paper. Given a finite abelian group H, the *generalized dihedral group* of H, written as $\mathscr{H}$, is the semidirect product of H and $\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting on H by inverting elements. Specifically, let H be a finite abelian group of order n with $H = \{h_0 = 1_H, h_1, h_2, \ldots, h_{n-1}\}$; then, the generalized dihedral group of H is defined as

$$\mathscr{H} = \left\langle H, b \,\middle|\, b^2 = 1, bhb = h^{-1}, h \in H \right\rangle = \left\{h_0 = 1_H, h_1, \ldots, h_{n-1}, b, bh_1, \ldots, bh_{n-1}\right\}. \tag{1.1}$$

It is obvious that $\mathscr{H}$ is reduced to the so-called dihedral group when H is a cyclic group.

The contribution of this paper is threefold. Firstly, in Sect. 3, we present a necessary and sufficient condition for the integrality of $Cay(\mathscr{H}, S)$, where S is any subset of $\mathscr{H}$ satisfying $1_H \notin S$ and $S = S^{-1}$; actually, each eigenvalue of the adjacency matrix $A(Cay(\mathscr{H}, S))$ can be presented explicitly in terms of the irreducible representations of H. Our method is quite different from those given in [17]. Consequently, several new infinite families of integral Cayley graphs over generalized dihedral groups are constructed and some previously known results in [17] may be extended naturally. Secondly, in Sect. 4, we use the Fourier transform of finite groups to obtain the decomposition formula of the distance matrix, which is used to give a necessary and sufficient condition for the distance integrity of $Cay(\mathscr{H}, S)$. Several infinite classes of distance integral Cayley graphs over generalized dihedral groups are constructed. Finally, in Sect. 5, we present some necessary and sufficient conditions for the equivalence of the integrity and distance integrity of Cayley graphs over generalized dihedral groups.

2 Preliminary results

We first restate some basic results from representation theory of finite groups, e.g., see [21,23]. We follow the notation and terminologies in [23] except if otherwise stated. Let G be a finite group, and let V be a finite-dimensional vector space over the complex field $\mathbb{C}$. Denote by $GL(V)$ the group of all bijective linear maps $T : V \to V$. A *representation* of G on V is a group homomorphism $\rho : G \to GL(V)$. The *degree* of ρ is the dimension of V. Suppose that V is a unitary space; that is, it is endowed with a Hermitian scalar product $\langle \cdot, \cdot \rangle_V$. A representation $\rho : G \mapsto GL(V)$ is *unitary* provided that $\rho(g)$ is a unitary operator for all $g \in G$, which means that $\langle \rho(g)v_1, \rho(g)v_2 \rangle_V = \langle v_1, v_2 \rangle_V$ for all $g \in G$ and $v_1, v_2 \in V$. It is well known that any finite-dimensional representation of a finite group can be unitarizable. Therefore, we consider only unitary representations.

Fix an orthonormal basis of V over $\mathbb{C}$. For each $g \in G$, the matrix $\mathfrak{X}(g)$ of $\rho(g)$ with respect to the orthonormal basis is a unitary matrix, and $\mathfrak{X} : g \mapsto \mathfrak{X}(g)$ defines a matrix representation of G called a *matrix representation afforded by* ρ. For every finite group G, we define the *trivial* (matrix) representation as the one-dimensional representation which sends every element $g \in G$ to $1 \in \mathbb{C}$. Two unitary representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ of G are said to be *unitarily equivalent*, denoted by $\rho_1 \sim \rho_2$, if there exists a unitary isomorphism of vector spaces $T : V_1 \to V_2$ such that $\rho_1(g) = T^{-1}\rho_2(g)T$ for all $g \in G$. Let $\widehat{G}$ denote the set of irreducible pairwise inequivalent unitary representations of G. The *character* $\chi_\rho : G \to \mathbb{C}$ of ρ is defined as $\chi_\rho(g) = Tr(\rho(g))$ for $g \in G$, where $Tr(\rho(g))$ is the trace of the representation $\rho(g)$ with respect to some orthonormal basis of V. A subspace $W \leqslant V$ is G-*invariant* if $\rho(g)w \in W$ for all $g \in G$ and $w \in W$. The trivial subspaces V and $\{0\}$ are always invariant. We say that a representation $\rho : G \to GL(V)$ is *irreducible* if V has no non-trivial invariant subspaces; otherwise, we say that it is *reducible*.

Regular representations of finite groups are useful in algebraic graph theory, and it plays an important role in our proofs of the main results. To introduce the regular representation of a finite group G, we need to recall the notion of group algebra $\mathbb{C}[G]$. Let $\mathbb{C}[G]$ denote the set of formal sums $\sum_{g\in G} a_g g$, where $a_g \in \mathbb{C}$ and G is any (not necessarily abelian) finite group; that is,

$$\mathbb{C}[G] = \left\{ \sum_{g\in G} a_g g \,\middle|\, a_g \in \mathbb{C} \right\}.$$

Obviously, $\mathbb{C}[G]$ is a complex algebra having a basis consisting of the set of group elements. The addition, scalar product and multiplication on $\mathbb{C}[G]$ are defined as

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g, \quad \lambda \cdot \left(\sum_{g\in G} a_g g \right) = \sum_{g\in G} (\lambda a_g) g$$

and

$$\left(\sum_{g\in G} a_g g \right) \cdot \left(\sum_{g\in G} b_g g \right) = \sum_{g\in G} \left(\sum_{h\in G} a_h b_{h^{-1}g} \right) g,$$

respectively, where $\lambda \in \mathbb{C}$. If $D = \sum_{g\in G} a_g g \in \mathbb{C}[G]$, define $D^{-1} = \sum_{g\in G} a_g g^{-1}$; if D is a subset of G, we identify D with $\sum_{d\in D} d \in \mathbb{C}[G]$. The *regular representation* ρ_{reg} of G on the vector space $\mathbb{C}[G]$ is defined by setting

$$\rho_{\mathrm{reg}}(g)\left(\sum_{s\in G} a_s s\right) = \sum_{s\in G} a_s gs = \sum_{t\in G} a_{g^{-1}t} t$$

for all $g \in G$ and $\sum_{s\in G} a_s s \in \mathbb{C}[G]$. The following result is well known, e.g., see [21] or [23].

Lemma 2.1 *Let G be a finite group with $\widehat{G} = \{\rho_1, \rho_2, \ldots, \rho_h\}$, and let ρ_{reg} be its regular representation. Then*

$$\rho_{\mathrm{reg}}(g) \sim d_1\rho_1(g) \bigoplus d_2\rho_2(g) \bigoplus \cdots \bigoplus d_h\rho_h(g),$$

for each $g \in G$, where d_i is the degree of ρ_i, $i = 1, 2, \ldots, h$.

Babai [4] noticed that the adjacency matrix of a Cayley graph over any finite groups can be expressed by the regular representation, as we reproduce below.

Lemma 2.2 [4] *Let G be a finite group with $S \subseteq G$ such that $1_G \notin S$ and $S^{-1} = S$. Then*

$$A(Cay(G, S)) = \sum_{s\in S} R(s),$$

where R is the matrix representation corresponding to ρ_{reg} with respect to the basis $\{g \mid g \in G\}$ of $\mathbb{C}[G]$.

For convenience, let $S_A(Cay(G, S))$ [resp. $S_D(Cay(G, S))$] be the set of all eigenvalues of $A(Cay(G, S))$ [resp. $D(Cay(G, S))$] and let $[\lambda]^k$ denote the eigenvalue λ with multiplicity k. Let $\rho_1, \rho_2, \ldots, \rho_k$ be all inequivalent unitary irreducible representations of G with $d_1, d_2, \ldots, d_k$ as their degrees, respectively. Denote by $R_i(g)$ the matrix representation corresponding to $\rho_i(g)$ for $g \in G$ and $1 \leqslant i \leqslant k$. Together with Lemmas 2.1–2.2, there exists an invertible matrix P such that

$$\begin{aligned} PA(Cay(G, S))P^{-1} &= P\left(\sum_{s\in S} R(s)\right)P^{-1} \\ &= d_1\sum_{s\in S} R_1(s) \bigoplus d_2\sum_{s\in S} R_2(s) \bigoplus \cdots \bigoplus d_k\sum_{s\in S} R_k(s). \end{aligned} \tag{2.1}$$

Consequently,

$$S_A(Cay(G, S)) = \left\{[\lambda_{1,1}]^{d_1}, \ldots, [\lambda_{1,d_1}]^{d_1}, \ldots, [\lambda_{k,1}]^{d_k}, \ldots, [\lambda_{k,d_k}]^{d_k}\right\}, \tag{2.2}$$

where $\lambda_{i,1}, \ldots, \lambda_{i,d_i}$ are all eigenvalues of the matrix $\sum_{s\in S} R_i(s)$ for $1 \leqslant i \leqslant k$.

We end this section with an elegant result due to Alperin and Peterson [3], which gave a necessary and sufficient condition for the integrality of Cayley graphs over finite abelian groups. Let G be a finite abelian group, and let $\mathcal{F}_G$ be the set consisting of all subgroups of G. Then the *Boolean algebra* $B(G)$ is the set whose elements are obtained by arbitrary finite intersections, unions and complements of the elements in $\mathcal{F}_G$. The minimal elements of $B(G)$ are called *atoms*. Denote by $\widetilde{B}(G)$ the set of all different atoms. A multi-subset S of G is called *integral* if $\chi(S) = \sum_{s\in S}\chi(s)$ is an integer for every irreducible character χ of G. Alperin and Peterson [3] not only showed that each element of $B(G)$ is the union of some atoms and each atom of $B(G)$ has the form $[g] = \{x \mid \langle x\rangle = \langle g\rangle, x \in G\}$ but also determined the integrality of Cayley graphs over abelian groups, which is list in the following.

Lemma 2.3 [3] *Let G be a finite abelian group and $S \subseteq G$. Then S is integral if and only if $S \in B(G)$.*

Lu et al. [17] used an approach similar to those given in [3] to extend the above lemma to multi-sets.

Lemma 2.4 [17] *Let G be a finite abelian group, and let T be a multi-subset of G. Then T is integral if and only if $T \in C(G)$, where $C(G) = \left\{\bigcup_{[g]\in\widetilde{B}(G)} m_g[g] \mid m_g \in \mathbb{N}\right\}$ with $\mathbb{N} = \{0, 1, 2, \ldots\}$ being the set of natural numbers.*

3 Integral Cayley graphs over generalized dihedral groups

The purpose of this section is to obtain a necessary and sufficient condition for the integrality of $Cay(\mathscr{H}, S)$, where $\mathscr{H}$ is the generalized dihedral group given as in (1.1) and S is a non-empty subset of $\mathscr{H}$ satisfying $1_H \notin S$ and $S = S^{-1}$. We first recall the irreducible representations of $\mathscr{H}$. One may be referred to [11] for details. Let H^2 be the set of squares in H; that is, $H^2 = \{h^2 \mid h \in H\}$. Denote by $H/H^2 = \{hH^2 \mid h \in H\}$ the quotient group with identity element H^2. Note that $(hH^2)^2 = h^2H^2 = H^2$ for any $h \in H$. Therefore, H/H^2 is an elementary abelian 2-group. By Sylow theorem, the order of H/H^2 is a power of 2, say 2^c for some nonnegative integer c. Now we list all the irreducible representations of $\mathscr{H}$ in the following.

• One-dimensional representations of $\mathscr{H}$: Since H/H^2 is an elementary abelian 2-group of order 2^c, it has 2^c one-dimensional representations. Each of these gives rise to a one-dimensional representation of H. For each such representation ρ of H, there are two corresponding one-dimensional representations of $\mathscr{H}$ whose restriction to H is ρ: One representation sends b to 1, and the other representation sends b to -1. Thus, we get a total of 2^{c+1} one-dimensional representations of $\mathscr{H}$.

• Two-dimensional irreducible representations of $\mathscr{H}$: There are $n - 2^c$ of these, described as follows. These two-dimensional representations arise from all the irreducible representations of H that do not contain H^2 in its kernel. Start with any representation χ of H that satisfies $\chi(h^2) \neq 1$ for some $h \in H$. Consider the following matrix representation R of $\mathscr{H}$ corresponding to the induced representation of χ of H:

$$R(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R(h) = \begin{pmatrix} \chi(h) & 0 \\ 0 & \chi(h^{-1}) \end{pmatrix}, \tag{3.1}$$

for each $h \in H$. Then this induced representation is irreducible. What's more, such two induced representations are equivalent if and only if they are complex conjugates of each other. Since these representations do not descend to H^2, it is not equal to its complex conjugate. Consequently, we obtain $\frac{n-2^c}{2}$ inequivalent two-dimensional irreducible representations.

Now we are ready to state and prove our main result in this section, which gives a necessary and sufficient condition for the integrality of Cayley graphs over generalized dihedral groups.

Theorem 3.1 *Let H be a finite abelian group and let $\mathscr{H}$ be its generalized dihedral group as given in (1.1). Denote by $S = S_1 \cup bS_2 \subseteq \mathscr{H}$ such that $1_H \notin S$ and $S^{-1} = S$, where S_1, S_2 are subsets of H. Then $Cay(\mathscr{H}, S)$ is integral if and only if $S_1 \in B(H)$ and $\chi(S_2 S_2^{-1})$ is a square number for all irreducible representations of H except for the ones satisfying $\chi(h^2) = 1$ for all $h \in H$.*

Proof From $(bh_j)^2 = h_j^{-1} h_j = 1_H$ for $h_j \in S_2$, we get $(bS_2)^{-1} = bS_2$. Thus, $S = S^{-1}$ if and only if $S_1 = S_1^{-1}$. By Lemma 2.2, one knows that the problem of computing the eigenvalues of the adjacency matrix $A(Cay(\mathscr{H}, S))$ can be fully converted into those of computing the eigenvalues of the matrix $\sum_{s \in S} R(s)$, where $R(s)$ is the representation matrix corresponding to $\rho_{\text{reg}}(s)$. It follows from representation theory that the matrix $\sum_{s \in S} R(s)$ is similar to a block diagonal matrix, as shown in (2.1). This suggests that for our purpose of determining the integrality of $Cay(\mathscr{H}, S)$ it is enough to ensure that the eigenvalues of the matrix $\sum_{s \in S} R_i(s)$ are integers, where R_i ranges over all the irreducible matrix representations of $\mathscr{H}$. Note, from the irreducible representation of $\mathscr{H}$ (as exhibited before), that if R_i is a one-dimensional representation of $\mathscr{H}$, then the eigenvalues of the matrix $\sum_{s \in S} R_i(s)$ are integers. Therefore, only the matrices $\sum_{s \in S} R_i(s)$ where R_i are two-dimensional irreducible representation of $\mathscr{H}$ require further attentions.

Now assume that R is a fixed two-dimensional irreducible representation of $\mathscr{H}$ given as in (3.1). We then have

$$\begin{aligned} \sum_{s \in S} R(s) &= \sum_{s \in S_1} R(s) + \sum_{s \in S_2} R(bs) \\ &= \sum_{s \in S_1} R(s) + \sum_{s \in S_2} R(b) R(s) \\ &= \begin{pmatrix} \chi(S_1) & \chi\left(S_2^{-1}\right) \\ \chi(S_2) & \chi\left(S_1^{-1}\right) \end{pmatrix} \\ &= \begin{pmatrix} \chi(S_1) & \chi\left(S_2^{-1}\right) \\ \chi(S_2) & \chi(S_1) \end{pmatrix}, \end{aligned} \tag{3.2}$$

where the last equality in (3.2) follows from the fact that $S_1 = S_1^{-1}$. Therefore,

$$\det\left(xI_2 - \sum_{s \in S} R(s)\right) = \begin{vmatrix} x - \chi(S_1) & -\chi\left(S_2^{-1}\right) \\ -\chi(S_2) & x - \chi(S_1) \end{vmatrix}$$

$$= (x - \chi(S_1))^2 - \chi(S_2)\chi\left(S_2^{-1}\right)$$
$$= x^2 - 2\chi(S_1)x + (\chi(S_1))^2 - \chi\left(S_2 S_2^{-1}\right).$$

Consequently, the eigenvalues of $\sum_{s \in S} R(s)$ are

$$x_1 = \chi(S_1) + \sqrt{\chi\left(S_2 S_2^{-1}\right)}, \quad x_2 = \chi(S_1) - \sqrt{\chi\left(S_2 S_2^{-1}\right)}. \tag{3.3}$$

If both x_1 and x_2 are integers, then by (3.3), $\chi(S_1) = \frac{x_1+x_2}{2}$ is a rational number. Since $\chi(S_1)$ is an algebraic number, $\chi(S_1)$ is thus forced to be an integer and so does $\sqrt{\chi(S_2 S_2^{-1})}$, which means $\chi(S_2 S_2^{-1})$ is a square number. Conversely, it is obvious that if $\chi(S_1)$ is an integer and $\chi(S_2 S_2^{-1})$ is a square number, then both x_1 and x_2 are integers. Consequently, it follows from Lemma 2.3 and (2.2) that $Cay(\mathscr{H}, S)$ is integral if and only if $S_1 \in B(H)$ and $\chi\left(S_2 S_2^{-1}\right)$ is a square number, where χ ranges over all irreducible representations of H except for the ones satisfying $\chi(h^2) = 1$ for all $h \in H$. This completes the proof. □

As a first corollary of Theorem 3.1, we reobtain a necessary and sufficient condition for the integrality of Cayley graphs over dihedral groups, which is the main result of [17]. Note that [17] used the irreducible characters of the dihedral groups D_n; we list them below. Let $H = \langle a \rangle$ be a cyclic group of order n; thus, $\mathscr{H}$ is reduced to the dihedral group $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$. The irreducible characters of D_n have been completely characterized, e.g., see [21].

Lemma 3.2 [21] *The irreducible characters of D_n are given in Table 1 if n is odd and in Table 2 otherwise, where ψ_i and ϕ_j are, respectively, the irreducible characters of degree one and two for* $1 \leqslant j \leqslant \lfloor \frac{n-1}{2} \rfloor$.

Table 1 Character table of D_n for odd n

	a^k	ba^k
ψ_1	1	1
ψ_2	1	− 1
ϕ_j	$2\cos\left(\frac{2kj\pi}{n}\right)$	0

Table 2 Character table of D_n for even n

	a^k	ba^k
ψ_1	1	1
ψ_2	1	−1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$
ϕ_j	$2\cos\left(\frac{2kj\pi}{n}\right)$	0

Let $S' = S'_1 \cup S'_2 \subseteq D_n \backslash \{1_{D_n}\}$ be such that $S' = S'^{-1}$, where $S'_1 \subseteq \langle a \rangle$ and $S'_2 \subseteq b\langle a \rangle$. Denote ϕ and χ by the irreducible characters of D_n and $\langle a \rangle$, respectively. Assume that $S'_2 = \{ba^{l_1}, ba^{l_2}, \ldots, ba^{l_t}\}$, where $\{l_1, l_2, \ldots, l_t\} \subseteq \{1, 2, \ldots, n\}$. Then it is routine to check that $S'^2_2 = \left\{a^{l_i - l_j} \mid i, j \in \{1, 2, \ldots, t\}\right\}$. On the one hand, after direct calculations, we have

$$\begin{aligned}\phi_h\left(S'^2_2\right) &= 2 \sum_{i,j\in\{1,2,\ldots,t\}} \cos \frac{2\pi h(l_i - l_j)}{n} \\ &= 4 \sum_{1\leqslant i<j\leqslant t} \cos \frac{2\pi h(l_j - l_i)}{n} + 2t, \end{aligned} \tag{3.4}$$

where the first equality in (3.4) follows by Lemma 3.2. On the other hand, note that $\chi_h(a) = \omega^h$, where ω denotes a primitive nth root of unity. Then

$$\begin{aligned}\chi_h\left(S'^2_2\right) &= \sum_{i,j\in\{1,2,\ldots,t\}} \omega^{h(l_i-l_j)} \\ &= \sum_{\substack{i\neq j \\ i,j\in\{1,2,\ldots,t\}}} \omega^{h(l_i-l_j)} + t \\ &= 2 \sum_{1\leqslant i<j\leqslant t} \cos \frac{2\pi h(l_j - l_i)}{n} + t\end{aligned}$$

for $1 \leqslant h \leqslant \lfloor \frac{n-1}{2} \rfloor$. Therefore,

$$\phi_h\left(S'^2_2\right) = 2\chi_h\left(S'^2_2\right). \tag{3.5}$$

By virtue of Theorem 3.1 and (3.5), we can immediately get the following corollary about the integrality of Cayley graphs over dihedral groups.

Corollary 3.3 [17] *Let $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group, and let $S' = S'_1 \cup S'_2 \subseteq D_n$ such that $1_{D_n} \notin S'$ and $S'^{-1} = S'$, where $S'_1 \subseteq \langle a \rangle$ and $S'_2 \subseteq b\langle a \rangle$. Then $Cay(D_n, S')$ is integral if and only if $S'_1 \in B(\langle a \rangle)$ and $2\phi_h(S'^2_2)$ is a square number for all $1 \leqslant h \leqslant \lfloor \frac{n-1}{2} \rfloor$.*

The following three corollaries are immediate consequences of Lemma 2.3 and Theorem 3.1, which can be used to generate infinite families of integral Cayley graphs over $\mathscr{H}$.

Corollary 3.4 *Let H be a finite abelian group, and let $\mathscr{H}$ be its generalized dihedral group. Let $S = S_1 \cup bS_2 \subseteq \mathscr{H}$ such that $1_H \notin S$ and $S^{-1} = S$, where $S_1, S_2 \subseteq H$ with $|S_2| = 1$. Then $Cay(\mathscr{H}, S)$ is integral if and only if $S_1 \in B(H)$.*

Proof We just note that $\chi(S_2 S_2^{-1}) = \chi(1) = 1$ holds for any irreducible representation χ of H. □

Corollary 3.5 *Let H be a finite abelian group, and let $\mathscr{H}$ be its generalized dihedral group. Denote $S = S_1 \cup bS_2 \subseteq \mathscr{H}$ such that $1_H \notin S$ and $S_i^{-1} = S_i$ for $i = 1, 2$, where $S_1, S_2 \subseteq H$. Then $Cay(\mathscr{H}, S)$ is integral if and only if $S_1, S_2 \in B(H)$.*

Proof In this case, we note that $\chi(S_2S_2^{-1}) = \chi(S_2)\chi(S_2^{-1}) = \chi(S_2)^2$. Then the desired result follows from Lemma 2.3 and Theorem 3.1. □

For all irreducible representations of H except for the ones satisfying $\chi(h^2) = 1$ for all $h \in H$, one hopes to characterize those subsets S_2 of H such that $\chi(S_2S_2^{-1})$ is a square number, which seems not to be an easy task. Here, we consider the simplest case. Suppose that H is a finite abelian group of odd order n and that $\chi(S_2S_2^{-1})$ is equal to a fixed square number for all non-trivial representation χ of H, say μ^2. Assume further that the value of μ^2 is less than the size of S_2, i.e., $\mu^2 < |S_2|$. It follows that

$$S_2S_2^{-1} = |S_2|1_H + \big(|S_2| - \mu^2\big)\big(H - 1_H\big).$$

This reveals that S_2 is a difference set in H with parameters $(n, |S_2|, |S_2| - \mu^2)$. We have thus arrived at the following corollary.

Corollary 3.6 *Let H be a finite abelian group of odd order n, and let $\mathscr{H}$ be its generalized dihedral group. Suppose S_1 and S_2 are two subsets of H satisfying $1_H \notin S_1$, $S_1 \in B(H)$ and S_2 is an $(n, |S_2|, \lambda)$-difference set of H. If $|S_2| - \lambda$ is a square number, then $Cay(\mathscr{H}, S)$ is integral.*

We mention that there are two well-known classes of difference sets meeting the requirements of Corollary 3.6. One is a particular class of the famous singer difference set: Let q be an even power of a prime number. Then there exists a $\big(q^2 + q + 1, q + 1, 1\big)$-difference set in the additive group of integers modulo $q^2 + q + 1$. The second class is the so-called twin prime power difference set: Let q and $q + 2$ be odd prime powers. Then there exists a difference set in the group $(\mathbb{F}_q, +) \oplus (\mathbb{F}_{q+2}, +)$ with parameters $\Big(q^2 + 2q, \frac{q^2+2q-1}{2}, \frac{q^2+2q-3}{4}\Big)$. Please refer to [7] for the constructions of such difference sets.

We conclude this section with the following corollary, which gives a necessary condition for the integrality of $Cay(\mathscr{H}, S)$.

Corollary 3.7 *Let H be a finite abelian group, and let $\mathscr{H}$ be its generalized dihedral group. Denote by $S = S_1 \cup bS_2 \subseteq \mathscr{H}$ such that $1_H \notin S$ and $S^{-1} = S$, where $S_1, S_2 \subseteq H$. If $Cay(\mathscr{H}, S)$ is integral, then $S_1 \in B(H)$ and $S_2S_2^{-1} \in C(H)$.*

4 Distance integral Cayley graphs over generalized dihedral groups

In this section, we study the distance integrality of $Cay(\mathscr{H}, S)$, where S is a non-empty subset of $\mathscr{H}$ satisfying $1_H \notin S$, $S = S^{-1}$ and $\langle S\rangle = \mathscr{H}$. A necessary and sufficient condition for the distance integrality of $Cay(\mathscr{H}, S)$ is derived, and some infinite families of distance integral Cayley graphs over generalized dihedral groups are constructed. To do this, we firstly establish a closed-form formula for the distance matrix of Cayley graphs over any finite groups by using the Fourier transform of finite groups.

For convenience, we begin to restate the main results about the Fourier transform of finite groups (see also [23]). We adopt the notation in [23]. Let G be a finite group, and let $L(G)$ be the vector space of all complex-valued functions defined on G. Then the set $\{\delta_g \mid g \in G\}$ is an orthogonal basis of $L(G)$, where $\delta_g(x) = \begin{cases} 1, & \text{if } g = x; \\ 0, & \text{if } g \neq x. \end{cases}$ Define a representation ρ_{reg} of G on $L(G)$ by setting

$$[\rho_{\text{reg}}(g)f](g_0) = f\left(g^{-1}g_0\right) \text{ for all } g, g_0 \in G \text{ and } f \in L(G).$$

This is indeed a representation:

$$[\rho_{\text{reg}}(g_1g_2)f](g_0) = f\left(g_2^{-1}g_1^{-1}x\right) = [\rho_{\text{reg}}(g_2)f]\left(g_1^{-1}g_0\right) = \rho_{\text{reg}}(g_1)[\rho_{\text{reg}}(g_2)f](g_0).$$

That is, $\rho_{\text{reg}}(g_1g_2) = \rho_{\text{reg}}(g_1)\rho_{\text{reg}}(g_2)$. The above representation ρ_{reg} is called the *regular representation*. Let $\widehat{G}$ be a fixed set of irreducible unitary pairwise inequivalent representation of G. Given two finite-dimensional vector spaces V and W over the complex field $\mathbb{C}$, we denote by $Hom(V, W)$ the space of all linear maps from V to W.

In what follows, set $\mathcal{A}(G) = \bigoplus_{\rho \in \widehat{G}} Hom(W_\rho, W_\rho)$, where W_ρ denotes the vector space corresponding to the representation ρ. For every $\rho \in \widehat{G}$, fix an orthonormal basis $\{v_1^\rho, v_2^\rho, \ldots, v_{d_\rho}^\rho\}$ in the representation space W_ρ. Define the element $T_{i,j}^\rho \in \mathcal{A}(G)$ by $T_{i,j}^\rho(w) = \delta_{\rho,\sigma}\langle w, v_j^\rho\rangle_{W_\rho} v_i^\rho$ for $w \in W_\sigma, \sigma \in \widehat{G}$ and $i, j = 1, 2, \ldots, d_\rho$, where $\delta_{\rho,\sigma}$ is the indicator function $\delta_{\rho,\sigma} = \begin{cases} 1, & \text{if } \rho = \sigma; \\ 0, & \text{if } \rho \neq \sigma. \end{cases}$ It has been proved that the set $\{T_{i,j}^\rho \mid \rho \in \widehat{G}, i, j = 1, 2, \ldots, d_\rho\}$ is an orthogonal basis of $\mathcal{A}(G)$.

The *Fourier transform* is the linear map $\mathcal{F} : L(G) \to \mathcal{A}(G)$ which satisfies $\mathcal{F}(f) = \bigoplus_{\rho \in \widehat{G}} \rho(f)$ for $f \in L(G)$. In correspondence with the orthonormal basis $\{v_1^\rho, v_2^\rho, \ldots, v_{d_\rho}^\rho\}$, we define $\varphi_{i,j}^\rho(g) = \langle \rho(g)v_j^\rho, v_i^\rho\rangle_{W_\rho}$. With the above notations, it has been showed that the Fourier transform has the following property.

Lemma 4.1 [23] *Let $\mathcal{F}$ be the Fourier transform from $L(G)$ to $\mathcal{A}(G)$. Then*

$$\mathcal{F}\overline{\varphi_{i,j}^\rho} = \frac{|G|}{d_\rho} T_{i,j}^\rho,$$

where $\overline{\varphi_{i,j}^\rho}$ is the conjugate of $\varphi_{i,j}^\rho$.

Given a Cayley graph $Cay(G, S)$ with $1_G \notin S = S^{-1}$ and $\langle S\rangle = G$, define a function $\ell_S : G \to \mathbb{C}$ such that

$$\ell_S(g) = \begin{cases} \min\{k \mid g = s_1s_2\ldots s_k, s_i \in S\}, & \text{if } g \neq 1; \\ 0, & \text{if } g = 1. \end{cases}$$

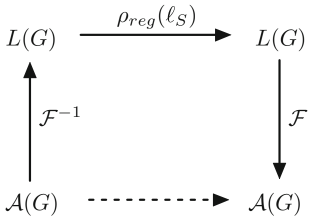

Fig. 1 Composition of transformations from $\mathcal{A}(G)$ to $\mathcal{A}(G)$

Then it is obvious that $d_{Cay(G,S)}(g, h) = \ell_S(gh^{-1})$ for $g, h \in G$. Note that

$$\left(\rho_{\text{reg}}(\ell_S)\right)(\delta_h) = \sum_{g\in G} \ell_S(g)\rho_{\text{reg}}(g)(\delta_h) = \sum_{g\in G} \ell_S(g)\delta_{gh} = \sum_{g\in G} \ell_S(gh^{-1})\delta_g.$$

As given in [9], the matrix of $\rho_{\text{reg}}(\ell_S)$ with respect to the basis $\{\delta_g \mid g \in G\}$ of $L(G)$ is exactly the distance matrix $D(Cay(G, S))$.

In the following, we use the composition of transformations as depicted in Fig. 1 to find the eigenvalues of the distance matrix $D(Cay(G, S))$. By Lemma 4.1, we have

$$\begin{aligned}
\left[\mathcal{F}\rho_{\text{reg}}(\ell_S)\mathcal{F}^{-1}\right] T^{\rho}_{i,j} &= \frac{d_\rho}{|G|}\mathcal{F}\rho_{\text{reg}}(\ell_S)\overline{\varphi^{\rho}_{i,j}} \\
&= \frac{d_\rho}{|G|}\mathcal{F}\left(\sum_{g\in G}\ell_S(g)\rho_{\text{reg}}(g)\overline{\varphi^{\rho}_{i,j}}\right) \\
&= \frac{d_\rho}{|G|}\mathcal{F}\left(\sum_{g\in G}\ell_S(g)\sum_{k=1}^{d_\rho}\varphi^{\rho}_{k,i}(g)\overline{\varphi^{\rho}_{k,j}}\right) \qquad (4.1)\\
&= \sum_{g\in G}\ell_S(g)\left(\sum_{k=1}^{d_\rho}\varphi^{\rho}_{k,i}(g)T^{\rho}_{k,j}\right) \\
&= \sum_{k=1}^{d_\rho}\left(\sum_{g\in G}\ell_S(g)\varphi^{\rho}_{k,i}(g)\right)T^{\rho}_{k,j}, \qquad (4.2)
\end{aligned}$$

where the third equality in (4.1) follows from the fact that

$$\begin{aligned}
\rho_{\text{reg}}(g)\overline{\varphi^{\rho}_{i,j}}(h) &= \overline{\varphi^{\rho}_{i,j}}(g^{-1}h) \\
&= \overline{\langle\rho(g^{-1}h)v^{\rho}_j, v^{\rho}_i\rangle} \\
&= \overline{\langle\rho(h)v^{\rho}_j, \rho(g)v^{\rho}_i\rangle} \\
&= \overline{\left\langle\rho(h)v^{\rho}_j, \sum_{k=1}^{d_\rho}\left\langle\rho(g)v^{\rho}_i, v^{\rho}_k\right\rangle v^{\rho}_k\right\rangle}
\end{aligned}$$

$$= \sum_{k=1}^{d_\rho} \langle \rho(g)v_i^\rho, v_k^\rho \rangle \overline{\langle \rho(h)v_j^\rho, v_k^\rho \rangle}$$

$$= \sum_{k=1}^{d_\rho} \varphi_{k,i}^\rho(g)\overline{\varphi_{k,j}^\rho}(h)$$

for any $h \in G$. Equality (4.2) indicates that $Hom(W_\rho, W_\rho)$ is an invariant subspace of $\mathcal{F}\rho_{\text{reg}}(\ell_S)\mathcal{F}^{-1}$ for each $\rho \in \widehat{G}$. Therefore, we have the following decomposition formula about the distance matrix of Cayley graphs over any finite groups.

Theorem 4.2 *Let G be a finite group with $S \subseteq G$ such that $1_G \notin S = S^{-1}$ and $\langle S \rangle = G$. Let $\rho_1, \rho_2, \ldots, \rho_h$ be all inequivalent irreducible unitary representations of G with $d_1, d_2, \ldots, d_h$ as their degrees, respectively. Then there exists an invertible matrix Q such that*

$$QD(Cay(G, S))Q^{-1} = d_1\Phi(\rho_1) \bigoplus d_2\Phi(\rho_2) \bigoplus \cdots \bigoplus d_h\Phi(\rho_h),$$

where $\Phi(\rho_k)$ denotes the $d_k \times d_k$ matrix whose (i, j)-entry is equal to $\sum_{g\in G} \ell_S(g)$ $\varphi_{i,j}^{\rho_k}(g)$ for $i, j = 1, 2, \ldots, d_k$ and $k = 1, 2, \ldots, h$. Consequently,

$$S_D(Cay(G, S)) = \left\{[\mu_{1,1}]^{d_1}, \ldots, [\mu_{1,d_1}]^{d_1}, \ldots, [\mu_{h,1}]^{d_h}, \ldots, [\mu_{h,d_h}]^{d_h}\right\},$$

where $\mu_{i,1}, \ldots, \mu_{i,d_i}$ are all eigenvalues of the matrix $\Phi(\rho_i)$ for $1 \leqslant i \leqslant h$.

We are now in a position to give a necessary and sufficient condition for the distance integrality of Cayley graphs over generalized dihedral groups.

Theorem 4.3 *Let H be a finite abelian group, and let $\mathscr{H}$ be its generalized dihedral group as given in (1.1). Denote by $S \subseteq \mathscr{H}$ such that $1_H \notin S = S^{-1}$ and $\langle S \rangle = \mathscr{H}$. Then $Cay(\mathscr{H}, S)$ is distance integral if and only if $\sum_{h\in H} \ell_S(h)h \in C(H)$ and $\chi\left[\sum_{h\in H} \ell_S(bh)h \cdot \sum_{h\in H} \ell_S(bh)h^{-1}\right]$ is a square number for each irreducible representation of H except for the ones satisfying $\chi(h^2) = 1$ for all $h \in H$.*

Proof Given a representation $\rho \in \widehat{\mathscr{H}}$, then $d_\rho = 1$ or $d_\rho = 2$. If $d_\rho = 1$, assume that $W_\rho = \{\alpha v_1^\rho | \alpha \in \mathbb{C}\}$. Then

$$\Phi(\rho) = \sum_{g\in G} \ell_S(g)\varphi_{1,1}^\rho(g) = \sum_{g\in G} \ell_S(g)\langle \rho(g)v_1^\rho, v_1^\rho \rangle = \sum_{g\in G} \ell_S(g)\rho(g).$$

Since both $\ell_S(g)$ and $\rho(g)$ are integers, $\Phi(\rho)$ is an integer. If $d_\rho = 2$, assume that $W_\rho = \{\beta v_1^\rho + \gamma v_2^\rho | \beta, \gamma \in \mathbb{C}\}$, where $\{v_1^\rho, v_2^\rho\}$ is an orthonormal basis corresponding to the induced representation (3.1). It follows from Theorem 4.2 that

$$\Phi(\rho) = \begin{pmatrix} \sum_{g\in G} \ell_S(g)\varphi_{1,1}^\rho(g) & \sum_{g\in G} \ell_S(g)\varphi_{1,2}^\rho(g) \\ \sum_{g\in G} \ell_S(g)\varphi_{2,1}^\rho(g) & \sum_{g\in G} \ell_S(g)\varphi_{2,2}^\rho(g) \end{pmatrix}. \tag{4.3}$$

In view of (3.1), we have

$$\begin{cases}\rho(b)v_1^\rho = v_2^\rho,\\ \rho(b)v_2^\rho = v_1^\rho,\\ \rho(h)v_1^\rho = \chi(h)v_1^\rho,\\ \rho(h)v_2^\rho = \chi(h^{-1})v_2^\rho\end{cases}$$

for each $h \in H$. Thus, we can obtain

$$\begin{cases}\varphi_{1,1}^\rho(h_i) = \langle \rho(h_i)v_1^\rho, v_1^\rho\rangle = \langle \chi(h_i)v_1^\rho, v_1^\rho\rangle = \chi(h_i),\\ \varphi_{1,1}^\rho(bh_i) = \langle \rho(bh_i)v_1^\rho, v_1^\rho\rangle = \langle \chi(h_i)\rho(b)v_1^\rho, v_1^\rho\rangle = \langle \chi(h_i)v_2^\rho, v_1^\rho\rangle = 0,\\ \varphi_{1,2}^\rho(h_i) = \langle \rho(h_i)v_2^\rho, v_1^\rho\rangle = \langle \chi(h_i^{-1})v_2^\rho, v_1^\rho\rangle = 0,\\ \varphi_{1,2}^\rho(bh_i) = \langle \rho(bh_i)v_2^\rho, v_1^\rho\rangle = \langle \chi(h_i^{-1})\rho(b)v_2^\rho, v_1^\rho\rangle = \langle \chi(h_i^{-1})v_1^\rho, v_1^\rho\rangle = \chi(h_i^{-1}),\\ \varphi_{2,1}^\rho(h_i) = \langle \rho(h_i)v_1^\rho, v_2^\rho\rangle = \langle \chi(h_i)v_1^\rho, v_2^\rho\rangle = 0,\\ \varphi_{2,1}^\rho(bh_i) = \langle \rho(bh_i)v_1^\rho, v_2^\rho\rangle = \langle \chi(h_i)\rho(b)v_1^\rho, v_2^\rho\rangle = \langle \chi(h_i)v_2^\rho, v_2^\rho\rangle = \chi(h_i),\\ \varphi_{2,2}^\rho(h_i) = \langle \rho(h_i)v_2^\rho, v_2^\rho\rangle = \langle \chi(h_i^{-1})v_2^\rho, v_2^\rho\rangle = \chi(h_i^{-1}),\\ \varphi_{2,2}^\rho(bh_i) = \langle \rho(bh_i)v_2^\rho, v_2^\rho\rangle = \langle \chi(h_i^{-1})\rho(b)v_2^\rho, v_2^\rho\rangle = \langle \chi(h_i^{-1})v_1^\rho, v_2^\rho\rangle = 0\end{cases} \tag{4.4}$$

for $i = 0, 1, \ldots, n-1$. By equalities (4.3) and (4.4), one has

$$\Phi(\rho) = \begin{pmatrix}\sum_{h\in H}\ell_S(h)\chi(h) & \sum_{h\in H}\ell_S(bh)\chi(h^{-1})\\ \sum_{h\in H}\ell_S(bh)\chi(h) & \sum_{h\in H}\ell_S(h)\chi(h^{-1})\end{pmatrix}.$$

Recall that $S = S^{-1}$, we get $\ell_S(h) = \ell_S(h^{-1})$ for all $h \in H$. Then

$$\sum_{h\in H}\ell_S(h)\chi(h^{-1}) = \sum_{h\in H}\ell_S(h^{-1})\chi(h^{-1}) = \sum_{h^{-1}\in H}\ell_S(h)\chi(h) = \sum_{h\in H}\ell_S(h)\chi(h).$$

This leads to

$$\Phi(\rho) = \begin{pmatrix}\sum_{h\in H}\ell_S(h)\chi(h) & \sum_{h\in H}\ell_S(bh)\chi(h^{-1})\\ \sum_{h\in H}\ell_S(bh)\chi(h) & \sum_{h\in H}\ell_S(h)\chi(h)\end{pmatrix}. \tag{4.5}$$

Therefore,

$$\begin{aligned}\det(xI_2 - \Phi(\rho)) &= \begin{vmatrix}x - \sum_{h\in H}\ell_S(h)\chi(h) & -\sum_{h\in H}\ell_S(bh)\chi(h^{-1})\\ -\sum_{h\in H}\ell_S(bh)\chi(h) & x - \sum_{h\in H}\ell_S(h)\chi(h)\end{vmatrix}\\ &= \left(x - \sum_{h\in H}\ell_S(h)\chi(h)\right)^2 - \sum_{h\in H}\ell_S(bh)\chi(h)\cdot\sum_{h\in H}\ell_S(bh)\chi(h^{-1})\end{aligned}$$

$$= \left[x - \chi\left(\sum_{h\in H}\ell_S(h)h\right)\right]^2 - \chi\left(\sum_{h\in H}\ell_S(bh)h\cdot\sum_{h\in H}\ell_S(bh)h^{-1}\right).$$

Consequently, the eigenvalues of $\Phi(\rho)$ are

$$x_1 = \chi\left(\sum_{h\in H}\ell_S(h)h\right) + \sqrt{\chi\left(\sum_{h\in H}\ell_S(bh)h\cdot\sum_{h\in H}\ell_S(bh)h^{-1}\right)}, \tag{4.6}$$

$$x_2 = \chi\left(\sum_{h\in H}\ell_S(h)h\right) - \sqrt{\chi\left(\sum_{h\in H}\ell_S(bh)h\cdot\sum_{h\in H}\ell_S(bh)h^{-1}\right)}. \tag{4.7}$$

If both x_1 and x_2 are integers, then by (4.6) and (4.7), $\chi\left(\sum_{h\in H}\ell_S(h)h\right) = \frac{x_1+x_2}{2}$ is a rational number. Note that $\chi\left(\sum_{h\in H}\ell_S(h)h\right)$ is an algebraic number, and $\chi\left(\sum_{h\in H}\ell_S(h)h\right)$ is thus forced to be an integer. Then

$$\chi\left[\sum_{h\in H}\ell_S(bh)h\cdot\sum_{h\in H}\ell_S(bh)h^{-1}\right]$$

is a square number from (4.6). Conversely, if $\chi\left(\sum_{h\in H}\ell_S(h)h\right)$ is an integer and

$$\chi\left[\sum_{h\in H}\ell_S(bh)h\cdot\sum_{h\in H}\ell_S(bh)h^{-1}\right]$$

is a square number, then both x_1 and x_2 are integers from (4.6) and (4.7).

Therefore, combining Lemma 2.4 and the arbitrariness of ρ yields that $Cay(\mathscr{H}, S)$ is distance integral if and only if $\sum_{h\in H}\ell_S(h)h$ is in $C(H)$ and $\chi\left[\sum_{h\in H}\ell_S(bh)(h)\cdot\sum_{h\in H}\ell_S(bh)h^{-1}\right]$ is a square number for each irreducible representation of H except for the ones satisfying $\chi(h^2) = 1$ for all $h \in H$. This completes the proof. □

By Theorem 4.3, we can obtain infinite families of distance integral Cayley graphs over generalized dihedral groups in the following two corollaries.

Corollary 4.4 *Let H be a finite abelian group, and let $\mathscr{H}$ be its generalized dihedral group. Let $S = S_1 \cup bS_2 \subseteq \mathscr{H}$ such that $1_H \notin S$ and $S^{-1} = S$, where $S_1, S_2 \subseteq H$ with $|S_2| = 1$. Then $Cay(\mathscr{H}, S)$ is distance integral if and only if $\sum_{h\in H}\ell_{S_1}(h)h \in C(H)$.*

Proof Assume that $S_2 = \{x_0\}$, then $S = S_1 \cup \{bx_0\}$. Fix an element $h \in H$, and assume that $\ell_S(bh) = r + 1$ with $bh = bx_0x_1x_2\ldots x_r$, where $x_1, x_2, \ldots, x_r \in S_1$. Then $x_0^{-1}h = x_1x_2\ldots x_r$. Thus, $\ell_{S_1}(x_0^{-1}h) \leqslant r = \ell_S(bh) - 1$. Similarly, $\ell_S(bh) \leqslant \ell_{S_1}(x_0^{-1}h) + 1$. Therefore, $\ell_S(bh) = \ell_{S_1}(x_0^{-1}h) + 1$ for every $h \in H$. Consequently,

$$\begin{aligned}\sum_{h\in H}\ell_S(bh)h\cdot\sum_{h\in H}\ell_S(bh)h^{-1} &= \sum_{h\in H}\ell_S(bh)h\cdot\sum_{h\in H}\ell_S(bh^{-1})h\\ &= \sum_{h\in H}(\ell_{S_1}(x_0^{-1}h)+1)h\cdot\sum_{h\in H}(\ell_{S_1}(x_0^{-1}h^{-1})+1)h\\ &= \left(\sum_{h\in H}\ell_{S_1}(x_0^{-1}h)h+\sum_{h\in H}h\right)\left(\sum_{h\in H}\ell_{S_1}(x_0^{-1}h^{-1})h+\sum_{h\in H}h\right).\end{aligned} \tag{4.8}$$

For any non-trivial irreducible representation χ of H, there exists $h'\in H$ such that $\chi(h')\neq 1$. Then

$$\chi(h')\chi\left(\sum_{h\in H}h\right)=\chi\left(\sum_{h\in H}hh'\right)=\chi\left(\sum_{h\in H}h\right).$$

Hence, $\chi\left(\sum_{h\in H}h\right)=0$. Expanding the right-hand side of (4.8) and after a short calculation, we have, for any non-trivial irreducible representation χ of H, that

$$\begin{aligned}\chi\left(\sum_{h\in H}\ell_S(bh)h\cdot\sum_{h\in H}\ell_S(bh)h^{-1}\right) &= \chi\left(\sum_{h\in H}\ell_{S_1}(x_0^{-1}h)h\cdot\sum_{h\in H}\ell_{S_1}(x_0^{-1}h^{-1})h\right)\\ &= \chi\left(\sum_{h\in H}\ell_{S_1}(h)h\cdot\sum_{h\in H}\ell_{S_1}(h)h^{-1}\right)\\ &= \chi\left(\sum_{h\in H}\ell_{S_1}(h)h\cdot\sum_{h\in H}\ell_{S_1}(h^{-1})h\right)\\ &= \left[\chi\left(\sum_{h\in H}\ell_{S_1}(h)h\right)\right]^2,\end{aligned} \tag{4.9}$$

where the last equality in (4.9) follows from the fact that $S_1=S_1^{-1}$. Then the desired result follows from Lemma 2.4 and Theorem 4.3. □

Corollary 4.5 *Let H be a finite abelian group, and let $\mathscr{H}$ be its generalized dihedral group. Denote by $S=S_1\cup bS_2\subseteq\mathscr{H}$ such that $1_H\notin S$ and $S_i^{-1}=S_i$ for $i=1,2$, where $S_1,S_2\subseteq H$. Then $Cay(\mathscr{H},S)$ is distance integral if and only if both $\sum_{h\in H}\ell_S(h)h$ and $\sum_{h\in H}\ell_S(bh)h$ are in $C(H)$.*

Proof First we show that $\ell_S(bh)=\ell_S(bh^{-1})$ for all $h\in H$. In fact, note that bh can be expressed as $bh=x_1x_2\ldots x_k$, where $x_i\in S_1$ or $x_i=bs_i\in bS_2$ for $1\leqslant i\leqslant k$. Then

$$bh^{-1}=b(bh)b=bx_1x_2\ldots x_kb=\begin{cases}(bx_1x_2\ldots x_{k-1}b)x_k^{-1}, & \text{if } x_k\in S_1;\\ (bx_1x_2\ldots x_{k-1}b)bs_k^{-1}, & \text{if } x_k=bs_k\in bS_2.\end{cases}$$

Iterating the above argument yields

$$bh^{-1} = x_1'x_2' \dots x_k', \text{ where } x_i' = \begin{cases} x_i^{-1}, & \text{if } x_i \in S_1; \\ bs_i^{-1}, & \text{if } x_i = bs_i \in bS_2 \end{cases} \text{ for } i = 1, 2, \dots, k.$$

We know from $S_i^{-1} = S_i$ for $i = 1, 2$ that $x_i' \in S$, and thus, $\ell_S(bh^{-1}) \leqslant \ell_S(bh)$. Similarly, $\ell_S(bh) \leqslant \ell_S(bh^{-1})$. Therefore, $\ell_S(bh^{-1}) = \ell_S(bh)$, which implies that

$$\sum_{h\in H} \ell_S(bh)h^{-1} = \sum_{h^{-1}\in H} \ell_S(bh^{-1})h = \sum_{h\in H} \ell_S(bh)h.$$

The desired result then follows from Lemma 2.4 and Theorem 4.3. □

The following corollary gives a necessary condition for the distance integrality of $Cay(\mathscr{H}, S)$.

Corollary 4.6 *Let H be a finite abelian group, and let $\mathscr{H}$ be its generalized dihedral group. Denote by $S \subseteq \mathscr{H}$ such that $1_H \notin S = S^{-1}$ and $\langle S\rangle = \mathscr{H}$. If $Cay(\mathscr{H}, S)$ is distance integral, then both $\sum_{h\in H} \ell_S(h)h$ and $\sum_{h\in H} \ell_S(bh)h \cdot \sum_{h\in H} \ell_S(bh)h^{-1}$ are in $C(H)$.*

5 Relationships between integral and distance integral Cayley graphs over generalized dihedral groups

In this section, we focus on the relations between the integral Cayley graphs and the distance integral Cayley graphs over generalized dihedral groups. Here, we first show that the integrity and distance integrity of Cayley graphs over generalized dihedral groups can be equivalent under some special conditions.

Theorem 5.1 *Let H be a finite abelian group, and let $\mathscr{H}$ be its generalized dihedral group. Let $S = S_1 \cup bS_2 \subseteq \mathscr{H}$ such that $1_H \notin S$ and $S^{-1} = S$, where $S_1, S_2 \subseteq H$ with $|S_2| = 1$. Then $Cay(\mathscr{H}, S)$ is integral if and only if $Cay(\mathscr{H}, S)$ is distance integral.*

Proof In view of Corollaries 3.4 and 4.4, it suffices for us to show that $S_1 \in B(H)$ if and only if $\sum_{h\in H} \ell_{S_1}(h)h \in C(H)$. Note that $S_1 = \{h \in H \mid \ell_{S_1}(h) = 1\}$. The sufficiency is thus obvious.

Suppose conversely that $\widetilde{B}(H) = \{[g_1], [g_2], \dots, [g_k]\}$ for some integer k. Let $\langle h_1\rangle = \langle h_2\rangle \in \widetilde{B}(H)$ with $\ell_S(h_1) = q$ and $ord(h_1) = t$. Then $h_2 = h_1^l$ for some integer l, which leads to $\gcd(l, t) = 1$. Recall that H is of order n, then t is a divisor of n (abbreviated $t \mid n$). Thus, there exists a surjective group homomorphism $f : \mathbb{Z}_n^* \to \mathbb{Z}_t^*$ such that $f(x \pmod n) = x \pmod t$, where $\mathbb{Z}_n^* = \{n' \mid \gcd(n', n) = 1\}$. Recall that $l \in \mathbb{Z}_t^*$. Then there exists $y \in \mathbb{Z}_n^*$ such that $f(y \pmod n) = l \pmod t = y \pmod t$. Therefore, $t \mid (l - y)$, which gives $h_2 = h_1^l = h_1^y$. Note that h_1 can be expressed as $h_1 = z_1z_2 \dots z_q$, where $z_i \in S_1$ for $1 \leqslant i \leqslant q$. Then $h_2 = z_1^y z_2^y \dots z_q^y$. Recall that $\gcd(y, n) = 1$, we thus have $\gcd(y, ord(z_i)) = 1$, leading to $z_i^y \in \langle z_i\rangle \in S_1$ for all $1 \leqslant i \leqslant q$. Therefore, $\ell_{S_1}(h_2) \leqslant \ell_{S_1}(h_1)$. In a similar way, $\ell_{S_1}(h_1) \leqslant \ell_{S_1}(h_2)$. Consequently, $\ell_{S_1}(h_1) = \ell_{S_1}(h_2)$ whenever $\langle h_1 \rangle = \langle h_2 \rangle$, and therefore, we conclude that $\sum_{h \in H} \ell_{S_1}(h)h \in C(H)$ as desired. □

Theorem 5.2 *Let H be a finite abelian group, and let $\mathscr{H}$ be its generalized dihedral group. Denote by $S = S_1 \cup bS_2 \subseteq \mathscr{H}$ such that $1_H \notin S$ and $S_i^{-1} = S_i$ for $i = 1, 2$, where $S_1, S_2 \subseteq H$. Then $Cay(\mathscr{H}, S)$ is integral if and only if $Cay(\mathscr{H}, S)$ is distance integral.*

Proof In view of Corollaries 3.5 and 4.5, it suffices to show that $S_1, S_2 \in B(H)$ if and only if $\sum_{h \in H} \ell_S(h)h \in C(H)$ and $\sum_{h \in H} \ell_S(bh)h \in C(H)$. Note that $S_1 = \{h \in H \mid \ell_S(h) = 1\}$ and $S_2 = \{h \in H \mid \ell_S(bh) = 1\}$. The sufficiency is thus obvious.

Conversely, assume that $\widetilde{B}(H) = \{[g_1], [g_2], \ldots, [g_w]\}$ for some integer w. Let $\langle h_1 \rangle = \langle h_2 \rangle \in \widetilde{B}(H)$ with $\ell_S(h_1) = r$. Just as we did in the proof of Theorem 5.1, there exists an integer k with $\gcd(k, n) = 1$ such that $h_2 = h_1^k$. Note that h_1 can be expressed as $h_1 = x_1 x_2 \ldots x_r$, where $x_i \in S_1$ or $x_i = bs_i \in bS_2$ for $1 \leqslant i \leqslant r$. Furthermore, assume that $x_i \in [g_j] \in \widetilde{B}(H)$ if $x_i \in S_1$ and $s_i \in [g_j] \in \widetilde{B}(H)$ if $x_i = bs_i \in bS_2$ for some $j \in [1, w]$. Moving all b contained in h_1 to the leftmost according to the same operation as in Corollary 4.5 yields

$$h_1 = x_1 x_2' \ldots x_r', \text{ where } x_i' \in \begin{cases} \{x_i, x_i^{-1}\}, & \text{if } x_i \in S_1; \\ \{s_i, s_i^{-1}\}, & \text{if } x_i = bs_i \in bS_2 \end{cases} \text{ for } i = 1, 2, \ldots, r.$$

Then $h_2 = h_1^k = (x_1')^k (x_2')^k \ldots (x_r')^k = x_1'' x_2'' \ldots x_r''$, where

$$x_i'' \in \begin{cases} \{x_i^k, x_i^{-k}\}, & \text{if } x_i \in S_1; \\ \{bs_i^k, bs_i^{-k}\}, & \text{if } x_i = bs_i \in bS_2. \end{cases}$$

Since $\gcd(k, n) = 1$, we get $\gcd(k, ord(x_i)) = 1$ or $\gcd(k, ord(s_i)) = 1$. Then $\{x_i^k, x_i^{-k}\} \subseteq [g_j] \subseteq S_1$ or $\{s_i^k, s_i^{-k}\} \subseteq [g_j] \subseteq S_2$ for $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant w$. Consequently, $\ell_S(h_2) \leqslant \ell_S(h_1)$. Similarly, $\ell_S(h_1) \leqslant \ell_S(h_2)$. Therefore, $\ell_S(h_1) = \ell_S(h_2)$ whenever $\langle h_1 \rangle = \langle h_2 \rangle$, implying that $\sum_{h \in H} \ell_S(h)h \in C(H)$. In a similar way, we have $\sum_{h \in H} \ell_S(bh)h \in C(H)$ as desired. This completes the proof. □

It is worth noting that the conditions $|S_2| = 1$ and $S_2 = S_2^{-1}$ in Theorems 5.1-5.2 cannot be omitted in general. We give a counterexample in the following.

Example 5.3 Let $D_{21} = \langle a, b \mid a^{21} = b^2 = 1, bab = a^{-1} \rangle$ be the dihedral group of order 42. Put $S_1 = \left\{a^7, a^{14}\right\}$, $S_2 = \left\{a^7, a^9, a^{14}, a^{15}, a^{18}\right\}$ and $S = S_1 \cup bS_2$. On the one hand, note that

$$\widetilde{B}(\langle a \rangle) = \Big\{\{1\}, \{a, a^2, a^4, a^5, a^8, a^{10}, a^{11}, a^{13}, a^{16}, a^{17}, a^{19}, a^{20}\}, \\ \{a^3, a^6, a^9, a^{12}, a^{15}, a^{18}\}, \{a^7, a^{14}\}\Big\}$$

and $\{a^7, a^9, a^{14}, a^{15}, a^{18}\}$ is a $(21, 5, 1)$-difference set of $\langle a\rangle$. By Corollary 3.6, $Cay(D_{21}, S)$ is integral. However, by direct calculations, we have

$$\ell_S\left(ba^7\right) = \ell_S\left(ba^9\right) = \ell_S\left(ba^{14}\right) = \ell_S\left(ba^{15}\right) = \ell_S\left(ba^{18}\right) = 1,$$
$$\ell_S(b) = \ell_S(ba) = \ell_S\left(ba^2\right) = \ell_S\left(ba^4\right) = \ell_S\left(ba^8\right) = \ell_S\left(ba^{11}\right) = \ell_S\left(ba^{16}\right) = 2,$$
$$\ell_S\left(ba^3\right) = \ell_S\left(ba^5\right) = \ell_S\left(ba^6\right) = \ell_S\left(ba^{10}\right) = \ell_S\left(ba^{12}\right) = \ell_S\left(ba^{13}\right) = \ell_S\left(ba^{17}\right)$$
$$= \ell_S\left(ba^{19}\right) = \ell_S\left(ba^{20}\right) = 3.$$

Therefore, $\chi_3\left(\sum_{k=0}^{11} \ell\left(ba^k\right) a^k \cdot \sum_{k=0}^{11} \ell_S\left(ba^k\right) a^{-k}\right) = 37$ is not a square number. Consequently, $Cay(D_{21}, S)$ is not distance integral by Theorem 4.3.

Although the conclusions of Theorems 5.1 and 5.2 may fail in general if we remove the assumptions $|S_2| = 1$ and $S_2 = S_2^{-1}$, we can still find a special kind of Cayley graphs over dihedral groups, which satisfy the integrality implies distance integrality and vice versa.

In the following, we discuss the distance integrality of Cayley graphs over dihedral groups D_p, where $p \geqslant 3$ is a prime. It is inspired by Lu et al. [17], who have completely determined all integral Cayley graphs over the dihedral group D_p. It turns out that $Cay(D_p, S)$ is integral if and only if it is distance integral.

We first establish the following result.

Theorem 5.4 *Let $D_p = \langle a, b \mid a^p = b^2 = 1, bab = a^{-1}\rangle$ with $p \geqslant 3$ being a prime. Let $S = S_1 \cup bS_2 \subseteq D_p$ such that $1 \notin S = S^{-1}$ and $\langle S\rangle = G$, where $S_1, S_2 \subseteq \langle a\rangle$. Then $Cay(D_p, S)$ is distance integral if and only if either $S_1 = \emptyset$ and $S_2 \in \left\{\langle a\rangle\backslash\{a^k\}, \langle a\rangle\right\}$ for $1 \leqslant k \leqslant p-1$ or $S_1 = \langle a\rangle\backslash\{1\}$ and $S_2 \in \left\{\{a^k\}, \langle a\rangle\backslash\{a^k\}, \langle a\rangle\right\}$ for $0 \leqslant k \leqslant p-1$.*

Proof Recall that $\chi_i\left(a^k\right) = \omega^{ki}$ are irreducible representations of $\langle a\rangle$ for $0 \leqslant i \leqslant p-1$, where ω is a primitive pth root of unity in the complex field. By Theorem 4.3, we know that $Cay(D_p, S)$ is distance integral if and only if $\sum_{k=1}^{p-1} \ell_S\left(a^k\right)\omega^{ki}$ is an integer and

$$\sum_{k=0}^{p-1} \ell_S\left(ba^k\right)\omega^{ki} \cdot \sum_{k=0}^{p-1} \ell_S\left(ba^k\right)\omega^{-ki}$$

is a square number for all $1 \leqslant i \leqslant \frac{p-1}{2}$.

Now assume that S_1 and S_2 satisfy the sufficient conditions of our result, and we aim to show that $Cay(D_p, S)$ is distance integral. If $S_1 = \emptyset$ and $S_2 = \langle a\rangle$, then $\ell_S\left(a^j\right) = 2$ for $1 \leqslant j \leqslant p-1$, $\ell_S\left(ba^{j'}\right) = 1$ for $0 \leqslant j' \leqslant p-1$, and thus,

$$\sum_{k=1}^{p-1} \ell_S\left(a^k\right)\omega^{ki} = 2\sum_{k=1}^{p-1} \omega^{ki} = -2, \quad \sum_{k=0}^{p-1} \ell_S\left(ba^k\right)\omega^{ki} \cdot \sum_{k=0}^{p-1} \ell_S\left(ba^k\right)\omega^{-ki} = 0.$$

If $S_1 = \emptyset$ and $S_2 = \langle a\rangle\backslash\left\{a^k\right\}$ for some $1 \leqslant k \leqslant p-1$, then it is routine to check that $\ell_S\left(a^j\right) = 2$ for $1 \leqslant j \leqslant p-1$, $\ell_S\left(ba^{j'}\right) = 1$ for $j' \neq k$ and $\ell_S\left(ba^k\right) = 3$, which leads to

$$\sum_{k=1}^{p-1} \ell_S\left(a^k\right)\omega^{ki} = 2\sum_{k=1}^{p-1}\omega^{ki}$$
$$= -2,\ \sum_{k=0}^{p-1} \ell_S\left(ba^k\right)\omega^{ki} \cdot \sum_{k=0}^{p-1} \ell_S\left(ba^k\right)\omega^{-ki} = 2\omega^k \cdot 2\omega^{-k} = 4.$$

In a similar way, if $S_1 = \langle a\rangle\backslash\{1\}$ and $S_2 \in \left\{\{a^k\}, \langle a\rangle\backslash\{a^k\}, \langle a\rangle\right\}$ for $0 \leqslant k \leqslant p-1$, then $\sum_{k=1}^{p-1} \ell_S\left(a^k\right)\omega^{ki} = \sum_{k=1}^{p-1}\omega^{ki} = -1$ is an integer and $\sum_{k=0}^{p-1} \ell_S\left(ba^k\right)\omega^{ki} \cdot \sum_{k=0}^{p-1} \ell_S\left(ba^k\right)\omega^{-ki} \in \{0, 1\}$ is a square number.

Assume conversely that $Cay(D_p, S)$ is distance integral. Note that $\widetilde{B}(\langle a\rangle) = \left\{\{1\}, \{a, a^2, \ldots, a^{p-1}\}\right\}$ by the fact that p is a prime number. It follows from Corollary 4.6 that

$$\sum_{k=1}^{p-1} \ell_S(a^k)a^k \in C(\langle a\rangle),\ \sum_{k=0}^{p-1} \ell_S\left(ba^k\right)a^k \cdot \sum_{k=0}^{p-1} \ell_S\left(ba^k\right)a^{-k} \in C(\langle a\rangle).$$

Then we have

$$\ell_S(a) = \ell_S\left(a^2\right) = \cdots = \ell_S\left(a^{p-1}\right) \tag{5.1}$$

and

$$\left[\sum_{k=0}^{p-1} \ell_S\left(ba^k\right)a^k\right] \cdot \left[\sum_{k=0}^{p-1} \ell_S\left(ba^k\right)a^{-k}\right] = \sum_{k=0}^{p-1}\left[\ell_S\left(ba^k\right)\right]^2 + m\left(a + a^2 + \cdots + a^{p-1}\right), \tag{5.2}$$

where m is a nonnegative integer. Taking the trivial representation of $\langle a\rangle$ through (5.2) yields

$$\left[\sum_{k=0}^{p-1} \ell_S\left(ba^k\right)\right]^2 = \sum_{k=0}^{p-1}\left[\ell_S\left(ba^k\right)\right]^2 + (p-1)m. \tag{5.3}$$

Recall that $\chi_i\left[\sum_{k=0}^{p-1} \ell_S\left(ba^k\right)a^k \cdot \sum_{k=0}^{p-1} \ell_S\left(ba^k\right)a^{-k}\right]$ is a square number, then there exists an integer t such that

$$t^2 = \sum_{k=0}^{p-1}\left[\ell_S\left(ba^k\right)\right]^2 - m. \tag{5.4}$$

Further on, we proceed by distinguishing the following two cases to show our result.

Case 1. $|S_1| \geqslant 1$. In this case, assume that $a^k \in S_1$ for some k. Then $\ell_S(a^k) = 1$. In view of (5.1), one has $\ell_S(a) = \ell_S(a^2) = \cdots = \ell_S(a^{p-1}) = 1$, which implies that $S_1 = \langle a \rangle \backslash \{1\}$. Thus, we get

$$\ell_S\left(ba^{k'}\right) = \begin{cases} 1, & \text{if } a^{k'} \in S_2; \\ 2, & \text{if } a^{k'} \notin S_2. \end{cases} \tag{5.5}$$

Assume that $|S_2| = x$, then $1 \leqslant x \leqslant p$ (based on the fact that $\langle S \rangle = G$). Substituting (5.5) into (5.3) and (5.4) yields

$$[x + 2(p-x)]^2 = x + 4(p-x) + (p-1)m, \;\; t^2 = x + 4(p-x) - m.$$

Then we obtain $p(x - t^2) = (x+t)(x-t)$. Therefore, $p \mid (x+t)$ or $p \mid (x-t)$. Note that $t^2 \leqslant 4p - 3x \leqslant p^2$, we have $x+t, x-t \in \{0, p, 2p\}$. If $x + t = 0$, then $x = t^2 = x^2$. Thus, we have $x = 1$. Similarly, all the possible cases lead to $x \in \{1, p-1, p\}$, which implies that $S_2 \in \left\{\{a^k\}, \langle a \rangle \backslash \{a^k\}, \langle a \rangle\right\}$ for $0 \leqslant k \leqslant p-1$, as desired.

Case 2. $|S_1| = 0$. In this case, as $\langle S \rangle = G$, we have $|S_2| \geqslant 2$. Assume that $a^k, a^j \in S_2$ for some $0 \leqslant k < j \leqslant p-1$. Then $\ell_S(a^{j-k}) = \ell_S(ba^k \cdot ba^j) = 2$. In view of (5.1), one has $\ell_S(a) = \ell_S(a^2) = \cdots = \ell_S(a^{p-1}) = 2$, which implies that

$$\ell_S\left(ba^{j'}\right) = \begin{cases} 1, & \text{if } a^{j'} \in S_2; \\ 3, & \text{if } a^{j'} \notin S_2. \end{cases} \tag{5.6}$$

Assume that $|S_2| = y$, then $2 \leqslant y \leqslant p$. Substituting (5.6) into (5.3) and (5.4) yields

$$[y + 3(p-y)]^2 = y + 9(p-y) + (p-1)m, \;\; t^2 = y + 9(p-y) - m.$$

Then we obtain $p(4y - t^2) = (2y+t)(2y-t)$. Therefore, $p \mid (2y+t)$ or $p \mid (2y-t)$. Note that $t^2 \leqslant 9p - 8y \leqslant (p+1)^2$, we have $2y+t, 2y-t \in \{0, p, 2p, 3p\}$. If $2y + t = 0$, then $4y = t^2 = 4y^2$, which is impossible since $y \geqslant 2$. Similarly, all the possible cases lead to $y \in \{p-1, p\}$, implying that $S_2 \in \left\{\langle a \rangle \backslash \{a^k\}, \langle a \rangle\right\}$ for $1 \leqslant k \leqslant p-1$. We are done. □

In order to compare the conditions for the integrality and distance integrality of $Cay(D_p, S)$, we restate [17, Theorem 4.2] below in our notations for convenience.

Lemma 5.5 [17] *Let $D_p = \langle a, b \mid a^p = b^2 = 1, bab = a^{-1} \rangle$ with $p \geqslant 3$ being a prime. Let $S = S_1 \cup bS_2 \subseteq D_p$ such that $1 \notin S$ and $S^{-1} = S$, where $S_1, S_2 \subseteq \langle a \rangle$. Then $Cay(D_p, S)$ is integral if and only if $S_1 \in \{\emptyset, \langle a \rangle \backslash \{1\}\}$ and $S_2 \in \left\{\{a^k\}, \langle a \rangle \backslash \{a^k\}, \langle a \rangle\right\}$ for $0 \leqslant k \leqslant p-1$.*

Comparing Theorem 5.4 with Lemma 5.5, we immediately arrive at the following result.

Corollary 5.6 *Let $D_p = \langle a, b \mid a^p = b^2 = 1, bab = a^{-1} \rangle$ with $p \geqslant 3$ being a prime. Denote by $S = S_1 \cup bS_2 \subseteq D_p$ such that $1 \notin S = S^{-1}$ and $\langle S \rangle = G$, where $S_1, S_2 \subseteq \langle a \rangle$. Then $Cay(D_p, S)$ is integral if and only if $Cay(D_p, S)$ is distance integral.*

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