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Weighted infinitesimal unitary bialgebras of rooted forests, symmetric cocycles and pre-Lie algebras

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Abstract

The concept of weighted infinitesimal unitary bialgebra is an algebraic meaning of the nonhomogenous associative Yang–Baxter equation. In this paper, we equip the space of decorated planar rooted forests with a coproduct which makes it a weighted infinitesimal unitary bialgebra. Further, we construct an infinitesimal unitary Hopf algebra on decorated planar rooted forests in the sense of Loday and Ronco. We then introduce the concept of symmetric 1-cocycle condition, which is derived from the dual of the Hochschild cohomology. We study the universal properties of the space of decorated planar rooted forests with the symmetric 1-cocycle, leading to the notation of a weighted Ω-cocycle infinitesimal unitary bialgebra. As an application, we obtain the initial object in the category of free cocycle infinitesimal unitary bialgebras on the undecorated planar rooted forests, which is the object studied in the well-known noncommutative Connes–Kreimer Hopf algebra. Finally, we construct a pre-Lie algebra on decorated planar rooted forests.

Keywords Rooted forest · Infinitesimal bialgebra · Cocycle condition · Pre-Lie algebra

Mathematics Subject Classification 16W99 · 05C05 · 16S10 · 16T10 · 16T30 · 17B60

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1 Introduction

Weighted infinitesimal unitary bialgebras first appeared in [39] and were further studied in [21,44,46,47], in order to give an algebraic meaning of nonhomogenous associative classical Yang–Baxter equations [39]. More precisely, a weighted infinitesimal unitary bialgebra is a module A, which is simultaneously an algebra (possibly without a unit) and a coalgebra (possibly without a counit) such that the coproduct Δ is a weighted derivation of A in the sense that

$$\Delta(ab) = a \cdot \Delta(b) + \Delta(a) \cdot b + \lambda(a \otimes b) \text{ for } a, b \in A,$$

where $\lambda \in \mathbf{k}$ is a fixed constant.

Parallel to the well-known fact that the solutions of a classical Yang–Baxter equation give rise to Lie bialgebras and quantum groups [11], Aguiar [1] introduced the associative Yang–Baxter equation (AYBE)

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0,$$

and showed that any solution r of AYBE in an algebra A endows A with an infinitesimal unitary bialgebra of weight zero, involving a principle derivation. This result was generalized by Ogievetsky and Popov in [39], by the concept of nonhomogenous associative classical Yang–Baxter equation

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = \lambda r_{13}.$$

In [39], Ogievetsky and Popov clarified an algebraic meaning of this equation, involving a coproduct given by

$$\Delta_r(a) := a \cdot r - r \cdot a - \lambda(a \otimes 1) \quad \text{for } a \in A. \tag{1}$$

Here, $r \in A \otimes A$ is a solution of the nonhomogenous associative classical Yang–Baxter equation. Note that Eq. (1) satisfies

$$\Delta_r(ab) = a \cdot \Delta_r(b) + \Delta_r(a) \cdot b + \lambda(a \otimes b) \quad \text{for } a, b \in A,$$

which is precisely the compatibility condition of a weighted infinitesimal unitary bialgebra, see [39,46] for more details.

We would like to emphasize that weighted infinitesimal unitary bialgebras give a uniform version of two infinitesimal bialgebras that have been studied intensely. The first one introduced by Joni and Rota [30] is aimed at giving an algebraic framework for the calculus of Newton divided differences. Aguiar [1] defined a notation of antipode for infinitesimal bialgebras, giving the notation of infinitesimal Hopf algebras. After that, there have been several interesting developments of infinitesimal bialgebras and infinitesimal Hopf algebras in mathematics and mathematics physics, including associative Yang–Baxter equations [1], Drinfeld's doubles [1], Lie bialgebras [2], pre-Lie algebras [4] and Dendriform algebras [4]. In combinatorics, Aguiar [3] showed a simple proof of the existence of the cd-index of polytopes, based on the theory of infinitesimal Hopf algebras. The second one was defined by Loday and Ronco [36] and further studied by Foissy [16,17], in the sense that

$$\Delta(ab) = a \cdot \Delta(b) + \Delta(a) \cdot b - a \otimes b \text{ for } a, b \in A.$$

Rooted forests are important objects studied in combinatorics and algebra. The connection of Hopf algebras with combinatorics was first discovered in the pioneering work of Joni and Rota [30]. One of the most prominent examples is the Connes–Kreimer Hopf algebra of rooted forests, which was introduced and studied extensively in [10,12,22,28,31,38]. Particularly, the Connes–Kreimer Hopf algebra can be used to treat a problem of renormalization in quantum field theory [7,9,25,31]. Many other Hopf algebras have been built on rooted forests, such as Foissy–Holtkamp [14,15,29], Grossman–Larson [22] and Loday–Ronco [35]. It has been observed that most of them possess universal properties which have an interesting application in renormalization [25,32]. It should be pointed out that Bruned, Hairer and Zambotti [8] used typed decorated rooted trees to give a description of a renormalisation procedure of stochastic PDEs, and a generalized Connes–Kreimer Hopf algebra on typed decorated tooted trees was studied by Foissy [18]. Thus, it would be interesting to construct weighted infinitesimal unitary bialgebras on some combinatorial objects, especially on various decorated rooted forests.

Inspired by the concept of multioperator group introduced by Higgins [27], Kurosh [33] proposed the notation of algebras with (one or more) linear operators. Later, Guo [24] constructed the free objects of such algebras in terms of some combinatorial objects. There such structure was called an Ω-operated algebra, where Ω is a set indexing the linear operators, see also [6,20,23,26]. We emphasize that the Connes–Kreimer Hopf algebra H_{RT} of planar rooted forests equipped with the grafting operation B^+ is an operated algebra. More generally, the space $H_{\mathrm{RT}}(\Omega)$ generated by decorated planar forests whose vertices are decorated by a set Ω, together with a set of grafting operations $\{B^+_\omega \mid \omega \in \Omega\}$, is the initial object in the category of Ω-operated algebras [32,43].

In the present paper along the line of operated algebras, we combine the Ω-operated algebra $(H_{\mathrm{RT}}(X, \Omega), \{B^+_\omega \mid \omega \in \Omega\})$ with weighted infinitesimal unitary bialgebras, leading to the concept of Ω-operated infinitesimal unitary bialgebra of weight λ. Moreover, we derive a symmetric 1-cocycle condition from the dual of the Hochschild cohomology. Involving this new 1-cocycle condition, we pose the concept of Ω-cocycle infinitesimal unitary bialgebras of weight λ. Based on these new concepts, we prove that the decorated planar rooted forests $H_{\mathrm{RT}}(X, \Omega)$ are the free objects in these categories provided suitable operations are equipped.

Pre-Lie algebras, also called Vinberg algebras, first appeared in the work of Vinberg on convex homogeneous cones [42] and also appeared independently at the same time in the study of the deformation and cohomology of associative algebras [19], see Bai's note [5] for their relations with some related algebraic structures. One interesting example is perturbative quantum field theory [31], where insertion of Feynman graphs into each other equips them with a pre-Lie algebraic structure. In 2004, Aguiar [4] gave a natural way to construct a pre-Lie algebra from an infinitesimal bialgebra of weight zero. Recently, we found that this construction does also hold for an arbitrary infinitesimal unitary bialgebra of weight λ [21]. As a consequence, by Theorem 3.11, a pre-Lie algebra structure on decorated planar rooted forests is built. This highlights the combinatorial nature of pre-Lie algebras and promotes their further study.

Structure of the paper In Sect. 2, we recall the concept of weighted infinitesimal unitary bialgebra and show that some well-known algebras possess a weighted infinitesimal unitary bialgebra.

In Sect. 3, we first recall the concepts of planar rooted forests and decorated planar rooted forests. By posing a symmetric version of a Hochschild 1-cocycle condition (Eq. (7)), we then construct a new coproduct on decorated planar rooted forests $H_{\mathrm{RT}}(X, \Omega)$ to equip it with a new coalgebraic structure (Theorem 3.10). Further $H_{\mathrm{RT}}(X, \Omega)$ can be turned into an infinitesimal unitary bialgebra of weight λ with respect to the concatenation product and the empty tree as its unit (Theorem 3.11). Viewing weighted infinitesimal bialgebras in the framework of operated algebras, we propose the concept of weighted Ω-cocycle infinitesimal unitary bialgebras (Definition 3.14 (a)), when a symmetric 1-cocycle condition is involved. Thanks to these concepts, we show that $H_{\mathrm{RT}}(X, \Omega)$ is the free Ω-cocycle infinitesimal unitary bialgebra of weight λ on a set X (Theorem 3.16). Finally, we obtain that the undecorated planar rooted forests are the free cocycle infinitesimal unitary bialgebra of weight λ on the empty set (Corollary 3.18).

In Sect. 4, we first recall the concept of infinitesimal unitary Hopf algebra in the sense of Loday and Ronco. We then construct an infinitesimal unitary counitary bialgebra of weight -1 on decorated planar rooted forests (Lemma 4.5). We prove that this new infinitesimal unitary counitary bialgebra is connected graded and so it is a infinitesimal unitary Hopf algebra (Theorem 4.8). We show that our construction is different from the one investigated by Foissy [16], see Remark 4.4.

In Sect. 5, by investigating the relationship between weighted infinitesimal unitary bialgebras and pre-Lie algebras (Lemma 5.3), we equip $H_{\mathrm{RT}}(X, \Omega)$ with a pre-Lie algebraic structure $(H_{\mathrm{RT}}(X, \Omega), \rhd_{\mathrm{RT}})$ and a Lie algebraic structure $(H_{\mathrm{RT}}(X, \Omega), [\_, \_]_{\mathrm{RT}})$ (Theorem 5.4).

Notation Throughout this paper, let $\mathbf{k}$ be a unitary commutative ring unless the contrary is specified, which will be the base ring of all modules, algebras, coalgebras, bialgebras, tensor products, as well as linear maps. By an algebra, we mean an associative $\mathbf{k}$-algebra (possibly without unit), and by a coalgebra, we mean a coassociative $\mathbf{k}$-coalgebra (possibly without counit). We use the Sweedler notation:

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}.$$

For an algebra A, $A \otimes A$ is viewed as an (A, A)-bimodule in the standard way

$$a \cdot (b \otimes c) := ab \otimes c \text{ and } (b \otimes c) \cdot a := b \otimes ca, \tag{2}$$

where $a, b, c \in A$.

2 Weighted infinitesimal unitary bialgebras and some examples

In this section, we first recall the concept of weighted infinitesimal (unitary) bialgebra [21], which generalizes simultaneously the one introduced by Joni and Rota [30] and the one initiated by Loday and Ronco [36]. We also show that some well-known algebras possess a weighted infinitesimal (unitary) bialgebra, via a construction of the suitable coproducts.

Definition 2.1 [21] Let λ be a given element of $\mathbf{k}$. An **infinitesimal bialgebra** (abbreviated ϵ-**bialgebra**) **of weight** λ is a triple (A, m, Δ), where

(a) (A, m) is an algebra (possibly without unit),
(b) (A, Δ) is a coalgebra (possibly without counit),

and the coproduct Δ satisfies the weighted derivation rule on A in the sense that

$$\Delta(ab) = a \cdot \Delta(b) + \Delta(a) \cdot b + \lambda(a \otimes b), \quad \forall a, b \in A. \tag{3}$$

If further $(A, m, 1)$ is a unitary algebra and (A, Δ, ε) is a counitary coalgebra, then the quintuple $(A, m, 1, \Delta, \varepsilon)$ is called an **infinitesimal unitary counitary bialgebra** (abbreviated ϵ-**unitary counitary bialgebra**) **of weight** λ.

We shall use the infix notation ϵ- interchangeably with the adjective "infinitesimal" throughout the rest of this paper.

Remark 2.2 (a) Let $(A, m, 1, \Delta)$ be an ϵ-unitary bialgebra of weight λ. Then, $\Delta(1) = -\lambda(1 \otimes 1)$ by taking $a = b = 1$ in Eq. (3).
(b) ϵ-bialgebras introduced by Joni and Rota [30] are ϵ-bialgebra of weight 0, and ϵ-bialgebras originated from Loday and Ronco [36] are ϵ-bialgebra of weight -1.
(c) Note that our base ring $\mathbf{k}$ is a unitary commutative ring, if the base ring $\mathbf{k}$ is further a field, then the ϵ-bialgebra of weight λ is equivalent to the one introduced by Loday and Ronco. Indeed, if the coproduct Δ satisfies Eq. (3) for $\lambda \neq 0$, then the linear map $\Delta' = -\Delta/\lambda$ satisfies the relation

$$\Delta'(ab) = a \cdot \Delta'(b) + \Delta'(a) \cdot b - a \otimes b.$$

The concept of ϵ-bialgebra morphism is given as usual.

Definition 2.3 [21] Let A and B be two ϵ-bialgebras of weight λ. A map $\phi : A \to B$ is called an **infinitesimal bialgebra morphism** (abbreviated ϵ-bialgebra morphism) if ϕ is an algebra morphism and a coalgebra morphism. The concept of **infinitesimal unitary bialgebra morphism** can be defined in the same way.

Example 2.4 Here are some examples of weighted ϵ-unitary bialgebras.

(a) Any unitary algebra $(A, \mu, 1)$ is an ϵ-unitary bialgebra of weight λ by taking

$$\Delta(a) = -\lambda(a \otimes 1) \text{ for } a \in A.$$

(b) [45, Example 2.4] The polynomial algebra $\mathbf{k}[x]$ is an ϵ-unitary bialgebra of weight λ with the coproduct defined by

$$\Delta(1) = -\lambda(1 \otimes 1) \text{ and } \Delta(x^n) = \sum_{i=0}^{n-1} x^i \otimes x^{n-1-i} + \lambda \sum_{i=1}^{n-1} x^i \otimes x^{n-i} \text{ for } n \geq 1.$$

(c) [47] The matrix algebra $M_n(\mathbf{k})$ is an ϵ-unitary bialgebra of weight zero with the coproduct defined by

$$\Delta_\epsilon(E_{ij}) := \begin{cases} \sum_{s=i}^{j-1} E_{is} \otimes E_{(s+1)j} & \text{if } i < j, \\ 0 & \text{if } i = j, \\ -\sum_{s=j}^{i-1} E_{is} \otimes E_{(s+1)j} & \text{if } i > j. \end{cases}$$

(d) [36, Section 2.3] Let V denote a vector space. Recall that the tensor algebra $T(V)$ over V is the tensor module,

$$T(V) = \mathbf{k} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots,$$

equipped with the associative multiplication called concatenation defined by

$$v_1 \cdots v_i \otimes v_{i+1} \cdots v_n \mapsto v_1 \cdots v_i v_{i+1} \cdots v_n \quad \text{for } 0 \leq i \leq n,$$

and with the convention that $v_1 v_0 = 1$ and $v_{n+1} v_n = 1$. It is a well-known free associative algebra. The tensor algebra $T(V)$ is an ϵ-unitary bialgebra of weight -1 with the coassociative coproduct defined by

$$\Delta(v_1 \cdots v_n) := \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n.$$

(e) [45, Example 2.4] The free algebra $\mathbf{k}\langle X\rangle$ generated by a set X (this algebra is isomorphic to tensor algebra $T(\mathbf{k}X)$) can be turned into an ϵ-unitary bialgebra of weight λ with the coproduct defined by

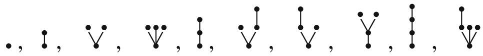
$$\Delta(x_1 x_2 \cdots x_n) := \sum_{i=1}^{n} x_1 \cdots x_{i-1} \otimes x_{i+1} \cdots x_n + \lambda \sum_{i=1}^{n-1} x_1 \cdots x_i \otimes x_{i+1} \cdots x_n.$$

3 Weighted infinitesimal unitary bialgebras of rooted forests

In this section, we first recall the concepts of planar rooted forests [41] and decorated planar rooted forests [14,21,24]. By a symmetric 1-cocycle condition, we define a coproduct on decorated planar rooted forests to equip it with a coalgebra structure, leading to the emergence of a weighted infinitesimal unitary bialgebra on it.

3.1 New decorated planar rooted forests

A **rooted tree** is a finite graph, connected and without cycles, with a special vertex called the **root**. A **planar rooted tree** is a rooted tree with a fixed embedding into the plane. The first few planar rooted trees are listed below:



where the root of a tree is on the bottom.

We now collect some basic definitions and facts on decorated rooted trees and forests that will be used in this paper. See [14,21,24,40] for more details.

(a) Let $\mathcal{T}$ denote the set of planar rooted trees and $M(\mathcal{T})$ the free monoid generated by $\mathcal{T}$ with the concatenation product, denoted by m_{RT} and usually suppressed. The **empty tree** in $M(\mathcal{T})$ is denoted by 1.

(b) An element in $M(\mathcal{T})$, called a **planar rooted forest**, is a noncommutative concatenation of planar rooted trees, denoted by $F = T_1 \cdots T_n$ with $T_1, \ldots, T_n \in \mathcal{T}$. Here, we use the convention that $F = 1$ when $n = 0$.

(c) Let Ω be a nonempty set and let X be a set whose elements are not in the set Ω. Let $\mathcal{T}(X, \Omega)$ (resp. $\mathcal{F}(X, \Omega)$) denote the vertex decorated planar rooted trees (resp. forests) whose **internal vertices** (vertices which are not leaves) are decorated by elements of Ω exclusively and **leaf vertices** are decorated by elements of $X \sqcup \Omega$. The only vertex of the tree $\bullet$ is regarded as a leaf vertex. The elements in $\mathcal{F}(X, \Omega)$ are called **new decorated planar rooted forests**.

(d) Define

$$H_{\mathrm{RT}}(X, \Omega) := \mathbf{k}\mathcal{F}(X, \Omega) = \mathbf{k}M(\mathcal{T}(X, \Omega))$$

to be the **free k-module** spanned by $\mathcal{F}(X, \Omega)$.

(e) For $\omega \in \Omega$, define

$$B_\omega^+ : H_{\mathrm{RT}}(X, \Omega) \to H_{\mathrm{RT}}(X, \Omega)$$

to be the linear **grafting operation** by taking 1 to $\bullet_\omega$ and sending a rooted forest in $H_{\mathrm{RT}}(X, \Omega)$ to its grafting with the new root decorated by ω.

(f) For $F = T_1 \cdots T_n \in \mathcal{F}(X, \Omega)$ with $n \geq 0$ and $T_1, \cdots, T_n \in \mathcal{T}(X, \Omega)$, we define $\mathrm{bre}(F) := n$ to be the **breadth** of F. Here, we use the convention that $\mathrm{bre}(1) = 0$ when $n = 0$.

(g) Denote $\bullet_X := \{\bullet_x \mid x \in X\}$ and set

$$\mathcal{F}_0 := M(\bullet_X) = S(\bullet_X) \sqcup \{1\},$$

where $M(\bullet_X)$ (resp. $S(\bullet_X)$) is the submonoid (resp. subsemigroup) of $\mathcal{F}(X, \Omega)$ generated by $\bullet_X$. Suppose that $\mathcal{F}_n$ has been defined for an $n \geq 0$, then define

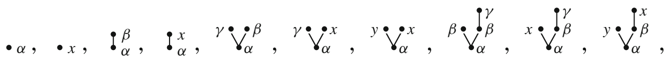
$$\mathcal{F}_{n+1} := M(\bullet_X \sqcup (\sqcup_{\omega \in \Omega} B_\omega^+(\mathcal{F}_n))).$$

Thus, we obtain $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and

$$\mathcal{F}(X, \Omega) = \varinjlim \mathcal{F}_n = \bigcup_{n=0}^{\infty} \mathcal{F}_n.$$

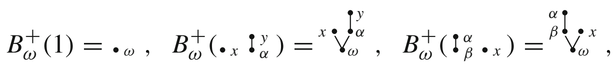
Now elements $F \in \mathcal{F}_n \setminus \mathcal{F}_{n-1}$ are said to have **depth** n, denoted by $\mathrm{dep}(F) = n$.

Example 3.1 The following are some examples in $\mathcal{T}(X, \Omega)$:



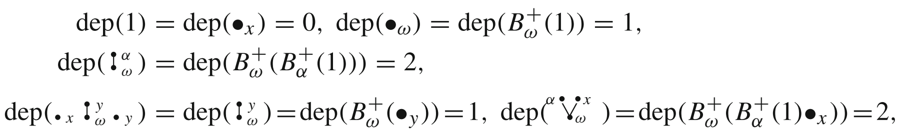
with $\alpha, \beta, \gamma \in \Omega$ and $x, y \in X$.

Example 3.2 The following are some grafting operations:



where $\alpha, \beta, \omega \in \Omega$ and $x, y \in X$.

Example 3.3 Here are some examples about the depths of some decorated planar rooted forests.



where $\alpha, \omega \in \Omega$ and $x, y \in X$.

Remark 3.4 [21,44] Now we give some special cases of our decorated planar rooted forests.

(a) If $X = \emptyset$ and Ω is a singleton set, then all decorated planar rooted forests in $\mathcal{F}(X, \Omega)$ have the same decoration and so can be identified with the planar rooted forests without decorations, which is the object studied in the well-known Foissy–Holtkamp Hopf algebra—the noncommutative version of Connes–Kreimer Hopf algebra [14,29].
(b) If $X = \emptyset$, then $\mathcal{F}(X, \Omega)$ was studied by Foissy [14,15], in which a decorated noncommutative version of Connes–Kreimer Hopf algebra was constructed.
(c) If Ω is a singleton set, then $\mathcal{F}(X, \Omega)$ was introduced and studied in [43] to construct a cocycle Hopf algebra on decorated planar rooted forests.

3.2 From Cartier–Quillen cohomologies to symmetric 1-cocycle conditions

Given an algebra A and a bimodule M over A, let $H^*(A, M)$ denote the **Hochschild cohomology** of A with coefficients in M which was defined from a complex with maps $A^{\otimes n} \to M$ as cochains, see [34] for more details. Let (C, Δ) be a coalgebra and (B, δ_G, δ_D) be a bicomodule over C. The **Cartier–Quillen cohomology** of C with coefficients in B is a dual notation of the Hochschild cohomology. Explicitly, it is a cohomology of the complex $\mathrm{Hom}_{\mathbf{k}}(B, C^{\otimes n})$ with the maps $b_n : \mathrm{Hom}_{\mathbf{k}}(B, C^{\otimes n}) \to \mathrm{Hom}_{\mathbf{k}}(B, C^{\otimes(n+1)})$ given by

$$\begin{aligned} b_n(L) =&(\mathrm{id} \otimes L) \circ \delta_G + \sum_{i=1}^{n} (-1)^i (\mathrm{id}_C^{\otimes(i-1)} \otimes \Delta \otimes \mathrm{id}_C^{\otimes(n-i)}) L \\ &+ (-1)^{n+1} (L \otimes \mathrm{id}) \circ \delta_D, \end{aligned}$$

where $L : B \to C^{\otimes n}$. In particular, a linear map $L : B \to C$ is the 1-**cocycle** for this cohomology precisely when it satisfies the following condition:

$$\Delta \circ L = (L \otimes \mathrm{id}) \circ \delta_D + (\mathrm{id} \otimes L) \circ \delta_G,$$

see [13,38] for more details. We consider the bicomodule (C, δ_G, δ_D) with $\delta_G(x) = \delta_D(x) = \Delta(x)$ for any $x \in C$. Then, the 1-cocycle is a linear endomorphism L of C which satisfies:

$$\Delta \circ L(x) = (L \otimes \mathrm{id}) \circ \Delta(x) + (\mathrm{id} \otimes L) \circ \Delta(x) \quad \text{for } x \in C. \tag{4}$$

We call Eq. (4) the **symmetric 1-cocycle condition**.

Remark 3.5 (a) When $L = B^+$, the symmetric 1-cocycle condition in Eq. (4) given by

$$\Delta(F) = \Delta B^+(\overline{F}) = (B^+ \otimes \mathrm{id})\Delta(\overline{F}) + (\mathrm{id} \otimes B^+)\Delta(\overline{F}) \quad \text{for } F = B^+(\overline{F}) \in \mathcal{F},$$

which is different from the well-known 1-cocycle condition,

$$\Delta(F) = \Delta B^+(\overline{F}) = B^+(\overline{F}) \otimes 1 + (\mathrm{id} \otimes B^+)\Delta(\overline{F}),$$

introduced and studied by Connes–Kreimer [10].

(b) Let $\mathbb{P}$ be a Hopf operad and let H be the algebra generated by finite rooted forests equipped with a linear endomorphism ϱ. Define $\sigma_i : H \to H, i = 1, 2$, by taking $\sigma_i(F) = q_i^n F$ for any forest F with n vertices, where $q_1, q_2 \in \mathbf{k}$. Moerdijk [38] showed that the pair

$$(\sigma_1, \sigma_2) = \sigma_1 \otimes \varrho + \varrho \otimes \sigma_2 : H \otimes H \to H \otimes H$$

giving a $\mathbb{P}[t]$-algebraic structure on $H \otimes H$, which induces a family of Hopf $\mathbb{P}$-algebra structures parameterized by $\sigma_i, i = 1, 2$. For $q_1 = 0, q_2 = 1$, one recovers the usual 1-cocycle relation. For $q_1 = q_2 = 1$, this gives the symmetric 1-cocycle condition used in the present paper, see [38, Example 3.6] for more details. However, the coproducts defined in this paper and the compatibilities with the coproducts are very different from Moerdijk's. It is a natural question to consider infinitesimal Hopf $\mathcal{P}$-algebras for any infinitesimal Hopf operad $\mathcal{P}$ in Moerdijk's direction.

3.3 Weighted infinitesimal unitary bialgebras on decorated planar rooted forests

In this subsection, we shall equip a weighted infinitesimal unitary bialgebraic structure on decorated planar rooted forests.

Let λ, μ be given elements of $\mathbf{k}$. We now define a new coproduct Δ_ϵ on $H_{\mathrm{RT}}(X, \Omega)$ by induction on depth. By linearity, we only need to define $\Delta_\epsilon(F)$ for basis elements $F \in \mathcal{F}(X, \Omega)$. For the initial step of $\mathrm{dep}(F) = 0$, we define

$$\Delta_\epsilon(F) := \begin{cases} -\lambda(1 \otimes 1) & \text{if } F = 1, \\ \mu(\bullet_x \otimes \bullet_x) - \lambda(\bullet_x \otimes 1 + 1 \otimes \bullet_x) & \text{if } F = \bullet_x \text{ for some } x \in X, \\ \bullet_{x_1} \cdot \Delta_\epsilon(\bullet_{x_2} \cdots \bullet_{x_m}) + \Delta_\epsilon(\bullet_{x_1}) \cdot (\bullet_{x_2} \cdots \bullet_{x_m}) & \\ +\lambda\, \bullet_{x_1} \otimes \bullet_{x_2} \cdots \bullet_{x_m} & \text{if } F = \bullet_{x_1} \cdots \bullet_{x_m} \text{ with } m \geq 2. \end{cases} \tag{5}$$

For the induction step of $\mathrm{dep}(F) \geq 1$, we reduce the definition to induction on breadth. If $\mathrm{bre}(F) = 1$, we write $F = B_\omega^+(\overline{F})$ for some $\omega \in \Omega$ and $\overline{F} \in \mathcal{F}(X, \Omega)$ and define

$$\Delta_\epsilon(F) = \Delta_\epsilon B_\omega^+(\overline{F}) := (B_\omega^+ \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) + (\mathrm{id} \otimes B_\omega^+)\Delta_\epsilon(\overline{F}). \tag{6}$$

In other words,

$$\Delta_\epsilon B_\omega^+ = (B_\omega^+ \otimes \mathrm{id})\Delta_\epsilon + (\mathrm{id} \otimes B_\omega^+)\Delta_\epsilon. \tag{7}$$

If $\mathrm{bre}(F) \geq 2$, we write $F = T_1 T_2 \cdots T_m$ with $m \geq 2$ and $T_1, \ldots, T_m \in \mathcal{T}(X, \Omega)$ and define

$$\Delta_\epsilon(F) = T_1 \cdot \Delta_\epsilon(T_2 \cdots T_m) + \Delta_\epsilon(T_1) \cdot (T_2 \cdots T_m) + \lambda T_1 \otimes T_2 \cdots T_m, \tag{8}$$

where the left and right actions are given in Eq. (2).

Example 3.6 Let $x, y \in X$ and $\alpha, \beta, \gamma \in \Omega$. Then,

$$\begin{aligned} \Delta_\epsilon(\bullet_\alpha) &= -\lambda(\bullet_\alpha \otimes 1 + 1 \otimes \bullet_\alpha), \\ \Delta_\epsilon(\mathbf{I}_{\alpha}^{x}) &= \mu(\bullet_x \otimes \mathbf{I}_{\alpha}^{x} + \mathbf{I}_{\alpha}^{x} \otimes \bullet_x) - \lambda(\mathbf{I}_{\alpha}^{x} \otimes 1 + \bullet_\alpha \otimes \bullet_x + \bullet_x \otimes \bullet_\alpha + 1 \otimes \mathbf{I}_{\alpha}^{x}), \\ \Delta_\epsilon(\bullet_x \mathbf{I}_{\alpha}^{x}) &= \mu(\bullet_x \bullet_x \otimes \mathbf{I}_{\alpha}^{x} + \bullet_x \mathbf{I}_{\alpha}^{x} \otimes \bullet_x + \bullet_x \otimes \bullet_x \mathbf{I}_{\alpha}^{x}) - \lambda(\bullet_x \mathbf{I}_{\alpha}^{x} \otimes 1 + \bullet_x \bullet_\alpha \otimes \bullet_x \\ &\quad + \bullet_x \bullet_x \otimes \bullet_\alpha + \bullet_x \otimes \mathbf{I}_{\alpha}^{x} + 1 \otimes \bullet_x \mathbf{I}_{\alpha}^{x}). \end{aligned}$$

To show $(H_{\mathrm{RT}}(X, \Omega), \Delta_\epsilon)$ is a coalgebra, we record the following two lemmas as a preparation.

Lemma 3.7 *Let* $\bullet_{x_1}\cdots\bullet_{x_m}\in H_{\mathrm{RT}}(X,\Omega)$ *with* $m\geq 1$ *and* $x_1,\ldots,x_m\in X$. *Then,*

$$\Delta_\epsilon(\bullet_{x_1}\cdots\bullet_{x_m})=\mu\sum_{i=1}^{m}\bullet_{x_1}\cdots\bullet_{x_i}\otimes\bullet_{x_i}\cdots\bullet_{x_m}-\lambda\sum_{i=0}^{m}\bullet_{x_1}\cdots\bullet_{x_i}\otimes\bullet_{x_{i+1}}\cdots\bullet_{x_m},$$

with the convention that $\bullet_{x_1}\bullet_{x_0}=1$ *and* $\bullet_{x_{m+1}}\bullet_{x_m}=1$.

Proof We prove the result by induction on $m\geq 1$. For the initial step of $m=1$, we have

$$\Delta_\epsilon(\bullet_{x_1})=\mu(\bullet_{x_1}\otimes\bullet_{x_1})-\lambda(\bullet_{x_1}\otimes 1+1\otimes\bullet_{x_1}),$$

and the result is true trivially. For the induction step of $m\geq 2$, we get

$$\begin{aligned}
\Delta_\epsilon(\bullet_{x_1}\cdots\bullet_{x_m}) &= \bullet_{x_1}\cdot\Delta_\epsilon(\bullet_{x_2}\cdots\bullet_{x_m})+\Delta_\epsilon(\bullet_{x_1})\cdot(\bullet_{x_2}\cdots\bullet_{x_m})\\
&\quad+\lambda(\bullet_{x_1}\otimes\bullet_{x_2}\cdots\bullet_{x_m})\quad\text{(by Eq. (5))}\\
&= \bullet_{x_1}\cdot\Delta_\epsilon(\bullet_{x_2}\cdots\bullet_{x_m})+\Big(\mu(\bullet_{x_1}\otimes\bullet_{x_1})-\lambda(\bullet_{x_1}\otimes 1+1\otimes\bullet_{x_1})\Big)\cdot(\bullet_{x_2}\cdots\bullet_{x_m})\\
&\quad+\lambda(\bullet_{x_1}\otimes\bullet_{x_2}\cdots\bullet_{x_m})\quad\text{(by Eq. (5))}\\
&= \bullet_{x_1}\cdot\Delta_\epsilon(\bullet_{x_2}\cdots\bullet_{x_m})+\mu(\bullet_{x_1}\otimes\bullet_{x_1}\cdots\bullet_{x_m})-\lambda(\bullet_{x_1}\otimes\bullet_{x_2}\cdots\bullet_{x_m})\\
&\quad-\lambda(1\otimes\bullet_{x_1}\bullet_{x_2}\cdots\bullet_{x_m})\\
&\quad+\lambda(\bullet_{x_1}\otimes\bullet_{x_2}\cdots\bullet_{x_m})\quad\text{(by Eq. (2))}\\
&= \bullet_{x_1}\cdot\Delta_\epsilon(\bullet_{x_2}\cdots\bullet_{x_m})+\mu(\bullet_{x_1}\otimes\bullet_{x_1}\cdots\bullet_{x_m})-\lambda(1\otimes\bullet_{x_1}\bullet_{x_2}\cdots\bullet_{x_m})\\
&= \bullet_{x_1}\cdot\left(\mu\sum_{i=2}^{m}\bullet_{x_2}\cdots\bullet_{x_i}\otimes\bullet_{x_i}\cdots\bullet_{x_m}-\lambda\sum_{i=1}^{m}\bullet_{x_2}\cdots\bullet_{x_i}\otimes\bullet_{x_{i+1}}\cdots\bullet_{x_m}\right)\\
&\quad+\mu(\bullet_{x_1}\otimes\bullet_{x_1}\cdots\bullet_{x_m})-\lambda(1\otimes\bullet_{x_1}\cdots\bullet_{x_m})\quad\text{(by the induction hypothesis)}\\
&= \mu\sum_{i=2}^{m}\bullet_{x_1}\bullet_{x_2}\cdots\bullet_{x_i}\otimes\bullet_{x_i}\cdots\bullet_{x_m}-\lambda\sum_{i=1}^{m}\bullet_{x_1}\bullet_{x_2}\cdots\bullet_{x_i}\otimes\bullet_{x_{i+1}}\cdots\bullet_{x_m}\\
&\quad+\mu(\bullet_{x_1}\otimes\bullet_{x_1}\cdots\bullet_{x_m})-\lambda(1\otimes\bullet_{x_1}\cdots\bullet_{x_m})\\
&= \mu\sum_{i=1}^{m}\bullet_{x_1}\cdots\bullet_{x_i}\otimes\bullet_{x_i}\cdots\bullet_{x_m}-\lambda\sum_{i=0}^{m}\bullet_{x_1}\cdots\bullet_{x_i}\otimes\bullet_{x_{i+1}}\cdots\bullet_{x_m},
\end{aligned}$$

as required. □

Lemma 3.8 *Let* $F_1, F_2\in H_{\mathrm{RT}}(X,\Omega)$. *Then,*

$$\Delta_\epsilon(F_1F_2)=F_1\cdot\Delta_\epsilon(F_2)+\Delta_\epsilon(F_1)\cdot F_2+\lambda(F_1\otimes F_2).$$

Proof It suffices to consider basis elements $F_1, F_2\in\mathcal{F}(X,\Omega)$ by linearity, and we shall use this technique tacitly in the later proofs of this paper. If $\mathrm{bre}(F_1)=0$ or

$\mathrm{bre}(F_2) = 0$, without loss of generality, letting $\mathrm{bre}(F_1) = 0$, then $F_1 = 1$ and by Eq. (5),

$$\begin{aligned}\Delta_\epsilon(F_1F_2) = \Delta_\epsilon(1F_2) &= \Delta_\epsilon(F_2) - \lambda(1\otimes F_2) + \lambda(1\otimes F_2)\\ &= \Delta_\epsilon(F_2) - \lambda(1\otimes 1)\cdot F_2 + \lambda(1\otimes F_2)\\ &= \Delta_\epsilon(F_2) + \Delta_\epsilon(1)\cdot F_2 + \lambda(1\otimes F_2)\\ &= 1\cdot\Delta_\epsilon(F_2) + \Delta_\epsilon(1)\cdot F_2 + \lambda(1\otimes F_2).\end{aligned}$$

If $\mathrm{bre}(F_1) \geq 1$ and $\mathrm{bre}(F_2) \geq 1$, we proceed to prove the result by induction on the sum of breadths $\mathrm{bre}(F_1) + \mathrm{bre}(F_2) \geq 2$. For the initial step of $\mathrm{bre}(F_1) + \mathrm{bre}(F_2) = 2$, we have $F_1 = T_1$ and $F_2 = T_2$ for some decorated planar rooted trees $T_1, T_2 \in \mathcal{T}(X, \Omega)$. Using Eq. (8), we have

$$\begin{aligned}\Delta_\epsilon(F_1F_2) = \Delta_\epsilon(T_1T_2) &= T_1\cdot\Delta_\epsilon(T_2) + \Delta_\epsilon(T_1)\cdot T_2 + \lambda(T_1\otimes T_2)\\ &= F_1\cdot\Delta_\epsilon(F_2) + \Delta_\epsilon(F_1)\cdot F_2 + \lambda(F_1\otimes F_2).\end{aligned}$$

For the induction step of $\mathrm{bre}(F_1) + \mathrm{bre}(F_2) \geq 3$, without loss of generality, we may suppose $\mathrm{bre}(F_2) \geq \mathrm{bre}(F_1) \geq 1$. If $\mathrm{bre}(F_1) = 1$ and $\mathrm{bre}(F_2) \geq 2$, we may write $F_1 = T_1$ for some decorated planar rooted trees $T_1 \in \mathcal{T}(X, \Omega)$. Applying Eq. (8), we have

$$\begin{aligned}\Delta_\epsilon(F_1F_2) = \Delta_\epsilon(T_1F_2) &= T_1\cdot\Delta_\epsilon(F_2) + \Delta_\epsilon(T_1)\cdot F_2 + \lambda(T_1\otimes F_2)\\ &= F_1\cdot\Delta_\epsilon(F_2) + \Delta_\epsilon(F_1)\cdot F_2 + \lambda(F_1\otimes F_2).\end{aligned}$$

If $\mathrm{bre}(F_1) \geq 2$, we can write $F_1 = T_1F_1'$ with $\mathrm{bre}(T_1) = 1$ and $\mathrm{bre}(F_1') = \mathrm{bre}(F_1)-1$. Then,

$$\begin{aligned}&\Delta_\epsilon(F_1F_2) = \Delta_\epsilon(T_1F_1'F_2)\\ &\quad= T_1\cdot\Delta_\epsilon(F_1'F_2) + \Delta_\epsilon(T_1)\cdot(F_1'F_2) + \lambda(T_1\otimes F_1'F_2) \quad \text{(by Eq. (8))}\\ &\quad= T_1\cdot\Big(F_1'\cdot\Delta_\epsilon(F_2) + \Delta_\epsilon(F_1')\cdot F_2 + \lambda(F_1'\otimes F_2)\Big) + \Delta_\epsilon(T_1)\cdot(F_1'F_2)\\ &\qquad+ \lambda(T_1\otimes F_1'F_2)\\ &\qquad\qquad\qquad \text{(by the induction hypothesis)}\\ &\quad= (T_1F_1')\cdot\Delta_\epsilon(F_2) + T_1\cdot\Delta_\epsilon(F_1')\cdot F_2 + \lambda(T_1F_1'\otimes F_2) + \Delta_\epsilon(T_1)\\ &\qquad\cdot(F_1'F_2) + \lambda(T_1\otimes F_1'F_2)\\ &\quad= (T_1F_1')\cdot\Delta_\epsilon(F_2) + \Big(T_1\cdot\Delta_\epsilon(F_1') + \Delta_\epsilon(T_1)\cdot F_1' + \lambda(T_1\otimes F_1')\Big)\\ &\qquad\cdot F_2 + \lambda(T_1F_1'\otimes F_2)\\ &\quad= (T_1F_1')\cdot\Delta_\epsilon(F_2) + \Delta_\epsilon(T_1F_1')\cdot F_2 + \lambda(T_1F_1'\otimes F_2)\\ &\quad= F_1\cdot\Delta_\epsilon(F_2) + \Delta_\epsilon(F_1)\cdot F_2 + \lambda(F_1\otimes F_2) \qquad \text{(by the induction hypothesis).}\end{aligned}$$

This completes the proof. □

The following lemma shows that $H_{\mathrm{RT}}(X, \Omega)$ is closed under the coproduct Δ_ϵ.

Lemma 3.9 *For* $F \in H_{\mathrm{RT}}(X, \Omega)$,

$$\Delta_\epsilon(F) \in H_{\mathrm{RT}}(X, \Omega) \otimes H_{\mathrm{RT}}(X, \Omega).$$

Proof By the methods of decoration of planar rooted forests introduced in Sect. 3.1, it is enough to show that $\Delta_\epsilon(F)$ for basis elements $F \in \mathcal{F}(X, \Omega)$ is a sum of tensor products of decorated planar rooted forests whose internal vertices are not decorated by X. Then, this result follows by the definition of Δ_ϵ and the induction on $\mathrm{dep}(F) \geq 0$. □

We now state our first main result in this subsection.

Theorem 3.10 *The pair* $(H_{\mathrm{RT}}(X, \Omega), \Delta_\epsilon)$ *is a coalgebra (without counit).*

Proof By Lemma 3.9, we only need to verify the coassociative law

$$(\mathrm{id} \otimes \Delta_\epsilon)\Delta_\epsilon(F) = (\Delta_\epsilon \otimes \mathrm{id})\Delta_\epsilon(F) \text{ for } F \in \mathcal{F}(X, \Omega), \tag{9}$$

which will be proved by induction on $\mathrm{dep}(F) \geq 0$. For the initial step of $\mathrm{dep}(F) = 0$, we have $F = \bullet_{x_1} \bullet_{x_2} \cdots \bullet_{x_m}$ for some $m \geq 0$, with the convention that $F = 1$ if $m = 0$. When $m = 0$, we have

$$\begin{aligned}(\mathrm{id} \otimes \Delta_\epsilon)\Delta_\epsilon(F) &= (\mathrm{id} \otimes \Delta_\epsilon)\Delta_\epsilon(1) = -\lambda 1 \otimes \Delta_\epsilon(1) = \lambda^2(1 \otimes 1 \otimes 1)\\ &= -\lambda\Delta_\epsilon(1) \otimes 1 = (\Delta_\epsilon \otimes \mathrm{id})\Delta_\epsilon(1).\end{aligned}$$

When $m \geq 1$, on the one hand,

$$\begin{aligned}
&(\mathrm{id} \otimes \Delta_\epsilon)\Delta_\epsilon(\bullet_{x_1} \bullet_{x_2} \cdots \bullet_{x_m})\\
&\quad= (\mathrm{id} \otimes \Delta_\epsilon)\Bigg(\mu \sum_{i=1}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_m}\\
&\qquad -\lambda \sum_{i=0}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_m}\Bigg) \quad \text{(by Lemma 3.7)}\\
&\quad= \mu \sum_{i=1}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \Delta_\epsilon(\bullet_{x_i} \cdots \bullet_{x_m}) - \lambda \bullet_{x_1} \cdots \bullet_{x_m} \otimes (-\lambda(1 \otimes 1))\\
&\qquad - \lambda \sum_{i=0}^{m-1} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \Delta_\epsilon(\bullet_{x_{i+1}} \cdots \bullet_{x_m})\\
&\quad= \mu \sum_{i=1}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \Bigg(\mu \sum_{j=i}^{m} \bullet_{x_i} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_m}\\
&\qquad -\lambda \sum_{j=i-1}^{m} \bullet_{x_i} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_m}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&- \lambda \sum_{i=0}^{m-1} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \left(\mu \sum_{j=i+1}^{m} \bullet_{x_{i+1}} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_m} \right. \\
&\left. -\lambda \sum_{j=i}^{m} \bullet_{x_{i+1}} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_m} \right) \\
&+ \lambda^2 (\bullet_{x_1} \cdots \bullet_{x_m} \otimes 1 \otimes 1) \\
&= \mu^2 \sum_{i=1}^{m} \sum_{j=i}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_m} \\
&- \lambda\mu \sum_{i=1}^{m} \sum_{j=i-1}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_m} \\
&- \lambda\mu \sum_{i=0}^{m-1} \sum_{j=i+1}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_m} \\
&+ \lambda^2 \sum_{i=0}^{m} \sum_{j=i}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_m} \\
&= \mu^2 \sum_{i=1}^{m} \sum_{j=i}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_m} \\
&- \lambda\mu \sum_{i=1}^{m} \sum_{j=i}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_m} \\
&- \lambda\mu \sum_{i=1}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes 1 \otimes \bullet_{x_i} \cdots \bullet_{x_m} \\
&- \lambda\mu \sum_{i=0}^{m-1} \sum_{j=i+1}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_m} \\
&+ \lambda^2 \sum_{i=0}^{m} \sum_{j=i}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_j} \\
&\quad \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_m} \quad \text{(by expanding the second term)}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(\Delta_\epsilon \otimes \mathrm{id}) \Delta_\epsilon (\bullet_{x_1} \bullet_{x_2} \cdots \bullet_{x_m}) \\
&= (\Delta_\epsilon \otimes \mathrm{id}) \left(\mu \sum_{i=1}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_m} \right. \\
&\left. -\lambda \sum_{i=0}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_m} \right) \quad \text{(by Lemma 3.7)}
\end{aligned}
$$

$$
\begin{aligned}
&= \mu \sum_{i=1}^{m} \Delta_\epsilon(\bullet_{x_1} \cdots \bullet_{x_i}) \otimes \bullet_{x_i} \cdots \bullet_{x_m} + \lambda^2(1 \otimes 1 \otimes \bullet_{x_1} \cdots \bullet_{x_m}) \\
&\quad - \lambda \sum_{i=1}^{m} \Delta_\epsilon(\bullet_{x_1} \cdots \bullet_{x_i}) \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_m} \\
&= \mu \sum_{i=1}^{m} \left(\mu \sum_{j=1}^{i} \bullet_{x_1} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_i} \right. \\
&\quad \left. - \lambda \sum_{j=0}^{i} \bullet_{x_1} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_i} \right) \otimes \bullet_{x_i} \cdots \bullet_{x_m} \\
&\quad - \lambda \sum_{i=1}^{m} \left(\mu \sum_{j=1}^{i} \bullet_{x_1} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_i} \right. \\
&\quad \left. - \lambda \sum_{j=0}^{i} \bullet_{x_1} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_i} \right) \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_m} \\
&\quad + \lambda^2(1 \otimes 1 \otimes \bullet_{x_1} \cdots \bullet_{x_m}) \\
&= \mu^2 \sum_{i=1}^{m} \sum_{j=1}^{i} \bullet_{x_1} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_m} \\
&\quad - \lambda\mu \sum_{i=1}^{m} \sum_{j=0}^{i} \bullet_{x_1} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_m} \\
&\quad - \lambda\mu \sum_{i=1}^{m} \sum_{j=1}^{i} \bullet_{x_1} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_m} \\
&\quad + \lambda^2 \sum_{i=0}^{m} \sum_{j=0}^{i} \bullet_{x_1} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_m} \\
&= \mu^2 \sum_{j=1}^{m} \sum_{i=1}^{j} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_m} \\
&\quad - \lambda\mu \sum_{j=1}^{m} \sum_{i=0}^{j} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_j} \otimes \bullet_{x_j} \cdots \bullet_{x_m} \\
&\quad - \lambda\mu \sum_{j=1}^{m} \sum_{i=1}^{j} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_m} \\
&\quad + \lambda^2 \sum_{j=0}^{m} \sum_{i=0}^{j} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_j} \otimes \bullet_{x_{j+1}} \cdots \bullet_{x_m}
\end{aligned}
$$

$$
\begin{aligned}
&\text{(by exchanging the index of } i \text{ and } j)\\
&= \mu^2 \sum_{i=1}^{m}\sum_{j=i}^{m} \bullet_{x_1}\cdots\bullet_{x_i} \otimes \bullet_{x_i}\cdots\bullet_{x_j} \otimes \bullet_{x_j}\cdots\bullet_{x_m}\\
&\quad - \lambda\mu \sum_{i=0}^{m-1}\sum_{j=i+1}^{m} \bullet_{x_1}\cdots\bullet_{x_i} \otimes \bullet_{x_{i+1}}\cdots\bullet_{x_j} \otimes \bullet_{x_j}\cdots\bullet_{x_m}\\
&\quad - \lambda\mu \sum_{j=1}^{m} \bullet_{x_1}\cdots\bullet_{x_j} \otimes 1 \otimes \bullet_{x_j}\cdots\bullet_{x_m}\\
&\quad - \lambda\mu \sum_{i=1}^{m}\sum_{j=i}^{m} \bullet_{x_1}\cdots\bullet_{x_i} \otimes \bullet_{x_i}\cdots\bullet_{x_j} \otimes \bullet_{x_{j+1}}\cdots\bullet_{x_m}\\
&\quad + \lambda^2 \sum_{i=0}^{m}\sum_{j=i}^{m} \bullet_{x_1}\cdots\bullet_{x_i}\\
&\qquad \otimes \bullet_{x_{i+1}}\cdots\bullet_{x_j} \otimes \bullet_{x_{j+1}}\cdots\bullet_{x_m} \quad \text{(by exchanging the summands).}
\end{aligned}
$$

Note that

$$
\sum_{i=1}^{m} \bullet_{x_1}\cdots\bullet_{x_i} \otimes 1 \otimes \bullet_{x_i}\cdots\bullet_{x_m} = \sum_{j=1}^{m} \bullet_{x_1}\cdots\bullet_{x_j} \otimes 1 \otimes \bullet_{x_j}\cdots\bullet_{x_m}\,.
$$

Thus,

$$
(\mathrm{id}\otimes\Delta_\epsilon)\Delta_\epsilon(\bullet_{x_1}\bullet_{x_2}\cdots\bullet_{x_m}) = (\Delta_\epsilon\otimes\mathrm{id})\Delta_\epsilon(\bullet_{x_1}\bullet_{x_2}\cdots\bullet_{x_m}).
$$

Suppose that Eq. (9) holds for $\mathrm{dep}(F) \le n$ for an $n \ge 0$, and consider the case of $\mathrm{dep}(F) = n+1$. We now apply the induction on breadth. Since $\mathrm{dep}(F) = n+1 \ge 1$, we have $F \ne 1$ and $\mathrm{bre}(F) \ge 1$. If $\mathrm{bre}(F) = 1$, then we may write $F = B^+_\omega(\overline{F})$ for some $\overline{F} \in \mathcal{F}(X,\Omega)$ and $\omega \in \Omega$. Hence,

$$
\begin{aligned}
&(\mathrm{id}\otimes\Delta_\epsilon)\Delta_\epsilon(F)\\
&\quad = (\mathrm{id}\otimes\Delta_\epsilon)\Delta_\epsilon(B^+_\omega(\overline{F}))\\
&\quad = (\mathrm{id}\otimes\Delta_\epsilon)\Big((B^+_\omega\otimes\mathrm{id})\Delta_\epsilon(\overline{F}) + (\mathrm{id}\otimes B^+_\omega)\Delta_\epsilon(\overline{F})\Big) \quad \text{(by Eq. (6))}\\
&\quad = (B^+_\omega\otimes\Delta_\epsilon)\Delta_\epsilon(\overline{F}) + (\mathrm{id}\otimes(\Delta_\epsilon B^+_\omega))\Delta_\epsilon(\overline{F})\\
&\quad = (B^+_\omega\otimes\Delta_\epsilon)\Delta_\epsilon(\overline{F}) + \Big(\mathrm{id}\otimes\big((B^+_\omega\otimes\mathrm{id})\Delta_\epsilon\\
&\qquad + (\mathrm{id}\otimes B^+_\omega)\Delta_\epsilon\big)\Big)\Delta_\epsilon(\overline{F}) \quad \text{(by Eq. (7))}\\
&\quad = (B^+_\omega\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes\Delta_\epsilon)\Delta_\epsilon(\overline{F}) + (\mathrm{id}\otimes B^+_\omega\otimes\mathrm{id})(\mathrm{id}\otimes\Delta_\epsilon)\Delta_\epsilon(\overline{F})\\
&\qquad + (\mathrm{id}\otimes\mathrm{id}\otimes B^+_\omega)(\mathrm{id}\otimes\Delta_\epsilon)\Delta_\epsilon(\overline{F})
\end{aligned}
$$

$$
\begin{aligned}
&= (B_\omega^+ \otimes \mathrm{id} \otimes \mathrm{id})(\Delta_\epsilon \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) + (\mathrm{id} \otimes B_\omega^+ \otimes \mathrm{id})(\Delta_\epsilon \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) \\
&\quad + (\mathrm{id} \otimes \mathrm{id} \otimes B_\omega^+)(\Delta_\epsilon \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) \\
&\qquad\qquad \text{(by the induction hypothesis)} \\
&= (B_\omega^+ \otimes \mathrm{id} \otimes \mathrm{id})(\Delta_\epsilon \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) + (\mathrm{id} \otimes B_\omega^+ \otimes \mathrm{id})(\Delta_\epsilon \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) \\
&\quad + (\Delta_\epsilon \otimes B_\omega^+)\Delta_\epsilon(\overline{F}) \\
&= \Big(((B_\omega^+ \otimes \mathrm{id})\Delta_\epsilon) \otimes \mathrm{id}\Big)\Delta_\epsilon(\overline{F}) + \Big(((\mathrm{id} \otimes B_\omega^+)\Delta_\epsilon) \otimes \mathrm{id}\Big)\Delta_\epsilon(\overline{F}) \\
&\quad + (\Delta_\epsilon \otimes B_\omega^+)\Delta_\epsilon(\overline{F}) \\
&= \Big(((B_\omega^+ \otimes \mathrm{id})\Delta_\epsilon + (\mathrm{id} \otimes B_\omega^+)\Delta_\epsilon) \otimes \mathrm{id}\Big)\Delta_\epsilon(\overline{F}) + (\Delta_\epsilon \otimes B_\omega^+)\Delta_\epsilon(\overline{F}) \\
&= \Big((\Delta_\epsilon B_\omega^+) \otimes \mathrm{id}\Big)\Delta_\epsilon(\overline{F}) + (\Delta_\epsilon \otimes B_\omega^+)\Delta_\epsilon(\overline{F}) \\
&= (\Delta_\epsilon \otimes \mathrm{id})(B_\omega^+ \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) + (\Delta_\epsilon \otimes \mathrm{id})(\mathrm{id} \otimes B_\omega^+)\Delta_\epsilon(\overline{F}) \\
&= (\Delta_\epsilon \otimes \mathrm{id})\Big((B_\omega^+ \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) + (\mathrm{id} \otimes B_\omega^+)\Delta_\epsilon(\overline{F})\Big) \\
&= (\Delta_\epsilon \otimes \mathrm{id})\Delta_\epsilon(F) \quad \text{(by Eq. (6))}.
\end{aligned}
$$

Assume that Eq. (9) holds for $\mathrm{dep}(F) = n + 1$ and $\mathrm{bre}(F) \leq m$, in addition to $\mathrm{dep}(F) \leq n$ by the first induction hypothesis. Consider the case when $\mathrm{dep}(F) = n+1$ and $\mathrm{bre}(F) = m + 1 \geq 2$. Then, $F = F_1F_2$ for some $F_1, F_2 \in \mathcal{F}(X, \Omega)$ with $0 < \mathrm{bre}(F_1), \mathrm{bre}(F_2) < \mathrm{bre}(F)$. Using the Sweedler notation, we write

$$
\Delta_\epsilon(F_1) = \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \text{ and } \Delta_\epsilon(F_2) = \sum_{(F_2)} F_{2(1)} \otimes F_{2(2)}.
$$

Thus,

$$
\begin{aligned}
&(\mathrm{id} \otimes \Delta_\epsilon)\Delta_\epsilon(F_1F_2) \\
&\quad = (\mathrm{id} \otimes \Delta_\epsilon)(F_1 \cdot \Delta_\epsilon(F_2) + \Delta_\epsilon(F_1) \cdot F_2 + \lambda(F_1 \otimes F_2)) \quad \text{(by Lemma 3.8)} \\
&\quad = (\mathrm{id} \otimes \Delta_\epsilon)\left(\sum_{(F_2)} F_1F_{2(1)} \otimes F_{2(2)} + \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)}F_2 \right. \\
&\qquad \left. +\lambda(F_1 \otimes F_2)\right) \quad \text{(by Eq. (2))} \\
&\quad = \sum_{(F_2)} F_1F_{2(1)} \otimes \Delta_\epsilon(F_{2(2)}) + \sum_{(F_1)} F_{1(1)} \otimes \Delta_\epsilon(F_{1(2)}F_2) + \lambda(F_1 \otimes \Delta_\epsilon(F_2)) \\
&\quad = \sum_{(F_2)} F_1F_{2(1)} \otimes F_{2(2)} \otimes F_{2(3)} + \sum_{(F_1)} F_{1(1)} \otimes \sum_{(F_2)} F_{1(2)}F_{2(1)} \otimes F_{2(2)} \\
&\qquad + \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \otimes F_{1(3)}F_2 + \lambda \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \otimes F_2 + \lambda F_1 \otimes \Delta_\epsilon(F_2)
\end{aligned}
$$

(by induction on breadth and Eq. (2))

$$
\begin{aligned}
&= \Bigg(\sum_{(F_2)} F_1F_{2(1)} \otimes F_{2(2)} \otimes F_{2(3)} \\
&\quad + \sum_{(F_1)} F_{1(1)} \otimes \sum_{(F_2)} F_{1(2)}F_{2(1)} \otimes F_{2(2)} + \lambda F_1 \otimes \Delta_\epsilon(F_2)\Bigg) \\
&\quad + \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \otimes F_{1(3)}F_2 + \lambda \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \otimes F_2 \\
&= \Bigg(\sum_{(F_2)} F_1F_{2(1)} \otimes F_{2(2)} \otimes F_{2(3)} \\
&\quad + \sum_{(F_2)}\sum_{(F_1)} F_{1(1)} \otimes F_{1(2)}F_{2(1)} \otimes F_{2(2)} + \lambda F_1 \otimes \Delta_\epsilon(F_2)\Bigg) \\
&\quad + \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \otimes F_{1(3)}F_2 + \lambda \Delta_\epsilon(F_1) \otimes F_2 \\
&= \Bigg(\sum_{(F_2)} F_1F_{2(1)} \otimes F_{2(2)} \otimes F_{2(3)} \\
&\quad + \sum_{(F_2)}\sum_{(F_1)} F_{1(1)} \otimes F_{1(2)}F_{2(1)} \otimes F_{2(2)} + \lambda \sum_{(F_2)} F_1 \otimes F_{2(1)} \otimes F_{2(2)}\Bigg) \\
&\quad + \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \otimes F_{1(3)}F_2 + \lambda \Delta_\epsilon(F_1) \otimes F_2 \\
&= \sum_{(F_2)} \Delta_\epsilon(F_1F_{2(1)}) \otimes F_{2(2)} + \sum_{(F_1)} \Delta_\epsilon(F_{1(1)}) \otimes F_{1(2)}F_2 + \lambda \Delta_\epsilon(F_1) \otimes F_2 \\
&\qquad \text{(by induction on breadth)} \\
&= (\Delta_\epsilon \otimes \mathrm{id})\Bigg(\sum_{(F_2)} F_1F_{2(1)} \otimes F_{2(2)} + \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)}F_2 + \lambda F_1 \otimes F_2\Bigg) \\
&= (\Delta_\epsilon \otimes \mathrm{id})(F_1 \cdot \Delta_\epsilon(F_2) + \Delta_\epsilon(F_1) \cdot F_2 + \lambda(F_1 \otimes F_2)) \quad \text{(by Eq. (2))} \\
&= (\Delta_\epsilon \otimes \mathrm{id})\Delta_\epsilon(F_1F_2) \quad \text{(by Lemma 3.8)}.
\end{aligned}
$$

This completes the induction on the breadth and hence the induction on the depth. □

Now we arrive at our second main result in this subsection.

Theorem 3.11 *The quadruple* $(H_{\mathrm{RT}}(X, \Omega),\ m_{RT},\ 1,\ \Delta_\epsilon)$ *is an* ϵ*-unitary bialgebra of weight* λ.

Proof Note that the triple $(H_{\mathrm{RT}}(X, \Omega),\ m_{RT},\ 1)$ is a unitary algebra. Then, the result follows from Lemma 3.8 and Theorem 3.10. □

3.4 Free Ω-cocycle infinitesimal unitary bialgebras

The main goal of this subsection is to understand weighted infinitesimal unitary bialgebras from the point view of operated algebras. This leads to the natural definition of a weighted Ω-cocycle infinitesimal unitary bialgebras. We show that $H_{\mathrm{RT}}(X, \Omega)$ is a free object in one of such categories.

Definition 3.12 [24, Section 1.2]

(a) An Ω-**operated monoid** is a monoid M together with a set of operators $P_\omega : M \to M, \omega \in \Omega$.
(b) An Ω-**operated unitary algebra** is a unitary algebra A together with a set of linear operators $P_\omega : A \to A, \omega \in \Omega$.

Definition 3.13 [21, Definition 3.17] Let λ be a given element of $\mathbf{k}$.

(a) An Ω-**operated** ϵ-**bialgebra of weight** λ is an ϵ-bialgebra H of weight λ together with a set of linear operators $P_\omega : H \to H, \omega \in \Omega$.
(b) Let $(H, \{P_\omega \mid \omega \in \Omega\})$ and $(H', \{P'_\omega \mid \omega \in \Omega\})$ be two Ω-operated ϵ-bialgebras of weight λ. A linear map $\phi : H \to H'$ is called an Ω-**operated** ϵ-**bialgebra morphism** if ϕ is a morphism of ϵ-bialgebras of weight λ and $\phi \circ P_\omega = P'_\omega \circ \phi$ for $\omega \in \Omega$.

By a symmetric 1-cocycle condition, we then propose

Definition 3.14 Let λ be a given element of $\mathbf{k}$.

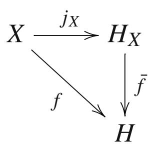
(a) An Ω-**cocycle** ϵ-**unitary bialgebra of weight** λ is an Ω-operated ϵ-unitary bialgebra $(H, m, 1_H, \Delta, \{P_\omega \mid \omega \in \Omega\})$ of weight λ satisfying the symmetric 1-cocycle condition:

$$\Delta P_\omega = (P_\omega \otimes \mathrm{id})\Delta + (\mathrm{id} \otimes P_\omega)\Delta \quad \text{for } \omega \in \Omega. \tag{10}$$

(b) The **free** Ω-**cocycle** ϵ-**unitary bialgebra of weight** λ **on a set** X is an Ω-cocycle ϵ-unitary bialgebra $(H_X, m_X, 1_{H_X}, \Delta_X, \{P_\omega \mid \omega \in \Omega\})$ of weight λ together with a set map $j_X : X \to H_X$ with the property that, for any Ω-cocycle ϵ-unitary bialgebra $(H, m, 1_H, \Delta, \{P'_\omega \mid \omega \in \Omega\})$ of weight λ and any set map $f : X \to H$ such that for any $x \in X$,

$$\Delta(f(x)) = \mu(f(x) \otimes f(x)) - \lambda(f(x) \otimes 1_H + 1_H \otimes f(x)),$$

there is a unique Ω-operated ϵ-unitary bialgebras unique morphism $\bar{f} : H_X \to H$ such that the diagram



commutes.

When Ω is a singleton set, we will omit it. From now on, our discussion takes place on $H_{\mathrm{RT}}(X,\Omega)=\mathbf{k}\mathcal{F}(X,\Omega)$. The following results generalize the universal properties which were studied in [10,13,24,38,43].

Lemma 3.15 [44, Theorem 4.5] *Let $j_X: X\hookrightarrow \mathcal{F}(X,\Omega)$, $x\mapsto \bullet_x$ be the natural embedding and m_{RT} be the concatenation product. Then, we have the following.*

(a) The quadruple $(\mathcal{F}(X,\Omega), m_{\mathrm{RT}}, 1, \{B_\omega^+\mid \omega\in\Omega\})$ together with the j_X is the free Ω-operated monoid on X.
(b) The quadruple $(H_{\mathrm{RT}}(X,\Omega), m_{\mathrm{RT}}, 1, \{B_\omega^+\mid \omega\in\Omega\})$ together with the j_X is the free Ω-operated unitary algebra on X.

Theorem 3.16 *Let $j_X: X\hookrightarrow \mathcal{F}(X,\Omega)$, $x\mapsto \bullet_x$ be the natural embedding and m_{RT} be the concatenation product. Then, the quintuple $(H_{\mathrm{RT}}(X,\Omega), m_{\mathrm{RT}}, 1, \Delta_\epsilon, \{B_\omega^+\mid \omega\in\Omega\})$ together with the j_X is the free Ω-cocycle ϵ-unitary bialgebra of weight λ on X.*

Proof By Theorem 3.11, $(H_{\mathrm{RT}}(X,\Omega), m_{\mathrm{RT}}, 1, \Delta_\epsilon)$ is an ϵ-unitary bialgebra of weight λ. Then, it follows from Eq. (6) that $(H_{\mathrm{RT}}(X,\Omega), m_{\mathrm{RT}}, 1, \Delta_\epsilon, \{B_\omega^+\mid \omega\in\Omega\})$ is an Ω-cocycle ϵ-unitary bialgebra of weight λ.

We next proceed to show the freeness. Let $(H, m, 1_H, \Delta, \{P_\omega\mid\omega\in\Omega\})$ be an Ω-cocycle ϵ-bialgebra of weight λ and $f: X\to H$ a set map such that

$$\Delta(f(x))=\mu(f(x)\otimes f(x))-\lambda(f(x)\otimes 1_H+1_H\otimes f(x)) \text{ for all } x\in X.$$

Particularly, $(H, m, 1_H, \{P_\omega\mid\omega\in\Omega\})$ is an Ω-operated unitary algebra. By Lemma 3.15 (b), there exists a unique Ω-operated unitary algebra morphism $\bar f: H_{\mathrm{RT}}(X,\Omega)\to H$ such that $\bar f\circ j_X=f$. It is sufficient to check the compatibility of the coproducts Δ and Δ_ϵ for which we verify

$$\Delta\bar f(F)=(\bar f\otimes\bar f)\Delta_\epsilon(F)\quad\text{for all } F\in\mathcal{F}(X,\Omega), \tag{11}$$

by induction on the depth $\mathrm{dep}(F)\geq 0$. For the initial step of $\mathrm{dep}(F)=0$, we have $F=\bullet_{x_1}\cdots\bullet_{x_m}$ for some $m\geq 0$, with the convention that $F=1$ when $m=0$. If $m=0$, then by Remark 2.2 (a) and Eq. (5),

$$\begin{aligned}\Delta\bar f(F)&=\Delta\bar f(1)=\Delta(1_H)=-\lambda(1_H\otimes 1_H)=-\lambda\bar f(1)\otimes\bar f(1)\\&=(\bar f\otimes\bar f)(-\lambda 1\otimes 1)=(\bar f\otimes\bar f)\Delta_\epsilon(1).\end{aligned}$$

If $m\geq 1$, then we have

$$\begin{aligned}\Delta\bar f(\bullet_{x_1}\cdots\bullet_{x_m})&=\Delta\Big(\bar f(\bullet_{x_1})\cdots\bar f(\bullet_{x_m})\Big)\\&=\cdots=\sum_{i=1}^{m}\Big(\bar f(\bullet_{x_1})\cdots\bar f(\bullet_{x_{i-1}})\Big)\cdot\Delta\Big(\bar f(\bullet_{x_i})\Big)\cdot\Big(\bar f(\bullet_{x_{i+1}})\cdots\bar f(\bullet_{x_m})\Big)\\&+\lambda\sum_{i=1}^{m-1}\bar f(\bullet_{x_1})\cdots\bar f(\bullet_{x_i})\otimes\bar f(\bullet_{x_{i+1}})\cdots\bar f(\bullet_{x_m})\quad(\text{by Eq. (3)})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\Big(\bar{f}(\bullet_{x_1})\cdots\bar{f}(\bullet_{x_{i-1}})\Big)\\
&\quad\cdot\Big(\mu\bar{f}(\bullet_{x_i})\otimes\bar{f}(\bullet_{x_i})-\lambda\bar{f}(\bullet_{x_i})\otimes 1_H-\lambda 1_H\otimes\bar{f}(\bullet_{x_i})\Big)\cdot\Big(\bar{f}(\bullet_{x_{i+1}})\cdots\bar{f}(\bullet_{x_m})\Big)\\
&\quad+\lambda\sum_{i=1}^{m-1}\bar{f}(\bullet_{x_1})\cdots\bar{f}(\bullet_{x_i})\otimes\bar{f}(\bullet_{x_{i+1}})\cdots\bar{f}(\bullet_{x_m})\\
&\qquad(\text{by } \Delta(\bar{f}(\bullet_{x_i}))=\Delta(f(x_i))=\mu(f(x)\otimes f(x))\\
&\quad-\lambda(f(x_i)\otimes 1_H+1_H\otimes f(x_i)))\\
&= \mu\sum_{i=1}^{m}\bar{f}(\bullet_{x_1})\cdots\bar{f}(\bullet_{x_i})\otimes\bar{f}(\bullet_{x_i})\cdots\bar{f}(\bullet_{x_m})-\lambda\sum_{i=1}^{m}\bar{f}(\bullet_{x_1})\cdot\cdots\bar{f}(\bullet_{x_i})\\
&\quad\otimes\bar{f}(\bullet_{x_{i+1}})\cdots\bar{f}(\bullet_{x_m})\\
&\quad-\lambda\sum_{i=1}^{m}\bar{f}(\bullet_{x_1})\cdot\cdots\bar{f}(\bullet_{x_{i-1}})\otimes\bar{f}(\bullet_{x_i})\cdots\bar{f}(\bullet_{x_m})\\
&\quad+\lambda\sum_{i=1}^{m-1}\bar{f}(\bullet_{x_1})\cdots\bar{f}(\bullet_{x_i})\otimes\bar{f}(\bullet_{x_{i+1}})\cdots\bar{f}(\bullet_{x_m})\\
&= \mu\sum_{i=1}^{m}\bar{f}(\bullet_{x_1})\cdots\bar{f}(\bullet_{x_i})\otimes\bar{f}(\bullet_{x_i})\cdots\bar{f}(\bullet_{x_m})-\lambda\sum_{i=1}^{m-1}\bar{f}(\bullet_{x_1})\cdot\cdots\bar{f}(\bullet_{x_i})\\
&\quad\otimes\bar{f}(\bullet_{x_{i+1}})\cdots\bar{f}(\bullet_{x_m})\\
&\quad-\lambda\bar{f}(\bullet_{x_1})\cdot\cdots\bar{f}(\bullet_{x_m})\otimes\bar{f}(1)-\lambda\sum_{i=1}^{m}\bar{f}(\bullet_{x_1})\cdot\cdots\bar{f}(\bullet_{x_{i-1}})\otimes\bar{f}(\bullet_{x_i})\cdots\bar{f}(\bullet_{x_m})\\
&\quad+\lambda\sum_{i=1}^{m-1}\bar{f}(\bullet_{x_1})\cdots\bar{f}(\bullet_{x_i})\otimes\bar{f}(\bullet_{x_{i+1}})\cdots\bar{f}(\bullet_{x_m})\\
&= \mu\sum_{i=1}^{m}\bar{f}(\bullet_{x_1})\cdots\bar{f}(\bullet_{x_i})\otimes\bar{f}(\bullet_{x_i})\cdots\bar{f}(\bullet_{x_m})-\lambda\bar{f}(\bullet_{x_1})\cdot\cdots\bar{f}(\bullet_{x_m})\otimes\bar{f}(1)\\
&\quad-\lambda\sum_{i=1}^{m}\bar{f}(\bullet_{x_1})\cdot\cdots\bar{f}(\bullet_{x_{i-1}})\otimes\bar{f}(\bullet_{x_i})\cdots\bar{f}(\bullet_{x_m})\\
&= \mu\sum_{i=1}^{m}\bar{f}(\bullet_{x_1})\cdots\bar{f}(\bullet_{x_i})\otimes\bar{f}(\bullet_{x_i})\cdots\bar{f}(\bullet_{x_m})-\lambda\bar{f}(\bullet_{x_1})\cdot\cdots\bar{f}(\bullet_{x_m})\otimes\bar{f}(1)\\
&\quad-\lambda\sum_{i=0}^{m-1}\bar{f}(\bullet_{x_1})\cdot\cdots\bar{f}(\bullet_{x_i})\otimes\bar{f}(\bullet_{x_{i+1}})\cdots\bar{f}(\bullet_{x_m})\\
&= (\bar{f}\otimes\bar{f})\left(\mu\sum_{i=1}^{m}\bullet_{x_1}\cdots\bullet_{x_i}\otimes\bullet_{x_i}\cdots\bullet_{x_m}-\lambda\sum_{i=0}^{m}\bullet_{x_1}\cdots\bullet_{x_i}\otimes\bullet_{x_{i+1}}\cdots\bullet_{x_m}\right)
\end{aligned}
$$

$$= (\bar{f} \otimes \bar{f})\Delta_\epsilon(\bullet_{x_1} \cdots \bullet_{x_m}) \quad \text{(by Lemma 3.7)}.$$

Suppose Eq. (11) holds for $\mathrm{dep}(F) \leq n$ for an $n \geq 0$ and consider the case of $\mathrm{dep}(F) = n + 1$. For this case, we apply the induction on the breadth $\mathrm{bre}(F)$. Since $\mathrm{dep}(F) = n + 1 \geq 1$, we have $F \neq 1$ and $\mathrm{bre}(F) \geq 1$. If $\mathrm{bre}(F) = 1$, we have $F = B^+_\omega(\overline{F})$ for some $\overline{F} \in \mathcal{F}(X, \Omega)$ and $\omega \in \Omega$. Then,

$$\begin{aligned}
\Delta \bar{f}(F) &= \Delta \bar{f}(B^+_\omega(\overline{F})) = \Delta P_\omega(\bar{f}(\overline{F})) \quad \text{(by } \bar{f} \text{ being an operated algebra morphism)}\\
&= (P_\omega \otimes \mathrm{id})\Delta(\bar{f}(\overline{F})) + (\mathrm{id} \otimes P_\omega)\Delta(\bar{f}(\overline{F})) \quad \text{(by Eq. (10))}\\
&= (P_\omega \otimes \mathrm{id})(\bar{f} \otimes \bar{f})\Delta_\epsilon(\overline{F})\\
&\quad + (\mathrm{id} \otimes P_\omega)(\bar{f} \otimes \bar{f})\Delta_\epsilon(\overline{F}) \text{ (by the induction hypothesis on dep}(F))\\
&= (P_\omega \bar{f} \otimes \bar{f})\Delta_\epsilon(\overline{F}) + (\bar{f} \otimes P_\omega \bar{f})\Delta_\epsilon(\overline{F})\\
&= (\bar{f} B^+_\omega \otimes \bar{f})\Delta_\epsilon(\overline{F})\\
&\quad + (\bar{f} \otimes \bar{f} B^+_\omega)\Delta_\epsilon(\overline{F}) \quad \text{(by } \bar{f} \text{ being an operated algebra morphism)}\\
&= (\bar{f} \otimes \bar{f})(B^+_\omega \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) + (\bar{f} \otimes \bar{f})(\mathrm{id} \otimes B^+_\omega)\Delta_\epsilon(\overline{F})\\
&= (\bar{f} \otimes \bar{f})\Big((B^+_\omega \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) + (\mathrm{id} \otimes B^+_\omega)\Delta_\epsilon(\overline{F})\Big)\\
&= (\bar{f} \otimes \bar{f})\Delta_\epsilon(B^+_\omega(\overline{F})) \quad \text{(by Eq. (6))}\\
&= (\bar{f} \otimes \bar{f})\Delta_\epsilon(F).
\end{aligned}$$

Assume Eq. (11) holds for $\mathrm{dep}(F) = n+1$ and $\mathrm{bre}(F) \leq m$, in addition to $\mathrm{dep}(F) \leq n$ by the first induction hypothesis, and consider the case when $\mathrm{dep}(F) = n + 1$ and $\mathrm{bre}(F) = m + 1 \geq 2$. Then, $F = F_1 F_2$ for some $F_1, F_2 \in \mathcal{F}(X, \Omega)$ with $0 < \mathrm{bre}(F_1), \mathrm{bre}(F_2) < m + 1$. By the Sweedler notation, we may write

$$\Delta_\epsilon(F_1) = \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \text{ and } \Delta_\epsilon(F_2) = \sum_{(F_2)} F_{2(1)} \otimes F_{2(2)}.$$

By the induction hypothesis on the breadth, we have

$$\begin{aligned}
\Delta(\bar{f}(F_1)) &= (\bar{f} \otimes \bar{f})\Delta_\epsilon(F_1) = \sum_{(F_1)} \bar{f}(F_{1(1)}) \otimes \bar{f}(F_{1(2)}),\\
\Delta(\bar{f}(F_2)) &= (\bar{f} \otimes \bar{f})\Delta_\epsilon(F_2) = \sum_{(F_2)} \bar{f}(F_{2(1)}) \otimes \bar{f}(F_{2(2)}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta \bar{f}(F) &= \Delta \bar{f}(F_1 F_2) = \Delta(\bar{f}(F_1)\bar{f}(F_2))\\
&= \bar{f}(F_1) \cdot \Delta(\bar{f}(F_2)) + \Delta(\bar{f}(F_1)) \cdot \bar{f}(F_2) + \lambda \bar{f}(F_1) \otimes \bar{f}(F_2) \quad \text{(by Eq. (3))}\\
&= \bar{f}(F_1) \cdot \left(\sum_{(F_2)} \bar{f}(F_{2(1)}) \otimes \bar{f}(F_{2(2)})\right) + \left(\sum_{(F_1)} \bar{f}(F_{1(1)}) \otimes \bar{f}(F_{1(2)})\right)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \bar{f}(F_2) + \lambda \bar{f}(F_1) \otimes \bar{f}(F_2) \\
&= \sum_{(F_2)} \bar{f}(F_1)\bar{f}(F_{2(1)}) \otimes \bar{f}(F_{2(2)}) + \sum_{(F_1)} \bar{f}(F_{1(1)}) \otimes \bar{f}(F_{1(2)})\bar{f}(F_2) \\
&\quad + \lambda \bar{f}(F_1) \otimes \bar{f}(F_2) \text{ (by Eq. (2))} \\
&= \sum_{(F_2)} \bar{f}(F_1 F_{2(1)}) \otimes \bar{f}(F_{2(2)}) + \sum_{(F_1)} \bar{f}(F_{1(1)}) \otimes \bar{f}(F_{1(2)} F_2) \\
&\quad + \lambda \bar{f}(F_1) \otimes \bar{f}(F_2) \\
&= (\bar{f} \otimes \bar{f})\Big(\sum_{(F_2)} F_1 F_{2(1)} \otimes F_{2(2)}\Big) + (\bar{f} \otimes \bar{f})\Big(\sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} F_2\Big) \\
&\quad + (\bar{f} \otimes \bar{f})(\lambda F_1 \otimes F_2) \\
&= (\bar{f} \otimes \bar{f}) \left(\sum_{(F_2)} F_1 F_{2(1)} \otimes F_{2(2)} + \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} F_2 + \lambda F_1 \otimes F_2\right) \\
&= (\bar{f} \otimes \bar{f}) \left(F_1 \cdot \sum_{(F_2)} F_{2(1)} \otimes F_{2(2)} \right. \\
&\quad \left. + \Big(\sum_{(F_1)} F_{1(1)} \otimes F_{1(2)}\Big) \cdot F_2 + \lambda F_1 \otimes F_2 \right) \quad \text{(by Eq. (2))} \\
&= (\bar{f} \otimes \bar{f})(F_1 \cdot \Delta_\epsilon(F_2) + \Delta_\epsilon(F_1) \cdot F_2 + \lambda(F_1 \otimes F_2)) \\
&= (\bar{f} \otimes \bar{f})\Delta_\epsilon(F_1 F_2) \quad \text{(by Lemma 3.8)} \\
&= (\bar{f} \otimes \bar{f})\Delta_\epsilon(F).
\end{aligned}
$$

This completes the induction on the breadth and hence the induction on the depth. □

Let $X = \emptyset$. Then, we obtain a freeness of $H_{RT}(\emptyset, \Omega)$, which is the infinitesimal version of decorated noncommutative Connes–Kreimer Hopf algebra by Remark 3.4 (b).

Corollary 3.17 *The quintuple* $(H_{RT}(\emptyset, \Omega), m_{\mathrm{RT}}, 1, \Delta_\epsilon, \{B^+_\omega \mid \omega \in \Omega\})$ *is the free* Ω*-cocycle* ϵ*-unitary bialgebra of weight* λ *on the empty set, that is, the initial object in the category of* Ω*-cocycle* ϵ*-unitary bialgebras of weight* λ*.*

Proof It follows from Theorem 3.16 (3.16) by taking $X = \emptyset$. □

Taking Ω to be singleton in Corollary 3.17, then all planar rooted forests are decorated by the same letter. In this case, planar rooted forests have no decorations that are precisely planar rooted forests in the classical Connes–Kreimer Hopf algebra under the noncommutative version which was introduced by Foissy [14] and Holtkamp [29].

Corollary 3.18 *Let* $\mathcal{F}$ *be the set of planar rooted forests without decorations. Then, the quintuple* $(\mathbf{k}\mathcal{F}, m_{\mathrm{RT}}, 1, \Delta_\epsilon, B^+)$ *is the free cocycle* ϵ*-unitary bialgebra of weight* λ *on the empty set, that is, the initial object in the category of* Ω*-cocycle* ϵ*-unitary bialgebras of weight* λ*.*

Proof It follows from Corollary 3.17 by taking Ω to be a singleton set. □

4 Infinitesimal unitary Hopf algebras of rooted forests

In this section, we first recall the concept of infinitesimal unitary Hopf algebra under the view of Loday and Ronco. We then construct an infinitesimal unitary Hopf algebra on decorated planar rooted forests which is different from the one investigated by Foissy [16].

4.1 An infinitesimal unitary Hopf algebra in the sense of Loday and Ronco

We first recall the convolution product on an infinitesimal unitary bialgebra in the sense of Loday and Ronco [36].

Let $A = (A, m, 1, \Delta, \varepsilon)$ be a ϵ-unitary counitary bialgebra of weight -1. If $f, g \in \mathrm{Hom}_{\mathbf{k}}(A, A)$, their **convolution product** is the map $f * g : A \to A$ defined to be the composition:

$$f * g := m(f \otimes g)\Delta.$$

Note that $1 \circ \varepsilon$ is the unit with respect to $*$. By the associativity of m and the coassociativity of Δ, we have $*$ is associative. The algebra $(\mathrm{Hom}_{\mathbf{k}}(A, A), *)$ is called **convolution algebra**. See [36, Section 2.4] for more details.

Definition 4.1 [16,36]

(a) An ϵ-unitary counitary bialgebra $A = (A, m, 1, \Delta, \varepsilon)$ of weight -1 is called an **infinitesimal unitary Hopf algebra** if the identity map $\mathrm{id} \in \mathrm{Hom}_{\mathbf{k}}(H, H)$ is invertible with respect to the convolution product $*$. This inverse is called the **antipode** of A and denoted by S.

(b) A bialgebra $(H, m, u, \Delta, \varepsilon)$ (either classical or infinitesimal unitary) is called **graded** if there are $\mathbf{k}$-submodules $H^{(n)}$, $n \geq 0$, of H such that

(i) $H = \bigoplus_{n \geq 0}^{\infty} H^n$;

(ii) $H^p H^q \subseteq H^{p+q}, \ p, q \geq 0$;

(iii) $\Delta(H^n) \subseteq \bigoplus_{p+q=n} H^p \otimes H^q, n \geq 0$.

H is called **connected graded** if in addition $H^0 = \mathbf{k}$ and $\ker\varepsilon = \bigoplus_{n \geq 1} H^n$. It is well known that a connected graded bialgebra is a Hopf algebra.

The ϵ-unitary Hopf algebra satisfies many properties analogous to those of a classical Hopf algebra [16,36].

Remark 4.2 Let $A = (A, m, 1, \Delta, \varepsilon)$ be an ϵ-unitary counitary bialgebra of weight -1.

(a) $\Delta(1) = 1 \otimes 1$. If we regard 1 as the unit map, then it is a coalgebra morphism.
(b) The counit ε is an algebra morphism.
(c) If A is connected graded, then it is an ϵ-unitary Hopf algebra in the sense of Loday and Ronco [16]. Its antipode S, inductively defined by

$$S(1) = 1 \text{ and } S(x) = -x - \sum_{(x)} x' S(x'') \text{ for } x \in \ker \varepsilon.$$

(d) If A is further an ϵ-unitary Hopf algebra with the antipode S, then

(i) $S(1) = 1$ and $\varepsilon \circ S = \varepsilon$.
(ii) $S(ab) = \varepsilon(a)S(b) + \varepsilon(b)S(a) - \varepsilon(a)\varepsilon(b)1$ for all $a, b \in A$.
(iii) $\Delta(S(a)) = S(a) \otimes 1 + 1 \otimes S(a) - \varepsilon(a)1 \otimes 1$ for all $a \in A$.

Particularly, for any $a \in A$, if $\varepsilon(a) = 0$, then $S(a)$ is a primitive element [16].

4.2 An infinitesimal unitary Hopf algebra on decorated rooted forests

By linearity, we only need to define $\Delta_\ell(F)$ for basis elements $F \in \mathcal{F}(X, \Omega)$. For the initial step of $\mathrm{dep}(F) = 0$, we define

$$\Delta_\ell(F) := \begin{cases} 1 \otimes 1 & \text{if } F = 1, \\ \bullet_x \otimes 1 + 1 \otimes \bullet_x & \text{if } F = \bullet_x \text{ for some } x \in X, \\ \bullet_{x_1} \cdot \Delta_\ell(\bullet_{x_2} \cdots \bullet_{x_m}) + \Delta_\ell(\bullet_{x_1}) \cdot (\bullet_{x_2} \cdots \bullet_{x_m}) & \\ - \bullet_{x_1} \otimes \bullet_{x_2} \cdots \bullet_{x_m} & \text{if } F = \bullet_{x_1} \cdots \bullet_{x_m} \text{ with } m \geq 2. \end{cases}$$

For the induction step of $\mathrm{dep}(F) \geq 1$, we reduce the definition to induction on breadth and define

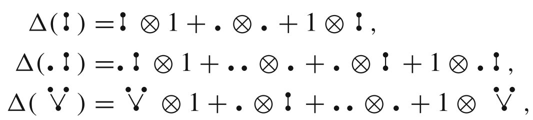
$$\Delta_\ell(F) := \begin{cases} (B^+_\omega \otimes \mathrm{id})\Delta_\ell(\overline{F}) + (\mathrm{id} \otimes B^+_\omega)\Delta_\ell(\overline{F}) & \text{if } F = B^+_\omega(\overline{F}), \\ T_1 \cdot \Delta_\ell(T_2 \cdots T_m) + \Delta_\ell(T_1) \cdot (T_2 \cdots T_m) & \\ -T_1 \otimes T_2 \cdots T_m & \text{if } F = T_1 \cdots T_m \text{ with } m \geq 2. \end{cases}$$

Example 4.3 Let $x, y \in X$ and $\alpha, \beta, \gamma \in \Omega$. Then,

$$\Delta_\ell(\bullet_\alpha) = \bullet_\alpha \otimes 1 + 1 \otimes \bullet_\alpha,$$
$$\Delta_\ell(\mathrm{I}^{x}_{\alpha}) = \mathrm{I}^{x}_{\alpha} \otimes 1 + \bullet_\alpha \otimes \bullet_x + \bullet_x \otimes \bullet_\alpha + 1 \otimes \mathrm{I}^{x}_{\alpha},$$
$$\Delta_\ell(\bullet_x \mathrm{I}^{x}_{\alpha}) = \bullet_x \mathrm{I}^{x}_{\alpha} \otimes 1 + \bullet_x \bullet_\alpha \otimes \bullet_x + \bullet_x \bullet_x \otimes \bullet_\alpha + \bullet_x \otimes \mathrm{I}^{x}_{\alpha} + 1 \otimes \bullet_x \mathrm{I}^{x}_{\alpha},$$
$$\Delta_\ell({}^{x}\mathrm{V}^{y}_{\alpha}) = {}^{x}\mathrm{V}^{y}_{\alpha} \otimes 1 + \mathrm{I}^{x}_{\alpha} \otimes \bullet_y + \bullet_\alpha \otimes \bullet_x \bullet_y + \bullet_x \bullet_y \otimes \bullet_\alpha + \bullet_x \otimes \mathrm{I}^{y}_{\alpha} + 1 \otimes {}^{x}\mathrm{V}^{y}_{\alpha}.$$

Remark 4.4 We emphasize that our coproduct is mainly different from the one studied by Foissy [16] who constructed an infinitesimal Hopf algebra under the view of Loday–Ronco on undecorated planar rooted forests. In his work,

$$\Delta(\bullet) = \bullet \otimes 1 + 1 \otimes \bullet,$$



which are different from the corresponding ones suppressed decorations in Example 4.3.

We now construct an ϵ-unitary bialgebra of weight λ with counit on $H_{\mathrm{RT}}(X, \Omega)$.

Lemma 4.5 *Let $(H_{\mathrm{RT}}(X, \Omega),\ m_{RT},\ 1,\ \Delta_\epsilon)$ be the ϵ-unitary bialgebra of weight λ in Theorem 3.11. Then, there is a counit on $H_{\mathrm{RT}}(X, \Omega)$ if, and only if, λ is invertible in* $\mathbf{k}$.

Proof For the necessary condition, by the counicity, we have

$$(\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Delta_\epsilon(1) = (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})(-\lambda 1 \otimes 1) = -\lambda\varepsilon_{\mathrm{RT}}(1)1 = 1,$$

so λ has an inverse in $\mathbf{k}$, namely $-\varepsilon_{\mathrm{RT}}(1)$. Conversely, for any forest $F \in \mathcal{T}(X, \Omega)$, define $\varepsilon_{\mathrm{RT}} : H_{\mathrm{RT}}(X, \Omega) \to \mathbf{k}$ by taking

$$\varepsilon_{\mathrm{RT}}(F) := \begin{cases} -\lambda^{-1}, & \text{if } F = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

and extending by $\mathbf{k}$-linearity. We now apply the induction on $\mathrm{dep}(F) \geq 0$ to check the counicity conditions:

$$(\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Delta_\epsilon(F) = \beta_l(F) \ \text{ and } \ (\mathrm{id} \otimes \varepsilon_{\mathrm{RT}})\Delta_\epsilon(F) = \beta_r(F), \tag{13}$$

where $\beta_l : H_{\mathrm{RT}}(X, \Omega) \to \mathbf{k} \otimes H_{\mathrm{RT}}(X, \Omega)$ is defined by $F \mapsto 1_{\mathbf{k}} \otimes F$ and $\beta_r : H_{\mathrm{RT}}(X, \Omega) \to H_{\mathrm{RT}}(X, \Omega) \otimes \mathbf{k}$ is given by $F \mapsto F \otimes 1_{\mathbf{k}}$.

For the initial step of $\mathrm{dep}(F) = 0$, we have $F = \bullet_{x_1} \bullet_{x_2} \cdots \bullet_{x_m}$ for some $m \geq 0$, with the convention that $F = 1$. When $m = 0$, by Eq. (12), we have

$$(\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Delta_\epsilon(F) = (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Delta_\epsilon(1) = -\lambda\varepsilon_{\mathrm{RT}}(1) \otimes 1 = 1_{\mathbf{k}} \otimes 1 = \beta_l(1).$$

When $m \geq 1$, by Lemma 3.7,

$$\begin{aligned}
&(\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Delta_\epsilon(F) \\
&= (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Big(\mu \sum_{i=1}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_i} \cdots \bullet_{x_m} - \lambda \sum_{i=0}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_m}\Big) \\
&= (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Big(- \lambda \sum_{i=0}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_m}\Big) \quad \text{(by Eq. (12))} \\
&= (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Big(- \lambda \sum_{i=1}^{m} \bullet_{x_1} \cdots \bullet_{x_i} \otimes \bullet_{x_{i+1}} \cdots \bullet_{x_m} - \lambda 1 \otimes \bullet_{x_1} \cdots \bullet_{x_m}\Big)
\end{aligned}$$

$$= -\lambda \varepsilon_{\mathrm{RT}}(1) \otimes \bullet_{x_1} \cdots \bullet_{x_m}$$
$$= 1_{\mathbf{k}} \otimes F = \beta_l(F) \quad \text{(by Eq. (12))}.$$

Suppose that Eq. (13) holds for $\mathrm{dep}(F) \leq n$ for an $n \geq 0$ and consider the case of $\mathrm{dep}(F) = n+1$. We next apply the induction on breadth. Since $\mathrm{dep}(F) = n+1 \geq 1$, we have $F \neq 1$ and $\mathrm{bre}(F) \geq 1$. When $\mathrm{bre}(F) = 1$, we may write $F = B_\omega^+(\overline{F})$ for some $\overline{F} \in \mathcal{F}(X, \Omega)$ and $\omega \in \Omega$. Then,

$$\begin{aligned}(\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Delta_\epsilon(F) &= (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Delta_\epsilon(B_\omega^+(\overline{F}))\\ &= (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Big((B_\omega^+ \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) + (\mathrm{id} \otimes B_\omega^+)\Delta_\epsilon(\overline{F})\Big) \quad \text{(by Eq. (6))}\\ &= (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})(\mathrm{id} \otimes B_\omega^+)\Delta_\epsilon(\overline{F}) \quad \text{(by Eq. (12))}\\ &= (\mathrm{id} \otimes B_\omega^+)(\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Delta_\epsilon(\overline{F})\\ &= (\mathrm{id} \otimes B_\omega^+)(1_{\mathbf{k}} \otimes \overline{F}) \quad \text{(by the induction hypothesis)}\\ &= 1_{\mathbf{k}} \otimes F = \beta_l(F).\end{aligned}$$

Assume that Eq. (13) holds for $\mathrm{dep}(F) = n+1$ and $\mathrm{bre}(F) \leq m$, in addition to $\mathrm{dep}(F) \leq n$ by the first induction hypothesis. Consider the case when $\mathrm{dep}(F) = n+1$ and $\mathrm{bre}(F) = m+1 \geq 2$. As in the proof of the coassociativity, let $F = F_1F_2$ for some $F_1, F_2 \in \mathcal{F}(X, \Omega)$ with $0 < \mathrm{bre}(F_1), \mathrm{bre}(F_2) < \mathrm{bre}(F)$. Thus,

$$\begin{aligned}(\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Delta_\epsilon(F) =\ & (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id})\Delta_\epsilon(F_1F_2)\\ =\ & (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id}))\Big(F_1 \cdot \Delta_\epsilon(F_2) + \Delta_\epsilon(F_1) \cdot F_2 + \lambda F_1 \otimes F_2\Big) \quad \text{(by Lemma 3.8)}\\ =\ & (\varepsilon_{\mathrm{RT}} \otimes \mathrm{id}) \left(\sum_{(F_2)} F_1F_{2(1)} \otimes F_{2(2)} + \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)}F_2 + \lambda F_1 \otimes F_2\right) \quad \text{(by Eq. (2))}\\ =\ & \sum_{(F_2)} \varepsilon_{\mathrm{RT}}(F_1F_{2(1)}) \otimes F_{2(2)} + \sum_{(F_1)} \varepsilon_{\mathrm{RT}}(F_{1(1)}) \otimes F_{1(2)}F_2 + \lambda\varepsilon_{\mathrm{RT}}(F_1) \otimes F_2\\ =\ & \sum_{(F_1)} \varepsilon_{\mathrm{RT}}(F_{1(1)}) \otimes F_{1(2)}F_2 \quad \text{(by Eq. (12))}\\ =\ & 1_{\mathbf{k}} \otimes F_1F_2 = \beta_l(F) \quad \text{(by the induction hypothesis)}.\end{aligned}$$

Similarly, we obtain that $(\mathrm{id} \otimes \varepsilon_{\mathrm{RT}})\Delta_\epsilon(F) = \beta_r(F)$ holds for $F \in \mathcal{F}(X, \Omega)$, completing the proof. □

We now give a graduation of $H_{\mathrm{RT}}(X, \Omega)$ by defining

$$H^n := H^n_{RT}(X, \Omega) := \mathbf{k}\{F \in \mathcal{F}(X, \Omega) \mid |V(F)| = n\} \text{ for } n \geq 0,$$

where $|V(F)|$ is the number of vertices of F. Then

$$H_{\mathrm{RT}}(X, \Omega) = \bigoplus_{n=0}^{\infty} H^n; \quad H^pH^q \subseteq H^{p+q},\ p, q \geq 0;$$

$$H^0 = \mathbf{k}; \quad B^+_\omega(H^n) \subseteq H^{n+1}, \omega \in \Omega. \tag{14}$$

Lemma 4.6 *Let $(H_{\mathrm{RT}}(X,\Omega), m_{RT}, 1, \Delta_\epsilon)$ be the ϵ-unitary bialgebra of weight λ in Theorem 3.11. Then, Δ_ϵ respects the graduation in the sense*

$$\Delta_\epsilon(H^n) \in \bigoplus_{p+q=n} H^p \otimes H^q \ \text{for}\ n \geq 0$$

if, and only if, $\mu = 0$.

Proof If $\mu \neq 0$, Δ_ϵ does not respect the graduation by Eq. (5). Conversely, consider $\mu = 0$. By linearity, we only need to verify

$$\Delta_\epsilon(F) \in \sum_{p+q=n} H^p \otimes H^q \ \text{ for } F \in \mathcal{F}(X,\Omega), \tag{15}$$

which will be proved by induction on $n \geq 0$. For the initial step of $n = 0$, we have $V(F) \leq 0$ and so $F = 1$. Then, Eq. (15) follows by Eq. (5).

For $k \geq 0$, assume that Eq. (15) holds for $n \leq k$ and consider the case of $n = k + 1 \geq 1$. If $\mathrm{bre}(F) = 1$, we have two cases to consider.

Case 1. $F = \bullet_x$ for some $x \in X$. By Eq. (5),

$$\Delta_\epsilon(F) = \Delta_\epsilon(\bullet_x) = -\lambda(1 \otimes \bullet_x + \bullet_x \otimes 1) \in H^1 \otimes H^1.$$

Case 2. $F = B^+_\omega(\overline{F})$ for some $\omega \in \Omega$ and some $\overline{F} \in \mathcal{F}(X,\Omega)$. By the induction hypothesis,

$$\Delta_\epsilon(\overline{F}) \in \sum_{p+q=k} H^p \otimes H^q.$$

Then, it follows from Eq. (14) that

$$\Delta_\epsilon(F) = \Delta_\epsilon(B^+_\omega(\overline{F})) = (B^+_\omega \otimes \mathrm{id})\Delta_\epsilon(\overline{F}) + (\mathrm{id} \otimes B^+_\omega)\Delta_\epsilon(\overline{F}) \in \sum_{p+q=k+1} H^p \otimes H^q,$$

as required.

If $\mathrm{bre}(F) \geq 2$, we may write $F = F_1F_2$ for some $F_1, F_2 \in \mathcal{F}(X,\Omega)$ and so

$$|V(F)| = |V(F_1)| + |V(F_2)| = k + 1, \tag{16}$$

which is greater than $|V(F_1)|$ and $|V(F_2)|$. By the induction hypothesis, we have

$$\Delta_\epsilon(F_1) \in \sum_{p_1+q_1=|V(F_1)|} H^{p_1} \otimes H^{q_1} \ \text{ and }\ \Delta_\epsilon(F_2) \in \sum_{p_2+q_2=|V(F_2)|} H^{p_2} \otimes H^{q_2}.$$

Thus,

$$
\begin{aligned}
\Delta_\epsilon(F) &= \Delta_\epsilon(F_1F_2)\\
&= F_1\cdot\Delta_\epsilon(F_1)+\Delta_\epsilon(F_1)\cdot F_2+\lambda F_1\otimes F_2 \quad \text{(by Lemma 3.8)}\\
&\in F_1\cdot\left(\sum_{p_2+q_2=|V(F_2)|} H^{p_2}\otimes H^{q_2}\right)+\left(\sum_{p_1+q_1=|V(F_1)|} H^{p_1}\otimes H^{q_1}\right)\cdot F_2\\
&\quad+\sum_{p+q=k+1} H^p\otimes H^q\\
&\subseteq \sum_{p_2+q_2=|V(F_2)|} H^{|V(F_1)|}H^{p_2}\otimes H^{q_2}+\sum_{p_1+q_1=|V(F_1)|} H^{p_1}\otimes H^{q_1}H^{|V(F_2)|}\\
&\quad+\sum_{p+q=k+1} H^p\otimes H^q\\
&\subseteq \sum_{p_2+q_2=|V(F_2)|} H^{|V(F_1)|+p_2}\otimes H^{q_2}+\sum_{p_1+q_1=|V(F_1)|} H^{p_1}\otimes H^{q_1+|V(F_2)|}\\
&\quad+\sum_{p+q=k+1} H^p\otimes H^q \quad \text{(by Eq. (14))}\\
&\subseteq \sum_{p+q=k+1} H^p\otimes H^q \quad \text{(by Eq. (16))}.
\end{aligned}
$$

This completes the induction. □

Remark 4.7 In general, the coproduct Δ_ϵ respects another graduation, counting the number of vertices decorated by elements of Ω.

When $\lambda = -1$ and $\mu = 0$, we have an ϵ-unitary Hopf algebra under the view of Loday and Ronco.

Theorem 4.8 *The quintuple* $(H_{\mathrm{RT}}(X,\Omega),\ m_{RT},1,\ \Delta_\ell,\varepsilon_{\mathrm{RT}})$ *is an* ϵ*-unitary Hopf algebra.*

Proof Taking $\lambda = -1$ in Lemma 4.5, the $(H_{\mathrm{RT}}(X,\Omega),\ m_{RT},1,\ \Delta_\ell,\varepsilon_{\mathrm{RT}})$ is an ϵ-unitary counitary bialgebra of weight -1. By Eq. (14) and Lemma 4.6, $(H_{\mathrm{RT}}(X,\Omega),\ m_{RT},1,\ \Delta_\ell,\varepsilon_{\mathrm{RT}})$ is connected and graded. Thus, the quintuple $(H_{\mathrm{RT}}(X,\Omega),\ m_{RT},1,\ \Delta_\ell,\varepsilon_{\mathrm{RT}})$ is an ϵ-unitary Hopf algebra. □

5 Pre-Lie algebras of decorated rooted forests

In this section, we recall the connection from weighted ϵ-bialgebras to pre-Lie algebras [21]. Using Theorem 3.11, we then construct a new pre-Lie algebraic structure on decorated planar rooted forests.

5.1 Pre-Lie algebras and infinitesimal unitary bialgebras

In this subsection, we first recall the concept of pre-Lie algebra and show the connection from weighted ϵ-bialgebras to pre-Lie algebras.

Definition 5.1 [37] A **(left) pre-Lie algebra** is a **k**-module A together with a binary operation $\rhd : A \otimes A \to A$ satisfying the left pre-Lie identity:

$$(a \rhd b) \rhd c - a \rhd (b \rhd c) = (b \rhd a) \rhd c - b \rhd (a \rhd c) \text{ for } a, b, c \in A.$$

The close relation between pre-Lie algebras and Lie algebras is characterized by the following result.

Lemma 5.2 [19, Theorem 1] *Let $(A, \rhd)$ be a pre-Lie algebra. Define for elements in A a new multiplication by setting*

$$[a, b] := a \rhd b - b \rhd a \text{ for } a, b \in A.$$

Then, $(A, [\_, \_])$ is a Lie algebra.

The following result captures the connection from weighted ϵ-bialgebras to pre-Lie algebras [21].

Lemma 5.3 [21] *Let (A, m, Δ) be an ϵ-bialgebra of weight λ. Define*

$$\rhd : A \otimes A \to A,\ a \otimes b \mapsto a \rhd b := \sum_{(b)} b_{(1)} a b_{(2)},$$

where $b_{(1)}, b_{(2)}$ are from the Sweedler notation $\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$. Then, $(A, \rhd)$ is a pre-Lie algebra.

5.2 A new pre-Lie algebras on decorated rooted forests

In this subsection, as an application of Theorem 5.3, we equip $H_{RT}(X, \Omega)$ with a pre-Lie algebraic structure $(H_{RT}(X, \Omega), \rhd_{RT})$ and a Lie algebraic structure $(H_{RT}(X, \Omega), [\_, \_]_{RT})$. We also give the combinatorial descriptions of $\rhd_{RT}$ and $[\_, \_]_{RT}$, respectively.

Theorem 5.4 *Let $H_{RT}(X, \Omega)$ be the ϵ-unitary bialgebra of weight λ in Theorem 3.11.*

(a) *The pair $(H_{RT}(X, \Omega), \rhd_{RT})$ is a pre-Lie algebra, where*

$$F_1 \rhd_{RT} F_2 := \sum_{(F_2)} F_{2(1)} F_1 F_{2(2)} \text{ for } F_1, F_2 \in H_{RT}(X, \Omega).$$

(b) *The pair $(H_{RT}(X, \Omega), [\_, \_]_{RT})$ is a Lie algebra, where*

$$[F_1, F_2]_{RT} := F_1 \rhd_{RT} F_2 - F_2 \rhd_{RT} F_1 \text{ for } F_1, F_2 \in H_{RT}(X, \Omega).$$

Proof By Theorems 3.11 and 5.3, $(H_{\mathrm{RT}}(X, \Omega), \rhd_{\mathrm{RT}})$ is a pre-Lie algebra. The remainder follows from Lemma 5.2. □

Example 5.5 Let $F_1 = \bullet_x$, $F_2 = \mathbf{l}^{\beta}_{\alpha}$, $F_3 = \bullet_y$. with $\alpha, \beta \in \Omega$ and $x, y \in X$. For the sake of simplicity, we consider the case of $\mu = 0$ and $\lambda = -1$. By Theorem 5.4, we have

$$\begin{aligned} F_1 \rhd_{\mathrm{RT}} F_2 &= \mathbf{l}^{\beta}_{\alpha} \bullet_x + \bullet_{\alpha} \bullet_x \bullet_{\beta} + \bullet_{\beta} \bullet_x \bullet_{\alpha} + \bullet_x \mathbf{l}^{\beta}_{\alpha}, \\ F_2 \rhd_{\mathrm{RT}} F_3 &= \bullet_y \mathbf{l}^{\beta}_{\alpha} + \mathbf{l}^{\beta}_{\alpha} \bullet_y. \end{aligned}$$

Moreover,

$$\begin{aligned} \Delta_{\epsilon}(F_3) = \Delta_{\epsilon}(\bullet_y) &= \bullet_y \otimes 1 + 1 \otimes \bullet_y, \\ \Delta_{\epsilon}(F_2 \rhd_{\mathrm{RT}} F_3) &= \bullet_y \mathbf{l}^{\beta}_{\alpha} \otimes 1 + \bullet_y \bullet_{\alpha} \otimes \bullet_{\beta} + \bullet_y \bullet_{\beta} \otimes \bullet_{\alpha} + \bullet_y \otimes \mathbf{l}^{\beta}_{\alpha} + 1 \otimes \bullet_y \mathbf{l}^{\beta}_{\alpha} \\ &\quad + \mathbf{l}^{\beta}_{\alpha} \bullet_y \otimes 1 + \mathbf{l}^{\beta}_{\alpha} \otimes \bullet_y + \bullet_{\alpha} \otimes \bullet_{\beta} \bullet_y + \bullet_{\beta} \otimes \bullet_{\alpha} \bullet_y + 1 \otimes \mathbf{l}^{\beta}_{\alpha} \bullet_y. \end{aligned}$$

Applying Theorem 5.4, we obtain

$$\begin{aligned} (F_1 \rhd_{\mathrm{RT}} F_2) \rhd_{\mathrm{RT}} F_3 &= \bullet_y \mathbf{l}^{\beta}_{\alpha} \bullet_x + \bullet_y \bullet_{\alpha} \bullet_x \bullet_{\beta} + \bullet_y \bullet_{\beta} \bullet_x \bullet_{\alpha} + \bullet_y \bullet_x \mathbf{l}^{\beta}_{\alpha} \\ &\quad + \mathbf{l}^{\beta}_{\alpha} \bullet_x \bullet_y + \bullet_{\alpha} \bullet_x \bullet_{\beta} \bullet_y + \bullet_{\beta} \bullet_x \bullet_{\alpha} \bullet_y + \bullet_x \mathbf{l}^{\beta}_{\alpha} \bullet_y, \\ F_1 \rhd_{\mathrm{RT}} (F_2 \rhd_{\mathrm{RT}} F_3) &= \bullet_y \mathbf{l}^{\beta}_{\alpha} \bullet_x + \bullet_y \bullet_{\alpha} \bullet_x \bullet_{\beta} + \bullet_y \bullet_{\beta} \bullet_x \bullet_{\alpha} + \bullet_y \bullet_x \mathbf{l}^{\beta}_{\alpha} + \bullet_x \bullet_y \mathbf{l}^{\beta}_{\alpha} \\ &\quad + \mathbf{l}^{\beta}_{\alpha} \bullet_y \bullet_x + \mathbf{l}^{\beta}_{\alpha} \bullet_x \bullet_y + \bullet_{\alpha} \bullet_x \bullet_{\beta} \bullet_y + \bullet_{\beta} \bullet_x \bullet_{\alpha} \bullet_y + \bullet_x \mathbf{l}^{\beta}_{\alpha} \bullet_y. \end{aligned}$$

Thus

$$F_1 \rhd_{\mathrm{RT}} (F_2 \rhd_{\mathrm{RT}} F_3) - (F_1 \rhd_{\mathrm{RT}} F_2) \rhd_{\mathrm{RT}} F_3 = \bullet_x \bullet_y \mathbf{l}^{\beta}_{\alpha} + \mathbf{l}^{\beta}_{\alpha} \bullet_y \bullet_x,$$

which is symmetric in $F_1 = \bullet_x$ and $F_2 = \mathbf{l}^{\beta}_{\alpha}$ and hence

$$\begin{aligned} (F_1 \rhd_{\mathrm{RT}} F_2) \rhd_{\mathrm{RT}} F_3 - F_1 \rhd_{\mathrm{RT}} (F_2 \rhd_{\mathrm{RT}} F_3) =& (F_2 \rhd_{\mathrm{RT}} F_1) \rhd_{\mathrm{RT}} F_3 \\ & - F_2 \rhd_{\mathrm{RT}} (F_1 \rhd_{\mathrm{RT}} F_3). \end{aligned}$$

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