

Type $D_n^{(1)}$ rigged configuration bijection

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Abstract We establish a bijection between the set of rigged configurations and the set of tensor products of Kirillov–Reshetikhin crystals of type $D_n^{(1)}$ in full generality. We prove the invariance of rigged configurations under the action of the combinatorial R -matrix on tensor products and show that the bijection preserves certain statistics (cocharge and energy). As a result, we establish the fermionic formula for type $D_n^{(1)}$. In addition, we establish that the bijection is a classical crystal isomorphism.

Keywords Rigged configurations · Crystal bases · Fermionic formula · Kyoto path model

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1 Introduction

In this paper, we establish a bijection between rigged configurations and paths for type $D_n^{(1)}$ and prove that it can be extended to a classical crystal isomorphism. Rigged configurations are combinatorial objects which were introduced by Kerov, Kirillov and Reshetikhin [14] through their insightful analysis of the Bethe Ansatz for quantum integrable systems. Observing that the number of Bethe vectors is equal to the number of irreducible components of the multiple tensor product of the vector representation of \mathfrak{sl}_2 , they constructed a bijection from rigged configurations to standard tableaux. This work was generalized to the symmetric tensor representations of \mathfrak{sl}_n in [19] and to rectangular shape ones in [21]. In these two works, the charge statistic is introduced for rigged configurations and it was shown to agree with Lascoux–Schützenberger’s charge [25] for tableaux.

Paths [3] (sometimes called Kyoto paths to avoid confusion with Littelmann’s path model [22]) also originated from quantum integrable systems; not from the Bethe ansatz, but from Baxter’s corner transfer matrix [1]. Thanks to Kashiwara’s crystal basis theory [10], the notion of a path was reformulated as an element of the tensor product of crystal bases of certain finite-dimensional modules of quantum affine algebras, called Kirillov–Reshetikhin (KR) modules [20] and then related with affine Lie algebra characters [12, 13]. In this paper, a path is a highest weight element in the crystal; that is, an element that is annihilated by the Kashiwara operator e_i for any index i of the Dynkin diagram of the affine Lie algebra except 0.

There are two physical methods, the corner transfer matrix method and the Bethe ansatz, which are used to analyze quantum integrable systems based on KR modules. Combinatorially, they give rise to a conjectural equality of generating functions $X = M$ of paths with the energy statistic X and of rigged configurations with the charge statistic M [8, 9]. Although the equality $X = M$ is yet to be proven bijectively in full generality except for type $A_n^{(1)}$, there is plenty of evidence for its correctness (for proofs in special cases, see below). In fact, it was shown to be true when $q = 1$ and the relevant affine algebra is of non-twisted type. In this case, X turns out to be a branching number of KR modules with respect to the quantized enveloping algebra corresponding to the underlying finite-dimensional simple Lie algebra. The relations between characters were shown to be Q -systems [7, 26]. They are known to imply the fermionic formula M at $q = 1$ in a weak sense, meaning that it may contain the binomial coefficient $\binom{p+m}{m}$ with $p < 0$ [8, 17]. This last gap was filled eventually in [5]. Naoi [27] showed $X = M$ for types $A_n^{(1)}$ and $D_n^{(1)}$ using fusion products of the current algebra and Demazure operators, but his proof is not bijective in nature.

One of the aims of the present paper is to prove $X = M$ for type $D_n^{(1)}$ in full generality by constructing an explicit bijection from paths to rigged configurations. To explain previous developments of this bijective method, we note that KR crystals are parameterized by two integers r, s , where r refers to a node of the Dynkin diagram

of the underlying simple Lie algebra, D_n in our case, and s is any positive integer. Let $B^{r,s}$ denote the corresponding KR crystal. The existence of $B^{r,s}$ and its combinatorial structure is known for all non-exceptional types [6,28,30]. Returning to the history of the bijective method, the simplest case when $B = (B^{1,1})^{\otimes k}$ is treated in [32] and the case when $\bigotimes_{i=1}^k B^{1,s_i}$ in [42], not only for type $D_n^{(1)}$ but also for non-exceptional types. In type $D_n^{(1)}$, the bijection for $\bigotimes_{i=1}^k B^{r_i,1}$ was constructed in [38] and for the single KR crystal $B = B^{r,s}$ in [35]. In fact, we rely on these two papers for many properties of combinatorial procedures used in this paper.

In [39], a crystal structure on rigged configurations of simply-laced types was defined. This led to the generalization of rigged configurations to unrestricted rigged configurations, which form the completion under the crystal operators. Unrestricted rigged configurations in type $A_n^{(1)}$ were characterized in [4]. This generalization turns out to be extremely powerful. In particular, the bijection Φ from paths to rigged configurations can be shown to extend to a crystal isomorphism.

Let us briefly explain how our bijection Φ from tensor products of KR crystals to unrestricted rigged configurations is constructed. We consider the general case $B = B^{r_k,s_k} \otimes \dots \otimes B^{r_2,s_1} \otimes B^{r_1,s_1}$. As is revealed in [35], we regard an element of the single KR crystal $B^{r,s}$ as a tableau, called a Kirillov–Reshetikhin tableau, of $r \times s$ rectangular shape. For $r_k \leq n - 2$, we then define three procedures ls, lb, lh on an element of B . The left-split ls splits off the leftmost column of the leftmost KR crystal B^{r_k,s_k} . The left-box lb splits off the lowest box of the leftmost column when the leftmost KR crystal is a column, that is $s_k = 1$. The left-hat lh deletes the box when the leftmost KR crystal is a box, that is $r_k = s_k = 1$. (If some r_i are n or $n - 1$, we use a “spin” version of the operator lh called left-hat-spin lh_s). These three operations on the path side correspond to γ, β, δ (resp., δ_s in the spin case) on rigged configurations. The bijection intertwines these operators and proceeds inductively on the total number of boxes $\sum_{i=1}^k r_i s_i$.

Our main results are threefold. Firstly, we show that the above bijection Φ is well-defined (see Theorem 4.2). At the same time, Lusztig’s involution \star on B is shown to be related to θ , which exchanges riggings and corriggings of rigged configurations. Secondly, we prove the R -invariance of rigged configurations (see Theorem 5.11). For the tensor product of KR crystals, we have a nontrivial bijection $R: B^{r_1,s_1} \otimes B^{r_2,s_2} \rightarrow B^{r_2,s_2} \otimes B^{r_1,s_1}$, called the combinatorial R -matrix, that commutes with all of the Kashiwara operators e_i, f_i . It can be applied to any two successive factors of a multiple tensor product of KR crystals. R -invariance means that this application of R does not have any effect on a rigged configuration. We prove this by using the combinatorial R -matrix involving the spin representation $B^{n,1}$ and to reduce the problem to the R -invariance for the type $A_n^{(1)}$ case, which was shown in [21]. In the proof, even though we consider the R -invariance for highest weight elements, we need to use the fact that the bijection is a classical crystal isomorphism [37] (see Theorem 4.3). Finally, we show that under the bijection $\theta \circ \Phi$, the coenergy statistic on a path is transferred to the cocharge on a rigged configuration (see Theorem 6.6), thereby proving the $X = M$ conjecture for type $D_n^{(1)}$ in a bijective fashion (see Corollary 6.7).

Let us address the question on why a bijective proof of the identity $X = M$ is very powerful. We give three reasons here. The first one is the computation of the image

of R . Although R is defined naturally in a representation-theoretical fashion, it is quite nontrivial combinatorially. However, denoting the bijection Φ from $B_1 \otimes B_2$ by $\Phi_{B_1 \otimes B_2}$, $R : B_1 \otimes B_2 \rightarrow B_2 \otimes B_1$ is simply realized as $\Phi_{B_2 \otimes B_1}^{-1} \circ \Phi_{B_1 \otimes B_2}$ thanks to the R -invariance. As a second reason, we mention the application to box–ball systems [44]. These are certain integrable dynamical systems formulated on the tensor product of KR crystals. Time evolution on the box–ball system is defined using R and considered to be nonlinear. However in [18], it was found that Φ linearizes its motion. Finally, as we mentioned before, the bijection in fact extends to a crystal isomorphism.

Rigged configurations of simply-laced types, and hence in particular type $D_n^{(1)}$, are of fundamental importance since those of non-simply-laced types can be constructed from simply-laced types by Dynkin diagram foldings. On the level of crystals, this is the virtual crystal construction carried out in [33, 34]. For rigged configurations, the virtual crystal construction is studied in [43]. In particular, the crystal operators on rigged configurations for simply-laced types [39] are extended to non-simply-laced types in [43]. The algorithm δ is known for arbitrary non-exceptional affine algebras [32], as well as type $E_6^{(1)}$ [31] and $D_4^{(3)}$ [41]. Moreover, δ was shown to commute with the virtual crystal construction for types $B_n^{(1)}$ and $A_{2n-1}^{(2)}$ in [43] and $D_4^{(3)}$ in [41], all of which are constructed as foldings of type $D_n^{(1)}$.

This paper is organized as follows. In Sect. 2, we review necessary facts from crystal base theory, define KR crystals, and prove some results about the affine crystal structure as well as properties of the left- and right-splitting map that we need. A review of type $D_n^{(1)}$ rigged configurations is given in Sect. 3. Section 4 contains the heart of this paper with a proof that the bijection Φ is well-defined. In Sect. 5, we prove the R -invariance of the rigged configuration bijection. We conclude in Sect. 6 with a proof that Φ preserves statistics (energy and cocharge), which implies the fermionic formula of [8]. Appendix is reserved for an example of the bijection.

2 Crystals and tableaux

2.1 Affine algebra of type $D_n^{(1)}$

We consider the Kac–Moody Lie algebra of affine type $D_n^{(1)}$ whose Dynkin diagram is depicted in Fig. 1. We denote the index set of the Dynkin diagram by $I = \{0, 1, \dots, n\}$ and set $I_0 = I \setminus \{0\}$.

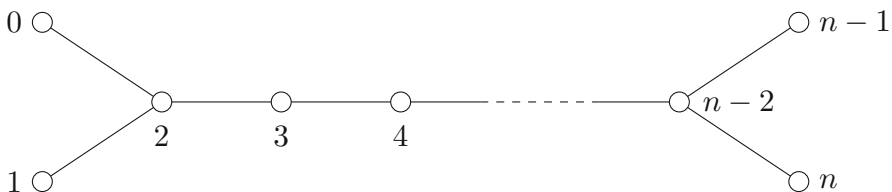


Fig. 1 The Dynkin diagram of type $D_n^{(1)}$

Let α_i, α_i^\vee , and Λ_i ($i \in I$) be the simple roots, simple coroots, and fundamental weights of $D_n^{(1)}$, respectively. Set $\varpi_i = \Lambda_i - \Lambda_0$ ($i \in I_0$), which are known as the level 0 fundamental weights. In particular, α_i, α_i^\vee , and ϖ_i ($i \in I_0$) can be identified with the simple roots, simple coroots, and fundamental weights of the underlying simple Lie algebra D_n . Using the standard orthonormal vectors ϵ_i ($i \in I_0$) in the weight lattice of type D_n , the simple roots are represented as

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \quad (1 \leq i \leq n - 1), \\ \alpha_n &= \epsilon_{n-1} + \epsilon_n, \end{aligned} \tag{2.1}$$

and the fundamental weights as

$$\begin{aligned} \varpi_i &= \epsilon_1 + \dots + \epsilon_i \quad (1 \leq i \leq n - 2), \\ \varpi_{n-1} &= (\epsilon_1 + \dots + \epsilon_{n-1} - \epsilon_n)/2, \\ \varpi_n &= (\epsilon_1 + \dots + \epsilon_{n-1} + \epsilon_n)/2. \end{aligned} \tag{2.2}$$

Let Q, Q^\vee , and P be the root, coroot, and weight lattices of type D_n , respectively. Let $\langle \cdot, \cdot \rangle: Q^\vee \times P \rightarrow \mathbb{Z}$ be the pairing such that $\langle \alpha_i^\vee, \varpi_j \rangle = \varpi_j(\alpha_i^\vee) = \delta_{i,j}$ is the Kronecker delta. Note that $\langle \alpha_i^\vee, \alpha_j \rangle = \alpha_j(\alpha_i^\vee) = A_{i,j}$ is the Cartan matrix of type D_n . The above can also be extended to the affine type $D_n^{(1)}$.

2.2 Crystals and Kashiwara–Nakashima tableaux

We refer to [10] for the basics of crystal basis theory. We denote by e_i and f_i the Kashiwara raising and lowering operators, respectively. For an element b of a crystal B , we use the following standard notation for the length of the i -strings through b :

$$\varepsilon_i(b) = \max\{m \geq 0 \mid e_i^m(b) \neq 0\}, \quad \varphi_i(b) = \max\{m \geq 0 \mid f_i^m(b) \neq 0\}.$$

They are related to the weight $\text{wt}(b)$ by $\langle \alpha_i^\vee, \text{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b)$.

For crystals B_1, B_2 of the same type, we can define their tensor product $B_2 \otimes B_1$ as follows. As a set, it is the Cartesian product $B_2 \times B_1$ of B_2 and B_1 . The action of the Kashiwara operators e_i, f_i on $b_2 \otimes b_1 \in B_2 \otimes B_1$ is given by

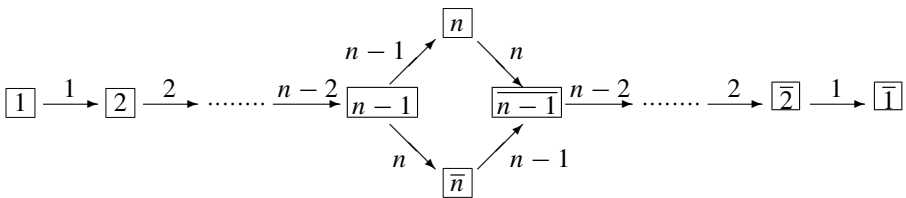
$$\begin{aligned} e_i(b_2 \otimes b_1) &= \begin{cases} e_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1), \\ b_2 \otimes e_i b_1 & \text{if } \varepsilon_i(b_2) \leq \varphi_i(b_1), \end{cases} \\ f_i(b_2 \otimes b_1) &= \begin{cases} f_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \\ b_2 \otimes f_i b_1 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \end{cases} \end{aligned}$$

where the result is declared to be 0 if either of its tensor factors are 0. The weight is defined as $\text{wt}(b_2 \otimes b_1) = \text{wt}(b_2) + \text{wt}(b_1)$. Note that this is opposite to the convention of Kashiwara for tensor products of crystals.

Let B_1 and B_2 be two crystals with index set I . A bijection $\psi : B_1 \rightarrow B_2$ is called a *crystal isomorphism* if it is a bijection that commutes with e_i and f_i by defining $\psi(0) = 0$.

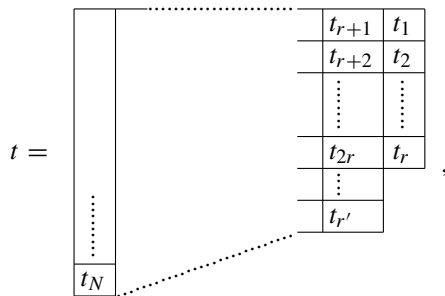
The crystals that we are concerned with in this paper are $U'_q(D_n^{(1)})$ -crystals and $U_q(D_n)$ -crystals. For a subset $J \subseteq I$, we also use the terminology J -crystal to mean the crystal over the quantized enveloping algebra corresponding to the Levi subalgebra associated with J . Hence, an I_0 -crystal is nothing but a $U_q(D_n)$ -crystal.

For a dominant integral weight λ , let $B(\lambda)$ be the crystal basis of the highest weight module of highest weight λ of $U_q(D_n)$. $B(\lambda)$ has a unique element u_λ satisfying $e_i u_\lambda = 0$ for all $i \in I_0$. We call u_λ the *highest weight element*. In order to perform explicit calculations on the crystal $B(\lambda)$, it is convenient to use the realization by tableaux, called *Kashiwara–Nakashima (KN) tableaux* [16]. They are defined for $U_q(\mathfrak{g})$ -crystals for Lie algebra \mathfrak{g} of type A_n, B_n, C_n and D_n . In all of these cases, we start by looking at $B(\varpi_1)$. In type D_n , the crystal graph $B(\varpi_1)$ is given as follows:



Here $b \xrightarrow{i} b'$ stands for $f_i b = b'$ or equivalently $b = e_i b'$. The weight is given by $\text{wt}(\boxed{i}) = \epsilon_i$ and $\text{wt}(\boxed{\bar{i}}) = -\epsilon_i$.

We now explain KN tableaux for $B(\lambda)$. Suppose $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$. Note that $\lambda_1 \geq \dots \geq \lambda_{n-1} \geq |\lambda_n| \geq 0$. We first assume that all λ_i are integers. The highest weight element u_λ corresponds to the tableau of partition shape $(\lambda_1, \dots, \lambda_{n-1}, |\lambda_n|)$ whose entries in the i th row are \bar{n} if $i = n$ and $\lambda_n < 0$, and i otherwise. Note that we use English convention for partitions and draw the Young diagram corresponding to a partition with the largest row on the top. For a given tableau



we introduce the so-called *column reading word* $t_N \dots t_2 t_1$ of t and regard it as an element of $B(\varpi_1)^{\otimes N}$ as

$$t \mapsto \boxed{t_N} \otimes \dots \otimes \boxed{t_2} \otimes \boxed{t_1}.$$

Via this identification, we introduce an action of Kashiwara operators on t using the tensor product rule. The whole set $B(\lambda)$ is generated from u_λ by applying f_i ($i \in I_0$).

Next we introduce a representation of elements of $B(s\varpi_n)$ and $B(s\varpi_{n-1})$. We consider $B(\varpi_n)$ and $B(\varpi_{n-1})$ first. As a set, they are given by

$$\begin{aligned} B(\varpi_n) &= \{(s_1, s_2, \dots, s_n) \mid s_i = \pm, s_1 s_2 \cdots s_n = +\}, \\ B(\varpi_{n-1}) &= \{(s_1, s_2, \dots, s_n) \mid s_i = \pm, s_1 s_2 \cdots s_n = -\}. \end{aligned} \tag{2.3}$$

The Kashiwara operators act by

$$e_i(s_1, \dots, s_n) = \begin{cases} (s_1, \dots, +, -, \dots, s_n) & \text{if } i \neq n, (s_i, s_{i+1}) = (-, +), \\ (s_1, \dots, s_{n-2}, +, +) & \text{if } i = n, (s_{n-1}, s_n) = (-, -), \\ 0 & \text{otherwise,} \end{cases} \tag{2.4a}$$

$$f_i(s_1, \dots, s_n) = \begin{cases} (s_1, \dots, -, +, \dots, s_n) & \text{if } i \neq n, (s_i, s_{i+1}) = (+, -), \\ (s_1, \dots, s_{n-2}, -, -) & \text{if } i = n, (s_{n-1}, s_n) = (+, +), \\ 0 & \text{otherwise.} \end{cases} \tag{2.4b}$$

The weight is given by

$$\text{wt}(s_1, \dots, s_n) = \frac{1}{2}(s_1\epsilon_1 + \cdots + s_n\epsilon_n). \tag{2.5}$$

In view of this, it is natural to associate with (s_1, \dots, s_n) a tableau whose shape has half width and height n . In the i th row, we put s_i . We call it a *spin column*.

For general s , we embed $B(s\varpi_n)$ (resp., $B(s\varpi_{n-1})$) into $B(\varpi_n)^{\otimes s}$ (resp., $B(\varpi_{n-1})^{\otimes s}$) by $c_s \cdots c_1 \mapsto c_s \otimes \cdots \otimes c_1$ where c_j are spin columns. In this way, we represent elements of $B(s\varpi_n)$ or $B(s\varpi_{n-1})$ by s spin columns. The highest weight elements are given by

$$u_{s\varpi_n} = \begin{matrix} \begin{array}{|c|c|c|c|c|} \hline + & + & + & \cdots & + \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline + & + & + & \cdots & + \\ \hline + & + & + & \cdots & + \\ \hline \end{array} , & u_{s\varpi_{n-1}} = \begin{matrix} \begin{array}{|c|c|c|c|} \hline + & + & + & \cdots & + \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline + & + & + & \cdots & + \\ \hline - & - & - & \cdots & - \\ \hline \end{array} . \end{matrix}$$

2.3 Kirillov–Reshetikhin crystals

In [28, 30], it was shown that Kirillov–Reshetikhin (KR) modules have crystal bases for any non-exceptional affine Lie algebra \mathfrak{g} . We call them *Kirillov–Reshetikhin (KR) crystals*. KR modules are finite-dimensional $U'_q(\mathfrak{g})$ -modules, where $U'_q(\mathfrak{g}) = U_q(\mathfrak{lg}, \mathfrak{g})$. The combinatorial structure of KR crystals is explicitly given in [6, 40], which we briefly recall in this section for type $D_n^{(1)}$.

KR crystals are parametrized by (r, s) ($r \in I_0, s \geq 1$). The KR crystal indexed by (r, s) is denoted by $B^{r,s}$. Since $U'_q(D_n^{(1)})$ contains $U_q(D_n)$ as a subalgebra, $B^{r,s}$ is decomposed into $U_q(D_n)$ -crystals. For $1 \leq r \leq n - 2$, we have

$$B^{r,s} \cong \bigoplus_{\lambda} B(\lambda) \quad \text{as } U_q(D_n)\text{-crystals} \tag{2.6}$$

where identifying the weight λ with the partition shape of the elements of $B(\lambda)$, the direct sum is taken over all Young diagrams obtained by removing vertical dominoes \square from the rectangular shape (s^r) . If $r = n - 1, n$, we have

$$B^{r,s} \cong B(s\varpi_r) \quad \text{as } U_q(D_n)\text{-crystals.} \tag{2.7}$$

To define the affine Kashiwara operators e_0 and f_0 , we first need a map σ that is the analogue of the Dynkin diagram automorphism that interchanges nodes 0 and 1; see [40]. We begin by recalling the notion of \pm -diagrams. A \pm -diagram P is a sequence of shapes $\lambda \subseteq \eta \subseteq \mu$ such that μ/η and η/λ are horizontal strips. We call λ and μ the inner and outer shapes of P , respectively. We depict P as a skew shape μ/λ , where we fill the boxes of μ/η with $-$ and those of η/λ with $+$. Next we define an involution \mathfrak{S} on \pm -diagrams, where $\mathfrak{S}(P)$ is the \pm -diagram that interchanges the number of columns of a given height h with only a $+$ and those of height h containing only $-$. In addition, it interchanges the number of columns of height $2 \leq h \leq r$ containing \mp with the number of columns containing no sign of height $h - 2$.

For $J \subseteq I$, we say an element b of a crystal is a J -highest weight element, if $e_i b = 0$ for all $i \in J$.

Proposition 2.1 ([6,40]) *There exists a bijection κ from \pm -diagrams to $\{2, \dots, n\}$ -highest weight elements in $B^{r,s}$ with $1 \leq r \leq n - 2$ as follows. Let $P = (\lambda \subseteq \eta \subseteq \mu)$. Then we construct $\kappa(P)$ as follows:*

- (i) *Start with shape μ and add a $\bar{1}$ in every cell that contains a $-$;*
- (ii) *Fill the remainder of the columns with $23 \cdots k$;*
- (iii) *As we read the \pm -diagram from bottom to top (in English convention), left to right, for every $+$ at height h that is encountered, do one of the following, moving in the current tableau from bottom to top and left to right:*
 - (a) *if we are at a $\bar{1}$, replace it by $\overline{h + 1}$;*
 - (b) *otherwise if one encounters a 2, replace the string $23 \cdots k$ with $12 \cdots h(h + 2) \cdots k$.*

We define $\sigma : B^{r,s} \rightarrow B^{r,s}$ for $1 \leq r \leq n - 2$ on $\{2, \dots, n\}$ -highest weight elements as $\sigma = \kappa \circ \mathfrak{S} \circ \kappa^{-1}$ and extend it to all elements in $B^{r,s}$ by making it a $\{2, \dots, n\}$ -crystal isomorphism. Explicitly, we have

$$\sigma = f_{\mathbf{a}^r} \circ \kappa \circ \mathfrak{S} \circ \kappa^{-1} \circ e_{\mathbf{a}}, \tag{2.8}$$

where

$$e_{\mathbf{a}} b = e_{a_1} \cdots e_{a_\ell} b \quad \text{for } \mathbf{a} = a_1 \cdots a_\ell \tag{2.9}$$

such that $e_a b$ is $\{2, \dots, n\}$ -highest weight and $f_a r b' = f_{a_\ell} \cdots f_{a_1} b'$. The map σ is an involution on $B^{r,s}$ [40, Definition 4.2]. Then we define

$$e_0 = \sigma \circ e_1 \circ \sigma, \tag{2.10a}$$

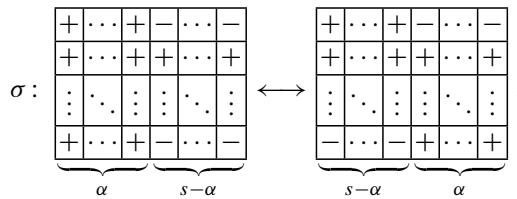
$$f_0 = \sigma \circ f_1 \circ \sigma. \tag{2.10b}$$

Let us introduce the following convention. By $c(i_1, \dots, i_\ell)$, we denote the spin column whose i_a th entry is $-$ for $1 \leq a \leq \ell$ and $+$ elsewhere. Let $c^t d^t$ stand for the tableau whose left t columns are c and right t' columns are d .

Note that when $r = n - 1, n$, we can define σ as an involutive map from $B^{n,s}$ to $B^{n-1,s}$ and vice versa [6, Definition 6.3]. This σ is also defined to be a $\{2, \dots, n\}$ -crystal isomorphism. A $\{2, \dots, n\}$ -highest weight element of $B(s\varpi_n)$ (resp., $B(s\varpi_{n-1})$) is given by the tableau $c()^\alpha c(1, n)^{s-\alpha}$ (resp., $c(n)^\alpha c(1)^{s-\alpha}$). The map σ is defined by [2, (2.7)]

$$\sigma : c()^\alpha c(1, n)^{s-\alpha} \longleftrightarrow c(n)^{s-\alpha} c(1)^\alpha, \tag{2.11}$$

or pictorially, we have



With this σ , we can again define the affine crystal operators e_0 and f_0 by (2.10).

In particular, we have $B^{r,1} \cong B(\varpi_r)$ as I_0 -crystals for $r = n - 1, n$ with the affine crystal operators, given in [38], explicitly as

$$e_0(s_1, \dots, s_n) = \begin{cases} (-, -, s_3, \dots, s_n) & \text{if } (s_1, s_2) = (+, +), \\ 0 & \text{otherwise,} \end{cases} \tag{2.12a}$$

$$f_0(s_1, \dots, s_n) = \begin{cases} (+, +, s_3, \dots, s_n) & \text{if } (s_1, s_2) = (-, -), \\ 0 & \text{otherwise.} \end{cases} \tag{2.12b}$$

We prepare two lemmas that will be used later. We use the notation $e_i^{\max} b = e_i^{\varepsilon_i(b)} b$. In the following lemmas, we note that we read columns of a KN tableau from bottom to top in accordance with our reading word. Spin columns are displayed in tuple notation as in (2.3). The following lemma will be used in Sect. 6.

Lemma 2.2 (1) *Let $2 \leq r \leq n - 2$. Let $b(\alpha) = c^{s-\alpha} c'^\alpha \in B^{r,s}$ ($0 \leq \alpha \leq s$), where $c = r \cdots 21$ and $c' = nr \cdots 31$. Then $\varepsilon_0(b(\alpha)) = 2s - \alpha$, $\varphi_0(b(\alpha)) = 0$, and $e_0^{\max} b(\alpha)$ is the tableau whose left α columns are $\overline{2}nr \cdots 3$ and right $(s - \alpha)$ columns are $\overline{1}2r \cdots 3$.*

(2) Let $b(\alpha) = c(n)^{s-\alpha}c(2)^\alpha \in B^{n-1,s}$ ($0 \leq \alpha \leq s$). Then $\varepsilon_0(b(\alpha)) = s - \alpha$, $\varphi_0(b(\alpha)) = 0$, and $e_0^{\max}b(\alpha) = c(2)^\alpha c(1, 2, n)^{s-\alpha}$. Columns here are spin columns.

Proof The $\{2, \dots, n\}$ -highest weight element $e_{(r+1)^\alpha \dots (n-1)^\alpha} f_1^\alpha b(\alpha)$ is the tableau whose left $(s - \alpha)$ columns are $r \cdots 21$ and right α columns are $(r + 1) \cdots 32$. Note that the Kashiwara operators appearing above commute with e_0, f_0 . Thus, (1) follows from [24, Lemma 9.4].

Next consider (2). Notice that $e_{\mathbf{a}}b(\alpha) = c(n)^s$ where $\mathbf{a} = (n - 1)^\alpha \cdots 2^\alpha$. Using (2.11), we have

$$\sigma(b(\alpha)) = (f_{\mathbf{a}^r} \circ \sigma \circ e_{\mathbf{a}})(c(n)^{s-\alpha}c(2)^\alpha) = c(1, n)^{s-\alpha}c(1, 2)^\alpha = b'.$$

Since $\varepsilon_1(b') = s - \alpha$ and $\varphi_1(b') = 0$, we have the result for ε_0 and φ_0 . We then have $e_1^{s-\alpha}b' = c(2, n)^{s-\alpha}c(1, 2)^\alpha = b''$. Noting $e_{\mathbf{a}'}b'' = c()^{s-\alpha}c(1, n)^\alpha$ where $\mathbf{a}' = n^{s-\alpha}(n - 1)^\alpha(n - 2)^s \cdots 2^s$ and calculating similarly, we obtain the desired result. \square

For the next lemma, which will be used in the proof of Proposition 5.7, we need to characterize the elements $b \in B^{r,s}$ for $1 \leq r \leq n - 2$ such that $\varepsilon_i(b) \leq \delta_{i,n}$ for $i \in I_0$. These are the elements that differ from the I_0 -highest weight element $u_{\bar{\lambda}}$ by the addition of a vertical strip whose (column) reading word is given by $w = \cdots n\bar{n}\bar{n}\bar{n}$, where $\bar{\lambda} := \text{wt}(b) - \text{wt}(w)$. Note that $\text{wt}(w) \in \{\pm \epsilon_n, 0\}$ and $\bar{\lambda} \in P^+$.

Example 2.3 Consider the $\{1, 2, 3, 4, 5, 6, 7\}$ -highest weight element

$$b = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & \bar{8} \\ \hline 2 & 2 & 2 & 8 \\ \hline 3 & \bar{8} & & \\ \hline 4 & 8 & & \\ \hline 5 & & & \\ \hline \bar{8} & & & \\ \hline \end{array} \in B(2\varpi_2 + \varpi_4 + \varpi_6) \subseteq B^{6,4}$$

of type $D_8^{(1)}$. We have

$$\begin{aligned} \text{wt}(b) &= 2\varpi_2 + \varpi_5 + \varpi_7 - \varpi_8, \\ w &= \bar{8}\bar{8}\bar{8}\bar{8}, \\ \text{wt}(w) &= -\epsilon_8, \\ \bar{\lambda} &= 2\varpi_2 + \varpi_5 = \text{wt}(b) + \epsilon_8. \end{aligned}$$

Using the notation in the proof of Lemma 2.4 below, we have $\xi = 4$ since $y_5 = 6$ and $y_i = i$ for $i \leq 4$.

Lemma 2.4 Consider $b \in B(\mu) \subseteq B^{r,s}$ for $1 \leq r \leq n - 2$ such that $\varepsilon_i(b) \leq \delta_{i,n}$ for all $i \in I_0$. Then f_0b is given by doing exactly one of the following:

- (i) Suppose there exists a column of height $h \geq 1$ with column reading word $\cdots n\bar{n}\bar{n}\bar{n}$. Then replace it with $\cdots n\bar{n}21$ of height $h + 2$ if this yields a valid tableau and $h < r$.
- (ii) Suppose there exists a column $\boxed{\bar{n}}$ of height 1 and a column of height $h \geq 1$ with column reading word $\bar{n}\bar{n} \cdots \bar{n}n1$. Then replace the largest column of height h with the column of height $h + 2$ with column reading $\bar{n}\bar{n}\bar{n} \cdots \bar{n}\bar{n}\bar{n}21$ and the column $\boxed{\bar{n}}$ with the column $\boxed{1}$ if this yields a valid tableau and $h < r$.
- (iii) In all other cases, slide in a vertical domino $\boxed{\frac{1}{2}}$ from the left at height 0 unless $\mu_1 = s$, in which case $f_0b = 0$.

Proof We use the notation introduced just before the lemma. Let y_i denote the heights of cells of $\mu/\bar{\lambda}$ read from top to bottom. Let ξ be the largest value such that $y_j = j$ for all $1 \leq j \leq \xi$ (which could be 0). The following element is the $\{2, \dots, n\}$ -highest weight element in the same component as b :

$$e_{\mathbf{a}}b := (e_{y_k} \cdots e_{n-2}e_{n'}) \cdots (e_{y_{\xi+1}}e_{y_{\xi+1}+1} \cdots e_{n-2}e_{n''}) \\ (e_{y_{\xi+1}} \cdots e_{n-2}e_{n''}) \cdots (e_{y_1+1} \cdots e_{n-2}e_n)b,$$

where $k = |\mu/\bar{\lambda}|$ and $n'' = n - 1$, n depending on the parity of ξ (equivalently r since the column heights of μ must also have the same parity) and $n''' = n, n - 1$, respectively (i.e., reversed parity of ξ). Let $P_b := \kappa^{-1}(e_{\mathbf{a}}b)$ be the corresponding \pm -diagram.

We first assume that there are no \bar{n} letters in the first row, that is, we are in Case (iii). Note that we have $e_{\mathbf{a}}b = u_{\mu}$, the corresponding \pm -diagram P_b is of outer shape μ with only a $+$ in all columns, and $\xi = 0$ (note that all of these conditions are equivalent). We first consider the case when $\mu_1 = s$. Then $\mathfrak{S}(P_b)$ is also of outer shape μ with only a $-$ in all columns. Thus, $\kappa(\mathfrak{S}(P_b))$ is given by columns of the form $23 \cdots h\bar{1}$, and we obtain $\sigma(b)$ by making the entry at height i an n (resp., \bar{n}) if there is a n (resp., \bar{n}) in the same column in b at height $i + 1$. Hence, there are no 1 nor $\bar{2}$ entries in $\sigma(b)$, and we have $f_1(\sigma(b)) = 0$. Therefore, $f_0b = 0$ as desired.

Now we consider the case when $\mu_1 < s$. Here, $\mathfrak{S}(P_b)$ contains $s - \mu_1$ columns with a \pm of height 2. Note that these are the only $+$ signs occurring in $\mathfrak{S}(P_b)$ since $e_{\mathbf{a}}b = u_{\mu}$. Thus, each of the $+$ signs in these columns changes either a 2 to a 1 or a $\bar{1}$ to a $\bar{2}$ when computing $\kappa(\mathfrak{S}(P_b))$, and we obtain n, \bar{n} in $\sigma(b)$ as in the previous case. Hence, f_1 changes the last 1 or $\bar{2}$ in the reading word in $\sigma(b)$. So $P_{f_1(\sigma(b))}$ differs from $\mathfrak{S}(P_b)$ by removing the leftmost $+$ sign, and therefore, f_0b differs from b by the addition of a column $\boxed{\frac{1}{x}}$, where $x = 2$ or \bar{n} is the rightmost entry in the second row of b , as desired.

Now assume we are in Case (ii), that is, there is a (necessarily unique) column $\boxed{\bar{n}}$ and a column $c = \cdots \bar{n}n1$ of height h (we pick the largest leftmost such column if several exist). Note that r must be odd, $h \leq \xi$, and $\mu_1 = s$. We first consider the case when $\xi < r$. Then P_b is of outer shape μ with only $+$ in every column except for a column of height ξ with no sign. There is exactly one column c' with a $+$ in $\mathfrak{S}(P_b)$, and the $+$ is at height $\xi + 1$. Hence, $\kappa(\mathfrak{S}(P_b))$ is the tableau where the leftmost column is of the form $\bar{\xi} + \bar{2}k \cdots 32$ and all other columns of height k are of the form

$\bar{1}k \cdots 32$ (or only contain $\bar{1}$ if of height 1). Next we apply the sequence f_{a^r} . This will change $\overline{\xi + 2}$ to $\chi + 1$, where χ is the height of the column to the right of c' . Note that $\chi = 1$ precisely when $\xi = h$ and the column to the left of c has height strictly greater than c (which is also precisely when Case (ii) applies and yields a valid tableau). It is straightforward to check that $\sigma(b)$ does not contain any additional 1 or $\bar{2}$ entries; more explicitly, the other changed entries either become n or \bar{n} . Thus, $f_1(\sigma(b))$ changes the $\bar{2}$ to a $\bar{1}$ if Case (ii) applies and $f_1(\sigma(b)) = 0$ otherwise. Therefore, it is easy to see that our claim follows using the fact that $P_{f_1(\sigma(b))}$ has a $-$ in all columns of shape μ' , which is the outer shape of $\mathfrak{S}(P_b)$. If $\xi = r$, then $\mathfrak{S}(P_b)$ contains no $+$ signs, and it is easy to see that $f_1(\sigma(b)) = 0$ as there are no 1 nor $\bar{2}$ entries in $\sigma(b)$.

Lastly, assume that there exists a column $\cdots n\bar{n}n\bar{n}$ of height ≥ 1 and we are not in Case (ii). We first consider Case (iii), where either $\xi > h$ or $\mu_1 - \mu_{h+1} > 1$ (if $\ell(\mu) \leq h$, we consider $\mu_{h+1} = 0$). Therefore, P_b has a $+$ in every nonempty column except the leftmost column of height ξ . Hence, the leftmost $+$ in $\mathfrak{S}(P_b)$ is at height $\xi + 1$, and if $\xi = r$, then there is no such $+$. Moreover, the remaining $s - \mu_1$ number of $+$ signs occurs at height 1. Thus, $\kappa(\mathfrak{S}(P_b))$ has a bottom left entry of $\overline{\xi + 2}$ if $\xi < r$, of $\bar{2}$ if $\mu_1 < s$ and $\xi = r$, or of $r + 1$ otherwise. Thus, from the description of \mathbf{a} , we have the bottom left entry x of $\sigma(b)$ as follows. If $\xi < r$, then $x = \chi + 1$, since it transforms under f_{y_j} for each $\chi < j < \xi$, where χ is the height of the column to the right of the column which contains the leftmost $+$ in $\mathfrak{S}(P_b)$, similar to above. If $\mu_1 < s$ and $\xi = r$, then $x = \bar{2}$. Otherwise $x = n$ or $x = \bar{n}$ depending on the parity of ξ . In addition, all 1 and $\bar{2}$ entries, of which there are $s - \mu_1$ many of them, are unchanged from $\kappa(\mathfrak{S}(P_b))$. For $j > \xi$, all other changed entries are \bar{n}, n in $\sigma(b)$ as above. If $\mu_1 = s$, then there are no additional $+$ signs in $\mathfrak{S}(P_b)$ other than the one at height $\xi + 1$, and hence, there are no 1, $\bar{2}$ entries in $\sigma(b)$. Therefore, $f_1(\sigma(b)) = 0$ and $f_0b = 0$ as desired. Now if $\mu_1 < s$, then f_1 changes the rightmost 1 or $\bar{2}$ in the reading word in $\sigma(b)$. Thus, $P_{f_1(\sigma(b))}$ differs from $\mathfrak{S}(P_b)$ by removing the leftmost $+$ sign in a column of height 1. Thus, it is easy to see that f_0b differs from b as claimed.

Now we consider Case (i), so that $\xi = h$ and $\mu_1 - \mu_{h+1} = 1$. The tableau $\sigma(b)$ is similar to the above except that the bottom left entry has to be $\bar{2}$ since it transforms under f_{y_j} for all j . Thus in this case, $P_{f_1(\sigma(b))}$ differs from $\mathfrak{S}(P_b)$ by removing the leftmost $+$ sign at height $h + 1$. Thus, it can be easily checked that f_0b is as claimed. □

2.4 Lusztig’s involution on $B^{r,s}$

Let w_0 be the longest element of the Weyl group of type D_n . There exists a type D_n Dynkin diagram automorphism $\tau : I_0 \rightarrow I_0$ satisfying

$$w_0\varpi_i = -\varpi_{\tau(i)} \quad \text{and} \quad w_0\alpha_i = -\alpha_{\tau(i)}.$$

In fact, τ is the identity if n is even, and interchanges $n - 1$ and n and fixes all other Dynkin nodes if n is odd.

On a $U_q(D_n)$ -crystal $B(\lambda)$, it is known [38,42] that there exists a unique involution, called *Lusztig’s involution*, $\star: B(\lambda) \rightarrow B(\lambda)$ satisfying

$$\text{wt}(b^\star) = w_0 \text{wt}(b), \quad (e_i b)^\star = f_{\tau(i)} b^\star, \quad (f_i b)^\star = e_{\tau(i)} b^\star. \tag{2.13}$$

As seen from (2.13), \star sends the I_0 -highest weight element of $B(\lambda)$ to the I_0 -lowest weight element, which is the element that satisfies $f_i b = 0$ for all $i \in I_0$. By defining $\tau(0) = 0$, we extend τ to the Dynkin diagram of type $D_n^{(1)}$ and the involution \star on the KR crystal $B^{r,s}$. For a crystal B , let B^\star be the crystal with the same set as B , but with the crystal structure given by (2.13). There is a natural isomorphism of crystals

$$(B_2 \otimes B_1)^\star \cong B_1^\star \otimes B_2^\star \tag{2.14}$$

such that $(b_2 \otimes b_1)^\star = b_1^\star \otimes b_2^\star$.

2.5 Left- and right-split on $B^{r,s}$

In [35], we defined the *filling map* fill on $B^{r,s}$ for $1 \leq r \leq n - 2, s \geq 1$. For an I_0 -highest weight element u_λ , $\text{fill}(u_\lambda)$ is a tableau of rectangular shape (s^r) which does not necessarily satisfy the conditions of KN tableaux in general. However, the filling map is necessary for the path to rigged configuration bijection.

The filling map can be defined inductively by cutting the leftmost column.

Definition 2.5 Let $\lambda = k_p \varpi_p + k_q \varpi_q + \sum_{0 \leq j < q} k_j \varpi_j$ ($p > q, k_p, k_q > 0, k_p + k_q + \sum_{0 \leq j < q} k_j = s$). Here we have set $\varpi_0 = 0$. We define the map, which we call *left-split*, $\text{ls}: B^{r,s} \rightarrow B^{r,1} \otimes B^{r,s-1}$ for $1 \leq r \leq n - 2, s \geq 2$ as follows. For λ define a pair (c, λ') of a column c of height r and a weight λ' by:

- (i) If $p = r$, then

$$c = r \cdots 21, \\ \lambda' = (k_r - 1)\varpi_r + k_q \varpi_q + \sum_{j < q} k_j \varpi_j.$$

- (ii) If $p < r$ and $k_p \geq 2$, then

$$c = \overline{p+1} \cdots \bar{r} p \cdots 21, \\ \lambda' = \varpi_r + (k_p - 2)\varpi_p + k_q \varpi_q + \sum_{j < q} k_j \varpi_j.$$

- (iii) If $p < r$ and $k_p = 1$, then

$$c = \overline{p+1} \cdots \bar{r} r \cdots q(r - p + q + 1) \cdots 21, \\ \lambda' = \varpi_{r-p+q} + (k_q - 1)\varpi_q + \sum_{j < q} k_j \varpi_j.$$

We remark that one can regard c as an element of the I_0 -crystal $B^{r,1}$ by embedding into $B(\varpi_1)^{\otimes r}$ via the column reading. For $u_\lambda \in B^{r,s}$ we define $\text{ls}(u_\lambda) = c \otimes u_{\lambda'} \in B^{r,1} \otimes B^{r,s-1}$. The image $\text{ls}(b)$ for an arbitrary element $b \in B^{r,s}$ is defined in such a way that e_i, f_i ($i \in I_0$) commute with ls .

For $B^{r,s}$ with $r = n - 1$ or n , the left-split map $\text{ls}: B^{r,s} \rightarrow B^{r,1} \otimes B^{r,s-1}$ is defined for the unique I_0 -highest weight element $u_{s\varpi_r}$ as $\text{ls}(u_{s\varpi_r}) = u_{\varpi_r} \otimes u_{(s-1)\varpi_r}$ and extended to any element again by the commutativity with e_i, f_i ($i \in I_0$).

The right-split map $\text{rs}: B^{r,s} \rightarrow B^{r,s-1} \otimes B^{r,1}$ is defined by $\text{rs} = \star \circ \text{ls} \circ \star$ using Definition 2.5 and (2.14). It also commutes with e_i, f_i ($i \in I_0$). For $u_{s\varpi_r} \in B^{r,s}$, the right-split map is given by $\text{rs}(u_{s\varpi_r}) = u_{(s-1)\varpi_r} \otimes u_{\varpi_r}$. For the explicit form of $\text{rs}(u_\lambda)$ for general $u_\lambda \in B^{r,s}$ when $r \leq n - 2$, we have the following proposition.

Proposition 2.6 *Let $\lambda = \varpi_p + \varpi_q + \mu$ where $p \geq q$ and $\mu = \sum_{a, j_a \leq q} \varpi_{j_a}$. For $u_\lambda \in B^{r,s}$ ($1 \leq r \leq n - 2, s \geq 2$), $\text{rs}(u_\lambda)$ is given by $t \otimes u_{\varpi_r}$ where t is represented as a KN tableau as follows. The first column is*

$$\overline{p+1} \cdots \overline{r-1} \bar{r} q \cdots 21$$

and the other part is the KN tableau for u_μ .

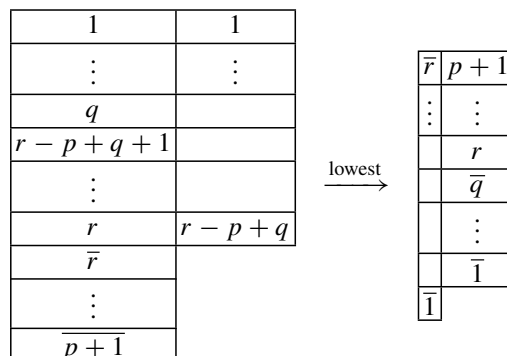
Proof For an I_0 -highest weight element u , let $u \xrightarrow{\text{lowest}} v$ represent that v is the I_0 -lowest weight element corresponding to u .

(1) Consider the case when $p = r$. Write $\lambda = \varpi_r + \mu$. Then as an I_0 -crystal, we can regard u_λ as $u_{\varpi_r} \otimes u_\mu$ where u_{ϖ_r} corresponds to the column c split in Definition 2.5 (i). Since $u_{\varpi_r} \otimes u_\mu \xrightarrow{\text{lowest}} v_{-\varpi_r} \otimes v_{-\mu}$, where v_ξ stands for the lowest weight element of weight ξ , we see that the leftmost column of u_λ^* is $\bar{1} \cdots \bar{r} - \bar{1} \bar{r}$ and the rest is $v_{-\mu}$ in the KN tableau representation. Hence, by applying $\star \circ \text{ls}$ we obtain $u_{\varpi_q + \mu}$, which is the desired result. \square

To prove the other cases, we need the following lemma.

Lemma 2.7 *Let $r > p \geq q$. We have:*

(1)



where the LHS is an I_0 -highest weight element of weight $\varpi_p + \varpi_q$, when viewed inside $B(\varpi_1)^{\otimes(p+q)}$. Note that these are not KN tableaux.

(2)

$$\begin{array}{|c|} \hline p+1 \\ \hline \vdots \\ \hline r \\ \hline \overline{q} \\ \hline \vdots \\ \hline \overline{1} \\ \hline \end{array} = e_{\mathbf{a}} \begin{array}{|c|} \hline \overline{r-p+q} \\ \hline \vdots \\ \hline \overline{1} \\ \hline \end{array}$$

where

$$\begin{aligned}
 \mathbf{a} &= \mathbf{b}_0 \mathbf{b}_1 \cdots \mathbf{b}_{r-p+1} \mathbf{c}_{n-r+p-q-1} \mathbf{c}_{n-r+p-q-2} \cdots \mathbf{c}_1, \\
 \mathbf{b}_i &= (r-i)(r-i+1) \cdots (n-i-2)(n-i-1)^2 \cdots (n-2)^2(n-1)n, \\
 \mathbf{c}_j &= (q+j)(q+j+1) \cdots (r-p+q+j-1),
 \end{aligned}$$

and for a word \mathbf{a} $e_{\mathbf{a}}$ is defined in (2.9). \mathbf{a} contains only letters larger than q .

(2) Consider the case when $r > p = q$. We have $\lambda = 2\varpi_p + \mu$. Then we can regard u_λ as $u_{2\varpi_p} \otimes u_\mu$, where $u_{2\varpi_p}$ corresponds to the LHS of Lemma 2.7 (1) with $p = q$ split two times by Definition 2.5 (ii) and (i). Then we have

$$u_\lambda = u_{2\varpi_p} \otimes u_\mu \xrightarrow{\text{lowest}} v_{-2\varpi_p} \otimes v_{-\mu} = \begin{array}{|c|c|} \hline \overline{r} & p+1 \\ \hline \vdots & \vdots \\ \hline & r \\ \hline & \overline{p} \\ \hline & \vdots \\ \hline \overline{1} & \overline{1} \\ \hline \end{array} \otimes v_{-\mu}.$$

Hence, the rest of u_λ by cutting the leftmost column is

$$\begin{array}{|c|} \hline p+1 \\ \hline \vdots \\ \hline r \\ \hline \overline{p} \\ \hline \vdots \\ \hline \overline{1} \\ \hline \end{array} \otimes v_{-\mu} = e_{\mathbf{a}} v_{-\varpi_r} \otimes v_{-\mu} = e_{\mathbf{a}} (v_{-\varpi_r} \otimes v_{-\mu}).$$

Here we have used Lemma 2.7 (2). By applying \star , we obtain

$$f_{\mathbf{a}}(u_{\varpi_r} \otimes u_{\mu}) = f_{\mathbf{a}} \left(\begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline r \\ \hline \end{array} \otimes u_{\mu} \right) = \begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline \frac{p}{\bar{r}} \\ \hline \vdots \\ \hline \overline{p+1} \\ \hline \end{array} \otimes u_{\mu}$$

as desired.

(3) Finally consider the case when $r > p > q$. We follow the same procedure as (2) and write $\lambda = \varpi_p + \varpi_q + \mu$. Then we can regard u_{λ} as $u_{\varpi_p + \varpi_q} \otimes u_{\mu}$ where $u_{\varpi_p + \varpi_q}$ corresponds to the LHS of Lemma 2.7 (1). Its left column is split by Definition 2.5 (iii). By using Lemma 2.7 (2) again, we have

$$u_{\lambda} = u_{\varpi_p + \varpi_q} \otimes u_{\mu} \xrightarrow{\text{lowest}} v_{-\varpi_p - \varpi_q} \otimes v_{-\mu} = \begin{array}{|c|} \hline \bar{r} \\ \hline \vdots \\ \hline \bar{1} \\ \hline \end{array} \otimes e_{\mathbf{a}}(v_{-\varpi_r - p + q} \otimes v_{-\mu}).$$

Applying \star , we obtain

$$f_{\mathbf{a}}(u_{\varpi_r - p + q} \otimes u_{\mu}) = f_{\mathbf{a}} \left(\begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline r - p + q \\ \hline \end{array} \otimes u_{\mu} \right) = \begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline \frac{q}{\bar{r}} \\ \hline \vdots \\ \hline \overline{p+1} \\ \hline \end{array} \otimes u_{\mu}.$$

□

2.6 Paths and various operations on them

Let B be a tensor product of KR crystals. We start with the definition of a path of B .

Definition 2.8 A path of B is an I_0 -highest weight element of the crystal B . The set of paths of B is denoted by $\mathcal{P}(B)$. We also set $\mathcal{P}(B, \lambda) = \{b \in \mathcal{P}(B) \mid \text{wt}(b) = \lambda\}$ for a dominant integral weight λ .

For later purposes, we introduce several operations on paths.

Definition 2.9 For a path $b = b_k \otimes b_{k-1} \otimes \dots \otimes b_1 \in B^{r_k, s_k} \otimes B^{r_{k-1}, s_{k-1}} \otimes \dots \otimes B^{r_1, s_1}$, we define the following operations:

(1) Suppose that $B^{r_k, s_k} = B^{1,1}$. Then we define the operation called *left-hat* by

$$\text{lh}(b) = b_{k-1} \otimes \cdots \otimes b_1 \in B^{r_{k-1}, s_{k-1}} \otimes \cdots \otimes B^{r_1, s_1}.$$

(1') Suppose that $B^{r_k, s_k} = B^{r_k, 1}$ and $r_k = n - 1, n$. Then we define the operation called *left-hat-spin* by

$$\text{lh}_s(b) = b_{k-1} \otimes \cdots \otimes b_1 \in B^{r_{k-1}, s_{k-1}} \otimes \cdots \otimes B^{r_1, s_1}.$$

(2) Suppose that $B^{r_k, s_k} = B^{r_k, 1}$ and $2 \leq r_k \leq n - 2$, so that b_k has the form

$$b_k = \begin{array}{|c|} \hline t_1 \\ \hline \vdots \\ \hline t_{r_k-1} \\ \hline t_{r_k} \\ \hline \end{array}. \text{ Then we define the operation called } \textit{left-box} \text{ by}$$

$$\text{lb}(b) = \boxed{t_{r_k}} \otimes \begin{array}{|c|} \hline t_1 \\ \hline \vdots \\ \hline t_{r_k-1} \\ \hline \end{array} \otimes b_{k-1} \otimes \cdots \otimes b_1 \in B,$$

where $B = B^{1,1} \otimes B^{r_k-1,1} \otimes B^{r_{k-1}, s_{k-1}} \otimes \cdots \otimes B^{r_1, s_1}$.

(3) Suppose that $s_k > 1$. We define the operation called *left-split* by

$$\text{ls}(b) = \text{ls}(b_k) \otimes b_{k-1} \otimes \cdots \otimes b_1 \in B,$$

where $B = B^{r_k, 1} \otimes B^{r_k, s_k-1} \otimes B^{r_{k-1}, s_{k-1}} \otimes \cdots \otimes B^{r_1, s_1}$. See Definition 2.5 for ls in the single KR crystal.

Next we define right analogues of lh , lh_s , lb , and ls . Let $B = B^{r_k, s_k} \otimes \cdots \otimes B^{r_1, s_1}$. For $b \in \mathcal{P}(B)$, we define $\diamond(b) = \text{high}(b_1^* \otimes \cdots \otimes b_k^*) \in \mathcal{P}(B^*)$. Here $\text{high}(b)$ denotes the highest weight element in the same I_0 -component as b .

- (1) We define *right-hat* $\text{rh}: B \otimes B^{1,1} \rightarrow B$ by $\text{rh} = \text{high} \circ \star \circ \text{lh} \circ \star = \diamond \circ \text{lh} \circ \diamond$.
- (1') We define *right-hat-spin* $\text{rh}_s: B \otimes B^{r,1} \rightarrow B$, where $r = n - 1, n$, by $\text{rh}_s = \text{high} \circ \star \circ \text{lh}_s \circ \star = \diamond \circ \text{lh}_s \circ \diamond$.
- (2) We define *right-box* $\text{rb}: B \otimes B^{r,1} \rightarrow B \otimes B^{r-1,1} \otimes B^{1,1}$ for $2 \leq r \leq n - 2$ by $\text{rb} = \star \circ \text{lb} \circ \star = \diamond \circ \text{lb} \circ \diamond$.
- (3) We define *right-split* $\text{rs}: B \otimes B^{r,s} \rightarrow B \otimes B^{r,s-1} \otimes B^{r,1}$ for $s \geq 2$ by $\text{rs} = \star \circ \text{ls} \circ \star = \diamond \circ \text{ls} \circ \diamond$.

Proposition 2.10 *When there are at least two KR crystals in the tensor product B , the left operation lx commutes with the right one ry for any pair of (x, y) where $x, y = \text{h}, \text{h}_s, \text{b}, \text{s}$ as long as they are well-defined.*

Proof If rh or rh_s are not involved, they clearly commute with each other since the left operation only changes the leftmost component and the right one does the rightmost one. Suppose the right operation is rh . It commutes with any left operations since

★ \circ lh \circ ★ only changes the rightmost component and high commutes with left operations. The case of rh_s is similar. □

It is known [12] that for KR crystals $B^{r,s}, B^{r',s'}$ there exists an isomorphism of crystals

$$R: B^{r,s} \otimes B^{r',s'} \longrightarrow B^{r',s'} \otimes B^{r,s},$$

called *combinatorial R-matrix*. Note that R commutes with e_i, f_i ($i \in I$).

3 Rigged configurations of type $D_n^{(1)}$

3.1 Definition of rigged configurations

We define two classes of rigged configurations of type $D_n^{(1)}$: *rigged configurations* and *unrestricted rigged configurations*. Rigged configurations will turn out to be in bijection with the paths of Definition 2.8. Unrestricted rigged configurations are obtained by defining the classical Kashiwara operators e_i and f_i for $i \in I_0$ on the rigged configurations and then considering the connected crystal components generated by the rigged configurations (which turn out to correspond to the I_0 -highest weight vectors).

Let us prepare the definition of several combinatorial objects which constitute *rigged configurations* and then give the defining conditions imposed on them. A rigged configuration consists of a sequence of partitions $\nu = (\nu^{(1)}, \dots, \nu^{(n)})$, called a *configuration*, together with riggings $J = (J^{(1)}, \dots, J^{(n)})$. More precisely, if the partition $\nu^{(a)} = (\nu_1^{(a)}, \dots, \nu_\ell^{(a)})$ has ℓ parts, then $J^{(a)} = (J_1^{(a)}, \dots, J_\ell^{(a)})$ is a sequence of integers called the *riggings*. The riggings are paired with the rows of the configuration as the multisets $(\nu^{(a)}, J^{(a)}) = \{(\nu_1^{(a)}, J_1^{(a)}), \dots, (\nu_\ell^{(a)}, J_\ell^{(a)})\}$ for each $a \in I_0$. We call the pairs $(\nu_i^{(a)}, J_i^{(a)})$ *strings* and associate the pair $(\nu^{(a)}, J^{(a)})$ to the node $a \in I_0$ of the Dynkin diagram of D_n . We denote by $m_i^{(a)}(\nu)$ the number of rows of length i of $\nu^{(a)}$. We identify two rigged configurations (ν, J) and $(\tilde{\nu}, \tilde{J})$ when $(\nu^{(a)}, J^{(a)}) = (\tilde{\nu}^{(a)}, \tilde{J}^{(a)})$ as multisets for all $a \in I_0$.

Rigged configurations also depend on the tensor product $B = B^{r_k, s_k} \otimes \dots \otimes B^{r_1, s_1}$. Let $L = \{L_i^{(a)}\}_{a \in I_0, i \in \mathbb{Z}_{>0}}$ be the number of components $B^{a,i}$ within B , also called the *multiplicity array* of B . We can define a configuration $\mu(L) = (\mu^{(1)}, \dots, \mu^{(n-1)}, \mu^{(n)})$ from L by $m_i^{(a)}(\mu) = L_i^{(a)}$. Thus, the rigged configuration (ν, J) is described by the configuration and a sequence of riggings

$$(\nu, J) = \left((\nu^{(1)}, J^{(1)}), \dots, (\nu^{(n-1)}, J^{(n-1)}), (\nu^{(n)}, J^{(n)}) \right)$$

together with additional data under certain constraints to be described below. We denote by $Q_i^{(a)}(\nu)$ the number of boxes in the first i columns of $\nu^{(a)}$:

$$Q_i^{(a)}(\nu) = \sum_{j \geq 1} \min(i, j) m_j^{(a)}(\nu).$$

Therefore, we have $Q_\infty^{(a)}(v) = |v^{(a)}|$, where $|v^{(a)}|$ is the total number of boxes in $v^{(a)}$. From the data $\mu(L)$ and v , we define the *vacancy number* $P_i^{(a)}(\mu(L), v)$ (usually abbreviated by $P_i^{(a)}(v)$) by the formula

$$\begin{aligned} P_i^{(a)}(v) &= Q_i^{(a)}(\mu(L)) + \sum_{b \in I_0} A_{a,b} Q_i^{(b)}(v) \\ &= Q_i^{(a)}(\mu(L)) - 2Q_i^{(a)}(v) + \sum_{\substack{b \in I_0 \\ b \sim a}} Q_i^{(b)}(v), \end{aligned} \tag{3.1}$$

where $A_{a,b}$ is the Cartan matrix of D_n and $a \sim b$ means that the vertices a and b are connected by a single edge in the Dynkin diagram.

Definition 3.1 Fix a multiplicity array L . Then (v, J) is a *rigged configuration* of type $D_n^{(1)}$ if all the strings $(v_i^{(a)}, J_i^{(a)})$ and the corresponding vacancy numbers satisfy the following condition for all $a \in I_0$ and $i \geq 1$

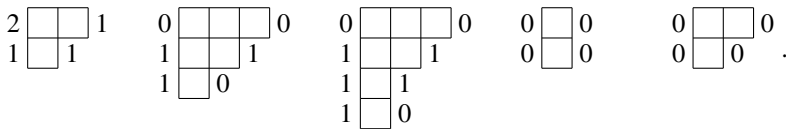
$$0 \leq J_i^{(a)} \leq P_{v_i^{(a)}}^{(a)}(v). \tag{3.2}$$

The *weight* λ of the rigged configuration is defined by the relation (sometimes called the (L, λ) -configuration condition)

$$\sum_{a \in I_0, i > 0} im_i^{(a)}(v)\alpha_a = \sum_{a \in I_0, i > 0} iL_i^{(a)}\varpi_a - \lambda. \tag{3.3}$$

The set of rigged configurations of weight λ and multiplicity array L is denoted by $\mathcal{RC}(L, \lambda)$. We also let $\mathcal{RC}(L) = \bigsqcup_{\lambda \in P^+} \mathcal{RC}(L, \lambda)$.

Example 3.2 The following object is a rigged configuration corresponding to the tensor product $B = B^{3,2} \otimes B^{3,1} \otimes B^{2,2} \otimes B^{1,2} \otimes B^{1,1}$ of type $D_5^{(1)}$:



Here we put the vacancy number (resp., rigging) on the left (resp., right) of the corresponding row of the configuration represented by a Young diagram. We order the riggings for rows of the same length in the same partition weakly decreasingly from top to bottom (since recall that we view $(v^{(a)}, J^{(a)}) = \{(v_1^{(a)}, J_1^{(a)}), \dots, (v_\ell^{(a)}, J_\ell^{(a)})\}$ as multisets).

Note that if we expand the weight λ by the basis ϵ_i , we can rewrite (3.3) as follows:

$$\lambda = \sum_{i \in I_0} \lambda_i \epsilon_i = \sum_{a \in I_0} (|\mu^{(a)}| \varpi_a - |v^{(a)}| \alpha_a). \tag{3.4}$$

Then we can use the expressions (2.1) and (2.2) to obtain the explicit expressions for the weight λ_i . We write the weight of the rigged configuration by $\text{wt}(v, J)$.

Following [39], we introduce the classical Kashiwara operators on the rigged configurations and use them to define the unrestricted rigged configurations. For the string $(i, x) \in (v^{(a)}, J^{(a)})$, we call the quantity $P_i^{(a)}(v) - x$ the *corriging*.

Definition 3.3 The *unrestricted rigged configurations* are obtained by all possible applications of the Kashiwara operators f_a ($a \in I_0$) on the rigged configurations. Here the Kashiwara operators f_a and e_a for $a \in I_0$ on unrestricted rigged configurations are defined as follows. Let x be the smallest rigging of $(v^{(a)}, J^{(a)})$.

- e_a : Let ℓ be the minimal length of the strings of $(v^{(a)}, J^{(a)})$ with the rigging x . If $x \geq 0$, define $e_a(v, J) = 0$. Otherwise $e_a(v, J)$ is obtained by replacing the string (ℓ, x) by $(\ell - 1, x + 1)$ while changing all other riggings to keep corrigings fixed.
- f_a : Let ℓ be the maximal length of the strings of $(v^{(a)}, J^{(a)})$ with the rigging x . Then $f_a(v, J)$ is obtained by the following procedure. If $x > 0$, add a string $(1, -1)$ to $(v^{(a)}, J^{(a)})$. Otherwise replace the string (ℓ, x) by $(\ell + 1, x - 1)$. Change other riggings to keep corrigings fixed. If the new rigging is strictly larger than the corresponding new vacancy number, define $f_a(v, J) = 0$.

Let $\text{RC}(L)$ denote the set of all unrestricted rigged configurations generated from $\mathcal{RC}(L)$ by the Kashiwara operators. Let

$$\text{RC}(L, \lambda) = \{(v, J) \in \text{RC}(L) \mid \text{wt}(v, J) = \lambda\}.$$

Example 3.4 Consider (v, J) in Example 3.2. Then $\text{wt}(v, J) = 3\varpi_1 + 3\varpi_4 + \varpi_5$,

$$f_4(v, J) = \begin{array}{c} 2 \\ 1 \end{array} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 1 \\ \hline \end{array} 1 \quad \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & 1 \\ \hline \square & 0 & \\ \hline \end{array} 0 \quad \begin{array}{c} 1 \\ 2 \\ 1 \\ 1 \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & 2 \\ \hline \square & & 1 \\ \hline \square & & 0 \\ \hline \end{array} 1 \quad \begin{array}{c} 0 \\ 0 \end{array} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \end{array} -1 \quad \begin{array}{c} 0 \\ 0 \end{array} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & 0 \\ \hline \end{array} 0,$$

and $f_2(v, J) = 0$.

3.2 Operations δ, β and γ

The rigged configuration bijection

$$\Phi: \mathcal{P}(B, \lambda) \rightarrow \mathcal{RC}(L, \lambda)$$

is a bijection between paths, the set of I_0 -highest weight elements in a tensor products of KR crystals B , and the set of rigged configurations. Here L is the multiplicity array of B . In this section, we prepare the main ingredients of the bijection. In fact, we will define the various maps not just on $\mathcal{RC}(L)$, but more generally on $\text{RC}(L)$. One of the basic operations is

$$\delta: (v, J) \longmapsto \{(v', J'), k\},$$

where (v, J) and (v', J') are rigged configurations and $k \in \{1, 2, \dots, n, \bar{n}, \dots, \bar{2}, \bar{1}\}$. In the description, we call a string (ℓ, x) of $(v^{(a)}, J^{(a)})$ *singular* if we have $x = P_\ell^{(a)}(v)$, that is, the rigging takes the maximal possible value. To begin with, we consider the generic case (non-spin case).

Definition 3.5 Let us consider a rigged configuration $(v, J) \in RC(L)$ corresponding to the tensor product of the form $B^{1,1} \otimes B'$. The map

$$\delta: (v, J) \longmapsto \{(v', J'), k\}$$

is defined by the following procedure. Set $\ell^{(0)} = 1$.

- (1) For $1 \leq a \leq n - 2$, suppose that $\ell^{(a-1)}$ is already determined. Then we search for the shortest singular string in $(v^{(a)}, J^{(a)})$ that is longer than or equal to $\ell^{(a-1)}$.
 - (a) If there exists such a string, set $\ell^{(a)}$ to be the length of the selected string and continue the process recursively. If there is more than one such string, choose any of them.
 - (b) If there is no such string, set $\ell^{(a)} = \infty, k = a$ and stop.
- (2) Suppose that $\ell^{(n-2)} < \infty$. Then we search for the shortest singular string in $(v, J)^{(n-1)}$ (resp., $(v, J)^{(n)}$) that is longer than or equal to $\ell^{(n-2)}$ and define $\ell^{(n-1)}$ (resp., $\ell^{(n)}$) as in part (1).
 - (a) If $\ell^{(n-1)} = \infty$ and $\ell^{(n)} = \infty$, set $k = n - 1$ and stop.
 - (b) If $\ell^{(n-1)} < \infty$ and $\ell^{(n)} = \infty$, set $k = n$ and stop.
 - (c) If $\ell^{(n-1)} = \infty$ and $\ell^{(n)} < \infty$, set $k = \bar{n}$ and stop.
 - (d) If $\ell^{(n-1)} < \infty$ and $\ell^{(n)} < \infty$, set $\ell_{(n-1)} = \max(\ell^{(n-1)}, \ell^{(n)})$ and continue.
- (3) For $1 \leq a \leq n - 2$, assume that $\ell_{(a+1)}$ is already defined. Then we search for the shortest singular string in $(v, J)^{(a)}$ that is longer than or equal to $\ell_{(a+1)}$ and has not yet been selected as $\ell^{(a)}$. Let $\ell_{(a)}$ be the length of this string if it exists and set $\ell_{(a)} = \infty$ otherwise. If $\ell_{(a)} = \infty$, set $k = \overline{a + 1}$ and stop. Otherwise continue. If $\ell_{(1)} < \infty$, set $k = \bar{1}$ and stop.
- (4) Once the process has stopped, remove the rightmost box of each selected row specified by $\ell^{(a)}$ or $\ell_{(a)}$. The result gives the output v' .
- (5) Define the new riggings J' as follows. For the rows that are not selected by $\ell^{(a)}$ or $\ell_{(a)}$, take the corresponding riggings from J . In order to define the remaining riggings, we use B' to compute all the vacancy numbers for v' . Then the remaining riggings are defined so that all the corresponding rows become singular with respect to the new vacancy number.

As we will see in Sect. 4, δ on the rigged configuration side corresponds to the left-hat operation of Definition 2.9 on the path side. Although the above definition only deals with paths of the form $B^{1,1} \otimes B'$, the essence of the rigged configuration bijection is contained in the operation δ which was originally discovered by [14] for type $A_n^{(1)}$ and generalized for type $D_n^{(1)}$ by [32].

The remaining maps involved in the rigged configuration bijection are the counterparts of left-box and left-split of Definition 2.9.

Table 1 The operations on rigged configurations and paths and their correspondence, where L is the multiplicity array of B

B	$RC(L)$
ls	γ
lb	β
lh	δ

Definition 3.6 Let $B = B^{r_k, s_k} \otimes B^{r_{k-1}, s_{k-1}} \otimes \dots \otimes B^{r_1, s_1}$ with multiplicity array L and $(\nu, J) \in RC(L)$.

- (1) If $B^{r_k, s_k} = B^{r, s}$ with $s > 1$, then γ replaces a length s row of $\mu^{(r)}$ by two rows of lengths $s - 1$ and 1 of $\mu^{(r)}$ and otherwise leaves (ν, J) unchanged.
- (2) If $B^{r_k, s_k} = B^{r, 1}$ with $1 < r \leq n - 2$, then β removes a length one row from $\mu^{(r)}$, adds a length one row to each of $\mu^{(1)}$ and $\mu^{(r-1)}$ and adds a length one singular string to each of $(\nu^{(a)}, J^{(a)})$ for $1 \leq a < r$.

We give a summary of the correspondence of the operations on the rigged configuration side and the path side in Table 1. It will be shown in Proposition 4.1 that these operations intertwine under Φ .

A formal definition of the rigged configuration bijection Φ and its inverse Φ^{-1} will be given in Sect. 4. Roughly speaking, the algorithm for Φ^{-1} is given by successive applications of the operators γ , β and δ and filling in the bottom leftmost unfilled entry in B with k on each application of δ , where k is as given in Definition 3.5. The algorithm for Φ is the reverse procedure of each step, where for δ^{-1} we add a box to the largest singular row at most as long as the previously selected row. However, the well-definedness of Φ and Φ^{-1} is a highly nontrivial fact and will be the subject of Sect. 4. A detailed example of Φ^{-1} is given in ‘‘Appendix.’’

Example 3.7 Consider the rigged configuration (ν, J) of Example 3.2. Then (ν, J) corresponds under Φ^{-1} to the following I_0 -highest weight element in B :

$$\Phi^{-1}(\nu, J) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline \end{array}.$$

Remark 3.8 We can perform the composition $\delta \circ \beta \circ \gamma$ at once (see, for example, [35, 37]). Suppose that we consider $B^{r, s}$ with $s > 1$. Then $\delta \circ \beta \circ \gamma$ is a modification of δ . Set $\ell^{(r-1)} = s$ and start from the corresponding part of Step (1) of Definition 3.5. For practical purposes, especially for hand computations, it is convenient to do $\delta \circ \beta \circ \gamma$ simultaneously. However, for the proof of several important properties of the rigged configuration bijection in Sect. 4, it is convenient to consider three separate operations γ , β and δ .

In [35, Proposition 3.3], the following result is proved with the help of [15].

Proposition 3.9 Let u_λ be the I_0 -highest weight element of $B^{r, s}$ ($1 \leq r \leq n - 2$) of weight λ . Denote by λ^c the Young diagram obtained as the complement of λ within the rectangle of height r and width s . Then $\Phi(u_\lambda)$ has the following form:

- $v^{(a)} = \lambda^c$ for $r \leq a \leq n - 2$. $v^{(n-1)} = v^{(n)}$ is obtained by replacing all dominoes \square of λ^c by \square .
- $v^{(a-i)}$ is obtained by removing the top i rows from λ^c . If i exceeds the length of λ^c , we understand that $v^{(a-i)} = \emptyset$.
- All riggings are 0.

Note that all the vacancy numbers of $\Phi(u_\lambda)$ are 0 so that the requirement (3.2) determines the riggings uniquely.

3.3 Dual operations on the rigged configurations

Definition 3.10 Let $(v, J) \in \mathcal{RC}(L)$. For each string $(v_i^{(a)}, J_i^{(a)}) \in (v^{(a)}, J^{(a)})$ of a given rigged configuration (v, J) , the operation θ is defined by

$$\theta: \mathcal{RC}(L) \longrightarrow \mathcal{RC}(L)$$

$$(v_i^{(a)}, J_i^{(a)}) \longmapsto (v_i^{(a)}, P_{v_i^{(a)}}^{(a)}(v) - J_i^{(a)}) \in (\tilde{v}^{(a)}, \tilde{J}^{(a)}),$$

where $\theta: (v, J) \longmapsto (\tilde{v}, \tilde{J})$. We then extend θ to all unrestricted rigged configurations $\mathcal{RC}(L)$ by extending it as a classical crystal automorphism.

Rather nontrivially, we will show in Proposition 4.1(7) that the operator \diamond on paths corresponds to the operation θ on rigged configuration under the bijection Φ . By using θ , we define the dual operations of γ, β and δ :

$$\begin{aligned} \tilde{\gamma} &:= \theta \circ \gamma \circ \theta, \\ \tilde{\beta} &:= \theta \circ \beta \circ \theta, \\ \tilde{\delta} &:= \theta \circ \delta \circ \theta. \end{aligned}$$

We provide direct descriptions of these operators in the case when $(v, J) \in \mathcal{RC}(L)$. Recall that for the string $(v_i^{(a)}, J_i^{(a)}) \in (v^{(a)}, J^{(a)})$, the quantity $P_{v_i^{(a)}}^{(a)}(v) - J_i^{(a)}$ is called the corigging. If the rigging is 0, we call such a string *cosingular*. Let $B = B^{r_k, s_k} \otimes B^{r_{k-1}, s_{k-1}} \otimes \dots \otimes B^{r_1, s_1}$, L of the corresponding multiplicity array, and $\mu = \mu(L)$.

- $\tilde{\gamma}$: If $B^{r_k, s_k} = B^{r, s}$ with $s > 1$, then $\tilde{\gamma}$ replaces a length s row of $\mu^{(r)}$ by two rows of lengths $s - 1$ and 1 of $\mu^{(r)}$ and add 1 to the rigging of each string in $(v^{(r)}, J^{(r)})$ of length strictly less than s .
- $\tilde{\beta}$: If $B^{r_k, s_k} = B^{r, 1}$, where $1 < r \leq n - 2$, then $\tilde{\beta}$ removes a length one row from $\mu^{(r)}$, adds a length one row to each of $\mu^{(1)}$ and $\mu^{(r-1)}$ and adds a length one cosingular string to each of $(v^{(a)}, J^{(a)})$ for $a < r$.
- $\tilde{\delta}$: Suppose $B^{r_k, s_k} = B^{1, 1}$. This is a corigging version of δ . Instead of selecting singular strings, it selects cosingular strings and makes them into cosingular strings of lengths shortened by 1. For unselected strings, it keeps coriggings constant by changing riggings.

Note that γ preserves riggings, whereas $\tilde{\gamma}$ preserves coriggings. Also note that $\tilde{\beta}$ preserves vacancy numbers since β preserves vacancy numbers [38, Lemma 4.2].

The following very important result is quite hard to prove.

Theorem 3.11 (Appendix C of [38]) *Let L be a multiplicity array with $L_1^{(1)} \geq 2$. Then on $\mathcal{RC}(L)$ the operators δ and $\tilde{\delta}$ commute:*

$$[\delta, \tilde{\delta}] = 0.$$

We will also need the following propositions.

Proposition 3.12 *We have the following relations on rigged configurations in $\mathcal{RC}(L)$:*

- (i) $[\delta, \tilde{\delta}] = [\delta, \tilde{\beta}] = [\delta, \tilde{\gamma}] = 0;$
- (ii) $[\beta, \tilde{\delta}] = [\beta, \tilde{\beta}] = [\beta, \tilde{\gamma}] = 0;$
- (iii) $[\gamma, \tilde{\delta}] = [\gamma, \tilde{\beta}] = [\gamma, \tilde{\gamma}] = 0.$

Proof The relation $[\delta, \tilde{\delta}] = 0$ is Theorem 3.11. The relation $[\delta, \tilde{\beta}] = 0$ is proven in [38, Lemma 5.4], whereas $[\delta, \tilde{\gamma}] = 0$ follows from the fact that $\tilde{\gamma}$ preserves coriggings, specifically singular strings stay singular. The relations $[\beta, \tilde{\beta}] = [\beta, \tilde{\gamma}] = 0$ follow from the fact that β and $\tilde{\beta}$ preserve vacancy numbers and that $\tilde{\gamma}$ preserves coriggings. The relation $[\gamma, \tilde{\gamma}] = 0$ follows from the fact that γ and $\tilde{\gamma}$ preserve riggings and coriggings, respectively. The remaining relations can be deduced by conjugation by θ . □

Proposition 3.13 *Let $1 \leq r \leq n - 2, s \geq 2$. For $u_\lambda \in B^{r,s}$ we have*

$$\Phi(rs(u_\lambda)) = \tilde{\gamma}(\Phi(u_\lambda)). \tag{3.5}$$

Proof Let $\lambda = k_p \varpi_p + k_q \varpi_q + \sum_{0 \leq j < q} k_j \varpi_j$ ($p > q, k_p, k_q > 0, k_p + k_q + \sum_{0 \leq j < q} k_j = s$). We prove the statement by dividing into the same cases as in Definition 2.5.

Consider Case (i), where $p = r$. In this case, $ls(u_\lambda) = c \otimes u_{\lambda - \varpi_r}$, where $c = r \cdots 21$ by Definition 2.5 (i). Hence, during the process of removing the leftmost column of u_λ , the corresponding rigged configuration never changes. Proposition 2.6 says that the left component of $rs(u_\lambda)$ is $u_{\lambda - \varpi_r}$. Therefore, the removing procedures of δ performed to obtain the corresponding rigged configuration is completely parallel from $u_{\lambda - \varpi_r}$ and from $rs(u_\lambda)$, which justifies (3.5).

Next consider Case (ii), where $p < r$ and $k_p \geq 2$. In this case, the left component of $ls(u_\lambda)$ is the same as the leftmost column of $rs(u_\lambda)$, which is $\overline{p+1} \cdots \overline{r} p \cdots 1$, the leftmost column of the right component of $ls(u_\lambda)$ is $r \cdots 1$, and the right $s - 2$ columns of $ls(u_\lambda)$ and the right $s - 2$ columns of the left component of $rs(u_\lambda)$ are the same. Hence, the applications of δ to remove the leftmost column of u_λ and the one of the left component of $rs(u_\lambda)$ are completely parallel and arrive at Case (i). Thus, the proof is done also in this case.

Finally, we consider Case (iii), where $p < r$ and $k_p = 1$. Divide further into two cases: (a) $s > 2$ and (b) $s = 2$. First consider Case (a), and set $\lambda = \varpi_p + \varpi_q + \varpi_{q'} + \dots$

($r > p > q \geq q'$). Just as in Case (ii), the right $s - 2$ columns of u_λ and the right $s - 2$ columns of the left component of $rs(u_\lambda)$ are the same. Hence, it is sufficient to show that the process of removing the leftmost column of the left component of $rs(u_\lambda)$ is parallel to removing the left two columns of u_λ . We list below the length of the row in which a box is removed by the i th application of δ along the leftmost column of the left component of $rs(u_\lambda)$. Here a stands for the position of the configuration when we move along it in the increasing order of a during the process of δ while \bar{a} the one when we move in the decreasing order, and ℓ is the length of the row in which a box is removed.

When $1 \leq i \leq p - q$:

if i is odd,

$$a = r - i + 1, \dots, r - 1 \rightarrow \ell = s - 2 \text{ and } a = r, \dots, \overline{q + i} \rightarrow \ell = s - 1;$$

if i is even,

$$a = r - i + 1, \dots, \bar{r} \rightarrow \ell = s - 2 \text{ and } a = \overline{r - 1}, \dots, \overline{q + i} \rightarrow \ell = s - 1.$$

When $p - q < i \leq r - q$, set $j = i - (p - q)$:

if i is odd,

$$a = r - i + 1, \dots, \overline{r - j} \rightarrow \ell = s - 2, a = r - j + 1, \dots, r - 1 \rightarrow \ell = s - 1 \text{ and } a = r, \dots, \overline{q + i} \rightarrow \ell = s;$$

if i is even,

$$a = r - i + 1, \dots, \overline{r - j} \rightarrow \ell = s - 2, a = r - j + 1, \dots, \bar{r} \rightarrow \ell = s - 1 \text{ and } a = \overline{r - 1}, \dots, \overline{q + i} \rightarrow \ell = s.$$

When $r - q < i \leq r - q'$:

$$a = r - i + 1, \dots, r - i + p - q \rightarrow \ell = s - 2.$$

When $r - q' < i \leq r$:

no box is removed.

In view of [35, Proposition 3.3], we see that the above deletions are in fact parallel to removing the left two columns of u_λ , and hence the proof is done.

For Case (b), set $\lambda = \varpi_p + \varpi_q$ ($r > p > q$) and $z = (p + q)/2$. We provide a similar list.

When $1 \leq i \leq (p - q)/2$:

$$a = r - i + 1, \dots, \overline{q + i} \rightarrow \ell = 1.$$

When $(p - q)/2 < i \leq r - z$, set $j = i - (p - q)/2$:

if i is odd,

$$a = r - i + 1, \dots, r - 1 \rightarrow \ell = 1 \text{ and } a = r, \dots, \overline{p + j} \rightarrow \ell = 2;$$

if i is even,

$$a = r - i + 1, \dots, \bar{r} \rightarrow \ell = 1 \text{ and } a = \overline{r - 1}, \dots, \overline{p + j} \rightarrow \ell = 2.$$

When $r - z < i \leq r$:

no box is removed.

□

3.4 Spin cases

Following [38], let us introduce the analogue of δ for the spin cases $B^{r,1}$ where $r = n - 1$ or n . Fix a multiplicity array L and let $\mu = \mu(L)$. We introduce the embedding of the unrestricted rigged configurations

$$\begin{aligned} \text{emb}: \text{RC}(L) &\longrightarrow \text{RC}(L') \\ (v, J) &\longmapsto (v', J'), \end{aligned}$$

where $v_i^{(a)} = 2v_i^{(a)}$, $J_i^{(a)} = 2J_i^{(a)}$, and L' is determined by $\mu_i^{(a)} = 2\mu_i^{(a)}$ for $\mu' = \mu(L')$. Note that emb is a similarity map with a scaling factor of 2 as described in [11, Theorem 3.1] in the following sense:

$$\text{emb}(e_i b) = e_i^2(\text{emb}(b)), \tag{3.6a}$$

$$\text{emb}(f_i b) = f_i^2(\text{emb}(b)), \tag{3.6b}$$

$$\text{wt}(\text{emb}(b)) = 2 \text{wt}(b). \tag{3.6c}$$

Remark 3.14 The image of emb is characterized by the condition that all parts of μ and ν as well as all riggings are even.

The analogue of δ for the spin case is denoted by δ_s . It corresponds to removing an entire spin column $B^{r,1}$ for $r = n - 1$ or n and is defined as

$$\delta_s = \text{emb}^{-1} \circ (\delta \circ \beta)^{n-2} \circ \delta \circ \bar{\beta} \circ \delta \circ \beta^{(r)} \circ \text{emb}, \tag{3.7}$$

where $\beta^{(r)}$ and $\bar{\beta}$ are given as follows:

- Define $\beta^{(n)}$ as the map which replaces a length two row of $\mu^{(n)}$ by a length one row, adds a length one row to each of $\mu^{(1)}$ and $\mu^{(n-1)}$, and adds a length one singular string to each of $(\nu^{(a)}, J^{(a)})$ for $a \leq n - 2$ and $a = n - 1$. The map $\beta^{(n-1)}$ is defined in the same way with n and $n - 1$ interchanged.
- Define $\bar{\beta}$ as the map which removes a length one row from each of $\mu^{(n-1)}$ and $\mu^{(n)}$, adds a length one row to each of $\mu^{(1)}$ and $\mu^{(n-2)}$, and adds a length one singular string to each of $(\nu^{(a)}, J^{(a)})$ for $a \leq n - 2$.

The following lemma is a straightforward computation.

Lemma 3.15 *The maps $\beta^{(r)}$ and $\bar{\beta}$ preserve the vacancy numbers.*

It was shown in [38] that δ_s is well-defined. Similarly, we define $\tilde{\delta}_s = \theta \circ \delta_s \circ \theta$.

Proposition 3.16 *We have the following relations on rigged configurations in $\mathcal{RC}(L)$:*

- (i) $[\delta_s, \tilde{\delta}_s] = [\delta_s, \tilde{\delta}] = [\tilde{\delta}_s, \delta] = 0$,
- (ii) $[\tilde{\delta}_s, \bar{\beta}] = [\tilde{\delta}_s, \tilde{\gamma}] = 0$,
- (iii) $[\tilde{\delta}_s, \beta] = [\tilde{\delta}_s, \gamma] = 0$.

Proof We note that θ commutes with emb and emb^{-1} . Additionally, we have

$$\text{emb} \circ \delta = \delta \circ \delta \circ \gamma \circ \text{emb}$$

as a similar statement and proof of [29, Lemma 3.5] holds for type $D_n^{(1)}$. Hence, the relations for (i) follow from Theorem 3.11, Proposition 3.12, and the definitions of $\tilde{\delta}_s$ and δ_s .

Fix an $1 \leq r \leq n$ such that $\mu^{(r)}$ has a row of length $s > 2$. Define $\tilde{\gamma}^e$ by replacing a length s row of $\mu^{(r)}$ by two rows of length $s - 2$ and 2 of $\mu^{(r)}$, leaving ν unchanged, and preserving all colabels. Also for $1 \leq r \leq n - 2$, define $\tilde{\beta}^e$ by removing a length 2 row from $\mu^{(r)}$, adding a length 2 row to each of $\mu^{(1)}$ and $\mu^{(r-1)}$, and adding a length 2 string with rigging 0 to each of $(\nu^{(a)}, J^{(a)})$ for $1 \leq a < r$. It is straightforward to verify that

$$\text{emb} \circ \tilde{\gamma} = \tilde{\gamma}^e \circ \text{emb}, \quad \text{emb} \circ \tilde{\beta} = \tilde{\beta}^e \circ \text{emb}.$$

We have $[\delta, \tilde{\gamma}^e] = [\beta, \tilde{\gamma}^e] = [\beta^{(m)}, \tilde{\gamma}^e] = [\bar{\beta}, \tilde{\gamma}^e] = 0$, for $m = n - 1, n$, since $\tilde{\gamma}^e$ preserves colabels and since Lemma 3.15 holds, similar to the proof of Proposition 3.12. Hence, $[\delta_s, \tilde{\gamma}] = 0$ follows from the definition of δ_s .

The proof of $[\delta, \tilde{\beta}^e] = 0$ follows the proof for $[\delta, \tilde{\beta}] = 0$ given in [38, Lemma 5.4]. We need to show that for the selected strings $\ell^{(a)}$ (resp., $s^{(a)}$) by δ in (ν, J) (resp., $\tilde{\beta}^e(\nu, J)$), we have $\ell^{(a)} = s^{(a)}$ for all $a \in I_0$. The case for $\bar{\ell}^{(a)} = \bar{s}^{(a)}$ is similar. First, we have that $\ell^{(a)} = 1$ if and only if $s^{(a)} = 1$ for any $a \in I_0$ as δ selects the smallest singular string and $\tilde{\beta}^e$ preserves the vacancy numbers and adds a string of length 2 . Note that we cannot have $s^{(a)} > \ell^{(a)}$ because $\tilde{\beta}^e$ preserves the vacancy numbers and possibly only adds a row of length 2 to $\nu^{(a)}$. Next, suppose a is minimal such that $2 = s^{(a)} < \ell^{(a)}$; thus, we must have $P_2^{(a)} = 0$ as the rigging of the string added by $\tilde{\beta}^e$ is 0 (i.e., $\tilde{\beta}^e$ adds a cosingular string of length 2). Hence, we must have $m_2^{(a)} = 0$. From the definition of the vacancy numbers, we have

$$-P_{i-1}^{(a)} + 2P_i^{(a)} - P_{i+1}^{(a)} = L_i^{(a)} - \sum_{b \in I_0} A_{ab} m_i^{(b)}, \tag{3.8}$$

and in particular for $i = 2$, we have

$$-P_1^{(a)} - P_3^{(a)} = L_2^{(a)} + \sum_{b \sim a} m_2^{(b)}.$$

Since $P_i^{(a)} \geq 0$ for all $i > 0$, we must have $P_1^{(a)} = L_2^{(a)} = m_2^{(b)} = 0$ for all $b \sim a$.

Recall from the definition of δ that $s^{(b)} \leq s^{(a)}$ for all $b < a$, and hence $s^{(a-1)} \leq 2$. If $s^{(a-1)} = 2$, then by the assumption a is minimal such that $s^{(a)} < \ell^{(a)}$, we have $2 = s^{(a-1)} = \ell^{(a-1)}$. However, this contradicts that $m_2^{(a-1)} = 0$, and therefore, we must have $s^{(a-1)} = \ell^{(a-1)} = 1$. This implies that $s^{(b)} = \ell^{(b)} = 1$ for all $b < a$. If $m_1^{(a)} > 0$, then $\ell^{(a)} = 1$ because $0 \leq x \leq P_1^{(a)} = 0$, which contradicts our assumption that $2 < \ell^{(a)}$. Hence, we have $m_1^{(a)} = 0$. Consider the case when $a = 1$, and (3.8) for $i = 1$ results in

$$0 = L_1^{(1)} + m_1^{(2)},$$

which is a contradiction since $L_1^{(1)} > 0$ as we are applying δ . Now consider the case $a > 1$. Hence, (3.8) for $i = 1$ results in

$$0 = m_1^{(a-1)} + m_1^{(a+1)} + L_1^{(a)},$$

and therefore, $m_1^{(a-1)} = 0$, which is a contradiction. Thus, we have $[\delta, \tilde{\beta}^e] = 0$.

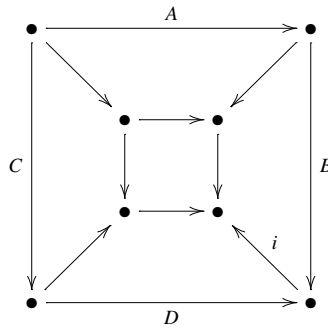
Next we have $[\beta^{(m)}, \tilde{\beta}^e] = [\bar{\beta}, \tilde{\beta}^e] = 0$ as all these maps preserve vacancy numbers. Thus, $[\delta_s, \tilde{\beta}^e] = 0$ from the definition of δ_s . This proves (ii).

The remaining relations follow from conjugation by θ . □

4 Proof of the well-definedness of the bijection

The main purpose of this section is to show that the bijection Φ is well-defined.

In the proof, we will use diagrams of the following kind as in [21]:



We regard this as a cube with front face given by the large square. Suppose that the square diagrams given by the faces of the cube except for the front face commute and i is the injective map. Then the front face also commutes since we have

$$i \circ B \circ A = i \circ D \circ C$$

by diagram chasing.

Proposition 4.1 *Let $B = B^{r_k, s_k} \otimes \dots \otimes B^{r_2, s_2} \otimes B^{r_1, s_1}$ be a tensor product of KR crystals with multiplicity array L . Then there exists a unique family of injections $\Phi: \mathcal{P}(B, \lambda) \rightarrow \mathcal{RC}(L, \lambda)$ such that the empty path maps to the empty rigged configuration and satisfies the following sequence of commutative diagrams.*

(1) Suppose $B = B^{1,1} \otimes B'$. Let $\text{lh}(B) = B'$ with multiplicity array $\text{lh}(L)$. Then the diagram

$$\begin{CD} \mathcal{P}(B, \lambda) @>\Phi>> \mathcal{RC}(L, \lambda) \\ @V\text{lh}VV @VV\delta V \\ \bigcup_{\mu} \mathcal{P}(\text{lh}(B), \mu) @>\Phi>> \bigcup_{\mu} \mathcal{RC}(\text{lh}(L), \mu) \end{CD}$$

commutes.

(1') Suppose $B = B^{r,1} \otimes B'$ for $r = n - 1$ or n . Let $\text{lh}_s(B) = B'$ with multiplicity array $\text{lh}_s(L)$. Then the diagram

$$\begin{CD} \mathcal{P}(B, \lambda) @>\Phi>> \mathcal{RC}(L, \lambda) \\ @V\text{lh}_sVV @VV\delta_s V \\ \bigcup_{\mu} \mathcal{P}(\text{lh}_s(B), \mu) @>\Phi>> \bigcup_{\mu} \mathcal{RC}(\text{lh}_s(L), \mu) \end{CD}$$

commutes.

(2) Suppose $B = B^{r,1} \otimes B'$ with $2 \leq r \leq n - 2$. Let $\text{lb}(B) = B^{1,1} \otimes B^{r-1,1} \otimes B'$ with multiplicity array $\text{lb}(L)$. Then the diagram

$$\begin{CD} \mathcal{P}(B, \lambda) @>\Phi>> \mathcal{RC}(L, \lambda) \\ @V\text{lb}VV @VV\beta V \\ \mathcal{P}(\text{lb}(B), \lambda) @>\Phi>> \mathcal{RC}(\text{lb}(L), \lambda) \end{CD}$$

commutes.

(3) Suppose $B = B^{r,s} \otimes B'$ with $s \geq 2$. Let $\text{ls}(B) = B^{r,1} \otimes B^{r,s-1} \otimes B'$ with multiplicity array $\text{ls}(L)$. Then the diagram

$$\begin{CD} \mathcal{P}(B, \lambda) @>\Phi>> \mathcal{RC}(L, \lambda) \\ @V\text{ls}VV @VV\gamma V \\ \mathcal{P}(\text{ls}(B), \lambda) @>\Phi>> \mathcal{RC}(\text{ls}(L), \lambda) \end{CD}$$

commutes.

(4) Suppose $B = B' \otimes B^{1,1}$. Let $\text{rh}(B) = B'$ with multiplicity array $\text{rh}(L)$. Then the diagram

$$\begin{array}{ccc} \mathcal{P}(B, \lambda) & \xrightarrow{\Phi} & \mathcal{RC}(L, \lambda) \\ \text{rh} \downarrow & & \downarrow \tilde{\delta} \\ \bigcup_{\mu} \mathcal{P}(\text{rh}(B), \mu) & \xrightarrow{\Phi} & \bigcup_{\mu} \mathcal{RC}(\text{rh}(L), \mu) \end{array}$$

commutes.

(4') Suppose $B = B' \otimes B^{r,1}$ for $r = n - 1$ or n . Let $\text{rh}_s(B) = B'$ with multiplicity array $\text{rh}_s(L)$. Then the diagram

$$\begin{array}{ccc} \mathcal{P}(B, \lambda) & \xrightarrow{\Phi} & \mathcal{RC}(L, \lambda) \\ \text{rh}_s \downarrow & & \downarrow \tilde{\delta}_s \\ \bigcup_{\mu} \mathcal{P}(\text{rh}_s(B), \mu) & \xrightarrow{\Phi} & \bigcup_{\mu} \mathcal{RC}(\text{rh}_s(L), \mu) \end{array}$$

commutes.

(5) Suppose $B = B' \otimes B^{r,1}$ with $2 \leq r \leq n - 2$. Let $\text{rb}(B) = B' \otimes B^{r-1,1} \otimes B^{1,1}$ with multiplicity array $\text{rb}(L)$. Then the diagram

$$\begin{array}{ccc} \mathcal{P}(B, \lambda) & \xrightarrow{\Phi} & \mathcal{RC}(L, \lambda) \\ \text{rb} \downarrow & & \downarrow \tilde{\beta} \\ \mathcal{P}(\text{rb}(B), \lambda) & \xrightarrow{\Phi} & \mathcal{RC}(\text{rb}(L), \lambda) \end{array}$$

commutes.

(6) Suppose $B = B' \otimes B^{r,s}$ with $s \geq 2$. Let $\text{rs}(B) = B' \otimes B^{r,s-1} \otimes B^{r,1}$ with multiplicity array $\text{rs}(L)$. Then the diagram

$$\begin{array}{ccc} \mathcal{P}(B, \lambda) & \xrightarrow{\Phi} & \mathcal{RC}(L, \lambda) \\ \text{rs} \downarrow & & \downarrow \tilde{\gamma} \\ \mathcal{P}(\text{rs}(B), \lambda) & \xrightarrow{\Phi} & \mathcal{RC}(\text{rs}(L), \lambda) \end{array}$$

commutes.

(7) The diagram

$$\begin{array}{ccc} \mathcal{P}(B, \lambda) & \xrightarrow{\Phi} & \mathcal{RC}(L, \lambda) \\ \diamond \downarrow & & \downarrow \theta \\ \mathcal{P}(B^*, \lambda) & \xrightarrow{\Phi} & \mathcal{RC}(L, \lambda) \end{array}$$

commutes.

Proof For $B = B^{r_k, s_k} \otimes \dots \otimes B^{r_2, s_2} \otimes B^{r_1, s_1}$, we set

$$\|B\| = \left(\sum_i r_i s_i, \sum_i (r_i - 1), \sum_i (s_i - 1) \right).$$

We prove the statement by induction on the lexicographic order of $\|B\|$. More precisely, at each induction level, we check that Φ is well-defined from (1), (1'), (2), (3) and show that this Φ satisfies (4)–(7) for the next induction step.

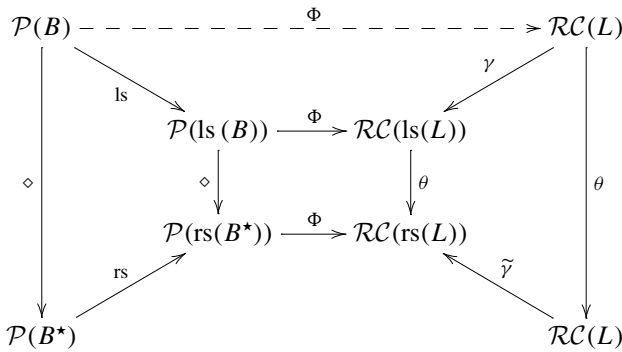
The well-definedness of (1) is shown in [32].

The well-definedness of (1') is shown in [38].

The well-definedness of (2) is shown in [38].

Now we prove the well-definedness of (3). When $B = B^{r, s}$ for $1 \leq r \leq n - 2$, the well-definedness of Φ and, in particular, (3) is established in [35, Theorem 5.9] by directly computing the bijection based on Proposition 3.9 and checking that the result agrees with the definition of the Kirillov–Reshetikhin tableau for the I_0 -highest weight element of $B^{r, s}$. When $B = B^{r, s}$ for $r = n - 1$ or n , both $\mathcal{P}(B^{r, s})$ and $\mathcal{RC}(L)$ consist of a single element and the property is easy to check.

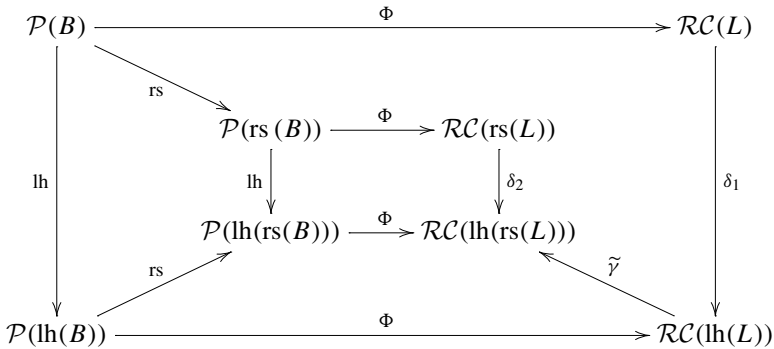
For more general $B = B^{r, s} \otimes B'$ with $s \geq 2$, consider the following diagram:



We see that the left face and right face commute by definition and the back face commutes by the induction hypothesis that (7) holds. We wish to show that $(\Phi \circ \text{Is})(\mathcal{P}(B)) \subseteq \gamma(\mathcal{RC}(L))$. If we have this relation, we can define $\Phi(\mathcal{P}(B)) = (\gamma^{-1} \circ \Phi \circ \text{Is})(\mathcal{P}(B))$ since γ is invertible on $\text{Im}(\gamma)$. Since \diamond and θ are bijections, the commutativity of the left, back and right faces of the above diagram shows that it is enough to prove that $(\Phi \circ \text{rs})(\mathcal{P}(B^*)) \subseteq \tilde{\gamma}(\mathcal{RC}(L))$.

Let us show $(\Phi \circ \text{rs})(\mathcal{P}(B)) \subseteq \tilde{\gamma}(\mathcal{RC}(L))$. For this, it is enough to check that the strings of $\nu^{(r)}$ of the image of $\Phi \circ \text{rs}$ have strictly positive riggings. We prove this claim by dividing into cases.

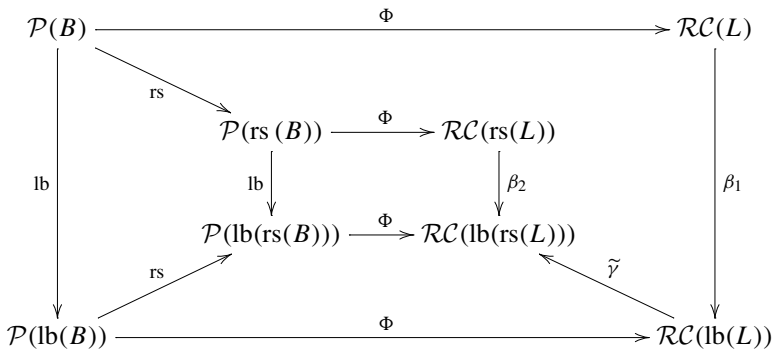
When $B = B^{1,1} \otimes B' \otimes B^{r,s}$, consider the following diagram:



The front face commutes by already proved (1), the left face commutes by definition of lh and rs, and the bottom face and the back face commute by induction hypothesis. There are two δ which we distinguish by denoting them by δ_1 and δ_2 . By diagram chasing, we see that $(\delta_2 \circ \Phi \circ \text{rs})(\mathcal{P}(B)) = (\tilde{\gamma} \circ \delta_1 \circ \Phi)(\mathcal{P}(B))$. Then by definition of $\tilde{\gamma}$, we see that the algorithm for δ_1 and δ_2 choose the same strings and the only difference is the fact that, during the process of δ_2 , the riggings of the strings $\{(l, x) \in (\nu^{(r)}, J^{(r)}) \mid l < s\}$ are greater by 1 compared with the case of δ_1 . Since the riggings of the elements of $\mathcal{RC}(L)$ are nonnegative, we see that the riggings for the strings of $\nu^{(r)}$ of the corresponding elements of $\mathcal{RC}(\text{rs}(L))$ which are shorter than s are strictly positive. Therefore, we have $(\Phi \circ \text{rs})(\mathcal{P}(B)) \subseteq \tilde{\gamma}(\mathcal{RC}(L))$ in this case.

For $B = B^{r',1} \otimes B' \otimes B^{r,s}$ when $r' = n - 1$ or n , the same arguments as above go through with δ and lh replaced by δ_s and lh_s , respectively.

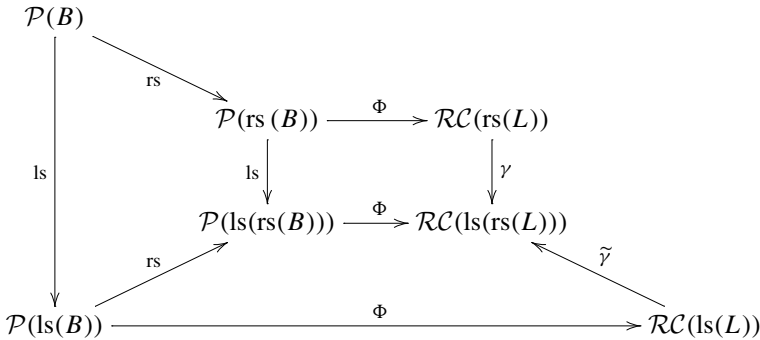
When $B = B^{r',1} \otimes B' \otimes B^{r,s}$ for $2 \leq r' \leq n - 2$, consider the following diagram:



The front face commutes by already proved (2), the left face commutes by definition of lb and rs, and the bottom face and the back face commute by induction hypothesis. In this case, we have $(\beta_2 \circ \Phi \circ \text{rs})(\mathcal{P}(B)) = (\tilde{\gamma} \circ \beta_1 \circ \Phi)(\mathcal{P}(B))$. Recall that β does not change riggings of untouched strings. Therefore, by definition of $\tilde{\gamma}$ we see that the riggings for strings of $\nu^{(r)}$ of the elements of $\mathcal{RC}(\text{rs}(L))$, which are shorter than s

are greater by 1 compared with the corresponding elements of $\mathcal{RC}(L)$. Therefore, we have $(\Phi \circ rs)(\mathcal{P}(B)) \subseteq \tilde{\gamma}(\mathcal{RC}(L))$ in this case.

Finally let us consider the case $B = B^{r',s'} \otimes B' \otimes B^{r,s}$. In this case, we have the following diagram:

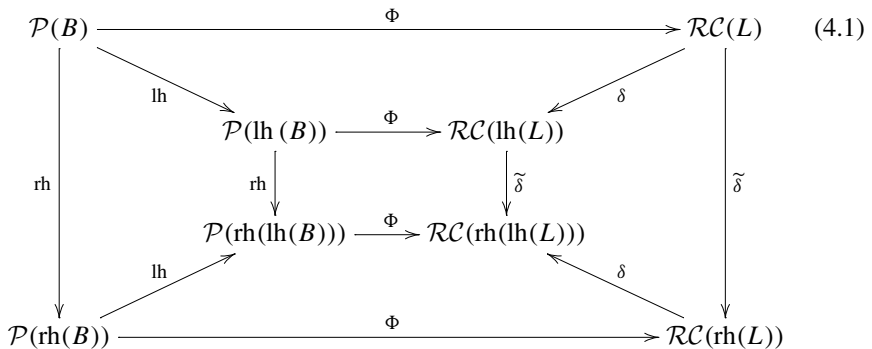


The left face commutes by definition of ls and rs , and the bottom face and the back face commute by induction hypothesis. In this case, we have $(\gamma \circ \Phi \circ rs)(\mathcal{P}(B)) = (\tilde{\gamma} \circ \Phi \circ ls)(\mathcal{P}(B))$. Since γ does not change any strings, we see that riggings for the strings of $\nu^{(r)}$ of the elements of $\mathcal{RC}(rs(L))$ which are shorter than s have strictly positive riggings. Therefore, we have $(\Phi \circ rs)(\mathcal{P}(B)) \subseteq \tilde{\gamma}(\mathcal{RC}(L))$ in this case. This completes the proof of (3).

We move to the proof of (4). We again divide into cases.

When $B = B^{1,1}$, this follows from a special case of [38].

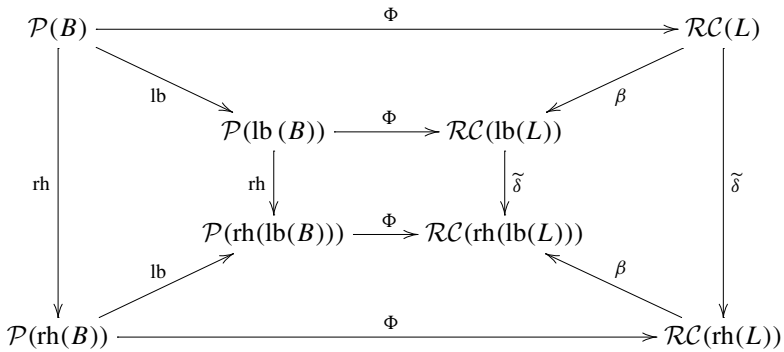
When $B = B^{1,1} \otimes B' \otimes B^{1,1}$, consider the following diagram:



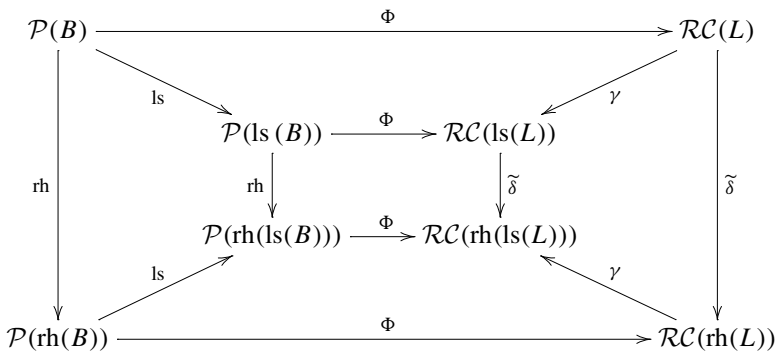
The top face commutes by (1). The left face commutes by definition of lh and rh , and the right face commutes by the fundamental relation $[\delta, \tilde{\delta}] = 0$ (Theorem 3.11). The back and bottom faces commute by induction hypothesis. Since δ is injective, the front face commutes.

When $B = B^{r',1} \otimes B' \otimes B^{1,1}$ for $r' = n - 1$ or n , we have the same commutative diagram as in (4.1) with lh and δ replaced by lh_s and δ_s , respectively. The commutativity $[\delta_s, \tilde{\delta}] = 0$ is shown in Proposition 3.16 (i).

When $B = B^{r',1} \otimes B' \otimes B^{1,1}$ for $2 \leq r' \leq n - 2$, consider the following diagram and similarly show the front face commutes, using Proposition 3.12 (ii):



When $B = B^{r',s'} \otimes B' \otimes B^{1,1}$ consider the following diagram and similarly show the front face commutes, using Proposition 3.12 (iii):

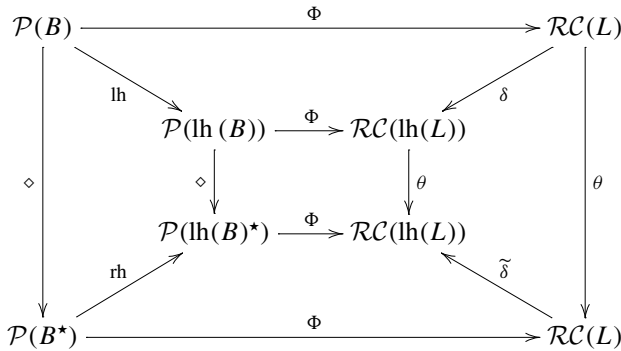


The proofs of (5) and (6) are parallel to the argument of (4). When $B = B^{r,1}$, (5) follows from [38]. When $B = B^{r,s}$, (6) follows from Proposition 3.13 for $r \leq n - 2$, and from the fact that both $\mathcal{P}(B^{r,s})$ and $\mathcal{RC}(L)$ consist of a single element for $r = n - 1, n$. We also need Propositions 3.12 and 3.16 (ii).

Case (4') follows from cases (4) and (5) by noting that δ_s is realized by a composition of the doubling map, a sequence of n compositions of $\delta \circ \beta$, and a halving map.

Finally, we prove (7). We again divide into cases.

When $B = B^{1,1} \otimes B'$, we can use the following diagram to show that the front face commutes:



The proofs when $B = B^{r,1} \otimes B'$ or $B = B^{r,s} \otimes B'$ are almost the same. □

We can now state the main result of this section.

Theorem 4.2 *Let B be a tensor product of KR crystals. Then $\Phi: \mathcal{P}(B, \lambda) \rightarrow \mathcal{RC}(L, \lambda)$ is a well-defined bijection.*

Proof By Proposition 4.1, Φ is a well-defined injection. By [5] or [27], we have $|\mathcal{P}(B, \lambda)| = |\mathcal{RC}(L, \lambda)|$, which proves that Φ is a bijection. □

In order to generalize the rigged configuration bijection to include non-highest weight elements, we invoke the following result stated in [37, Thm. 4.1].

Theorem 4.3 ([37]) *The rigged configuration bijection $\Phi: \mathcal{P}(B) \rightarrow \mathcal{RC}(L)$ can be extended to a bijection $\Phi: B \rightarrow \mathcal{RC}(L)$ by requiring Φ to be a classical crystal isomorphism:*

$$[\Phi, e_i] = [\Phi, f_i] = 0 \quad (i \in I_0). \tag{4.2}$$

Example 4.4 Consider the rigged configuration (v, J) in Example 3.2, $f_4(v, J)$ from Example 3.4, and the tensor product of KR crystal elements in Example 3.7. We have

$$\Phi^{-1}(f_4(v, J)) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 5 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline \end{array} = f_4 \Phi^{-1}(v, J).$$

5 R-invariance of rigged configurations

In this section, we prove that the combinatorial R -matrix on unrestricted rigged configurations is the identity under the bijection Φ . We do so in several steps. First we show that the combinatorial R -matrix $R: B^{n,1} \otimes B^{r,s} \rightarrow B^{r,s} \otimes B^{n,1}$ corresponds to

the identity map under Φ (see Proposition 5.7 which follows from Lemmas 5.2 and 5.5 and Proposition 5.6). The general statement (see Theorem 5.11) can then be deduced by passing a spin column through enough times to reduce to the type $A_n^{(1)}$ case which was proven in [21].

We begin with the R -matrix $R: B^{n,1} \otimes B^{r,s} \rightarrow B^{r,s} \otimes B^{n,1}$ for $1 \leq r \leq n - 2$ in type $D_n^{(1)}$. Let $b \otimes u_\eta \in B^{n,1} \otimes B^{r,s}$ be an I_0 -highest weight element, where $u_\eta \in B(\eta) \subseteq B^{r,s}$ is the unique I_0 -highest weight element of highest weight η . Note that, for fixed η , the element $b \otimes u_\eta$ is uniquely specified by the weight $\lambda = \text{wt}(b \otimes u_\eta)$ since the multiplicity of any weight space in $B(\varpi_n) \cong B^{n,1}$ (as classical crystals) is at most 1.

Let us introduce the following notation. Write $\lambda = \sum_{i \in I_0} m_i \varpi_i$ in terms of the classical fundamental weights ϖ_i , where all m_i are nonnegative integers since λ is dominant. Define $\bar{\lambda} = \sum_{i \in I_0 \setminus \{n-1, n\}} m_i \varpi_i$. We can interpret $\bar{\lambda}$ as a partition (where each fundamental weight ϖ_i for $1 \leq i \leq n - 2$ contributes a column of height i). Then let $\bar{\lambda}^c$ be the complement of the partition $\bar{\lambda}$ in the $r \times s$ rectangle. Similarly, η^c is the complement of η (interpreted as a partition) in the $r \times s$ rectangle. The skew partition $\bar{\lambda}^c / \eta^c$ can have at most one box in each row. Denote the cells in $\bar{\lambda}^c / \eta^c$ from top to bottom (in English convention for partitions) by c_1, c_2, \dots, c_ℓ . Finally, $\hat{\eta}$ is obtained by replacing all \boxplus of η^c by \square (recall that this is well-defined since η is obtained from the $r \times s$ rectangle by removing vertical dominoes).

Note that $m_{n-1} + m_n = 1$ due to the fact that $r \leq n - 2$ and $b \in B^{n,1}$. Also, $\ell + m_{n-1}$ is even since $b \in B^{n,1}$ (it would be odd for $b \in B^{n-1,1}$).

We now define the configuration that we will show under the bijection Φ corresponds to the I_0 -highest weight elements $b \otimes u_\eta \in B^{n,1} \otimes B^{r,s}$ for $1 \leq r \leq n - 2$. Recall that $b \otimes u_\eta$ is uniquely determined by the two weights η and λ . Let $v = \Gamma(\eta, \lambda)$ be the following configuration:

- (i) For $r \leq a \leq n - 2$, $v^{(a)} = \bar{\lambda}^c$.
- (ii) For $1 \leq a < r$, $v^{(a)}$ is obtained from $v^{(n-2)}$ by removing $r - a$ rows starting from longer rows.
- (iii) If $m_{n-1} = 0$, then $v^{(n-1)}$ (resp., $v^{(n)}$) is obtained from $\hat{\eta}$ by adding the cells c_1, c_3, \dots (resp., c_2, c_4, \dots) to the same row length as in $\bar{\lambda}^c / \eta^c$.
 If $m_{n-1} = 1$, then $v^{(n-1)}$ (resp., $v^{(n)}$) is obtained from $\hat{\eta}$ by adding the cells c_2, c_4, \dots (resp., c_1, c_3, \dots) to the same row length as in $\bar{\lambda}^c / \eta^c$.

Let us examine the vacancy numbers for elements $v \in \Gamma(\eta, \lambda)$. It is not hard to check that $P_i^{(a)}(v) = 0$ for all $1 \leq a \leq n - 2$ and $i \geq 0$. Furthermore, from weight considerations we have $P_i^{(n-1)}(v) = m_{n-1}$ and $P_i^{(n)}(v) = 1 - m_{n-1}$ for large i . Call $1 \leq h_1 \leq h_2 \leq \dots \leq h_\ell$ the ordered row labels of the cells in $v^{(n-1)} / \hat{\eta}$ and $v^{(n)} / \hat{\eta}$. Starting with the largest row in $v^{(n-1)}$ (resp., $v^{(n)}$), the vacancy number switches from 1 to 0 (or 0 to 1) whenever one of the row lengths h_i is crossed. In particular, $P_i^{(n-1)}(v) + P_i^{(n)}(v) = 1$ for all $i \geq 0$.

Example 5.1 We illustrate the construction of $\Gamma(\eta, \lambda)$ with two examples. Consider type $D_{10}^{(1)}$ and $B = B^{10,1} \otimes B^{8,5}$. Let $b = (+, -, +, +, +, -, +, +, +, +)$ and

$$u_\eta = \begin{matrix} \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 \\ 7 & 7 & 7 \\ 8 & 8 & 8 \end{matrix} \end{matrix},$$

where $\eta = \varpi_2 + \varpi_6 + 3\varpi_8$ and $\lambda = \varpi_1 + \varpi_5 + 3\varpi_8 + \varpi_{10}$, and hence, $m_{10} = 1$ and $m_9 = 0$. Thus, we have as partitions $\eta^c = 221111$ and $\bar{\lambda}^c = 2221111$, and so

$$\bar{\lambda}^c / \eta^c = \begin{matrix} \square & \square \\ \square & \square \\ \square & \mathbf{1} \\ \square \\ \square \\ \square \\ \mathbf{2} \end{matrix},$$

where we have indicated the cells c_1 and c_2 in the construction by blue letters 1 and 2, respectively. Then $\Phi(b \otimes u_\eta)$ is equal to

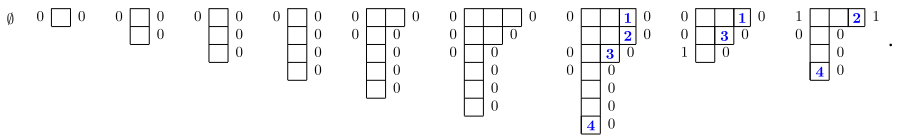
$$\emptyset \quad 0 \square 0 \quad 0 \begin{matrix} \square \\ \square \end{matrix} 0 \quad 0 \begin{matrix} \square & 0 \\ \square & 0 \\ \square & 0 \end{matrix} 0 \quad 0 \begin{matrix} \square & 0 \\ \square & 0 \\ \square & 0 \\ \square & 0 \end{matrix} 0 \quad 0 \begin{matrix} \square & 0 \\ \square & 0 \\ \square & 0 \\ \square & 0 \\ \square & 0 \end{matrix} 0 \quad 0 \begin{matrix} \square & 0 \\ \square & 0 \\ \square & \mathbf{1} \\ \square & 0 \\ \square & 0 \\ \square & 0 \\ \square & \mathbf{2} \end{matrix} 0 \quad 0 \begin{matrix} \square & 0 \\ \square & 0 \\ \square & \mathbf{1} \\ \square & 0 \end{matrix} 0 \quad 1 \begin{matrix} \square & 0 \\ \square & 0 \\ \square & 0 \\ \square & \mathbf{2} \end{matrix} 0,$$

where again we labelled the cells added to η^c in $v^{(n-2)}$ and η^c in $v^{(n-1)}$ and $v^{(n)}$ by blue letters 1 and 2.

Next we consider $b' = (+, -, +, +, +, -, -, -, +, +)$ and $\mu := \text{wt}(b' \otimes u_\eta) = \varpi_1 + \varpi_5 + \varpi_6 + 2\varpi_8 + \varpi_{10}$. Note that $\lambda \neq \mu$ and

$$\bar{\mu}^c / \eta^c = \begin{matrix} \square & \square & \mathbf{1} \\ \square & \square & \mathbf{2} \\ \square & \mathbf{3} \\ \square \\ \square \\ \mathbf{4} \end{matrix}.$$

Then $\Phi(b' \otimes u_\eta)$ is



Lemma 5.2 Consider the I_0 -highest weight element $b \otimes u_\eta \in B^{n,1} \otimes B^{r,s}$ of weight λ with $1 \leq r \leq n - 2$, where u_η is the I_0 -highest weight element of highest weight η . Then the rigged configuration

$$(v, J) = \Phi(b \otimes u_\eta)$$

satisfies $v = \Gamma(\eta, \lambda)$ and $J_i^{(a)} = 0$ for all $1 \leq a \leq n - 2$ and $i \geq 0$. $J_i^{(n-1)}$ and $J_i^{(n)}$ are determined as follows. Since $P_i^{(n-1)}(v) + P_i^{(n)}(v) = 1$ either $J_i^{(n-1)}$ or $J_i^{(n)}$ can contain 1's and the other riggings must be 0. The number of 1's in $J_i^{(n-1)}$ (or $J_i^{(n)}$) is equal to the number of vertical dominoes in column i of $\bar{\lambda}^c / \eta^c$.

Proof Let us define $d = \text{emb}(b)$. Let x be the letter in d at height h . Define $\delta' = \beta^{-1} \circ \delta^{-1}$ unless $h = 1$ in which case $\delta' = \delta^{-1}$. Thus, δ' under the bijection Φ corresponds to adding the letter x to the step in the path using the algorithm for δ^{-1} , except we terminate after adding the last box to $(v^{(h)}, J^{(h)})$. Let w_i denote the row length of η^c corresponding to the row in $\bar{\lambda}^c / \eta^c$ containing the cell c_i as given in the construction $\Gamma(\eta, \lambda)$ (recall $\ell = |\bar{\lambda}^c / \eta^c|$). Set $w_0 = \infty$ and $w_i = 0$ for all $i > \ell$. Define $\tilde{w}_i = w_{\ell-i}$. In particular, we have $w_i \leq w_{i+1}$ for all $0 \leq i < \ell$. Define $\mu(x) = \#\{x \leq x' \leq n - 2 \mid x' \in d\}$.

First note that $\Phi(u_\eta)$ is given by Proposition 3.9. Next we want to add b , which by the definition of δ_s involves the doubling map emb . Hence, we need to double $\Phi(u_\eta)$. Denote the rigged configuration before having added the letter x by (\bar{v}, \bar{J}) and $(\bar{v}', \bar{J}') = \delta'(\bar{v}, \bar{J})$. Suppose we are adding the letter $x \leq n - 1$ to d at height h . \square

Claim We obtain $\delta'(\bar{v}, \bar{J})$ from (\bar{v}, \bar{J}) by adding a box to a (singular) string of length $2\tilde{w}_{a-h}$ to $\bar{v}^{(a)}$ for $h \leq a < x$. Note that this is the $(a - h + 1)$ th cell from the bottom of $\bar{\lambda}^c / \eta^c$. Moreover, all riggings are 0.

We prove the claim by induction on h . Suppose the letter y was added at height $h - 1$ in the previous step (which might be vacuous if x is at height 1 in which case we set $y = 0$). Note that $P_i^{(a)}(\bar{v}) > 0$ for all i such that $2\tilde{w}_{a-h} < i \leq 2\tilde{w}_{a-h+1}$ and $h - 1 \leq a \leq y < x$.

Observe that the number of minus signs appearing in b before position x is $x - h$. The application of δ' adds a box to the row of length $2\tilde{w}_{x-h-1}$ in $\bar{v}^{(x-1)}$. This follows from the fact that no previous application of δ' changed $\bar{v}^{(x-1)}$ (our induction hypothesis), the description of \bar{v} as given in Proposition 3.9, and that all rows strictly longer than $2\tilde{w}_{x-h-1}$ are non-singular. Next δ' adds a box to a row of length $2\tilde{w}_{x-h-2}$ in $\bar{v}^{(x-2)}$ since all rows of length strictly between $2\tilde{w}_{x-h-2}$ and $2\tilde{w}_{x-h-1}$ are non-singular,

$\bar{v}^{(x-2)}$ is the same as $\bar{v}^{(x-1)}$ with the top row removed, and we must select a row i such that $\bar{v}_i^{(x-2)} \leq 2\tilde{w}_{x-h-1}$ by the definition of δ' . A similar argument holds for $\bar{v}^{(a)}$ for the remaining $h \leq a < x - 1$ for the application of δ' . From (3.1), the resulting riggings are all 0, and $P_i^{(a)}(\bar{v}') > 0$ for all i such that $2\tilde{w}_{a-h-1} < i \leq 2\tilde{w}_{a-h}$ and $h \leq a \leq x$.

For $x = n, \bar{n}$, the application of δ' is similar to the above except we do not change $\bar{v}^{(n)}, \bar{v}^{(n-1)}$, respectively. Next suppose we are adding \bar{x} at height h with $1 \leq x \leq n - 1$.

Claim We obtain $\delta'(\bar{v}, \bar{J})$ from (\bar{v}, \bar{J}) by the following steps:

- (i) add a box to the largest odd length string in $\bar{v}^{(a)}$ for $x \leq a \leq n - 2$,
- (ii) for both $\bar{v}^{(n-1)}, \bar{v}^{(n)}$, either add to the largest odd length row or the row of length $2w_{\mu(x)}$ depending on if there is an odd length row or not, respectively, and
- (iii) add a box to a row of length $2\tilde{w}_{a-h}$ in $\bar{v}^{(a)}$ for $h \leq a < n - 1$.

We prove the claim by using induction on \bar{x} (i.e., as x decreases). Note that the largest singular row in $\bar{v}^{(x)}$ has length $2w_{\mu(x)} + 1$, and it is the first row of $\bar{v}^{(x)}$.

By our induction assumption and the definition of δ' , we add a box to a row of length $2w_{\mu(x)} + 1$ in all $\bar{v}^{(a)}$ for $x \leq a \leq n - 2$. Thus, step (i) follows. For step (ii), let $a = n - 1, n$. Note that if there exists an odd length row, then from our induction, there is exactly one such row and it is of length $2w_{\mu(x)} + 1$. Therefore, we add a box to this row. Otherwise all rows are currently of even length, so we add a box to a row of length $2w_{\mu(x)}$. Note that $P_{2w_{\mu(x)}}^{(a)}(\bar{v}) = 0$ because we have only added boxes to $\bar{v}^{(n-2)}$ in rows of length at least $2w_{\mu(x)}$ (i.e., the resulting rows are strictly larger and don't contribute to $P_{2w_{\mu(x)}}^{(a)}(\bar{v})$). The proof of step (iii) of the claim is similar to the proof of the previous claim for $x \leq n - 2$.

From the aforementioned claims, it immediately follows that we obtain the desired configuration of $\Gamma(\eta, \lambda)$ after applying emb^{-1} .

Let us now turn to the statement about the riggings. A straightforward check from the above claim shows that all riggings are 0 in $\bar{v}^{(a)}$ for $a \leq n - 2$. Next, each removable domino in $\bar{\lambda}^c / \eta^c$ corresponds to two consecutive minus signs in b , which in turns corresponds to two barred letters \bar{x} and $\overline{x + 1}$. Note that when adding \bar{x} , the resulting vacancy number $P_{2w_{\mu(x)+2}}^{(a)}(\bar{v}') = 2$ for either $a = n - 1$ or $a = n$ since the vertical domino implies we have an additional (as compared with $\Phi(u_\eta)$) contribution of 6 boxes in $\bar{v}^{(n-2)}$, but only 2 boxes in $\bar{v}^{(a)}$. Hence, the corresponding rigging is 2. From the above description of δ' , we do not change the corresponding row again when we add the remaining letters; thus, $\text{emb}^{-1}(\bar{v}, \bar{J}) = \Phi(b \otimes u_\eta)$ is the desired rigged configuration. □

Example 5.3 We consider $B = B^{8,1} \otimes B^{6,5}$ in type $D_8^{(1)}$ and fix $\lambda = \varpi_1 + 2\varpi_6 + \varpi_7$. We now look at the set $\{b \otimes u_\eta \in \mathcal{P}(B) \mid \text{wt}(b \otimes u_\eta) = \lambda\}$. We start with $\eta = \varpi_2 + 2\varpi_6$ and $b = (+, -, +, +, +, +, +, -)$. Thus, we have

$$\Phi(b \otimes u_\eta) = \begin{matrix} \emptyset & 0 & \square & 0 & 0 & \begin{matrix} \square & 0 \\ \square & 0 \end{matrix} & 0 & \begin{matrix} \square & 0 \\ \square & 0 \\ \square & 0 \end{matrix} & 0 & \begin{matrix} \square & 0 \\ \square & 0 \\ \square & 0 \\ \square & 0 \end{matrix} & 1 & \begin{matrix} \square & 0 \\ \square & 0 \end{matrix} & 0 & \begin{matrix} \square & 0 \\ \square & 0 \\ \square & 0 \end{matrix} \end{matrix} .$$

Next we consider $\eta' = \varpi_4 + 2\varpi_6$ and $b' = (+, -, -, -, +, +, +, -)$, and we have

$$\Phi(b' \otimes u_{\eta'}) = \emptyset \quad 0 \quad \begin{array}{|c|} \hline 0 \\ \hline \end{array} \quad 0 \quad \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array} \quad 0 \quad \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad 0 \quad \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad 1 \quad \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline \end{array} \quad 0 \quad \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} .$$

Finally, we take $\eta'' = 3\varpi_6$ and $b'' = (+, -, -, -, -, -, +, -)$, so that

$$\Phi(b'' \otimes u_{\eta''}) = \emptyset \quad 0 \quad \begin{array}{|c|} \hline 0 \\ \hline \end{array} \quad 0 \quad \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array} \quad 0 \quad \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad 0 \quad \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad 1 \quad \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \quad 0 \quad \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} .$$

Let us now turn our attention to I_0 -highest weight elements in $B^{r,s} \otimes B^{n,1}$ for $1 \leq r \leq n - 2$ of weight λ . They are of the form $b \otimes u$ with $u := u_{\varpi_n} = (+, \dots, +) \in B^{n,1}$ being (the unique) I_0 -highest weight element and $b \in B(\mu) \subseteq B^{r,s}$ for some μ . Since $\varphi_i(u) = 0$ for $1 \leq i \leq n - 1$ and $\varphi_n(u) = 1$, we have that the part of b as a KN tableau without the letters n and \bar{n} is highest weight of highest weight $\bar{\lambda}$ and the skew shape $\mu/\bar{\lambda}$ is a vertical strip. In b , the vertical strip $\mu/\bar{\lambda}$ contains the letter n and \bar{n} in alternating order. Define η as follows. In column i of b , let x be the number of removable n and \bar{n} pairs and y the number of addable n and \bar{n} pairs. Form a new tableau with the same $\bar{\lambda}$, but x and y interchanged for each column i . Then η is the shape of this tableau.

Example 5.4 Consider type $D_{10}^{(1)}$ and

1	1	1
2	2	$\overline{10}$
3	3	$\overline{10}$
4	4	$\overline{10}$
5	5	
6	$\overline{10}$	
7		
$\overline{10}$		

 \otimes

+
+
+
+
+
+
+
+
+
+

 $= b \otimes u \in B^{8,3} \otimes B^{10,1}.$

Thus, we have $\text{wt}(b \otimes u) = \lambda = \varpi_1 + \varpi_5 + \varpi_7 + \varpi_9$ and $\eta = \varpi_2 + \varpi_6 + \varpi_8$.

We use (4.2) in order to simplify the proof of the following lemma (instead of proving it directly using Φ).

Lemma 5.5 *Consider the I_0 -highest weight element $b \otimes u \in B^{r,s} \otimes B^{n,1}$ of weight λ with $1 \leq r \leq n - 2$ and η as defined above. Then the rigged configuration*

$$(\nu, J) = \Phi(b \otimes u)$$

is the same as in Lemma 5.2.

Proof Recall that $\Phi(u)$ is the empty rigged configuration $v^{(a)} = \emptyset$. Therefore, by the combinatorial procedure of Φ , we see that the only difference between $\Phi(b \otimes u)$ and $\Phi(b)$ is that the riggings of $v^{(n)}$ for the latter case is smaller than those for the former case by 1 since there is a contribution to the vacancy number by the tensor factor $B^{n,1}$ to $P_i^{(n)}(v)$. Thus, it is enough to show that for $(\tilde{v}, \tilde{J}) = \Phi(b)$, we have $\tilde{v} = \Gamma(\eta, \lambda)$ and $\tilde{J}_i^{(a)} = 0$ whenever $J_i^{(a)} = 1$ in Lemma 5.2 and $\tilde{J}_i^{(a)} = -1$ otherwise for $a = 1, 2$.

We are going to prove the lemma by induction on $|\mu/\bar{\lambda}|$. In fact, for simplicity we prove the statement simultaneously for the case stated as well as with all letters n and \bar{n} as well as u_{ϖ_n} and $u_{\varpi_{n-1}}$ interchanged. The claim for $\Phi(b \otimes u_{\varpi_{n-1}})$ is the same as in Lemma 5.2 with $(v, J)^{(n-1)}$ and $(v, J)^{(n)}$ interchanged. The base case $|\mu/\bar{\lambda}| = 0$ follows directly from Proposition 3.9.

Now suppose that $|\mu/\bar{\lambda}| > 0$ and assume that the topmost cell in $\mu/\bar{\lambda}$ is filled with \bar{n} (resp., n). Suppose this cell is at height h . Consider

$$b' = e_h e_{h+1} \cdots e_{n-2} e_n(b) \quad (\text{resp., } b' = e_h e_{h+1} \cdots e_{n-2} e_{n-1}(b)).$$

Compared to b , the element b' has a letter h instead of \bar{n} (resp., n) in this cell, so that $\bar{\lambda}' = \bar{\lambda} + \epsilon_h$. Hence, by induction hypothesis $\Phi(b')$ is given as stated by the extension of the lemma. Furthermore, by Theorem 4.3 the Kashiwara operators e_i and Φ commute, so that it suffices to check that

$$e_h e_{h+1} \cdots e_{n-2} e_n(\tilde{v}, \tilde{J}) \quad (\text{resp., } e_h e_{h+1} \cdots e_{n-2} e_{n-1}(\tilde{v}, \tilde{J}))$$

is indeed $(\tilde{v}', \tilde{J}') = \Phi(b')$ as stated in the extension of the lemma. Recall the action of e_i on rigged configurations as given in Definition 3.3. The smallest string with rigging -1 in $(\tilde{v}, \tilde{J})^{(n)}$ (resp., $(\tilde{v}, \tilde{J})^{(n-1)}$) is of length h , so that e_n (resp., e_{n-1}) removes a box from this string. By the form of (\tilde{v}, \tilde{J}) as stated in the lemma, it is not hard to see that $e_{i+1} e_{i+2} \cdots e_{n-2} e_n(\tilde{v}, \tilde{J})$ (resp., $e_{i+1} e_{i+2} \cdots e_{n-2} e_{n-1}(\tilde{v}, \tilde{J})$) has a negative rigging -1 of smallest length in a string of length h in the i th rigged partition for each $h \leq i \leq n - 2$. This shows that $\tilde{v}^{(n-2)} = (\bar{\lambda}')^c$ as required and all $\tilde{v}^{(a)}$ with $1 \leq a < n - 2$ are as desired. The cells c'_1, c'_2, \dots in the construction of $\tilde{v}^{(n-1)}$ and $\tilde{v}^{(n)}$ are obtained from $(\bar{\lambda}')^c / (\eta')^c$. There are two cases to consider:

- (1) e_n (resp., e_{n-1}) removes one of the cells c_i in $(\tilde{v}, \tilde{J})^{(a)}$ for $a = n$ (resp., $a = n - 1$).
- (2) e_n (resp., e_{n-1}) does not remove any of the cells c_i in $(\tilde{v}, \tilde{J})^{(a)}$ for $a = n$ (resp., $a = n - 1$).

Recall that the cells c_j are those in $\bar{\lambda}^c / \eta^c$ or alternatively $\eta / \bar{\lambda}$.

In Case (1), the cell containing the letter \bar{n} (resp., n) at height h , which is also c_i , is part of η and not part of a removable (n, \bar{n}) -pair. Hence, $\eta' = \eta$. This implies that the list c'_1, c'_2, \dots is the same as the list c_1, c_2, \dots , but with c_i missing. If in the algorithm for (\tilde{v}, \tilde{J}) , the cell c_1 is added to the a th rigged partition for $a = n - 1$ or n , then in (\tilde{v}', \tilde{J}') , the cell c'_1 is added to the opposite rigged partition. All addable (n, \bar{n}) -pairs at height $< h$ come in pairs of the same row length. Hence, still cells are added to the

same rows. The cell c_i is missing from the list c'_j and for $j \geq i$, c'_j adds the same cell as c_{j+1} to the same rigged partition, which proves the claim.

In Case (2), we have $\eta' = \eta + \epsilon_h + \epsilon_{h+1}$ since the \bar{n} (resp., n) removed by the string of e_j is part of a removable (n, \bar{n}) -pair. Hence, $(\bar{\lambda}')^c / (\eta')^c$ compared to $\bar{\lambda}^c / \eta^c$ has an extra cell at height $h + 1$, which implies that the sequence c'_j compared to c_j has an extra cell c'_i . As in Case (1), the cells at height $< h$ come in pairs and hence it does not matter whether one first adds them to the a th rigged partitions for $a = n - 1$ or n or vice versa. Since c'_j contains an extra cell c_i , c'_j for $j > i$ adds the same cell as c_{j-1} to the same rigged partition, which proves the claim. \square

To deal with the combinatorial R -matrices between two spin cases, we recall the following result by Mohammad [2].

Proposition 5.6 ([2]) *For $r = n - 1, n$, the I_0 -highest weight decomposition of $B^{n,1} \otimes B^{r,s}$ and $B^{r,s} \otimes B^{n,1}$ is multiplicity free. As a consequence, the combinatorial R -matrix*

$$R: B^{n,1} \otimes B^{r,s} \rightarrow B^{r,s} \otimes B^{n,1}$$

is uniquely determined by being an I_0 -crystal isomorphism.

By combining these results, we now establish the R -invariance of the rigged configuration bijection for the case of $B^{n,1} \otimes B^{r,s}$ as follows:

Proposition 5.7 *Consider an I_0 -highest element $s_1 \otimes b_1$ of $B^{n,1} \otimes B^{r,s}$. Suppose that*

$$R: s_1 \otimes b_1 \mapsto b_2 \otimes s_2 \in B^{r,s} \otimes B^{n,1}$$

under the combinatorial R -matrix. Then we have

$$\Phi(s_1 \otimes b_1) = \Phi(b_2 \otimes s_2).$$

Proof If $r = n - 1, n$, this follows from Proposition 5.6 and Theorem 4.3.

Thus assume $r \leq n - 2$. By Lemmas 5.2 and 5.5, it suffices to show that under the combinatorial R -matrix, an I_0 -highest weight element in $B^{n,1} \otimes B^{r,s}$ corresponding to the tuple (η, λ) goes to the I_0 -highest weight element in $B^{r,s} \otimes B^{n,1}$ with the same tuple (η, λ) as given in the corresponding lemmas. We do so by induction on $|\lambda|$ (from largest to smallest) by either increasing $|\eta|$ or decreasing $|\eta/\lambda|$.

Note that there is a unique element of weight $s\varpi_r + \varpi_n$ in both $B^{n,1} \otimes B^{r,s}$ and $B^{r,s} \otimes B^{n,1}$, so that $R(u_{\varpi_n} \otimes u_{s\varpi_r}) = u_{s\varpi_r} \otimes u_{\varpi_n}$. We set up the induction in several steps. Let $s_1 \otimes u_\eta \in B^{n,1} \otimes B^{r,s}$ be a highest weight element different from $u_{\varpi_n} \otimes u_{s\varpi_r}$. Perform one of the following steps to obtain a new $s'_1 \otimes u_{\eta'}$ closer to $u_{\varpi_n} \otimes u_{s\varpi_r}$ by induction:

Step 1 Suppose $\varphi_0(s_1 \otimes u_\eta) > 0$. Then let $s'_1 \otimes u_{\eta'}$ be the I_0 -highest weight element in the same component as $f_0(s_1 \otimes u_\eta)$.

Step 2 Suppose $\varphi_0(s_1 \otimes u_\eta) = 0$. Note that in this case $\eta_1 = s - 1$ or s . Set $b_1 = u_\eta$ and repeat the following steps until $\varphi_0(s_1 \otimes b_1) > 0$. Then let $s'_1 \otimes u_{\eta'}$ be the I_0 -highest weight element in the same component as $f_0(s_1 \otimes b_1)$.

- (i) Let $1 \leq h < n - 1$ be minimal such that $\varphi_h(s_1 \otimes b_1) > 0$.
- (ii) Compute $s'_1 \otimes b'_1 = f_h(s_1 \otimes b_1)$.
- (iii) Reset $s_1 \otimes b_1$ to $s'_1 \otimes b'_1$.

Step 3 If neither Step 1 nor Step 2 applies or yields a $c = s_1 \otimes b_1$ with $\varphi_0(c) > 0$, we have $\eta = (s - 1)\varpi_r + \varpi_k$ for some $0 \leq k < r$ and $s_1 = (+, \dots, +)$ or $(+, \dots, +, -, +, \dots, +, -)$ with one minus in position n and the other in position k or r . Note that, for fixed weight λ , these elements minimize $|\eta|$ and there is a unique I_0 -highest weight element corresponding to (η, λ) with minimal $|\eta|$. On the $B^{r,s} \otimes B^{n,1}$ side, these are precisely the I_0 -highest weight elements $b_2 \otimes u_{\varpi_n}$ with $b_2 \in B(s\varpi_r) \subseteq B^{r,s}$ (since minimization of $|\eta|$ corresponds to maximization of $|\mu|$). Due to the uniqueness for a given weight class, they must map to each other under R .

We are now going to check for Steps 1 and 2 inductively, that if an I_0 -highest weight element $s_1 \otimes u_\eta$ characterized by the tuple (η, λ) goes to (η', λ') under a given step, then $b_2 \otimes u_{\varpi_n}$ corresponding to the same (η, λ) also goes to (η', λ') under the same step.

We begin with Step 1. Suppose $f_0(s_1 \otimes u_\eta) = s_1 \otimes f_0(u_\eta)$. Take $s_1 \otimes u_\eta$ of weight λ . By the combinatorial rules for f_0 on elements in $B^{r,s}$ provided in [40], we have that η' (resp., λ) is η (resp., λ) with a vertical domino added to rows 1 and 2. More precisely, $\eta'_i = \eta_i + 1$ for $i = 1, 2$. Note also that $\eta_1 < s$ (and in particular r even) and $\bar{\lambda}_1 < \eta_1$ if $\eta_1 = s - 1$ since otherwise f_0 does not apply. Now take $b_2 \otimes u_{\varpi_n}$ corresponding to (η, λ) with $b_2 \in B(\mu)$. The action of f_0 is given by Lemma 2.4. If case Lemma 2.4(i) applies, we have $\mu_i = \eta_i = \bar{\lambda}_i + 1$ and $\eta'_i = \mu'_i + 1 = \bar{\lambda}'_i + 1$ for $i = 1, 2$, so that indeed $\eta'_i = \eta_i + 1$. Lemma 2.4(ii) does not apply since r cannot be odd. If Lemma 2.4(iii) applies, we have $\mu_i = \bar{\lambda}_i + 1 = \eta_i + 1$ and $\mu'_i = \bar{\lambda}'_i + 1 = \eta'_i + 1$ with $\mu'_i = \mu_i + 1$ for $i = 1, 2$, so that again $\eta'_i = \eta_i + 1$. This proves the claim for Step 1 with $f_0(s_1 \otimes u_\eta) = s_1 \otimes f_0(u_\eta)$.

Step 1 with $f_0(s_1 \otimes u_\eta) = f_0(s_1) \otimes u_\eta$ can only apply when $\eta_1 = s$ and s_1 has $-$ in positions 1 and 2. Under f_0 these two minus in s_1 are turned into $+$, so that $\lambda'_i = \lambda_i + 1$ for $i = 1, 2$, $\lambda'_i = \lambda_i$ for $i \geq 3$, and $\eta' = \eta$. When r is even, the correspondence gives $\mu_i = \bar{\lambda}_i = s - 1$ for $i = 1, 2$. Then Lemma 2.4(iii) applies and yields $\mu'_i = s = \bar{\lambda}'_i = \eta'_i$ for $i = 1, 2$ as desired. When r is odd, the correspondence yields $\eta_1 = s, \bar{\lambda}_1 = s - 1$, and $\mu_1 = s$. Lemma 2.4(i) or (ii) applies, and it can again be easily checked that $\eta' = \eta$, proving Step 1 with $f_0(s_1 \otimes u_\eta) = f_0(s_1) \otimes u_\eta$.

Next consider Step 2. The algorithm of always picking the smallest applicable f_h and applying it has the following effect on $s_1 \otimes u_\eta$. It raises the entries in the shortest columns of u_η and/or lowers the lowest minus signs in s_1 until f_0 is applicable. We are going to consider several cases:

- (a) No minus sign in s_1 moves to positions 1 and 2.
- (b) Only one minus sign in s_1 moves to position 1 or 2.
- (c) Two minus signs in s_1 move to positions 1 and 2.

First consider Case (a). Note that $\varepsilon_0(s_1) = 1$, so that the sequence of f_h must raise the entries in at least the two shortest columns in order to achieve $\varphi_0(s_1 \otimes u_\eta) > 0$. Let $h_1 \leq h_2 < r$ be the height of the two rightmost columns in η . Then the algorithm for Step 2 adds a vertical domino to the leftmost column of height h_2 in η and does

the same to $\bar{\lambda}$. On the other side $b_2 \otimes u_{\varpi_n}$ with $b_2 \in B(\mu)$, the subpart of b_2 of shape $\bar{\lambda}$ is a tableau with i in row i , whereas $\mu/\bar{\lambda}$ is filled with the letters n and \bar{n} . Since we only apply f_h with $1 \leq h \leq n - 2$, the letters n and \bar{n} never change in b_2 . By the correspondence between (η, λ) and μ as described before Example 5.4, the vertical strip $\mu/\bar{\lambda}$ has boxes at height h with $h_1 < h \leq h_2$ and b_2 contains letters n and \bar{n} in these cells. The steps in the algorithm add a vertical strip at height h_2 to $\bar{\lambda}$ and as a consequence push any letters n and/or \bar{n} at height $h_2 + 1$ and $h_2 + 2$ one column to the right. Hence, $\mu' = \mu + \epsilon_{h_2+1} + \epsilon_{h_2+2}$ which implies that the correspondence between (η', λ') and μ' also holds, proving the claim.

Next consider Case (b). Let $h_1 \leq h_2 < r$ again be the height of the two rightmost columns in η . The spin column s_1 must have precisely one minus sign in position $1 \leq h \leq h_2$, which is either in position h_1 or position h_2 . First consider that it is in position $h_2 > h_1$. Then the algorithm for Step 2 adds a vertical domino to the column of height h_1 in η and moves the minus sign in s_1 from position h_2 to position $h_1 + 2$. This means $\eta' = \eta + \epsilon_{h_1+1} + \epsilon_{h_1+2}$ and $\bar{\lambda}' = \bar{\lambda} + \epsilon_{h_2} + \epsilon_{h_1+1}$. On the $b_2 \otimes u_{\varpi_n}$ side, the vertical strip $\mu/\bar{\lambda}$ has boxes at heights $h_1 < h \leq h_2 - 2$ and h_2 . The algorithm for Step 2 on this side again changes $\bar{\lambda}' = \bar{\lambda} + \epsilon_{h_2} + \epsilon_{h_1+1}$. The letters n and \bar{n} in b_2 in the rightmost column move up one, and the letter \bar{n} at height h_2 moves from the second to last to last column. This implies $\eta' = \eta + \epsilon_{h_1+1} + \epsilon_{h_1+2}$ as before, proving the claim.

Next consider Case (b) with a minus in s_1 at height $h_1 \leq h_2 < r$. In this case, the algorithm for Step 2 adds a vertical domino at height h_2 to η and $\bar{\lambda}$, so that $\eta' = \eta + \epsilon_{h_2+1} + \epsilon_{h_2+2}$ and $\bar{\lambda}' = \bar{\lambda} + \epsilon_{h_2+1} + \epsilon_{h_2+2}$. On the $b_2 \otimes u_{\varpi_n}$ side, the vertical strip $\mu/\bar{\lambda}$ has boxes at heights $h_1 \leq h \leq h_2$. The algorithm for Step 2 also changes $\bar{\lambda}' = \bar{\lambda} + \epsilon_{h_2+1} + \epsilon_{h_2+2}$ and pushes any n and \bar{n} at height $h_2 + 1$ and h_2 one column to the right. This implies that indeed $\eta' = \eta + \epsilon_{h_2+1} + \epsilon_{h_2+2}$ as desired.

Finally consider Case (c). In this case, there must be at least two minus signs in s_1 in positions $1 \leq i \leq h_2$, where $h_1 \leq h_2 < r$ are again the heights of the rightmost columns in η . The algorithm for Step 2 removes the lowest two minus signs in s_1 . Let $i_1 < i_2$ be the positions of these two minus signs. Then $\bar{\lambda}' = \bar{\lambda} + \epsilon_{i_1} + \epsilon_{i_2}$ and $\eta' = \eta$. On the $b_2 \otimes u_{\varpi_n}$ side, the vertical strip $\mu/\bar{\lambda}$ has boxes at height $i_1 \leq i \leq i_2 - 2$ and i_2 if $i_1 \leq i_2 - 2$ and at height i_1 and $i_1 + 1$ if $i_2 = i_1 + 1$. The algorithm of Step 2 also changes $\bar{\lambda}' = \bar{\lambda} + \epsilon_{i_1} + \epsilon_{i_2}$. In addition, it moves the letters n and \bar{n} in b_2 up by one in the last column of b_2 and n at height i_2 one column to the right if $i_1 \leq i \leq i_2 - 2$. When $i_2 = i_1 + 1$, it moves the letters n and \bar{n} in the last column of b_2 up by two. Comparing μ' and $\bar{\lambda}'$ shows that indeed $\eta' = \eta$ as desired. □

Finally, we prepare the following proposition to prove the R -invariance. Let $u \in B^{n,1}$ be the I_0 -highest weight element. For $b \in B$, we define $\mathbb{T}(b)$ by $R(b \otimes u) = u' \otimes \mathbb{T}(b)$. Here we view $\mathbb{T}: B \rightarrow B$ as an operator by passing through u via the combinatorial R -matrix.

Proposition 5.8 *Suppose $r_2 \leq r_1$. Let $b_2 \otimes b_1 \in B^{r_2, s_2} \otimes B^{r_1, s_1}$ be an I_0 -highest weight element. Then there exists an integer N such that $\mathbb{T}^N(b_2 \otimes b_1)$ belongs to $B(s_2 \varpi_{r_2}) \otimes B(s_1 \varpi_{r_1})$ and the right factor is the I_0 -highest weight element. Moreover,*

- (i) *if $r_2 \leq n - 2$ and $r_1 \leq n - 1$, then the left factor does not contain barred letters;*

(ii) otherwise, the left factor is the I_0 -highest weight element.

Proof We first assume $r_1, r_2 \leq n - 2$. Suppose $b_2 \otimes b_1 \in B^{r_2, s_2} \otimes B^{r_1, s_1}$ is I_0 -highest weight. Then $b_1 = u_\mu$ where u_μ is the I_0 -highest weight element of $B(\mu) \subset B^{r_1, s_1}$. In view of the description of the combinatorial $R: B^{r_1, s_1} \otimes B^{n, 1} \rightarrow B^{n, 1} \otimes B^{r_1, s_1}$ in terms of the pairs (η, λ) as described in the proof of Proposition 5.7, successive applications of R take this u_μ to $u_{s_1 \varpi_{r_1}}$. Thus, we can assume $b_1 = u_{s_1 \varpi_{r_1}}$ and

$$e_i^{s_1 \delta_{i, r_1} + 1} b_2 = 0 \text{ for } i \in I_0. \tag{5.1}$$

We also know $R(u_{s_1 \varpi_{r_1}} \otimes u) = u \otimes u_{s_1 \varpi_{r_1}}$. So we consider $R(b_2 \otimes u)$ next. Consider $\text{high}(b_2 \otimes u)$. The left component may contain n, \bar{n} pairs. If so, $R: B^{r_2, s_2} \otimes B^{n, 1} \rightarrow B^{n, 1} \otimes B^{r_2, s_2}$ removes them. Therefore, we can assume that there do not exist any n, \bar{n} pairs. Once all n, \bar{n} pairs are removed, suppose $b_2 \in B(\lambda)$. Then the right component of $R(b_2 \otimes u)$ belongs to $B(\lambda')$, where λ' is strictly greater than λ in the dominance order. Hence, we can assume $\lambda = s_2 \varpi_{r_2}$. From (5.1), the first barred letter in the inverse column reading word of b_2 is \bar{r}_1 . Let $R(b_2 \otimes u) = c \otimes \tilde{b}_2 \in B^{n, 1} \otimes B^{r_2, s_2}$. Since c contains a $-$, the number of barred letters in \tilde{b}_2 is at least one smaller than that of b_2 . Therefore, the claim follows.

A similar argument when $r_2 \leq n - 2$ and $r_1 = n - 1$ finishes the proof of (i). Case (ii) is easier. □

Example 5.9 Consider type $D_8^{(1)}$, $B = B^{4,4} \otimes B^{6,3}$ and

$$b_2 \otimes b_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 7 & 4 \\ \hline 3 & 5 & & \\ \hline 4 & 8 & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 4 & 4 & 4 \\ \hline 5 & & \\ \hline 6 & & \\ \hline \end{array}.$$

Then we have

$$\mathbb{T}(b_2 \otimes b_1) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 7 \\ \hline 3 & 3 & 7 & \\ \hline 4 & 5 & 7 & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 4 & 4 & 4 \\ \hline 5 & 5 & \\ \hline 6 & 6 & \\ \hline \end{array}, \quad u' = (+, -, -, -, -, -, +, -),$$

$$\mathbb{T}^2(b_2 \otimes b_1) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 7 \\ \hline 4 & 6 & 7 & 6 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 4 & 4 & 4 \\ \hline 5 & 5 & 5 \\ \hline 6 & 6 & 6 \\ \hline \end{array}, \quad u' = (+, -, -, +, +, -, -, +),$$

$$\mathbb{T}^3(b_2 \otimes b_1) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 4 & 7 & 7 & 7 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 4 & 4 & 4 \\ \hline 5 & 5 & 5 \\ \hline 6 & 6 & 6 \\ \hline \end{array}, \quad u' = (+, +, -, +, +, +, -, +),$$

where u' is defined by $R(b \otimes u) = u' \otimes \mathbb{T}(b)$ with $b = b_2 \otimes b_1$. This illustrates Case (i) of Proposition 5.8.

Example 5.10 Consider type $D_8^{(1)}$, $B = B^{4,4} \otimes B^{8,3}$ and

$$b_2 \otimes b_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 8 \\ \hline 2 & 2 & 2 & 7 \\ \hline 3 & & & \\ \hline 4 & & & \\ \hline \end{array} \otimes u_{3\varpi_8}.$$

Then we have

$$\mathbb{T}(b_2 \otimes b_1) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & \\ \hline 4 & 4 & \\ \hline \end{array} \otimes u_{3\varpi_8}, \quad u' = (+, +, -, -, +, +, -, -),$$

$$\mathbb{T}^2(b_2 \otimes b_1) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & \\ \hline 4 & 4 & 4 & \\ \hline \end{array} \otimes u_{3\varpi_8}, \quad u' = (+, +, -, -, +, +, -, -),$$

$$\mathbb{T}^3(b_2 \otimes b_1) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 4 & 4 & 4 & 4 \\ \hline \end{array} \otimes u_{3\varpi_8}, \quad u' = (+, +, -, -, +, +, +, +).$$

This illustrates Case (ii) of Proposition 5.8.

The following property is one of the most profound properties of the rigged configuration bijection.

Theorem 5.11 (*R*-invariance) *Let $B = B^{r_k, s_k} \otimes \dots \otimes B^{r_2, s_2} \otimes B^{r_1, s_1}$ and let B' be an arbitrary reordering of the factors of B . For a given element $b \in B$, suppose that $b' \in B'$ is isomorphic to b under the combinatorial R -matrix:*

$$R: b \cong b'.$$

Then the corresponding unrestricted rigged configurations are invariant:

$$\Phi(b) = \Phi(b').$$

Proof We divide the proof into three steps.

Step 1 By Theorem 4.3, we can assume that b is an I_0 -highest weight element. Suppose that $b = b_k \otimes b_{k-1} \otimes b_{k-2} \otimes \cdots \otimes b_1 \in B$. By the recursive structure of the algorithm of the rigged configuration bijection Φ , we see that it is enough to consider the case $b' = \tilde{b}_{k-1} \otimes \tilde{b}_k \otimes b_{k-2} \otimes \cdots \otimes b_1$, where $R: b_k \otimes b_{k-1} \cong \tilde{b}_{k-1} \otimes \tilde{b}_k$. Then the relationship between the \diamond duality and the θ operation established in Proposition 4.1(7) shows that it is enough to consider the case $k = 2$. Therefore, we concentrate on the case when $b = b_2 \otimes b_1$ is I_0 -highest weight.

Step 2 Let $u \in B^{n,1}$ be the I_0 -highest weight element and $b = b_2 \otimes b_1 \in B$ be I_0 -highest weight. We shall show that

$$\Phi(b \otimes u) = \Phi(u' \otimes \mathbb{T}(b)), \tag{5.2}$$

where $R: b \otimes u \cong u' \otimes \mathbb{T}(b)$. By Proposition 5.7, we have

$$\Phi(b_2 \otimes b_1 \otimes u) = \Phi(b_2 \otimes u'' \otimes b'_1),$$

where $R: b_1 \otimes u \cong u'' \otimes b'_1$. Then by Proposition 4.1(7), we have

$$\Phi(\widehat{u} \otimes \widehat{b}_1 \otimes \widehat{b}_2) = \Phi(\widehat{b}'_1 \otimes \widehat{u}'' \otimes \widehat{b}_2),$$

where $(b_2 \otimes b_1 \otimes u)^\diamond = \widehat{u} \otimes \widehat{b}_1 \otimes \widehat{b}_2$ and $(b_2 \otimes u'' \otimes b'_1)^\diamond = \widehat{b}'_1 \otimes \widehat{u}'' \otimes \widehat{b}_2$. Since $\widehat{u}'' \in B^{n,1}$, again by Proposition 5.7 we have

$$\Phi(\widehat{b}'_1 \otimes \widehat{u}'' \otimes \widehat{b}_2) = \Phi(\widehat{b}'_1 \otimes \widehat{b}'_2 \otimes \widehat{u}'),$$

where $R: \widehat{u}'' \otimes \widehat{b}_2 \cong \widehat{b}'_2 \otimes \widehat{u}'$. Note that $(\widehat{b}'_1 \otimes \widehat{b}'_2 \otimes \widehat{u}')^\diamond = u' \otimes \mathbb{T}(b)$ since \diamond is involutive. Therefore by Proposition 4.1(7), we obtain (5.2).

Step 3 Let $b_2 \otimes b_1 \in B$ be I_0 -highest weight and $R(b_2 \otimes b_1) = \tilde{b}_1 \otimes \tilde{b}_2$. Then we have

$$b_2 \otimes b_1 \otimes u \simeq u' \otimes \mathbb{T}(b_2 \otimes b_1) \text{ and } \tilde{b}_1 \otimes \tilde{b}_2 \otimes u \simeq u' \otimes \mathbb{T}(\tilde{b}_1 \otimes \tilde{b}_2).$$

Thus by Step 2, we have

$$\Phi(b_2 \otimes b_1 \otimes u) = \Phi(u' \otimes \mathbb{T}(b_2 \otimes b_1)) \text{ and } \Phi(\tilde{b}_1 \otimes \tilde{b}_2 \otimes u) = \Phi(u' \otimes \mathbb{T}(\tilde{b}_1 \otimes \tilde{b}_2)).$$

Since $\Phi(b_2 \otimes b_1 \otimes u) = \Phi(\tilde{b}_1 \otimes \tilde{b}_2 \otimes u)$ is equivalent to $\Phi(b_2 \otimes b_1) = \Phi(\tilde{b}_1 \otimes \tilde{b}_2)$, the claim is reduced to showing $\Phi(\mathbb{T}(b_2 \otimes b_1)) = \Phi(\mathbb{T}(\tilde{b}_1 \otimes \tilde{b}_2))$. Thanks to Proposition 5.8, we are left to show $\Phi(b_2 \otimes b_1) = \Phi(\tilde{b}_1 \otimes \tilde{b}_2)$ when either $b_2 \otimes b_1$ or $\tilde{b}_1 \otimes \tilde{b}_2$ has the property described in the proposition. If $r_1, r_2 \leq n - 2$, this R -matrix is nothing but type A by [24, Proposition 9.1] and the R -invariance of type A case is already treated in [21, Lemma 8.5]. If $r_1 = n$, by Proposition 5.8 both factors are I_0 -highest weight and the images of Φ are empty. When $r_1 = n - 1$, then by symmetry we can interchange in all arguments $n - 1$ and n (including in the definition of \mathbb{T}), so that this case reduces to the $r_1 = n$ case. \square

Theorem 5.11 means that the rigged configuration bijection gives an explicit algorithm to compute the combinatorial R -matrices. It is noteworthy that the rigged configuration bijection gives not only the combinatorial R -matrices for twofold tensor products but also much more general reorderings of multiple tensor products by only computing Φ and Φ^{-1} . In the following, we give one such example.

Example 5.12 Consider the following element of $B = B^{3,3} \otimes B^{2,4} \otimes B^{2,2}$ of type $D_5^{(1)}$:

$$b = \begin{array}{|c|c|c|} \hline 1 & 1 & \bar{2} \\ \hline 2 & 2 & \\ \hline 5 & 5 & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & \bar{3} \\ \hline 2 & 3 & 4 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & 4 \\ \hline \end{array}.$$

The corresponding unrestricted rigged configuration is

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & 0 & & \\ \hline & 0 & & \\ \hline \end{array} -1 \quad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & -2 \\ \hline & & & -2 \\ \hline & & -1 & \\ \hline & & -1 & \\ \hline & & -1 & \\ \hline \end{array} -2 \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & 0 & \\ \hline & & & 0 & \\ \hline & & 1 & & \\ \hline & & 1 & & \\ \hline & & 1 & & \\ \hline \end{array} 3 \quad \begin{array}{|c|c|c|} \hline & & \\ \hline & 0 & \\ \hline & 0 & \\ \hline \end{array} 0 \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} -2.$$

From this unrestricted rigged configuration, we can obtain the image in $B' = B^{2,2} \otimes B^{2,4} \otimes B^{3,3}$ as follows:

$$b' = \begin{array}{|c|} \hline 1 \\ \hline 5 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & \bar{3} \\ \hline 2 & 2 & 5 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline 3 & 4 & 2 \\ \hline \end{array}.$$

Then by Theorem 5.11, we have $R : b \cong b'$. In this way, we can obtain the combinatorial R -matrices as $R = \Phi_{B'}^{-1} \circ \Phi_B$.

6 Energy function and fermionic formula

6.1 Involution ζ and energy function

In [23], the involution ζ on $B^{r,s}$ ($1 \leq r \leq n - 2, s \geq 1$) was shown to exist, which is the unique map satisfying

$$\zeta(e_i b) = e_{n-i} \zeta(b), \tag{6.1a}$$

$$\zeta(f_i b) = f_{n-i} \zeta(b), \tag{6.1b}$$

for any $i \in I, b \in B^{r,s}$. We need the following lemma later.

Lemma 6.1 For $u_\lambda \in B(\lambda) \subseteq B^{r,s}$, $\zeta(u_\lambda)$ belongs to $B(s\overline{\omega}_r)$ and is given as follows. To λ associate a tuple $\mu = (\lambda'_s, \lambda'_{s-1}, \dots, \lambda'_1, r, \dots, r)$ with $2s$ entries, where λ' is the transpose of the partition λ . Denote the j th entry of μ by μ_j . Let t be the KN tableau for $\zeta(u_\lambda)$. Then the k th column of t is given by

$$\overline{n - \mu_{2k-1} + 1} \cdots \overline{n - 1} \overline{n} \cdots \overline{n} \overline{n} (n - \mu_{2k}) \cdots (n - r + 2)(n - r + 1).$$

Proof Since u_λ is the unique element of $B^{r,s}$ of weight λ satisfying $e_i u_\lambda = 0$ for $i = 1, \dots, n$, the element $\zeta(u_\lambda)$ should be the unique element of $-w_0^{A_{n-1}} \lambda$ satisfying $e_i \zeta(u_\lambda) = 0$ for $i = 0, 1, \dots, n - 1$, where $w_0^{A_{n-1}}$ is the longest element of the Weyl group of $A_{n-1} \subseteq D_n$. One checks that t is an allowed KN tableau [16] of the desired weight. Hence, it is sufficient to show that $e_i t = 0$ for $i = 0, 1, \dots, n - 1$. For $i = 1, \dots, n - 1$, this is immediate. To see $e_0 t = 0$, calculate the $\{2, \dots, n\}$ -highest weight element in the same component as t , which is the tableau whose columns are all $23 \cdots r + 1$, and hence $\sigma(t) = t$ with σ as in (2.8). In view of the definition of e_0 given in (2.10), we have $e_0 t = 0$, since $e_1 t = 0$. \square

We need an involution ζ also on $B^{n-1,s}$ and $B^{n,s}$. When n is odd, it becomes a map from $B^{n-1,s}$ to $B^{n,s}$ and vice versa. To incorporate this situation, we introduce η defined to be 0 if n is even and 1 otherwise. Then there exists a map $\zeta : B^{n-1,s} \rightarrow B^{n-1+\eta,s}$ (resp., $B^{n,s} \rightarrow B^{n-\eta,s}$) satisfying (6.1). In fact, it is characterized by (6.1) and defining the image of $u_{s\overline{\omega}_{n-1}}$ (resp., $u_{s\overline{\omega}_n}$) to be the unique element in $B^{n-1+\eta,s}$ (resp., $B^{n-\eta,s}$) of weight $-w_0^{A_{n-1}}(s\overline{\omega}_{n-1})$ (resp., $-w_0^{A_{n-1}}(s\overline{\omega}_n)$).

For a multiple tensor product of KR crystals $B = B^{r_k, s_k} \otimes \cdots \otimes B^{r_1, s_1}$, we define

$$\zeta(b) = \zeta(b_k) \otimes \cdots \otimes \zeta(b_1)$$

for $b = b_k \otimes \cdots \otimes b_1 \in B$

Proposition 6.2 For any I_0 -highest weight element u_λ of a single KR crystal $B^{r,s}$, we have $(rs \circ \zeta)(u_\lambda) = (\zeta \circ rs)(u_\lambda)$.

Proof The case when $r = n - 1$ or n is straightforward. Both $rs \circ \zeta$ and $\zeta \circ rs$ send $u_{s\overline{\omega}_{n-1}}$ (resp., $u_{s\overline{\omega}_n}$) to $v_{-(s-1)w_0\overline{\omega}_{n-1}} \otimes v_{-w_0\overline{\omega}_{n-1}}$ (resp., $v_{-(s-1)w_0\overline{\omega}_n} \otimes v_{-w_0\overline{\omega}_n}$) where w_0 is the longest element of $A_{n-1} \subseteq D_n$ and v_τ stands for the unique element in the crystal of weight τ .

Suppose $r \leq n - 2$. In this case, $(rs \circ \zeta)(u_\lambda)$ is given by cutting the rightmost column of $\zeta(u_\lambda)$. From Proposition 2.6, we see $rs(u_\lambda) = t \otimes u_{\overline{\omega}_r}$, where the tableau t is described in the proposition. We use all the notation for t from Proposition 2.6. Hence, $(\zeta \circ rs)(u_\lambda) = \zeta(t) \otimes \zeta(u_{\overline{\omega}_r})$. By Lemma 6.1, the right components of $(rs \circ \zeta)(u_\lambda)$ and $(\zeta \circ rs)(u_\lambda)$ agree. We prove that the left components also agree by decreasing induction on p , where p is as in Proposition 2.6 for t . If $p = r$, the claim is true, since in this case t is I_0 -highest weight and the agreement can be shown by Lemma 6.1. Suppose $p < r$. Let t' be the tableau that differs from t only in the first column with $\overline{p + 1} \cdots \overline{r - 1} \overline{r} \cdots \overline{21}$ replaced by $\overline{p + 3} \cdots \overline{r - 1} \overline{r} \cdots \overline{21}$. These

tableaux are connected in the crystal graph as $t' = e_{\mathbf{a}_2} f_0 f_{\mathbf{a}_1} t$ where

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{b}(2, 1 + \mu_{\geq 2}) \cdots \mathbf{b}(p - 2, 1 + \mu_{\geq p-2}) \mathbf{b}(p, 1 + \mu_{\geq p}), \\ \mathbf{a}_2 &= \mathbf{b}(p, \mu_{\geq p}) \cdots \mathbf{b}(4, \mu_{\geq 4}) \mathbf{b}(2, \mu_{\geq 2}), \end{aligned}$$

$\mathbf{b}(j, a) = j^a (j - 1)^a (j + 1)^a j^a$ and $\mu_{\geq c} = \sum_{j=c}^{\ell(\mu)} \mu_j$. (See (2.9) for the definition of $e_{\mathbf{a}}$. The definition of $f_{\mathbf{a}}$ is similar.) In fact, the tableau $f_{\mathbf{a}_1} t$ has the following form. The first column is

$$\overline{1} \overline{2} \overline{p} + \overline{3} \cdots \overline{r} - \overline{1} \overline{r} (q + 2) \cdots 43,$$

and the entries in the other part of t are shifted by 2. Thus, f_0 removes $\overline{1} \overline{2}$ from the first column.

Since $\zeta(e_{\mathbf{a}_2} f_0 f_{\mathbf{a}_1} t) = e_{n-\mathbf{a}_2} f_n f_{n-\mathbf{a}_1} t$, where for $\mathbf{a} = a_1 \cdots a_m$, $n - \mathbf{a}$ stands for $(n - a_1) \cdots (n - a_m)$, we are left to show that $e_{n-\mathbf{a}_2} f_n f_{n-\mathbf{a}_1} t$ agrees with the first $s - 1$ columns of $\zeta(u_\lambda)$ of Lemma 6.1 with λ'_1 replaced by $\lambda'_1 + 2$. This can be checked directly. \square

In Sect. 2.6, we defined the combinatorial R -matrix $R: B^{r_2, s_2} \otimes B^{r_1, s_1} \rightarrow B^{r_1, s_1} \otimes B^{r_2, s_2}$. In addition, there exists a function $\overline{H}: B^{r_2, s_2} \otimes B^{r_1, s_1} \rightarrow \mathbb{Z}$, called the *local energy function*, unique up to a global additive constant. It is constant on I_0 -components and satisfies for all $b_2 \in B^{r_2, s_2}$ and $b_1 \in B^{r_1, s_1}$ with $R(b_2 \otimes b_1) = b'_1 \otimes b'_2$

$$\overline{H}(e_0(b_2 \otimes b_1)) = \overline{H}(b_2 \otimes b_1) + \begin{cases} 1 & \text{if } \varepsilon_0(b_2) > \varphi_0(b_1) \text{ and } \varepsilon_0(b'_1) > \varphi_0(b'_2), \\ -1 & \text{if } \varepsilon_0(b_2) \leq \varphi_0(b_1) \text{ and } \varepsilon_0(b'_1) \leq \varphi_0(b'_2), \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

We shall normalize the local energy function by the condition $\overline{H}(u_{s_2 \varpi_{r_2}} \otimes u_{s_1 \varpi_{r_1}}) = 0$.

For a crystal $B^{r, s}$, the *intrinsic energy* $\overline{D}_{B^{r, s}}: B^{r, s} \rightarrow \mathbb{Z}$ is defined as follows. When $r \leq n - 2$, we define $\overline{D}_{B^{r, s}}(b) = (rs - |\lambda|)/2$ for $b \in B(\lambda) \subseteq B^{r, s}$ (see (2.6)). When $r = n - 1, n$, there is only one I_0 -component (see (2.7)) and we set $\overline{D}_{B^{r, s}}(b) = 0$ for $b \in B^{r, s}$. On the tensor product $B = B_k \otimes \cdots \otimes B_1$ of single KR crystals B_j , there is an intrinsic energy function $\overline{D}_B: B \rightarrow \mathbb{Z}$ defined in [9, (3.8)] and [33, Section 2.14] by

$$\overline{D}_B = \sum_{1 \leq i < j \leq k} \overline{H}_i R_{i+1} R_{i+2} \cdots R_{j-1} + \sum_{j=1}^k \overline{D}_{B_j} \pi_1 R_1 R_2 \cdots R_{j-1}. \tag{6.3}$$

Here R_i and \overline{H}_i denote the combinatorial R -matrix and local energy function, respectively, acting on the i th and $(i + 1)$ th tensor factors counting from the right. Also π_1 is the projection onto the rightmost tensor factor. The intrinsic energy \overline{D}_B is constant on any I_0 -component of B . Let B' be the same tensor product of KR crystals as B except that the i th and $(i + 1)$ th positions are interchanged. Let $b' = b'_k \otimes \cdots \otimes b'_1 \in B'$ be such that $b'_{i+1} \otimes b'_i = R(b_{i+1} \otimes b_i)$ and $b'_j = b_j$ for $j \neq i, i + 1$. Then we

have $\overline{D}_B(b) = \overline{D}_{B'}(b')$. Similarly, let B' be any reordering of the tensor factors of B . The image b' of b under the compositions of the combinatorial R -matrices does not depend on the order of applications due to the Yang–Baxter equation, and we also have $\overline{D}_B(b) = \overline{D}_{B'}(b')$, see [9, Prop. 3.9] and [33, Prop. 2.15]. In what follows we often drop the subscript B from $\overline{D}_B(b)$.

In [23,24], we proved the following formula on the intrinsic energy \overline{D} under the assumption that the KR crystals B^{r_j, s_j} appearing in B satisfy $r_j \leq n - 2$.

Proposition 6.3 ([23,24]) *For an I_0 -highest weight element b in B , we have*

$$\overline{D}(b) = \overline{D}(\zeta(b)) + \frac{|B| - |\lambda(b)|}{2}, \tag{6.4}$$

where $|B| = \sum_{j=1}^k r_j s_j$, $\lambda(b)$ is the weight of b and $|\mu| = \sum_{j=1}^n \mu_j$ if $\mu = \sum_{j=1}^n \mu_j \epsilon_j$.

However, if one extends the definition of $|B|$ suitably and examines the proof of [23, Prop. 3.1] carefully, one understands that it is valid even when B contains $B^{n-1, s}$ or $B^{n, s}$. In fact, setting $|B| = \sum_{j=1}^k |B_j|$ where

$$|B^{r, s}| = |s\varpi_r| = \begin{cases} rs & \text{if } r \leq n - 2, \\ (n - 2)s/2 & \text{if } r = n - 1, \\ ns/2 & \text{if } r = n, \end{cases}$$

one checks that the above proposition remains true. When the weight $\lambda(b)$ of an element $b \in B$ satisfies $|\lambda(b)| = |B|$, we call b maximal. Note that if $b = b_k \otimes \dots \otimes b_1$ is maximal, then b_j is maximal for any j . If $b \in B^{r, s}$ is a maximal I_0 -highest weight element, then $b = u_{s\varpi_r}$. Recall that $\text{high}(b)$ is the I_0 -highest weight element in the same I_0 -component as b .

Lemma 6.4 *We use the notation $b(\alpha)$ in Lemma 2.2 indicating the dependences of r, s explicitly as $b^{r, s}(\alpha)$. If $r \leq n - 1$ and $\alpha \leq s$, then we have*

$$\overline{H}(b^{n-1, s'}(\alpha) \otimes u_{s\varpi_r}) = \alpha.$$

Proof Suppose $2 \leq r \leq n - 2$. By weight consideration, we see $R(b^{n-1, s'}(\alpha) \otimes u_{s\varpi_r}) = b^{r, s}(\alpha) \otimes u_{s'\varpi_{n-1}}$. Using Lemma 2.2 and (6.2), one finds

$$\overline{H}(b^{n-1, s'}(\alpha) \otimes u_{s\varpi_r}) - \alpha = \overline{H}(e_0^{\max}(b^{n-1, s'}(\alpha) \otimes u_{s\varpi_r})).$$

Since $e_0^{\max}(b^{n-1, s'}(\alpha) \otimes u_{s\varpi_r})$ belongs to the same I_0 -component as $u_{s'\varpi_{n-1}} \otimes u_{s\varpi_r}$, we obtain the desired result.

The cases $r = 1$ or $n - 1$ are similar. □

Proposition 6.5 *Suppose the rightmost factor of B is a KR crystal $B^{r,s}$ such that $s \geq 2$. For an I_0 -highest weight element $b \in B$, we have*

$$\overline{D}(b) = \overline{D}(\text{rs}(b)). \tag{6.5}$$

Proof First we reduce the proposition to the case when b maximal. Substituting $\text{rs}(b)$ into b in (6.4), using Proposition 6.2 and noting that rs preserves the weight, we obtain $\overline{D}(b) - \overline{D}(\text{rs}(b)) = \overline{D}(\zeta(b)) - \overline{D}(\text{rs}(\zeta(b)))$. Since rs commutes with e_i, f_i for $i \in I_0$ and \overline{D} is constant on I_0 -components, we have

$$\overline{D}(b) - \overline{D}(\text{rs}(b)) = \overline{D}(\text{high} \circ \zeta(b)) - \overline{D}(\text{rs}(\text{high} \circ \zeta(b))).$$

From Proposition 6.3, we know that applying $\text{high} \circ \zeta$ finitely many times sends any b to a maximal element. Hence, we have reduced the proposition to showing (6.5) for all maximal elements of B .

Next we reduce the proposition to showing a certain relation for the energy function \overline{H} . Suppose b is a maximal I_0 -highest weight element in B . Then the rightmost factor b_1 of b is also an I_0 -highest weight element and maximal. Hence, $b_1 = u_{s\varpi_r}$. Denote it simply by u . Let $\text{rs}(u) = u_1 \otimes u_0 \in B^{r,s-1} \otimes B^{r,1}$, then $u_1 = u_{(s-1)\varpi_r}, u_0 = u_{\varpi_r}$, as in Sect. 2.5. So we have

$$\begin{aligned} b &= b_k \otimes \cdots \otimes b_2 \otimes u, \\ \text{rs}(b) &= b_k \otimes \cdots \otimes b_2 \otimes u_1 \otimes u_0. \end{aligned}$$

For consistency, we regard the rightmost factor of $\text{rs}(b)$ as the 0th and the left adjacent one as the first factor. Define $u_1^{(0)} = \pi_0 R_0(u_1 \otimes u_0)$ and for $j \geq 2$

$$b_j^{(0)} = \pi_0 R_0 R_1 \cdots R_{j-1}(\text{rs}(b)), \quad b_j^{(i)} = \pi_i R_i R_{i+1} \cdots R_{j-1}(b) \text{ for } i = 1, 2.$$

For the definition of R_i and π_i , see the explanations after (6.3). With this notation, one calculates

$$\overline{D}(\text{rs}(b)) - \overline{D}(b) = \sum_{j=2}^k \left(\overline{H}(b_j^{(2)} \otimes u_1) + \overline{H}(\tilde{b}_j^{(2)} \otimes u_0) - \overline{H}(b_j^{(2)} \otimes u) \right).$$

Here $\tilde{b}_j^{(2)}$ is the right component of $R(b_j^{(2)} \otimes u_1)$, and we have used

$$\begin{aligned} u_1^{(0)} &= u_{(s-1)\varpi_r} \\ b_j^{(0)} &= b_j^{(1)} = u_{s_j \varpi_{r_j}} && \text{for } b_j \in B^{r_j, s_j}, \\ \overline{H}(u_1 \otimes u_0) &= 0 = \overline{D}(u_{s' \varpi_{r'}}) && \text{for } u_{s' \varpi_{r'}} \in B^{r', s'}. \end{aligned}$$

Finally, suppose $b \otimes u_{s\varpi_r}$ is a maximal I_0 -highest weight element of $B^{r',s'} \otimes B^{r,s}$ and let \tilde{b} be the right factor of $R(b \otimes u_{(s-1)\varpi_r})$. To complete the proof, it is sufficient to show that

$$\overline{H}(b \otimes u_{(s-1)\varpi_r}) + \overline{H}(\tilde{b} \otimes u_{\varpi_r}) = \overline{H}(b \otimes u_{s\varpi_r}). \tag{6.6}$$

We show this by dividing into the following cases:

- (i) $r, r' \leq n - 2$, (ii) $r \leq n - 2, r' = n$, (iii) $r \leq n - 2, r' = n - 1$,
- (iv) $r = r' = n$, (v) $r = n, r' = n - 1$, (vi) $r = r' = n - 1$.

For (i), recall that the value of the energy function for maximal elements for type $D_n^{(1)}$ is equal to that for type $A_{n-1}^{(1)}$ [24, Section 9]. Since (6.6) is true for type $A_{n-1}^{(1)}$, the proof is done for this case.

For (ii), since b is maximal, it should agree with $u_{s'\varpi_{r'}}$ and also $\tilde{b} = u_{s'\varpi_{r'}}$. Hence, both sides of (6.6) are 0. (iv) and (v) are similar.

For (iii), suppose $r \geq 3$ first. The possibilities for b are $\hat{b}(\alpha)$ for some $0 \leq \alpha \leq \min(s, s')$, where $\hat{b}(\alpha)$ is the tableau whose left $(s' - \alpha)$ half columns are $12 \dots (n - 1)\bar{n}$ and right α half columns are $1 \dots (r - 1)(r + 1) \dots n\bar{r}$. Suppose $\alpha \leq s - 1$. If $b = \hat{b}(\alpha)$, then $\tilde{b} = \hat{b}(0)$. Since $f_{\mathbf{a}}(\hat{b}(\alpha) \otimes u_{s\varpi_r}) = b^{n-1,s'}(\alpha) \otimes u_{s\varpi_r}$, where $\mathbf{a} = (2^\alpha 3^\alpha \dots (r - 1)^\alpha)$, we have $\alpha + 0 = \alpha$ for (6.6) from Lemma 6.4. Now suppose $\alpha = s$. If $b = \hat{b}(s)$, then $\tilde{b} = \hat{b}(1)$. Noting that $f_{(n-1)\dots(r+1)r}(\hat{b}(s) \otimes u_{(s-1)\varpi_r}) = \hat{b}(s - 1) \otimes u_{(s-1)\varpi_r}$, we have $(\alpha - 1) + 1 = \alpha$ for (6.6) this time. The cases $r = 1, 2$ are similar. (vi) is also similar. \square

6.2 Preservation of statistics

Similar to the intrinsic energy, there is a statistic called *cocharge* on the set of rigged configurations given by $cc(v, J) = cc(v) + \sum_{a,i} |J^{(a,i)}|$, where

$$cc(v) = \frac{1}{2} \sum_{a,b \in J} \sum_{j,k \geq 1} (\alpha_a | \alpha_b) \min(j, k) m_j^{(a)} m_k^{(b)}. \tag{6.7}$$

Let $\tilde{\Phi} = \theta \circ \Phi$, where $\theta: \mathcal{RC}(L, \lambda) \rightarrow \mathcal{RC}(L, \lambda)$ with $\theta(v, J) = (v, \tilde{J})$ being the function that complements the riggings, meaning that \tilde{J} is obtained from J by complementing all partitions $J^{(a,i)}$ in the $m_i^{(a)} \times p_i^{(a)}$ rectangle.

Theorem 6.6 *Let $B = B^{r_k, s_k} \otimes \dots \otimes B^{r_1, s_1}$ be a tensor product of KR crystals and λ a dominant integral weight. The bijection $\tilde{\Phi}: \mathcal{P}(B, \lambda) \rightarrow \mathcal{RC}(L(B), \lambda)$ preserves the statistics, that is $\overline{D}(b) = cc(\tilde{\Phi}(b))$ for all $b \in \mathcal{P}(\lambda, B)$.*

Proof We define a map

$$\text{col}: B = B^{r_k, s_k} \otimes \dots \otimes B^{r_1, s_1} \longrightarrow (B^{r_k, 1})^{\otimes s_k} \otimes \dots \otimes (B^{r_1, 1})^{\otimes s_1}$$

as follows. If $s_1 > 1$, apply rs to B first. Exchange B^{r_1, s_1-1} and $B^{r_1, 1}$ by using the combinatorial R -matrix so that we have $B^{r_1, 1} \otimes B^{r_1, s_1-1}$ at the rightmost two factors. If $s_1 - 1 > 1$, apply again rs and R so that we have $(B^{r_1, 1})^{\otimes 2} \otimes B^{r_1, s_1-2}$ at the rightmost three factors. Continuing this procedure, we have $(B^{r_1, 1})^{\otimes s_1}$ at the rightmost s_1 factors. Next applying a sequence of combinatorial R -matrices, bring B^{r_2, s_2} to the rightmost factor. We then apply a similar procedure to the rightmost B^{r_2, s_2} and bring $(B^{r_2, 1})^{\otimes s_2}$ back to the original position. At this point, we have $(B^{r_2, 1})^{\otimes s_2} \otimes (B^{r_1, 1})^{\otimes s_1}$ at the rightmost $(s_1 + s_2)$ factors. Continuing these, we arrive at the image of col .

Since rs and R commute with e_i, f_i ($i \in I_0$), col restricts to the map from $\mathcal{P}(B, \lambda)$ to $\mathcal{P}(\text{col}(B), \lambda)$. By Proposition 6.5 and the invariance of \overline{D} by the combinatorial R -matrix, we have $\overline{D}(b) = \overline{D}(\text{col}(b))$ for $b \in \mathcal{P}(B, \lambda)$.

Finally, note that for $(v, J) \in \mathcal{RC}(L(B), \lambda)$ we have $cc(v, J) = cc(\gamma(v, J))$. In view of $\tilde{\Phi} \circ rs = \gamma \circ \tilde{\Phi}$ and Theorem 5.11, the proof is reduced to the case when all the components of B are of column type $B^{r_j, 1}$. The corresponding statement is proven in [38, Theorem 5.1]. \square

An immediate corollary of Theorem 4.2 and Theorem 6.6 is the equality

$$\sum_{b \in \mathcal{P}(B, \lambda)} q^{\overline{D}(b)} = \sum_{(v, J) \in \mathcal{RC}(L(B), \lambda)} q^{cc(v, J)}.$$

The left-hand side is in fact the one-dimensional configuration sum

$$X(B, \lambda) = \sum_{b \in \mathcal{P}(B, \lambda)} q^{\overline{D}(b)}.$$

The right-hand side can be simplified slightly by observing that the generating function of partitions in a box of width p and height m is the q -binomial coefficient

$$\begin{bmatrix} m + p \\ m \end{bmatrix} = \frac{(q)_{p+m}}{(q)_m (q)_p},$$

where $(q)_m = (1 - q)(1 - q^2) \cdots (1 - q^m)$. Hence, the right-hand side becomes the *fermionic formula*

$$M(B, \lambda) = \sum_{v \in C(L(B), \lambda)} q^{cc(v)} \prod_{\substack{i \geq 1 \\ a \in I_0}} \begin{bmatrix} m_i^{(a)} + P_i^{(a)} \\ m_i^{(a)} \end{bmatrix},$$

where $m_i^{(a)}$ and $P_i^{(a)}$ are as defined in Sect. 3. This proves the following result conjectured in [8, 9].

Corollary 6.7 *Let $B = B^{r_k, s_k} \otimes \dots \otimes B^{r_1, s_1}$ be a tensor product of KR crystals of type $D_n^{(1)}$ and λ a dominant integral weight. Then we have*

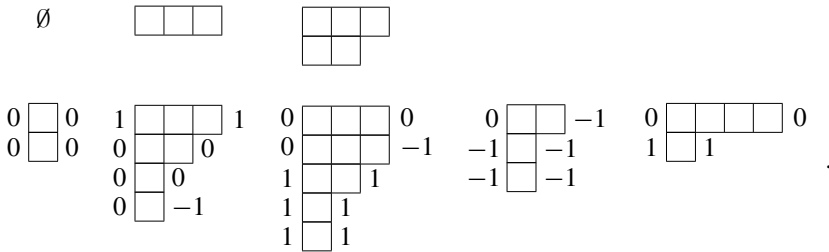
$$X(B, \lambda) = M(B, \lambda).$$

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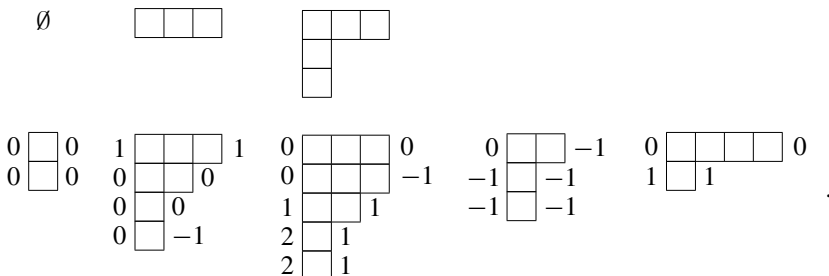
7 Appendix: Example of the rigged configuration bijection

In this section, we provide an example of the algorithm Φ^{-1} . The reader may easily infer from the following example the meaning of the correspondence between the operators summarized in Table 1.

Example 7.1 Consider the unrestricted rigged configuration $r_1 \in RC(L(B))$ for $B = B^{3,2} \otimes B^{3,3} \otimes B^{2,3}$ of type $D_5^{(1)}$:



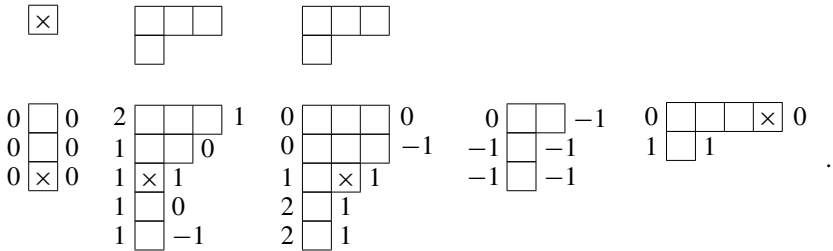
In the above diagram, we show the partition $\mu^{(a)}$ as defined in Sect. 3.1 over the corresponding rigged partition $(v^{(a)}, J^{(a)})$ in order to make it easier to see the operations γ and β . In Φ^{-1} , let us remove the $B^{3,2}$ part first. We begin by applying γ and obtain $r_2 := \gamma(r_1)$ which looks as follows:



The changes are the shape of $\mu^{(3)}$ and the resulting change of the vacancy numbers for $P_1^{(3)}(v)$ which makes the length 1 strings of $(v, J)^{(3)}$ non-singular.¹ This operation corresponds to

$$\text{ls: } B^{3,2} \longrightarrow B^{3,1} \otimes B^{3,1}. \tag{7.1}$$

Then $r_3 := \beta(r_2)$ looks as follows:

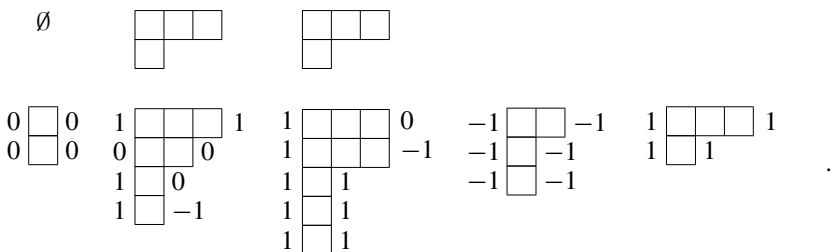


This corresponds to

$$\text{lb: } B^{3,1} \longrightarrow B^{1,1} \otimes B^{2,1}. \tag{7.2}$$

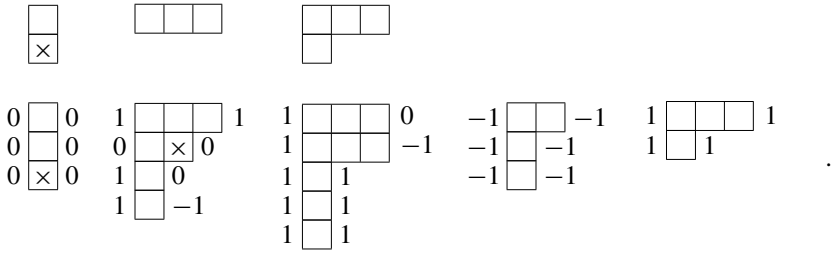
Note that the vacancy numbers for $v^{(3)}$ do not change. Since β adds length 1 singular strings to $(v, J)^{(1)}$ and $(v, J)^{(2)}$, applying δ removes the boxes with “x” in the above diagram.² Then δ gives the following rigged configuration $r_4 := \delta(r_3)$ together with the

output letter $\bar{5}$ which fills the bottom left corner of $B^{3,2}$ as



Since the above δ determines $B^{1,1}$ of (7.2), we start to apply $\gamma, \beta,$ and δ corresponding to $B^{2,1}$ of (7.2). Since $\gamma(r_4) = r_4$, the unrestricted rigged configuration $r_5 := \beta(r_4)$ looks as follows:

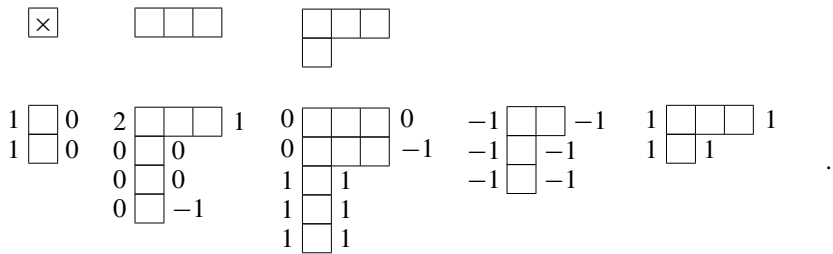
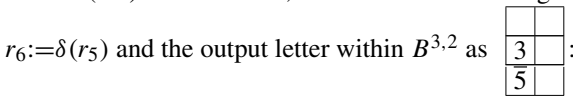
¹ In general, if we consider γ for $B^{r,s}$, the strings in $(v, J)^{(r)}$ which are shorter than s become non-singular.
² In general, if we consider β for $B^{r,1}$, the next δ removes length 1 singular strings of $(v, J)^{(1)}, (v, J)^{(2)}, \dots, (v, J)^{(r-1)}$ added by β .



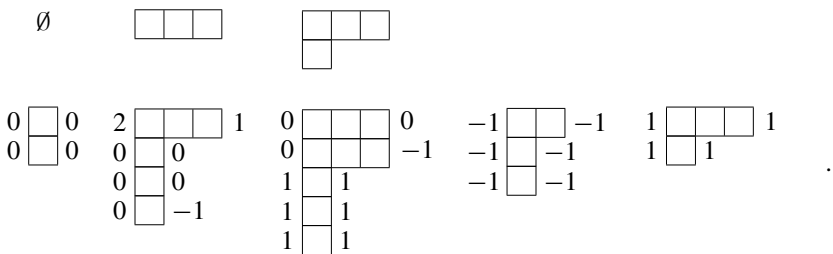
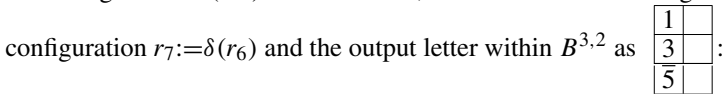
This corresponds to

$$lb : B^{2,1} \longrightarrow B^{1,1} \otimes B^{1,1}. \tag{7.3}$$

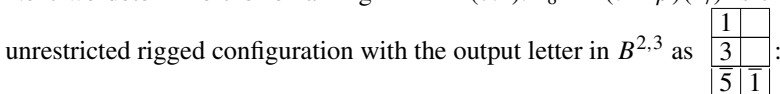
Then δ removes the boxes with “ \times ” in the above diagram which determines one of $B^{1,1}$ in (7.3). As the result, we obtain the following unrestricted rigged configuration

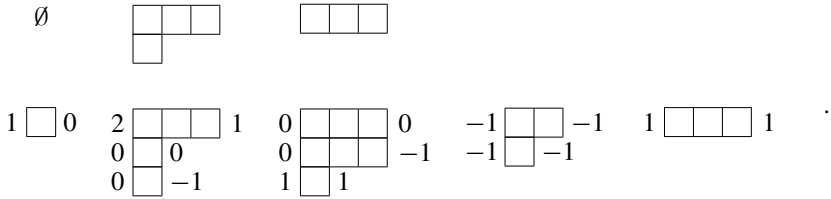


The next δ removes the box with “ \times ” in the above diagram and determines the remaining $B^{1,1}$ in (7.3). As the result, we obtain the following unrestricted rigged

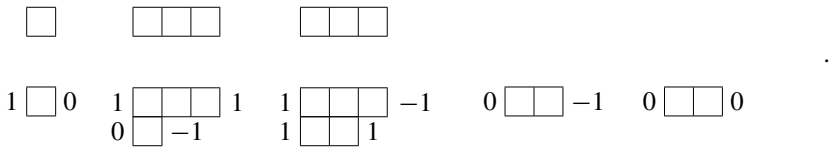


Next we determine the remaining $B^{3,1}$ in (7.1). $r_8 = (\delta \circ \beta)(r_7)$ is the following

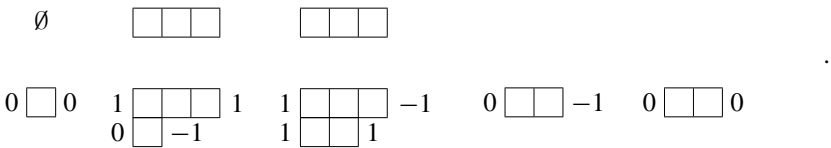




$r_9 = (\delta \circ \beta)(r_8)$ is the following unrestricted rigged configuration with the output letter in $B^{2,3}$ as $\begin{matrix} \overline{1} & \square \\ \overline{3} & \overline{3} \\ \overline{5} & \overline{1} \end{matrix}$:



$r_{10} = \delta(r_9)$ is the following unrestricted rigged configuration with the output letter in $B^{2,3}$ as $\begin{matrix} \overline{1} & \overline{1} \\ \overline{3} & \overline{3} \\ \overline{5} & \overline{1} \end{matrix}$:



For the KN tableau representation of the rectangular tableau, we need to apply the inverse of the filling map of Definition 2.5 to obtain $\begin{matrix} \overline{1} & \overline{3} \\ \overline{3} & \square \\ \overline{5} & \square \end{matrix}$. If we further determine $B^{3,3}$ and $B^{2,3}$ in this order, we obtain the empty rigged configuration and the following path

$$\Phi^{-1}(r_1) = \begin{matrix} \overline{1} & \overline{3} \\ \overline{3} & \square \\ \overline{5} & \square \end{matrix} \otimes \begin{matrix} \overline{1} & \overline{1} & \overline{1} \\ \overline{2} & \overline{3} & \overline{4} \\ \overline{3} & \overline{5} & \overline{3} \end{matrix} \otimes \begin{matrix} \overline{1} & \overline{1} & \overline{3} \\ \overline{2} & \overline{2} & \overline{5} \end{matrix}.$$

Summary As we see in the above example, the algorithm Φ^{-1} is recursively defined as follows. Suppose that we consider the unrestricted rigged configuration (ν, J) associated with the tensor product of type

$$B = B^{r,s} \otimes B'.$$

Then we determine $b \in B^{r,s}$ by the following procedure. In correspondence with the operation

$$\text{ls}: B^{r,s} \longrightarrow B^{r,1} \otimes B^{r,s-1}, \tag{7.4}$$

we apply γ to the unrestricted rigged configuration. Then apply lb on $B^{r,1}$ in (7.4)

$$\text{lb}: B^{r,1} \longrightarrow B^{1,1} \otimes B^{r-1,1}, \tag{7.5}$$

which on the rigged configuration side corresponds to β . Finally we use lh and δ to remove $B^{1,1}$ of (7.5). In order to process the remaining $B^{r-1,1}$ of (7.5), we apply β and δ repeatedly for $(r - 1)$ -times. We fill the leftmost column of $B^{r,s}$ from bottom to top by the letters obtained by δ during this procedure. In order to determine the remaining $B^{r,s-1}$ of (7.4), we repeat the same procedure for the first column and fill the remaining columns of $B^{r,s}$ from left to right. Once $B^{r,s}$ is fully determined, we proceed to the leftmost rectangle of B' and repeat the same procedure used for $B^{r,s}$. In this manner, we obtain the filling of shape B , which we denote by $b \in B$. Then we define

$$\Phi^{-1} : (v, J) \longmapsto b.$$

The inverse procedure Φ is obtained by reversing all the steps of Φ^{-1} .

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