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Pieri and Littlewood–Richardson rules for two rows and cluster algebra structure

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Abstract We study an algebra encoding a twice-iterated Pieri rule for the representations of the general linear group and prove that it has the structure of a cluster algebra. We also show that its cluster variables invariant under a unipotent subgroup generate the highest weight vectors of irreducible representations occurring in the decomposition of the tensor product of two irreducible representations of the general linear group one of whom is labeled by a Young diagram with less than or equal to two rows.

Keywords Cluster algebra · Highest weight vectors · Tensor product decomposition · Pieri rules · Littlewood–Richardson rules · Branching rules

Mathematics Subject Classification 05E10 · 20G05 · 13F50 · 13F60

1 Introduction

1.1 Main results

Let us consider the complex general linear group GL_d, the group of $d \times d$ invertible matrices over the complex number field $\mathbb{C}$, and its maximal unipotent subgroup U_d consisting of unit upper triangular matrices. For a Young diagram λ, we write V_d^λ for the irreducible polynomial representation of GL_d labeled by λ and write $\ell(\lambda)$

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for the number of rows in λ. If λ has only one row of length k, then we often write $\lambda = (k)$. Note that the symmetric power $Sym^k\mathbb{C}^d$ of $\mathbb{C}^d$ can be taken as an irreducible representation of GL_d labeled by the Young diagram (k).

The Pieri rule in representation theory describes the decomposition of the tensor product $V_d^\lambda \otimes V_d^{(k)}$ and also, using the reciprocity explained in [10,13], the decomposition of an irreducible representation V_d^λ of GL_d as a representation of its subgroup GL_{d-1}. See, for example, [6,7,18].

In this paper, we study an algebra, called the *twice-iterated Pieri algebra* and denoted by $\mathcal{A}_{n,m}$, encoding the Pieri rule iterated twice. We investigate its explicit cluster algebra structure and relation with the tensor product decomposition problem.

Main Theorem (1) *The twice-iterated Pieri algebra $\mathcal{A}_{n,m}$ admits the structure of a cluster algebra of type A_r where $r = m - 1$ if $m \leq n$ and $r = n$ if $m \geq n + 1$.*

(2) *The cluster variables invariant under $U_2 \subset GL_2$ generate the highest weight vectors of the isomorphic copies of V_m^λ occurring in the decomposition*

$$V_m^\mu \otimes V_m^\nu = \bigoplus_\lambda \left(V_m^\lambda\right)^{\oplus c_{\mu\nu}^\lambda} \text{ where } \ell(\mu) \leq \min(n, m) \text{ and } \ell(\nu) \leq 2. \tag{1.1}$$

The case (1.1) is important in the tensor product decomposition problem in that, by applying the decomposition rule for this case repeatedly, one can obtain the full description of the celebrated Littlewood–Richardson rule for all polynomial representations of GL_m. See [11, §§6-7].

1.2 Twice-iterated Pieri algebra

In Sect. 2, using classical invariant theory, we construct and study the twice-iterated Pieri algebra $\mathcal{A}_{n,m}$, which provides collective descriptions of (i) the decomposition of the tensor products of the form

$$V_m^\mu \otimes V_m^{(a)} \otimes V_m^{(b)} = \bigoplus_\lambda \left(V_m^\lambda\right)^{\oplus p(\lambda,\mu,a,b)} \tag{1.2}$$

for Young diagrams μ with less than or equal to $\min(n, m)$ rows and nonnegative integers a and b and (ii) the restriction of GL_{n+2} irreducible representations to the subgroup GL_{n+1} and further to GL_n

$$V_{n+2}^\lambda \downarrow_{GL_n}^{GL_{n+2}} = \bigoplus_\mu \left(V_n^\mu\right)^{\oplus q(\lambda,\mu)} \tag{1.3}$$

for Young diagrams λ with less than or equal to $\min(n + 2, m)$ rows.

The algebra $\mathcal{A}_{n,m}$ provides an example of an algebra with straightening laws (ASL), or more generally Hodge algebra. Its finite presentation is well compatible with the combinatorics of Young tableaux, and such combinatorial properties can be explained in the context of a flat degeneration of $\mathcal{A}_{n,m}$ to an affine semigroup ring defined over a distributive lattice. Also, with an appropriate condition on n and m, the algebra $\mathcal{A}_{n,m}$ can carry the branching information for other classical groups as well. We refer the reader to [15,16,22] for these directions.

1.3 Cluster algebra

In Sects. 3 and 4, using the presentation of the algebra $\mathcal{A}_{n,m}$ given in Sect. 2, we prove that it admits another nice algebraic structure, the structure of a *cluster algebra*, which makes an interesting connection between highest weight vectors in the branching decomposition and highest weight vectors in the tensor product decomposition.

Cluster algebras, introduced by Fomin and Zelevinsky, are commutative algebras with generators called *cluster variables* and relations constructed via involutive operations called *mutations*. To define a (skew-symmetric) cluster algebra (of geometric type) $\mathcal{A} = \mathcal{A}(\mathbf{z}_0, Q_0)$, we need initial data $(\mathbf{z}_0, Q_0)$ consisting of a quiver Q_0 and a set $\mathbf{z}_0$ of certain elements in the algebra indexed by the vertices of Q_0. The elements of $\mathbf{z}_0$ form only a subset of the generating set of $\mathcal{A}$, and the full set of generators and relations of the algebra can be obtained by mutating the initial data. It is known that the coordinate ring of any Bruhat cell is isomorphic to a so-called upper cluster algebra [1], and that the multi-homogeneous coordinate ring of any partial flag algebra has the structure of a cluster algebra [5,19].

In this paper, while the algebra we consider is closely related to the coordinate rings mentioned above, we will take a more direct computational approach to obtain explicit expressions of its cluster variables. One of the main difficulties in proving our results is to find appropriate initial data which can produce all the cluster variables in a systematic way. To find such data, we will modify the poset structure of the generating set for $\mathcal{A}_{n,m}$ studied in [15] and obtain an initial quiver by gluing small quivers associated with the Grassmannian of two-dimensional subspaces in $\mathbb{C}^4$. We will explicitly compute all the cluster variables for the algebra $\mathcal{A}_{n,m}$.

1.4 Tensor product algebra

Finally, in Sect. 5, we focus on the cluster variables of $\mathcal{A}_{n,m}$ invariant under the maximal unipotent subgroup U_2 of GL_2 and show that they generate the highest weight vectors of the isomorphic copies of V_m^λ occurring in the decomposition of the tensor product (1.1).

More precisely, we show that the U_2-invariant ring $\mathcal{A}_{n,m}^{U_2}$ is indeed a special case of *the* GL_m *tensor product algebras* studied by Howe et al. [11,12], and prove that the U_2-invariant cluster variables of the algebra $\mathcal{A}_{n,m}$ form a generating set of the GL_m tensor product algebra encoding the Littlewood–Richardson rule for the tensor products of the form (1.1). It would be an interesting problem to find other classes of the tensor product algebras and associated highest weight vectors have the kind of algebraic interpretations as our results.

2 Twice-iterated Pieri algebra

In this section, we define the twice-iterated Pieri algebra and study its finite presentation.

2.1 Twice-iterated Pieri algebra

Once and for all, we fix two integers $n \geq 1$ and $m \geq 2$. Let us consider the coordinate ring $\mathbb{C}[\mathsf{M}_{n+2,m}]$ of the space

$$\mathsf{M}_{n+2,m} \cong \mathbb{C}^{n+2} \otimes \mathbb{C}^{m}$$

of complex $(n+2) \times m$ matrices. We will fix the coordinates x_{ij} of the space $\mathsf{M}_{n+2,m}$ and then identify the coordinate ring of $\mathsf{M}_{n+2,m}$ with the polynomial ring in $nm + 2m$ indeterminates. We let $GL_{n+2} \times GL_m$ act on the ring by

$$((g_1, g_2) \cdot f)(X) = f(g_1^t X g_2) \tag{2.1}$$

for $(g_1, g_2) \in GL_{n+2} \times GL_m$, $f \in \mathbb{C}[\mathsf{M}_{n+2,m}]$ and $X \in \mathsf{M}_{n+2,m}$.

In considering the restriction of GL_{n+2} to its subgroup GL_n and the action of GL_2 on $\mathbb{C}[\mathsf{M}_{n+2,m}]$, we use the following embedding of GL_n and GL_2 in GL_{n+2}: for $Y \in GL_n$ and $Z \in GL_2$,

$$\begin{bmatrix} Y & 0 \\ 0 & I_2 \end{bmatrix} \in GL_{n+2} \text{and} \begin{bmatrix} I_n & 0 \\ 0 & Z \end{bmatrix} \in GL_{n+2} \tag{2.2}$$

where I_d is the $d \times d$ identity matrix.

Now let U_d be the maximal unipotent subgroup of GL_d consisting of upper triangular matrices with 1's on the diagonal. We shall consider the ring of polynomials $f \in \mathbb{C}[\mathsf{M}_{n+2,m}]$ such that

$$((u_1, u_2) \cdot f)(X) = f(X)$$

for $X \in \mathsf{M}_{n+2,m}$ and $(u_1, u_2) \in U_n \times U_m$.

Definition 2.1 The twice-iterated Pieri algebra $\mathcal{A}_{n,m}$ is the ring of polynomials in $\mathbb{C}[\mathsf{M}_{n+2,m}]$ invariant under the subgroup $U_n \times U_m$ of $GL_{n+2} \times GL_m$

$$\mathcal{A}_{n,m} = \mathbb{C}[\mathsf{M}_{n+2,m}]^{U_n \times U_m}$$

with respect to the action (2.1).

Howe and Lee [10] studied the module structure of a polynomial ring over multiple copies of the general linear groups. Focusing on a case with three copies of GL_m, we have

$$\begin{aligned}\mathbb{C}[\mathsf{M}_{n+2,m}]^{U_n\times 1} &= \mathbb{C}[\mathsf{M}_{n,m}]^{U_n\times 1}\otimes\mathbb{C}[\mathsf{M}_{1,m}]\otimes\mathbb{C}[\mathsf{M}_{1,m}] \\ &\cong \bigoplus_{\mu,a,b} V_m^{\mu}\otimes V_m^{(a)}\otimes V_m^{(b)}\end{aligned} \tag{2.3}$$

as a $GL_m \times GL_m \times GL_m$ module. Here, we used the fact that as a GL_q-module,

$$\mathbb{C}[\mathsf{M}_{p,q}]^{U_p\times 1} \cong \bigoplus_{\eta}(V_p^{\eta})^{U_p}\otimes V_q^{\eta} \cong \bigoplus_{\eta} V_q^{\eta} \tag{2.4}$$

where the summation runs over Young diagrams η with not more than $\min(p, q)$ rows. See, for example, [6,7]. Therefore, the summation in (2.3) runs over Young diagrams μ with not more than $\min(n, m)$ rows and nonnegative integers a and b. Then, by taking U_m-invariant polynomials in the algebra $\mathbb{C}[\mathsf{M}_{n+2,m}]^{U_n\times 1}$, we obtain the twice-iterated Pieri algebra $\mathcal{A}_{n,m}$ consisting of the highest weight vectors of irreducible representations V_m^{λ} of GL_m in the tensor products $V_m^{\mu}\otimes V_m^{(a)}\otimes V^{(b)}$. Since the multiplicities of irreducible representations in such tensor products can be described by applying the Pieri rule twice, we call $\mathcal{A}_{n,m}$ the twice-iterated Pieri algebra.

2.2 Homogeneous components

On the other hand, from the decomposition of the polynomial ring

$$\mathbb{C}[\mathsf{M}_{n+2,m}] \cong \bigoplus_{\lambda} V_{n+2}^{\lambda}\otimes V_m^{\lambda}$$

as a $GL_{n+2}\times GL_m$ module, by taking $U_n \times U_m$ invariants, we have

$$\mathcal{A}_{n,m} \cong \bigoplus_{\lambda}(V_{n+2}^{\lambda})^{U_n}\otimes (V_m^{\lambda})^{U_m}$$

where the summation runs over λ with $\ell(\lambda) \le \min(n + 2, m)$. Since the U_n-invariant vectors in V_{n+2}^{λ} are the highest weight vectors of GL_n-irreducible representations V_n^{μ}, our twice-iterated Pieri algebra $\mathcal{A}_{n,m}$ also carries the branching rules under the restriction of GL_{n+2} to its subgroup GL_n.

To study the ring structure of $\mathcal{A}_{n,m}$ in this context, let us write T_d for the maximal torus of GL_d consisting of diagonal matrices. Since U_d is normalized by T_d in GL_d, we can consider the $T_n \times T_m$-eigenspaces

$$W(\mu,\lambda) = \left\{f \in \mathcal{A}_{n,m} : ((s,t)\cdot f)(X) = s^{\mu}t^{\lambda} f(X)\ \text{for}\ (s,t)\in T_n\times T_m\right\}$$

where $s^{\mu} = \prod_j s_j^{\mu_j}$ and $t^{\lambda} = \prod_j t_j^{\lambda_j}$. Here, s and t are diagonal matrices in T_n and T_m with entries $s_1,\ldots,s_n$ and $t_1,\ldots,t_m$, respectively; $\mu = (\mu_1,\ldots,\mu_n)$ and $\lambda = (\lambda_1,\ldots,\lambda_m)$ are Young diagrams identified with dominant weights for GL_n and

GL_m, respectively. Then, we have the decomposition

$$\mathcal{A}_{n,m} = \bigoplus_{(\mu,\lambda)} W(\mu, \lambda),$$

which provides a multi-grading structure on the algebra $\mathcal{A}_{n,m}$. With highest weight theory (see, for example, [6, § 3.2]), it is straightforward to see that $W(\mu, \lambda)$ consists of the highest weight vectors of the isomorphic copies of the irreducible representation V_n^μ of GL_n appearing in the irreducible representation V_{n+2}^λ of GL_{n+2}. As a consequence, the dimension of the space $W(\mu, \lambda)$ equals the multiplicity of V_n^μ in V_{n+2}^λ. See [15].

Moreover, with an appropriate condition on n and m, the algebra $\mathcal{A}_{n,m}$ also describes the branching rules for other classical groups. For example, when $n = \ell - 1$ and $m = \ell$, the algebra $\mathcal{A}_{n,m}$ carries the information on the decomposition of every irreducible representation of the symplectic group $Sp_{2\ell}$ as a representation of its subgroup $Sp_{2\ell-2}$. For further details in this direction, we refer the reader to [15,16,22].

We remark that one can also construct an algebra encoding both tensor products and branchings of Pieri type for the general linear groups at the same time. It turns out such an algebra, called the double Pieri algebra, carries a nice algebraic structure with interesting connections with tensor product decomposition problems for other classical groups. These results will be discussed in separate articles. See [9].

2.3 Standard monomial basis

In [15], one of the present authors investigated explicit presentations of algebras encoding branching rules for classical groups. Here, we summarize and modify some of the results relevant for us.

For $X \in M_{n+2,m}$ and a subset $I = \{r_1, r_2, \ldots, r_h\}$ of $\{1, 2, \ldots, n+2\}$, we let $\delta_I(X)$ denote the minor of $X = (x_{ij})$ with row indices $r_1, r_2, \ldots, r_h$ and column indices $1, 2, \ldots, h$:

$$\delta_I(X) = \det \begin{bmatrix} x_{r_1 1} & x_{r_1 2} & \cdots & x_{r_1 h} \\ x_{r_2 1} & x_{r_2 2} & \cdots & x_{r_2 h} \\ \vdots & \vdots & \ddots & \vdots \\ x_{r_h 1} & x_{r_h 2} & \cdots & x_{r_h h} \end{bmatrix} \tag{2.5}$$

In considering subsets $I = \{r_1, r_2, \ldots, r_h\}$ of $\{1, \ldots, n+2\}$ for δ_I, we assume that their entries are in increasing order, i.e., $r_1 < r_2 < \cdots < r_h$. Let us focus on the following subsets of $\{1, \ldots, n+2\}$

$$I_{4j+1} := \{1, 2, \ldots, j, n+2\}, \qquad I_{4j+2} := \{1, 2, \ldots, j, n+1\},$$
$$I_{4j+3} := \{1, 2, \ldots, j, n+1, n+2\}, \qquad I_{4j} := \{1, 2, \ldots, j-1, j\}.$$

for $0 \leq j \leq n$, with the conventions $I_1 = \{n+2\}$, $I_2 = \{n+1\}$, and $I_3 = \{n+1, n+2\}$.

Definition 2.2 Writing $\mathcal{L}^j$ for the collection

$$\mathcal{L}^j = \left\{ I_{4j+1}, I_{4j+2}, I_{4j+3}, I_{4(j+1)}, I_{4(j+1)+1}, I_{4(j+1)+2} \right\}, \tag{2.6}$$

we define $\mathcal{L}_{n,m}$ as follows:

(1) if $m \leq n$, then

$$\mathcal{L}_{n,m} = \{I_{4m}\} \cup \bigcup_{0 \leq j \leq m-2} \mathcal{L}^j;$$

(2) if $m = n + 1$, then

$$\mathcal{L}_{n,m} = \bigcup_{0 \leq j \leq n-1} \mathcal{L}^j;$$

(3) if $m \geq n + 2$, then

$$\mathcal{L}_{n,m} = \{I_{4n+3}\} \cup \bigcup_{0 \leq j \leq n-1} \mathcal{L}^j.$$

Notation 2.3 *When there is no danger of confusion, we denote by δ_i the minor δ_{I_i} whose rows are indexed by $I_i \in \mathcal{L}_{n,m}$.*

In order to describe a $\mathbb{C}$-basis for the twice-iterated Pieri algebra, we define the *standard monomials* for $\mathcal{A}_{n,m}$ to be monomials in δ_ℓ's which are not divisible by $\delta_{4j+3}\delta_{4(j+1)}$ for any j. Then, for a standard monomial

$$\delta_{\ell_1}\delta_{\ell_2}\dots\delta_{\ell_r},$$

we can define its *shape* to be the skew Young diagram λ/μ where λ is, after the reordering of the product so that $|I_{\ell_1}| \geq |I_{\ell_2}| \geq \dots \geq |I_{\ell_r}|$, the transpose of the Young diagram $(|I_{\ell_1}|, |I_{\ell_2}|, \dots, |I_{\ell_r}|)$ and $\mu = (d_1, \dots, d_n)$ where d_i is the number of times i's appearing in the sets $I_{\ell_1}, \dots, I_{\ell_r}$.

Proposition 2.4 (1) *The twice-iterated Pieri algebra $\mathcal{A}_{n,m}$ is generated by*

$$\mathcal{G}_{n,m} := \left\{\delta_i : I_i \in \mathcal{L}_{n,m}\right\}.$$

(2) *The standard monomials for $\mathcal{A}_{n,m}$ form a $\mathbb{C}$-basis for the space $\mathcal{A}_{n,m}$. More precisely, the standard monomials $\mathcal{A}_{n,m}$ of shape λ/μ form a $\mathbb{C}$-basis of the space $W(\lambda, \mu)$.*

We can identify each standard monomial of shape λ/μ with a semistandard Young tableau of shape λ/μ by concatenating the indices I_ℓ's of the factors δ_ℓ's in the standard monomial. This can be understood as a skew version of the well-known correspondence between weight vectors of an irreducible GL_n module with highest weight λ and semistandard Young tableaux of shape λ [15]. In what follows, we sketch the proof of the above results.

2.4 Straightening laws

When proving Proposition 2.4, we find that the following poset structure on the set $\mathcal{L}_{n,m}$ is useful. First, for each j, we impose a partial order on $\mathcal{L}^j$ defined in (2.6) using the Hasse diagram in Fig. 1 whose nodes increase from bottom to top.

Next, we want to define a poset structure on the set $\mathcal{L}_{n,m}$ (Definition 2.2) using the ordinal sum of posets $\mathcal{L}^j$. Recall that the ordinal sum $P \oplus Q$ of two posets P and Q is the poset on the union $P \cup Q$ such that $x \le y$ in $P \oplus Q$ if $x, y \in P$ and $x \le y$ in P, or $x, y \in Q$ and $x \le y$ in Q, or $x \in P$ and $y \in Q$ [20, § 3.2].

Definition 2.5 We define the following poset structure on the set $\mathcal{L}_{n,m}$.

(1) If $m \le n$, then

$$\mathcal{L}_{n,m} = \{I_{4m}\} \oplus \mathcal{L}^{m-2} \oplus \mathcal{L}^{m-1} \oplus \cdots \oplus \mathcal{L}^1 \oplus \mathcal{L}^0.$$

(2) If $m = n + 1$, then

$$\mathcal{L}_{n,m} = \mathcal{L}^{n-1} \oplus \mathcal{L}^{n-2} \oplus \cdots \oplus \mathcal{L}^1 \oplus \mathcal{L}^0.$$

(3) If $m \ge n + 2$, then

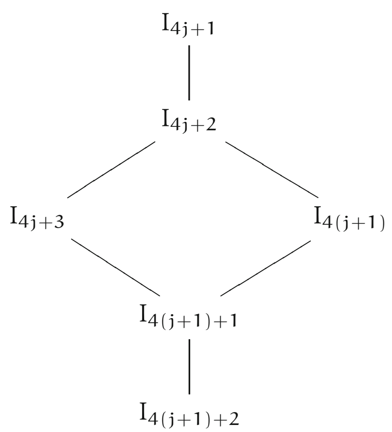
$$\mathcal{L}_{n,m} = \{I_{4n+3}\} \oplus \mathcal{L}^{n-1} \oplus \mathcal{L}^{n-2} \oplus \cdots \oplus \mathcal{L}^1 \oplus \mathcal{L}^0.$$

Then, for each incomparable pair $(I_{4j+3}, I_{4(j+1)})$ in $\mathcal{L}_{n,m}$, we find a relation called *straightening law*. That is,

$$\delta_{I_{4j+3}}\delta_{I_{4(j+1)}} = \delta_{I_{4(j+1)+1}}\delta_{I_{4j+2}} - \delta_{I_{4(j+1)+2}}\delta_{I_{4j+1}}. \quad (2.7)$$

First, it is easy to verify that δ_i for $I_i \in \mathcal{L}_{n,m}$ are invariant under the action of $U_n \times U_m$. Then, we observe that for each monomial in δ_i's, by using the above identities, we

Fig. 1 Hasse diagram for $\mathcal{L}^j$



can replace the factors $\delta_{I_{4j+3}}\delta_{I_{4(j+1)}}$ with linear combinations of standard monomials. By applying such replacements as many times as necessary, we can express every monomial in δ_i's as a linear combination of standard monomials, thereby 'straightening out' all nonstandard monomials. Finally, by using a counting argument obtained from representation theory or by showing directly that standard monomials are linearly independent, we can prove Proposition 2.4. For more details, see [15].

3 Cluster algebra structure: base cases

In this section, we show that some small twice-iterated Pieri algebras admit the structure of a cluster algebra. Recall that we fix two integers $n \geq 1$ and $m \geq 2$ for the algebra $\mathcal{A}_{n,m}$.

Theorem 3.1 (1) *The twice-iterated Pieri algebras* $\mathcal{A}_{1,m}$ *with* $m \geq 2$ *and* $\mathcal{A}_{2,2}$ *admit the structure of a cluster algebra of type* A_1.
(2) *The twice-iterated Pieri algebras* $\mathcal{A}_{2,m}$ *with* $m \geq 3$ *admit the structure of a cluster algebra of type* A_2.

Let us briefly review some basic concepts of (skew-symmetric) cluster algebras (of geometric type). For details, we refer the reader [2–4,17,21]. Then, for each case, we will define an initial seed and show that all the cluster variables obtained by sequences of mutations form a generating set of the algebra $\mathcal{A}_{n,m}$.

3.1 Cluster algebras

For two positive integers $c \leq d$, let F be a field of rational functions in c variables over $\mathbb{C}(z_{c+1}, \ldots, z_d)$. A *seed* is a pair $(\mathbf{z}, Q)$ consisting of a generating set $\mathbf{z} = \{z_1, \ldots, z_d\}$ of F and a finite quiver Q without loops or 2-cycles with vertex set $\{1, \ldots, d\}$. The vertices i of Q will be called *mutable* if $1 \leq i \leq c$ and *frozen* if $c + 1 \leq i \leq d$. The variables $z_i \in \mathbf{z}$ are associated with the vertices i of Q, and they will be called (mutable or frozen) cluster variables.

A cluster algebra is a commutative ring generated inside F with generators obtained from an initial seed via iterative processes of seed mutations described below. For a mutable vertex k of Q, by applying the *mutation* μ_k *at* k to $(\mathbf{z}, Q)$ we obtain a new pair $\mu_k(\mathbf{z}, Q) = (\mathbf{z}', Q')$ consisting of a set $\mathbf{z}'$ and a quiver $Q' = \mu_k(Q)$. Here, $\mathbf{z}'$ is obtained from $\mathbf{z}$ by replacing the element z_k with

$$z'_k = \frac{\prod_{i\to k} z_i + \prod_{k\to j} z_j}{z_k} \text{(exchange relation)} \tag{3.1}$$

where the products runs for all arrows $i \to k$ and $k \to j$, respectively, and $Q' = \mu_k(Q)$ is defined as follows: (i) for each subquiver $i \to k \to j$, add an arrow $i \to j$, (ii) reverse all arrows starting from or ending at k, (iii) remove the arrows in a maximal set of pairwise disjoint 2 cycles. It is known that $\mu_k(\mathbf{z}, Q)$ is again a seed and that μ_k is an involution.

In describing mutations, we often find matrices are more convenient than quivers. Let $B = (b_{ij})$ be the antisymmetric matrix associated with Q, i.e., $b_{ij} = -b_{ji}$ is the number of arrows from vertex i to vertex j in Q. Here we note that the rows and columns of B are labeled by the vertices of Q. If we write $\mu_k(B) = (b'_{ij})$ for the matrix associated with $\mu_k(Q)$, then we have

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases} \tag{3.2}$$

A cluster algebra is of finite-type X if the quiver under consideration is mutation-equivalent to an orientation of a finite-type Dynkin diagram of type X. We say an algebra admits the structure of a cluster algebra with an initial seed $(\mathbf{z}_0, Q_0)$ if the cluster variables obtained by applying all possible sequences of mutations to $(\mathbf{z}_0, Q_0)$ form a generating set for the algebra.

3.2 Case $n = 1$

The poset structure of the generating set $\mathcal{G}_{1,2} = \{\delta_i : I_i \in \mathcal{L}_{1,2}\}$ for the algebra $\mathcal{A}_{1,2}$ can be illustrated via the Hasse diagram in Fig. 1 with $j = 0$. These six generators satisfy the following relation:

$$\delta_3\delta_4 = \delta_2\delta_5 - \delta_1\delta_6. \tag{3.3}$$

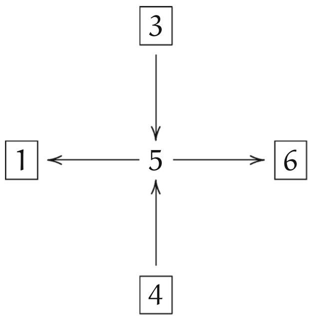
We define the initial quiver Q_0 for the algebra $\mathcal{A}_{1,2}$ as shown in Fig. 2 with one mutable variable $z_5 = \delta_5$, therefore of type A_1, and four frozen variables $z_1 = \delta_1$, $z_3 = \delta_3$, $z_4 = \delta_4$ and $x_6 = \delta_6$. Then, there is only one possible mutation, and after the mutation at 5, we obtain a new quiver $\mu_5(Q_0)$ in Fig. 3 with the new cluster variable

$$z'_5 = (z_1z_6 + z_3z_4)/z_5,$$

which can be, using (3.3), simplified to $z'_5 = \delta_2$.

Consequently, all the cluster variables we have are nothing but the elements in the generating set $\mathcal{G}_{1,2}$ of the algebra $\mathcal{A}_{1,2}$. We also note that, from this presentation,

Fig. 2 Initial quiver for $\mathcal{A}_{1,2}$



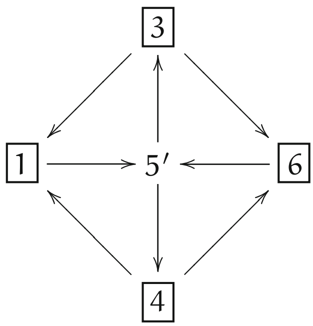


Fig. 3 After the mutation at 5

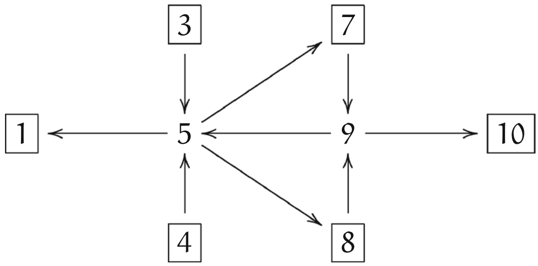


Fig. 4 Initial quiver for $\mathcal{A}_{2,3}$

this algebra is isomorphic to the homogeneous coordinate ring for the Grassmannian $Gr(2,4)$ of two-dimensional subspaces in $\mathbb{C}^4$.

Next, if $n = 1$ and $m > 2$, then in addition to the elements in $\mathcal{G}_{1,2}$, there is one more generator δ_7 in $\mathcal{A}_{1,m}$. By taking $z_7 = \delta_7$ as an additional frozen variable, we can easily see that the algebra $\mathcal{A}_{1,m}$ is again isomorphic to a cluster algebra of type A_1.

3.3 Case $n = 2$

Now let us first consider the algebra $\mathcal{A}_{2,3}$. The poset structure of the generating set $\mathcal{G}_{2,3}$ may be realized as the ordinal sum of two Hasse diagrams in Fig. 1 with $j = 0$ and $j = 1$:

$$\mathcal{L}_{2,3} = \mathcal{L}^1 \oplus \mathcal{L}^0. \tag{3.4}$$

There are ten generators with two relation among them:

$$\delta_3\delta_4 = \delta_2\delta_5 - \delta_1\delta_6 \quad \text{and} \quad \delta_7\delta_8 = \delta_6\delta_9 - \delta_5\delta_{10}. \tag{3.5}$$

Motivated by the realization of the generating set for $\mathcal{A}_{2,3}$ given in (3.4), we want to construct a quiver by gluing two quivers for $\mathcal{A}_{1,2}$. Let us define the initial quiver Q_0 for the algebra $\mathcal{A}_{2,3}$ as shown in Fig. 4, which is of type A_2. The set $\mathbf{z}_0$ of the initial cluster variables consists of two mutable variable $z_5 = \delta_5$ and $z_9 = \delta_9$ and six frozen variables $z_\ell = \delta_\ell$ for $\ell = 1, 3, 4, 7, 8, 10$.

Next, let us consider the following mutations of the initial quiver Q_0

$$\mu_5(Q_0), \quad (\mu_9 \circ \mu_5)(Q_0), \quad \text{and} \quad \mu_9(Q_0),$$

which provide new cluster variables

$$z_5' := \mu_5(z_5) = (z_3z_4z_9 + z_1z_7z_8)/z_5, \;\; z_9' := (\mu_9 \circ \mu_5)(z_9) = (z_5' + z_1z_{10})/z_9,$$
$$\text{and} \;\; z_9'' := \mu_9(z_9) = (z_7z_8 + z_5z_{10})/z_9.$$

expressed as Laurent polynomials in initial variables $z_i \in \mathbf{z}_0$.

After simplifying them using the identities (3.5), we end up with

$$z_5' = (\delta_2\delta_9 - \delta_1\delta_{10}), \;\; z_9' = \delta_2,$$
$$\text{and} \;\; z_9'' = \delta_6,$$

which are polynomials in the generators of $\mathcal{A}_{2,3}$.

Using the fact that cluster algebras of type A_2 can have only five cluster variables, we know that there are no other cluster variables we can obtain by other sequences of mutations. Therefore, all the cluster variables the initial seed $(\mathbf{z}_0, Q_0)$ can produce are the elements in the generating set $\mathcal{G}_{2,3}$ for the algebra $\mathcal{A}_{2,3}$ and one additional element $(\delta_2\delta_9 - \delta_1\delta_{10})$ which is a polynomial in the elements of $\mathcal{G}_{2,3}$. This shows that the twice-iterated Pieri algebra $\mathcal{A}_{2,3}$ is isomorphic to a cluster algebra of type A_2 with the initial seed $(\mathbf{z}_0, Q_0)$.

Finally, when $n = 2$ and $m > 3$, in addition to the elements in $\mathcal{G}_{2,3}$, there is one more generator δ_{11} for $\mathcal{A}_{2,m}$. By taking $z_{11} = \delta_{11}$ as an additional frozen variable, we can easily see that the algebra $\mathcal{A}_{2,m}$ has the structure of a cluster algebra of type A_2. If $n = 2$ and $m = 2$, by comparing the generators and straightening laws of $\mathcal{A}_{2,2}$ with those of $\mathcal{A}_{1,3}$, it is straightforward to see that $\mathcal{A}_{2,2}$ has the structure of a cluster algebra of type A_1.

4 Cluster algebra structure: general case

In this section, we prove the following general statement.

Theorem 4.1 *The twice-iterated Pieri algebra $\mathcal{A}_{n,m}$ admits the structure of a cluster algebra of type A_r where $r = m - 1$ if $m \leq n$ and $r = n$ if $m \geq n + 1$.*

From Definition 2.5, our generating set of $\mathcal{A}_{n,m}$ for $m \geq n + 2$ consists of the generators of $\mathcal{A}_{n,n+1}$ with one additional element which is not involved with any straightening laws. Similarly, the presentation of $\mathcal{A}_{n,m}$ for $m \leq n$ can be obtained from that of $\mathcal{A}_{m-1,m}$. Therefore, all the cases can be derived from the case $m = n + 1$.

Recall that if $m = n + 1$, the algebra $\mathcal{A}_{n,m}$ is a subalgebra of the polynomial ring $\mathbb{C}[x_{ij}]$ generated by the following minors (see Eq. (2.5) and Notation 2.3):

$$\{\delta_i : 1 \leq i \leq 4n + 2\},$$

and that δ_i's satisfy

$$\delta_{4j-1}\delta_{4j} = \delta_{4j-2}\delta_{4j+1} - \delta_{4j-3}\delta_{4j+2}$$

for $1 \leq j \leq n$.

To prove Theorem 4.1 for $m = n + 1$, we will show that starting from a subset of the generating set $\mathcal{G}_{n,m}$ of the algebra $\mathcal{A}_{n,m}$, it is possible to produce all the missing generators by mutations, and that also all the other cluster variables are in fact polynomials in δ_i's. Therefore, the cluster variables form a complete generating set of the algebra $\mathcal{A}_{n,m}$.

4.1 Initial seed

Let us consider the quiver Q_0 given in Fig. 5 with the repeating middle part as shown in Fig. 6 and the collection $\mathbf{z}_0$ of initial cluster variables z_i indexed by the vertices i of Q_0. More precisely, $\mathbf{z}_0$ consists of

(1) n mutable variables z_{4j+1} for $1 \leq j \leq n$,
(2) $(2n + 2)$ frozen variables
 (a) z_1 and z_{4n+2},
 (b) z_{4j-1} and z_{4j} for $1 \leq j \leq n$.

In Figs. 5 and 6, the vertices associated with frozen variables are put in rectangular boxes.

In computing cluster variables, we need to keep track of various mutations, and it is more convenient to work with matrices corresponding to mutated quivers. Using (3.2), we obtain the matrix B_0 corresponding to the initial quiver Q_0 in Fig. 7. Here, since arrows between two frozen vertices do not affect seed mutation, we ignore the corresponding data and consider the $(3n + 2) \times n$ matrix B_0.

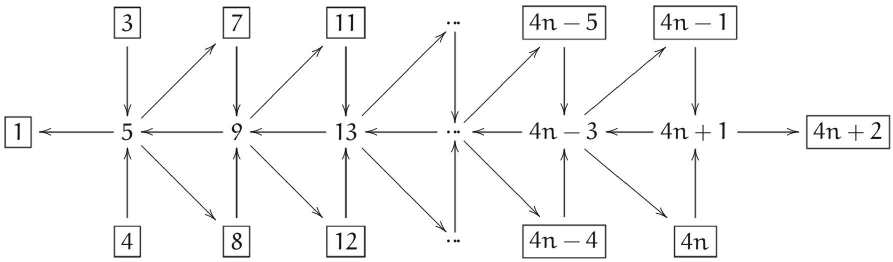
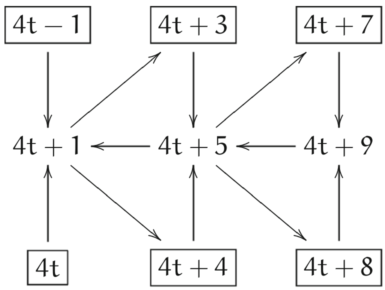


Fig. 5 Initial quiver Q_0

Fig. 6 Repeating middle part of Q_0



$$
\begin{array}{r|ccccccc|}
 & 5 & 9 & 13 & \cdots & 4n-7 & 4n-3 & 4n+1 \\
\boxed{1} & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
5 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\
9 & 1 & 0 & -1 & \cdots & 0 & 0 & 0 \\
13 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
4n-7 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\
4n-3 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 \\
4n+1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\boxed{4n+2} & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\
\boxed{3} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\boxed{7} & -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\boxed{11} & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\boxed{4n-5} & 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\
\boxed{4n-1} & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
\boxed{4} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\boxed{8} & -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\boxed{12} & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\boxed{4n-4} & 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\
\boxed{4n} & 0 & 0 & 0 & \cdots & 0 & -1 & 1
\end{array}
$$

Fig. 7 Matrix B_0 for Q_0

4.2 Mutations

For each ℓ with $1 \leq \ell \leq n$, we define the following sequences of mutations

$$\mu_{[k,\ell]} = \mu_{4k+1} \circ \cdots \circ \mu_{4\ell+5} \circ \mu_{4\ell+1} \quad \text{for} \quad k = \ell, \ell+1, \ldots, n.$$

Lemma 4.2 *When the mutation $\mu_{[k,\ell]}$ is applied to the initial seed $(\mathbf{z}_0, Q_0)$, the entries of the column labeled by the vertex $(4k+1)$ in the matrix $\mu_{[k,\ell]}(B_0) = (b^{\dagger}_{ij})$ are all zero except the following entries:*

(1) if $k = \ell < n$,

$$b^{\dagger}_{4\ell-3,4\ell+1} = b^{\dagger}_{4\ell+3,4\ell+1} = b^{\dagger}_{4\ell+4,4\ell+1} = 1,$$
$$b^{\dagger}_{4\ell-1,4\ell+1} = b^{\dagger}_{4\ell,4\ell+1} = b^{\dagger}_{4\ell+5,4\ell+1} = -1;$$

(2) if $k = \ell = n$,

$$b^{\dagger}_{4n-3,4n+1} = b^{\dagger}_{4n+2,4n+1} = 1 \text{ and } b^{\dagger}_{4n-1,4n+1} = b^{\dagger}_{4n,4n+1} = -1;$$

(3) if $\ell + 1 \leq k \leq n - 1$,

$$b^{\dagger}_{4k+3,4k+1} = b^{\dagger}_{4k+4,4k+1} = b^{\dagger}_{4\ell-3,4k+1} = 1,$$
$$b^{\dagger}_{4k-3,4k+1} = b^{\dagger}_{4k+5,4k+1} = -1;$$

(4) if $\ell + 1 \leq k = n$,

$$b^{\dagger}_{4\ell-3,4n+1} = b^{\dagger}_{4n+2,4n+1} = 1 \text{and} b^{\dagger}_{4n-3,4n+1} = -1.$$

Proof (1) & (2) These two cases with $k = \ell$ follow directly from the definition of the mutation $\mu_{[\ell,\ell]} = \mu_{4\ell+1}$ at the vertex $(4\ell + 1)$. We obtain the column $(4\ell + 1)$ of $\mu_{4\ell+1}(B_0)$ simply by changing the sign of the entries in the column $(4\ell + 1)$ of B_0.

(3) Let us prove the case $\ell+1 \leq k \leq n-1$ by induction. When $k = \ell+1$, the entries of the column labeled by $(4\ell + 5)$ in the matrix $\mu_{[\ell+1,\ell]}(B_0)$ can be verified by direct computation. Now, assuming that the statement is true for k such that $\ell+1 \leq k \leq n-2$, let us keep track of the entries in the column labeled by $(4k + 5)$ in the matrices $\mu_{[k-1,\ell]}(B_0)$, $\mu_{[k,\ell]}(B_0)$, and $\mu_{[k+1,\ell]}(B_0)$.

Since the column labeled by $(4k+5)$ in B_0 is not affected by the mutation $\mu_{[k-1,\ell]}$ yet, in the matrix $\mu_{[k-1,\ell]}(B_0) = (b_{ij})$, we have the following nonzero entries:

$$\begin{aligned} b_{4k+3,4k+5} &= b_{4k+4,4k+5} = b_{4k+9,4k+5} = 1, \\ b_{4k+1,4k+5} &= b_{4k+7,4k+5} = b_{4k+8,4k+5} = -1. \end{aligned} \tag{4.1}$$

By the induction hypothesis, the column labeled by $(4k + 1)$ in $\mu_{[k,\ell]}(B_0)$ contains nonzero entries in the rows labeled by $(4k + 3)$, $(4k + 4)$, $(4\ell - 3)$, $(4k - 3)$, and $(4k + 5)$. Because the mutation μ_{4k+1} changes only the sign of the entries in the column labeled by $(4k + 1)$, we know that the nonzero entries in the column labeled by $(4k + 1)$ in $\mu_{[k-1,\ell]}(B_0) = (b_{ij})$ are

$$\begin{aligned} b_{4k+3,4k+1} &= b_{4k+4,4k+1} = b_{4\ell-3,4k+1} = -1, \\ b_{4k-3,4k+1} &= b_{4k+5,4k+1} = 1. \end{aligned} \tag{4.2}$$

Now we apply μ_{4k+1} to $\mu_{[k-1,\ell]}(B_0)$ and use the identities (4.1) and (4.2) to see that nonzero entries of the matrix $\mu_{[k,\ell]}(B_0) = (b'_{ij})$ are

$$b'_{4k+7,4k+5} = b'_{4k+8,4k+5} = b'_{4\ell-3,4k+5} = -1,$$
$$b'_{4k+1,4k+5} = b'_{4k+9,4k+5} = 1.$$

Finally, we apply μ_{4k+5} to obtain the nonzero entries in the column $(4k + 5)$ of the matrix $\mu_{[k+1,\ell]}(B_0) = (b''_{ij})$:

$$b''_{4k+7,4k+5} = b''_{4k+8,4k+5} = b''_{4\ell-3,4k+5} = 1$$
$$b''_{4k+1,4k+5} = b''_{4k+9,4k+5} = -1.$$

This shows that the statement is true for $k+1$.

(4) For the case $\ell+1 \leq k = n$, we note that the column labeled by $(4n+1)$ in B_0 is not affected by the mutation $\mu_{[n-2,\ell]}$ yet. Therefore, in $\mu_{[n-2,\ell]}(B_0) = (b_{ij})$ we have

$$b_{4n,4n+1} = b_{4n-1,4n+1} = 1 \quad \text{and} \quad b_{4n+2,4n+1} = b_{4n-3,4n+1} = -1. \tag{4.3}$$

Also, by setting $k = n-1$ in the result (3), we know that the nonzero entries in the column labeled by $(4n-3)$ in the matrix $\mu_{[n-1,\ell]}(B_0) = \mu_{4n-3}(\mu_{[n-2,\ell]}(B_0))$ are in the rows $(4\ell-3)$, $(4n-1)$, $4n$, $(4n-7)$, and $(4n+1)$. By exchanging the sign of them, we have the following in $\mu_{[n-2,\ell]}(B_0)$:

$$\begin{aligned} b_{4n-1,4n-3} = b_{4n,4n-3} = b_{4\ell-3,4n-3} = -1, \\ b_{4n-7,4n-3} = b_{4n+1,4n-3} = 1. \end{aligned} \tag{4.4}$$

Now we apply μ_{4n-3} to $\mu_{[n-2,\ell]}(B_0)$ and use the identities (4.3) and (4.4) to compute the following nonzero entries in $\mu_{[n-1,\ell]}(B_0) = (b'_{ij})$:

$$b'_{4n-3,4n+1} = 1 \text{and} b'_{4n+2,4n+1} = b'_{4\ell-3,4n+1} = -1,$$

and then finally, by applying μ_{4n+1} to $\mu_{[n-1,\ell]}(B_0)$, in $\mu_{[n,\ell]}(B_0) = (b''_{ij})$ we have

$$b''_{4\ell-3,4n+1} = b''_{4n+2,4n+1} = 1 \quad \text{and} \quad b''_{4n-3,4n+1} = -1.$$

□

Using the above computation of the matrix $\mu_{[k,\ell]}(B_0)$, we can further compute the cluster variables generated by the mutations $\mu_{[k,\ell]}$.

4.3 Cluster variables

Let $\hat{\mathbf{z}} = \{z_i : 1 \leq i \leq 4n+2\}$ be the set of indeterminates satisfying

$$z_{4j-1}z_{4j} = z_{4j-2}z_{4j+1} - z_{4j-3}z_{4j+2}$$

for $1 \leq j \leq n$. We note that

$$\hat{\mathbf{z}} = \mathbf{z}_0 \cup \{z_{4j+2} : 0 \leq j \leq n-1\}. \tag{4.5}$$

Proposition 4.3 *In considering the cluster algebra with the initial seed $(\mathbf{z}_0, Q_0)$, the cluster variables generated by the $(n+1-\ell)$ sequences $\mu_{[k,\ell]}$ of mutations for $\ell \leq k \leq n$ are*

$$\det\begin{bmatrix} z_2 & z_1 \\ z_{10} & z_9 \end{bmatrix}, \det\begin{bmatrix} z_2 & z_1 \\ z_{14} & z_{13} \end{bmatrix}, \ldots, \det\begin{bmatrix} z_2 & z_1 \\ z_{4n+2} & z_{4n+1} \end{bmatrix}, \; z_2 \text{ when} \ell = 1;$$

$$\det\begin{bmatrix} z_6 & z_5 \\ z_{14} & z_{13} \end{bmatrix}, \ldots, \det\begin{bmatrix} z_6 & z_5 \\ z_{4n+2} & z_{4n+1} \end{bmatrix}, \; z_6 \text{ when} \ell = 2;$$

$$\vdots$$

$$\det\begin{bmatrix} z_{4n-6} & z_{4n-7} \\ z_{4n+2} & z_{4n+1} \end{bmatrix}, \; z_{4n-6} \quad \text{when} \ell = n-1;$$

$$z_{4n-2} \quad \text{when} \ell = n.$$

Proof We prove it by induction. For the base case $k = \ell$, if $k = \ell < n$, then we apply $\mu_{4\ell+1}$ to the initial seed. From the case (1) in Lemma 4.2, we have

$$z'_{4\ell+1} = \left(z_{4\ell-3}z_{4\ell+3}z_{4\ell+4} + z_{4\ell-1}z_{4\ell}z_{4\ell+5}\right)/z_{4\ell+1},$$

and using substitutions

$$z_{4\ell+3}z_{4\ell+4} = z_{4\ell+2}z_{4\ell+5} - z_{4\ell+1}z_{4\ell+6} \text{ and}$$
$$z_{4\ell-1}z_{4\ell+5} = z_{4\ell-2}z_{4\ell+1} - z_{4\ell-3}z_{4\ell+2}$$

we obtain $z'_{4\ell+1} = z_{4\ell-2}z_{4\ell+5} - z_{4\ell-3}z_{4\ell+6}$. If $k = \ell = n$, then using the case (2) in Lemma 4.2, we have

$$z'_{4n+1} = \left(z_{4n-3}z_{4n+2} + z_{4n-1}z_{4n}\right)/z_{4n+1}.$$

From the identity $z_{4n-1}z_{4n} = z_{4n+1}z_{4n-2} - z_{4n+2}z_{4n-3}$, we conclude that $z'_{4n+1} = z_{4n-2}$.

Next, assuming that the statement is true for $\ell + 1 \leq k \leq n-2$, we want to verify the cases $k = n-1$ and $k = n$. From the case (3) of Lemma 4.2,

$$z'_{4(n-1)+1} = \left(z_{4\ell-3}z_{4n-1}z_{4n} + z'_{4n-7}z_{4n+1}\right)/z_{4n-3}$$

and after substitutions

$$z_{4n-1}z_{4n} = z_{4n-2}z_{4n+1} - z_{4n-3}z_{4n+2} \text{ and}$$
$$z'_{4(n-2)+1} = z_{4\ell-2}z_{4n-3} - z_{4\ell-3}z_{4n-2}$$

we obtain $z'_{4(n-1)+1} = z_{4\ell-2}z_{4n+1} - z_{4n+2}z_{4\ell-3}$. Similarly, from the case (4) of Lemma 5.1, we can compute

$$\begin{aligned} z'_{4n+1} &= (z_{4\ell-3}z_{4n+2} + z'_{4n-3})/z_{4n+1} \\ &= (z_{4\ell-3}z_{4n+2} + (z_{4\ell-2}z_{4n+1} - z_{4\ell-3}z_{4n+2}))/z_{4n+1} \\ &= z_{4\ell-2}. \end{aligned}$$

□

Note that the number of cluster variables in the statement together with the initial mutable variables is exactly the number of cluster variables for a cluster algebra of type A_n, namely $n(n+3)/2$. Therefore, the cluster variables given in Proposition 4.3 and the initial mutable variables make a complete list of the cluster variables. Moreover, the set of these cluster variables contains all the generators of the algebra and some polynomials in the generators. This shows that the twice-iterated Pieri algebra $\mathcal{A}_{n,n+1}$ admits the structure of a cluster algebra of type A_n.

5 GL_n tensor product algebras

In this section, we study the cluster variables of the twice-iterated Pieri algebra $\mathcal{A}_{n,m}$ in the context of the decomposition of the tensor product of two irreducible representations of GL_m.

5.1 GL_m tensor product algebra

In [11,12], Howe et al. studied a family of algebras, called the GL_m *tensor product algebras*, encoding the decomposition of tensor products of two irreducible representations of GL_m

$$V_m^\mu \otimes V_m^\nu = \bigoplus_\lambda (V_m^\lambda)^{\oplus c_{\mu\nu}^\lambda} \tag{5.1}$$

with conditions on the number of rows in μ and ν. Here, the multiplicity $c_{\mu\nu}^\lambda$ of V_m^λ is the Littlewood–Richardson coefficient. In particular, such algebras are multi-graded by triples (λ, μ, ν) of dominant weights of GL_m or Young diagrams with not more than m rows, and the (λ, μ, ν)-homogeneous component consists of the highest weight vectors of the isomorphic copies of V_m^λ occurring in the tensor product $V_m^\mu \otimes V_m^\nu$. Explicit expressions of such highest weight vectors labeled by the Littlewood–Richardson tableaux (LR tableaux) of shape λ/μ and content ν are investigated in [11,12].

The subring of the twice-iterated Pieri algebra $\mathcal{A}_{n,m}$ consisting of the polynomials invariant under the maximal unipotent subgroup U_2 of GL_2, with respect to (2.1) and (2.2), is indeed an example of the GL_m tensor product algebra. Note that

$$\mathcal{A}_{n,m}^{U_2} = \mathbb{C}[\mathrm{M}_{n+2,m}]^{U_n \times U_2 \times U_m} = \mathbb{C}[\mathrm{M}_{n,m} \oplus \mathrm{M}_{2,m}]^{U_n \times U_2 \times U_m},$$

and then, using (2.4), we have

$$\begin{aligned} \mathcal{A}_{n,m}^{U_2} &\cong \left(\mathbb{C}[\mathrm{M}_{n,m}]^{U_n} \otimes \mathbb{C}[\mathrm{M}_{2,m}]^{U_2}\right)^{U_m} \\ &\cong \left(\bigoplus_\mu V_m^\mu \otimes \bigoplus_\nu V_m^\nu\right)^{U_m} \cong \bigoplus_{\mu,\nu} \left(V_m^\mu \otimes V_m^\nu\right)^{U_m} \end{aligned} \tag{5.2}$$

where μ and ν are Young diagrams with not more than $\min(n, m)$ and $\min(2, m)$ rows, respectively. Then, the U_m-invariant vectors in the tensor product $V_m^\mu \otimes V_m^\nu$ are exactly the highest weight vectors of the irreducible components occurring in the decomposition of $V_m^\mu \otimes V_m^\nu$.

5.2 U_2-invariant cluster variables

Let us consider the action of the subgroup U_2 of GL_2, as given in (2.1) and (2.2), on the last two rows of $(x_{ij}) \in \mathsf{M}_{n+2,m}$. The following result can be verified by straightforward computations.

Lemma 5.1 *Among the cluster variables for the twice-iterated Pieri algebra $\mathcal{A}_{n,m}$, after setting $z_i = \delta_i$ in (4.5), the following are invariant under the action of U_2.*

(1) δ_{4j} *for* $1 \le j \le m$ *if* $m \le n$; *for* $1 \le j \le n$ *if* $m \ge n+1$.
(2) δ_{4j+2} *for* $0 \le j \le m-1$ *if* $m \le n$; *for* $1 \le j \le n$ *if* $m \ge n+1$.
(3) δ_{4j+3} *for* $0 \le j \le m-2$ *if* $m \le n+1$; *for* $1 \le j \le n$ *if* $m \ge n+2$.
(4) $\delta_{4(k+1)+1}\delta_{4(\ell-1)+2} - \delta_{4(k+1)+2}\delta_{4(\ell-1)+1}$ *for* $1 \le \ell \le k \le m-2$ *if* $m \le n$; *for* $1 \le \ell \le k \le n-1$ *if* $m \ge n+1$.

Our next task is to show that these U_2-invariant cluster variables form a generating set of the GL_m tensor product algebra $\mathcal{A}_{n,m}^{U_2}$. In [8,12], Howe et al. constructed highest weight vectors f_T of the isomorphic copies of V_m^λ in the decomposition of $V_m^\mu \otimes V_m^\nu$ attached to LR tableaux T on the skew Young diagram λ/μ with content ν. They showed that these vectors f_T form a $\mathbb{C}$-basis for the (λ, μ, ν)-homogeneous component of the GL_m tensor product algebra, and therefore, every element in the GL_m tensor product algebra can be expressed as a linear combination of such highest weight vectors attached to LR tableaux. Moreover, for each LR tableau T there is an associated monomial m_T such that m_T is equal to the initial monomial $in(f_T)$ of the highest weight vector f_T with respect to a certain monomial order. For our case with $\ell(\nu) \le 2$, it is

$$m_T = \prod_{i \ge 1} x_{ii}^{\mu_i} \cdot \prod_{j \ge 1} \left(y_{1j}^{\alpha_j} y_{2j}^{\beta_j} \right)$$

where α_j and β_j are the numbers of boxes in the jth row of T filled with 1's and 2's, respectively. Note that one can obtain m_T directly from T by replacing the empty boxes, boxes with 1's, and boxes with 2's in the ith row of T with x_{ii}'s, y_{1i}'s, and y_{2i}'s, respectively. For instance,

$$\mathsf{T} = \begin{array}{|c|c|c|c|c|c|c|} \hline \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & 1 & 1 & 2 & 2 \\ \cline{1-6} \ & 1 & 2 \\ \cline{1-3} 2 & 2 \\ \cline{1-2} \end{array} \leftrightarrow \begin{array}{|c|c|c|c|c|c|c|} \hline x_{11} & x_{11} & x_{11} & x_{11} & y_{11} & y_{11} & y_{11} \\ \hline x_{22} & x_{22} & y_{12} & y_{12} & y_{22} & y_{22} \\ \cline{1-6} x_{33} & y_{13} & y_{23} \\ \cline{1-3} y_{24} & y_{24} \\ \cline{1-2} \end{array}$$

which represents the monomial

$$m_T = x_{11}^4 x_{22}^2 x_{33}^1 y_{11}^3 y_{12}^2 y_{22}^2 y_{13}^1 y_{23}^1 y_{24}^2.$$

They further showed that the initial monomial $in(h)$ of any element h in the GL_m tensor product algebra equals to the monomial m_T associated with some LR tableau T. See [8,12] for more details.

In order to apply these results to our $\mathcal{A}_{n,m}^{U_2} \subset \mathbb{C}[M_{n+2,m}]$, first we write y_{cd} for $x_{n+c,d}$ for $c = 1$ and 2, so that the coordinates of the space $M_{n+2,m}$ are given as

$$M_{n+2,m} = \left\{ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \\ y_{11} & y_{12} & \cdots & y_{1m} \\ y_{21} & y_{22} & \cdots & y_{2m} \end{bmatrix} \right\}.$$

For a monomial order, let us use the graded lexicographic order based on $x_{ab} > y_{cd}$ for all x_{ab} and y_{cd}; $x_{ab} > x_{cd}$ and $y_{ab} > y_{cd}$ if and only if $b < d$ or $b = d$ and $a < c$. With respect to this order, the initial monomials of the U_2-invariant cluster variables are

(1) $in(\delta_{4j}) = \prod_{i=1}^{j} x_{ii}$,
(2) $in(\delta_{4j+2}) = y_{1,j+1} \cdot \prod_{i=1}^{j} x_{ii}$,
(3) $in(\delta_{4j+3}) = y_{1,j+1} \cdot y_{2,j+2} \cdot \prod_{i=1}^{j} x_{ii}$,
(4) $in(\delta_{4(k+1)+1}\delta_{4(\ell-1)+2} - \delta_{4(k+1)+2}\delta_{4(\ell-1)+1}) = y_{1,\ell} \cdot y_{2,k+2} \cdot \prod_{i=1}^{k+1} x_{ii} \prod_{i=1}^{\ell-1} x_{ii}$.

Theorem 5.2 *The U_2-invariant cluster variables for $\mathcal{A}_{n,m}$ form a generating set of the GL_m tensor product algebra $\mathcal{A}_{n,m}^{U_2}$.*

Proof It is enough to show that the initial monomials of elements $h \in \mathcal{A}_{n,m}^{U_2}$ are the products of the initial monomials of the U_2 invariant cluster variables. Then, the U_2-invariant cluster variables form a SAGBI basis for $\mathcal{A}_{n,m}^{U_2}$, and therefore, they generate the algebra $\mathcal{A}_{n,m}^{U_2}$. Using the result of [8,12], since the initial monomial $in(h)$ of $h \in \mathcal{A}_{n,m}^{U_2}$ is equal to a monomial attached to some LR tableau, we want to show that for each LR tableau T, its associated monomial m_T is the product of the initial monomials of the U_2-invariant cluster variables.

Recall that each of the LR tableaux T accounting for the multiplicity $c_{\mu\nu}^{\lambda}$ of V_m^{λ} in the decomposition of $V_m^{\mu} \otimes V_m^{\nu}$ in (5.2) satisfies, in addition to the semistandardness condition, the Yamanouchi condition. That is, the number of 1's in its first r rows is at least as large as the number of 2's in the first $r + 1$ rows for all r.

If a LR tableau T contains a column having both 1's and 2's or a column with only empty boxes, then the monomial m_T attached to T is divisible by $in(\delta_{4j+3})$ or $in(\delta_{4j'})$ for some j or j'. Now after removing all such columns of T, if there is a column with

2's in the pth row, then by the Yamanouchi condition, there should be 1's in the qth row for some $q < p$. Let us focus on two columns containing 1's and 2's in the pth and qth rows, respectively. If $q = p-1$, then the monomial corresponding to this pair of columns equals to $in(\delta_{4q-1}) \cdot in(\delta_{4q})$. Note that δ_{4q-1} and δ_{4q} are invariant cluster variables under U_2. If $q < p-1$, then the monomial corresponding to T can be further divisible by the initial monomial of $\Delta_{pq} = \delta_{4(p-1)+1}\delta_{4(q-1)+2} - \delta_{4(p-1)+2}\delta_{4(q-1)+1}$. After removing pairs of columns in T corresponding to the initial monomials of Δ_{pq}'s, the only possible columns are those with only 1's and this shows that m_T can be further divisible by $in(\delta_{4j+2})$'s. Therefore, the monomial m_T for LR tableau T is the product of the initial monomials of U_2 invariant cluster variables, and such cluster variables form a SAGBI basis for the algebra $\mathcal{A}_{n,m}$. □

Let us illustrate our argument in the proof in the language of Young tableaux. For $m = 4$ and $n = 3$, the U_2-invariant cluster variables for the twice-iterated Pieri algebra $\mathcal{A}_{n,m}$ are

(1) $\delta_4, \delta_8, \delta_{12}$;
(2) $\delta_2, \delta_6, \delta_{10}, \delta_{14}$;
(3) $\delta_3, \delta_7, \delta_{11}$;
(4) $\Delta_{31} = \delta_9\delta_2 - \delta_{10}\delta_1$, $\Delta_{41} = \delta_{13}\delta_2 - \delta_{14}\delta_1$, $\Delta_{42} = \delta_{13}\delta_6 - \delta_{14}\delta_5$.

Using (2.5) and Notation 2.3, we can identify the determinants δ_i of submatrices of

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{24} & y_{24} \end{bmatrix}$$

with column tableaux whose empty box in the jth row, 1's, and 2's in the column tableaux indicating the jth, 4th and 5th rows of X, respectively. Then, the U_2-invariant cluster variables can be drawn as

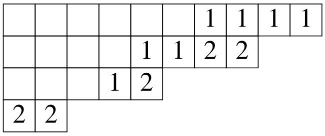
$$\delta_4 = \begin{array}{|c|}\hline \\ \hline\end{array},\ \delta_8 = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array},\ \delta_{12} = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array},\ \delta_2 = \begin{array}{|c|}\hline 1 \\ \hline\end{array},\ \delta_6 = \begin{array}{|c|}\hline \\ \hline 1 \\ \hline\end{array},\ \delta_{10} = \begin{array}{|c|}\hline \\ \hline \\ \hline 1 \\ \hline\end{array},\ \delta_{14} = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline 1 \\ \hline\end{array},$$

$$\delta_3 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array},\ \delta_7 = \begin{array}{|c|}\hline \\ \hline 1 \\ \hline 2 \\ \hline\end{array},\ \delta_{11} = \begin{array}{|c|}\hline \\ \hline \\ \hline 1 \\ \hline 2 \\ \hline\end{array},\ \Delta_{31} = \begin{array}{|c|}\hline \\ \hline \\ \hline 2 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline 1 \\ \hline\end{array} - \begin{array}{|c|}\hline \\ \hline \\ \hline 1 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline 2 \\ \hline\end{array},$$

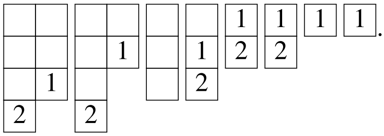
$$\Delta_{41} = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline 2 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline 1 \\ \hline\end{array} - \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline 1 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline 2 \\ \hline\end{array},\ \Delta_{42} = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline 2 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline \\ \hline 1 \\ \hline\end{array} - \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline 1 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline \\ \hline 2 \\ \hline\end{array}.$$

Note that, with this tableau notation, the first terms of the U_2-invariant cluster variables are LR tableaux, and every LR tableau associated with the multiplicity of

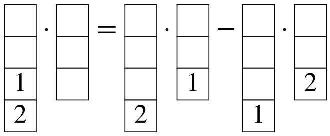
V_4^λ in the decomposition of $V_4^\mu \otimes V_4^\nu$ for μ and ν such that $\ell(\mu) \leq 3$ and $\ell(\nu) \leq 2$ can be realized as a concatenation of those LR tableaux appearing in the U_2-invariant cluster variables. For example, the following LR tableau

$T =$


can be, after its columns are rearranged, considered as the concatenation of smaller LR tableaux



We observe that the first factor appears in the expression of the U_2-invariant $\delta_{11}\delta_{12} = \delta_{13}\delta_{10} - \delta_{14}\delta_9$ by (2.7). That is,



and its related monomial is exactly $in(\delta_{11}\delta_{12}) = in(\delta_{13}\delta_{10})$. This shows that the monomial m_T associated with the LR tableau T can be realized as the product of the initial monomials of U_2-invariant cluster variables

$$m_T = in(\delta_{11}\delta_{12}) \cdot in(\Delta_{42}) \cdot in(\delta_{12}) \cdot in(\delta_7) \cdot in(\delta_3^2) \cdot in(\delta_2^2).$$

In fact, using a similar argument, we can explicitly compute a SAGBI basis for the GL_m tensor product algebras of more general types. We refer the reader to [14].

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