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Regular maps with nilpotent automorphism group

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Abstract A 2-cell decomposition of a closed orientable surface is called a *regular map* if its automorphism group acts transitively on the set of all its darts (or arcs). It is well known that the group $G = \mathrm{Aut}^+(\mathcal{M})$ of all orientation-preserving automorphisms of such a map $\mathcal{M}$ is a finite quotient of the free product $\Gamma = \mathbb{Z} * C_2$. In this paper we investigate the situation where G is nilpotent and the underlying graph of the map is simple (with no multiple edges). By applying a theorem of Labute (Proc Amer Math Soc 66:197–201, 1977) on the ranks of the factors of the lower central series of Γ (via the associated Lie algebra), we prove that the number of vertices of any such map is bounded by a function of the nilpotency class of the group G. Moreover, we show that for a fixed nilpotency class c there is exactly one such simple regular map $\mathcal{M}_c$ attaining the bound, and that this map is universal, in the sense that every simple regular map $\mathcal{M}$ for which $\mathrm{Aut}^+(\mathcal{M})$ is nilpotent of class at most c is a quotient of $\mathcal{M}_c$. In particular, there are finitely many such quotients for any given value of c, and

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every regular map $\mathcal{M}$, whether simple or non-simple, for which $\mathrm{Aut}^+(\mathcal{M})$ is nilpotent of class at most c, is a cyclic cover of exactly one of them.

Keywords Regular map · Graph embedding · Nilpotent group

Mathematics Subject Classification 57M15 · 05C10 · 05E18 · 57M60

1 Introduction

Regular maps are cellular decompositions of closed surfaces exhibiting the largest possible number of symmetries, or the largest number of these that preserve orientation. The study of regular maps grew out of work in the late 1800s by Dyck and Burnside and others on group actions on surfaces, and was progressed substantially by Brahana [1]. Since those earliest times, regular maps have been closely related to their automorphism groups.

The main reason behind this is the fact that the group of all orientation-preserving automorphisms of any regular map on an orientable surface having p-gonal faces, with q faces meeting at each vertex, is a smooth finite quotient of the ordinary triangle group

$$\Delta(q, 2, p) = \langle\, a, b \mid a^q = b^2 = (ab)^p = 1 \,\rangle,$$

with 'smooth' (here) meaning that the orders of the elements a, b and ab are preserved, or equivalently, that the kernel is torsion-free. Conversely, any smooth homomorphism from $\Delta(q, 2, p)$ to a finite group G gives rise to a unique regular map with p-gonal faces, q at each vertex, whose orientation-preserving automorphism group is isomorphic to G.

In this paper we address the problem of classification of regular maps with a given automorphism group. This problem has received relatively little attention in comparison to classifying regular maps on a given surface, or with a given underlying graph. Complete classification results are very rare and, aside from the folklore result describing regular maps with abelian automorphism groups, most of them are related to finite simple groups—see [11,12,19], for example.

Here we take another viewpoint, and concentrate our attention on orientably-regular maps for which the orientation-preserving automorphism group is nilpotent. We will call these *nilpotent maps*. Our attention is restricted to regular maps on orientable surfaces, because the only non-orientable regular maps with nilpotent automorphism group are cycles of length 2^n (for each $n \geq 1$) embedded in the projective plane, and their geometric duals; see [22, p. 450] or [9, Theorem 3.4].

In contrast to the abelian case, the class of nilpotent maps is rich. This class includes, for example, regular embeddings of the complete bipartite graphs $K_{n,n}$ and regular embeddings of the n-dimensional cubes, when n is a power of 2; indeed, the automorphism groups of all such maps are 2-groups (since the numbers of vertices and the orders of vertex-stabilisers are powers of 2). Recently completed classifications of these two families of maps (in [3,6–8]) show that both complete bipartite maps and

n-cube maps exhibit a great variety of forms, and suggest that achieving a complete classification of nilpotent regular maps may be a very difficult problem.

The first thorough study of nilpotent regular maps was undertaken by Malnič, Nedela and Škoviera in [16]. Two fundamental observations were made in the latter study: first, a nilpotent regular map with non-abelian automorphism group must be bipartite (see [16, Theorem 3.4]), and second, every nilpotent regular map can be uniquely decomposed into a direct product of a regular map whose automorphism is a 2-group and a spherical regular map consisting of a single vertex and odd number of semi-edges (see [16, Theorem 3.2]). Moreover, it is easy to see that the second factor adds only edges that are parallel to the edges of the first factor. It follows that nilpotent maps tend to have multiple edges, and those with simple underlying graph must arise from 2-groups.

It is also known that every orientably-regular map $\mathcal{M}$ with multiple edges gives rise to two smaller regular maps: a quotient regular map $\mathcal{M}'$ which has the same number of vertices as $\mathcal{M}$ but with simple underlying graph, and a regular embedding $\mathcal{M}''$ of the dipole graph which has two vertices and the same multiplicity as $\mathcal{M}$; see [23, Theorem 3] or [9, Theorem 2.1]. Moreover, if $\mathcal{M}$ is nilpotent, then both $\mathcal{M}'$ and $\mathcal{M}''$ are also nilpotent. It follows that the problem of classifying nilpotent maps splits into the problem of classifying nilpotent maps with simple underlying graph, the problem of classifying nilpotent dipoles, and the problem of determining how these two types of regular maps may be combined.

As shown in [16], nilpotent dipoles are not difficult to classify. In fact, for every integer $\mu \geq 2$, up to isomorphism there are only one, two, or four nilpotent dipole maps with edge-multiplicity μ; see [16, Theorem 4.6]. Hence the main part of our classification problem involves understanding nilpotent maps with simple underlying graph.

Our main objective in this paper is to show that for any positive integer c, there are only finitely many regular maps with simple underlying graph, such that the orientation-preserving automorphism group of the map is nilpotent of class c. In fact, we prove that the number of vertices of any such map is bounded by a function of c. Also we give an exact formula for the maximum number of vertices of a simple nilpotent map of class c, and show that this maximum is achieved by exactly one nilpotent simple map $\mathcal{M}_c$, and that every other simple nilpotent map of the given class c is a quotient of $\mathcal{M}_c$.

We do this in Sect. 3, after giving some further background in Sect. 2, and to complete the paper we give all nilpotent simple maps of class 2 to 4 in Sect. 4.

2 Preliminaries

In this section we provide some further background, first on orientably-regular maps, and second on nilpotent groups.

2.1 Orientably-regular maps

A *map* is a cellular embedding of a connected graph or multigraph into a closed surface, giving a decomposition of the surface into 0-cells (the *vertices*), 1-cells (the *edges*), and 2-cells (called the *faces* of the map). Note that the faces must be simply-

connected (that is, homeomorphic to a unit disc in $\mathbb{R}^2$), which forces connectedness of the underlying graph. A map is said to be *orientable* if its carrier surface is orientable.

An *automorphism* of a map $\mathcal{M}$ is a permutation of the cells of $\mathcal{M}$ that sends vertices to vertices, edges to edges, and faces to faces, and preserves incidence between them. The group of all automorphisms of $\mathcal{M}$ is denoted by $\mathrm{Aut}(\mathcal{M})$. In the orientable case, an automorphism may preserve or reverse orientation, but in this paper, we will deal with only orientable maps, and we will consider only orientation-preserving automorphisms. These form a group $\mathrm{Aut}^+(\mathcal{M})$, which has index 1 or 2 in $\mathrm{Aut}(\mathcal{M})$, depending on whether the map is *chiral* or *reflexible*, respectively.

It is easy to see that by connectedness, every orientation-preserving automorphism is uniquely determined by its effect on any given ordered edge (or *arc*, or *dart*) of $\mathcal{M}$, and so $\mathrm{Aut}^+(\mathcal{M})$ acts semi-regularly on the set $D_\mathcal{M}$ of all darts of $\mathcal{M}$. It follows that the largest conceivable number of orientation-preserving automorphisms is equal to $|D_\mathcal{M}|$, and that this maximum is achieved if and only if the action of $\mathrm{Aut}^+(\mathcal{M})$ is transitive (and hence regular) on $D_\mathcal{M}$. Thus $\mathcal{M}$ is said to be *regular* (or to be more precise, *orientably-regular*) if $\mathrm{Aut}^+(\mathcal{M})$ is transitive on $D_\mathcal{M}$, or equivalently, if $|\mathrm{Aut}^+(\mathcal{M})| = |D_\mathcal{M}|$.

When $\mathcal{M}$ is orientably-regular, the group $\mathrm{Aut}^+(\mathcal{M})$ also acts transitively on vertices, and on edges, and on faces of $\mathcal{M}$. It follows that every vertex has the same degree/valency q, and every face has the same size p, and in this case, we say that $\mathcal{M}$ has *type* $\{p, q\}$.

Next, it is well known that every orientably-regular map $\mathcal{M}$ can be identified with a triple $(G; a, b)$, where G is a finite group and $\{a, b\}$ is a generating pair for G with $b^2 = 1$. The easiest way to do this is to take $G = \mathrm{Aut}^+(\mathcal{M})$, and then let a be an automorphism of $\mathcal{M}$ that fixes a vertex v and induces a single-step rotation of its q neighbours, consistently with the orientation of the surface, and let b be an automorphism of $\mathcal{M}$ that interchanges v with one of its neighbours, reversing the edge e between them. In that case, the product ab preserves a face f containing e (and v), and induces a single-step rotation of the p vertices of f, again consistently with the orientation of the surface. Then a and b satisfy the defining relations of the ordinary triangle group $\Delta(q, 2, p)$, namely $a^q = b^2 = (ab)^p = 1$, and also by connectedness, they generate G.

Any such triple $(G; a, b)$ may be called an *algebraic orientably-regular map*. From any such triple we can construct an orientably-regular map $\mathcal{M}$, as follows. The vertices, edges, and faces of $\mathcal{M}$ may be taken as the (right) cosets in G of the subgroups $V = \langle a \rangle$, $E = \langle b \rangle$ and $F = \langle ab \rangle$, respectively, with incidence between given by non-empty intersection of cosets. In particular, the number of vertices of $\mathcal{M}$ is equal to the index of $V = \langle a \rangle$ in G. Furthermore, the group G acts naturally on $\mathcal{M}$ by right multiplication of cosets, preserving incidence, and indeed $G = \mathrm{Aut}^+(\mathcal{M})$.

If the underlying graph of this map $\mathcal{M}$ has multiple edges, then some non-trivial subgroup of $V = \langle a \rangle$ is preserved by b, and so the core of V in G is non-trivial. The converse also holds, and therefore the map $M = (G; a, b)$ has simple underlying graph if and only if $V = \langle a \rangle$ is core-free in G.

Also the *genus* of the carrier surface of $\mathcal{M}$ is the non-negative integer g given by the Euler-Poincaré formula $2 - 2g = |G|(1/q - 1/2 + 1/p)$.

An easy example of an algebraic orientably-regular map $\mathcal{M} = (G;\, a, b)$ occurs when G is a cyclic group of order n, generated by the element a, and b is the identity element of G. This map has a single vertex and n darts, with each dart being self-reverse.

Now let $\mathcal{M}_1 = (G_1;\, a_1, b_1)$ and $\mathcal{M}_2 = (G_2;\, a_2, b_2)$ be algebraic orientably-regular maps. If there exists a group epimorphism $\phi\colon G_1 \to G_2$ taking $a_1 \mapsto a_2$ and $b_1 \mapsto b_2$, then we call ϕ a *map homomorphism* from $\mathcal{M}_1$ to $\mathcal{M}_2$, and say that $\mathcal{M}_2$ is a *quotient* of $\mathcal{M}_1$, and $\mathcal{M}_1$ is a *cover* of $\mathcal{M}_2$. (Note: we are not requiring the valency of vertices to be preserved.)

It is sometimes helpful to extend the symbol for the type of an orientably-regular map $\mathcal{M}$ by adding the size of a Petrie polygon. A *Petrie path* is a walk in the underlying graph of $\mathcal{M}$ that is like a 'zig-zag', having the property that any two but no three consecutive edges belong to a face of $\mathcal{M}$, and a *Petrie polygon* is a closed Petrie path. An orientably-regular map $\mathcal{M} = (G;\, a, b)$ of type $\{p, q\}$ with Petrie polygons of size r is said to have type $\{p, q\}_r$. It is easy to see that r is twice the order of the commutator $[a, b] = a^{-1}bab$.

The *mirror image* of an orientably-regular map $\mathcal{M}$ is obtained by reversing its orientation. In particular, if $\mathcal{M} = (G;\, a, b)$ then the mirror image of $\mathcal{M}$ is isomorphic to $(G;\, a^{-1}, b)$, and has the same type as $\mathcal{M}$. Similarly, the (geometric) *dual* of $\mathcal{M}$ is obtained by interchanging the roles of vertices and faces. In particular, when $\mathcal{M} = (G;\, a, b)$ has type $\{p, q\}_r$, the dual of $\mathcal{M}$ is also orientably-regular, of type $\{q, p\}_r$, and is isomorphic to $(G;\, ab, b)$. The *mirror-dual* of $\mathcal{M}$ is the mirror image of the dual of $\mathcal{M}$ (or equivalently, the dual of the mirror image of $\mathcal{M}$), and also has the same type as the dual of $\mathcal{M}$.

For more information about regular maps and the associated groups, see [5]. For more specific details of some of the above properties and constructions, see [13,17].

2.2 Nilpotent groups

We use standard concepts and notation from group theory. We let $|H|$ denote the order of a subgroup H of a group G, and let $|G : H|$ denote its index in G, and we let $|x|$ denote the order of an element x of G. Also we use $[x, y]$ as an abbreviation for the *commutator* $x^{-1}y^{-1}xy$ of an ordered pair (x, y) of elements of G, and $[H, K]$ for the subgroup generated by all commutators $[x, y]$ with $x \in H$ and $y \in K$, when H and K are subgroups of G.

We define *higher-order commutators* inductively by setting

$$[x_1, x_2, \ldots, x_{i+1}] = [[x_1, x_2, \ldots, x_i], x_{i+1}] \text{ for } i \geq 1.$$

In the next section, we will use commutator identities (such as $[a, b]^2 = [a, b^2][b, [a, b]]$) in the lead-up to the proofs of our main theorems.

Next, for any group G, set $G_0 = G$ and then define $G_{n+1} = [G, G_n]$, for all $n \geq 0$, so that $G = G_0 \geq G_1 \geq G_2 \geq \cdots$ is the *lower central series* of G, with each G_n normal in G. (Note that some authors denote G_n by $\gamma_{n+1}(G)$ or $L_{n+1}(G)$ or similar.)

It is easy to prove that if G is generated by a subset X, then G_n is generated by all conjugates in G of elements of $[x_1, x_2, \ldots, x_{n+1}]$ with $x_i \in X$ for $1 \leq i \leq n+1$,

or equivalently, by G_{n+1} and all such commutators; see [10, Hilfsatz III.1.11] for example.

The group G is said to be *nilpotent* if G_c is trivial for some c, and then the smallest c for which this happens is called the *nilpotency class* of G. Also (in general) for each $n \geq 0$, we may consider the quotient G/G_n. For example, $G/G_1 = G/[G, G]$ is the abelianisation of G, and if $G_{n-1} \neq G_n$, then G/G_n is called the *maximal class n nilpotent quotient* of G.

Every finite p-group is nilpotent, and every direct product of nilpotent groups is nilpotent. In fact, an equivalent definition for finite groups is that G is nilpotent if and only if G is the direct product of its Sylow subgroups; see [10, Hauptsatz III.2.3].

3 Nilpotent maps

In this section we prove our main theorems. We establish an upper bound on the number of vertices of a nilpotent regular map of class c. In doing so, we derive an exact formula for the maximum number of vertices in such a map, and show that there is exactly one nilpotent map $\mathcal{M}_c$ of class c with simple underlying graph attaining this maximum, and derive some of its properties.

We begin with the observation that in proving the bound we may restrict our attention to maps whose automorphism group is a 2-group. The following lemma is essentially a rephrasing of Theorem 3.2 from [16].

Lemma 3.1 *Let $\mathcal{M} = (G; a, b)$ be an orientably-regular map for which $G = \text{Aut}^+(\mathcal{M})$ is nilpotent. Then $\mathcal{M}$ has a quotient $\mathcal{M}' = (H; a', b')$ with the property that H is a 2-group and $\mathcal{M}'$ has the same number of vertices as $\mathcal{M}$.*

Proof Since G is nilpotent, G is isomorphic to the direct product of its Sylow subgroups, and so can be expressed as a direct product $H \times K$, where H is the Sylow 2-subgroup of G and K is the Hall $2'$-subgroup of G (of odd order).

Now the canonical generators of $G = \text{Aut}^+(\mathcal{M})$ can be written as $a = a'a''$ and $b = b'b''$, where $\langle a', b' \rangle = H$ and $\langle a'', b'' \rangle = K$, but since $b^2 = 1$ and K has odd order, we must have $b'' = 1$, and therefore K is cyclic, generated by a''. It follows that the subgroup H determines an orientably-regular map $\mathcal{M}' = (H; a', b')$ whose automorphism is a 2-group. Also since H and K have relatively prime orders, we find that $|G|/|\langle a \rangle| = |H||K|/(|\langle a' \rangle||\langle a'' \rangle|) = |H|/|\langle a' \rangle|$, and so the maps $\mathcal{M}$ and $\mathcal{M}'$ have the same number of vertices.

Finally, it is easy to see that $\mathcal{M}'$ is isomorphic to $(G/K; aK, bK)$, and hence a quotient of $\mathcal{M}$, as required. □

Now let $\Gamma = \mathbb{Z} * C_2$ be the free product of the infinite cyclic group with the cyclic group of order 2, with presentation $\langle x, y \mid y^2 = 1 \rangle$.

Clearly, every finite homomorphic image G of this group Γ gives rise to an algebraic map $\mathcal{M} = (G; a, b)$, where a and b are the images of x and y, respectively. When investigating a bound for the number of vertices of $(G; a, b)$ with G nilpotent, we may restrict our attention to the case where the vertex-stabiliser $V = \langle a \rangle$ is core-free in G. By the proof of Lemma 3.1, in that case G is a 2-group.

Before we proceed further, we introduce two infinite families of nilpotent quotients of Γ:

Example 3.2 For each $m \geq 2$, the dihedral group $D_m = \langle a, b \mid a^m = b^2 = (ab)^2 = 1 \rangle$ is a quotient of Γ via the epimorphism taking (x, y) to (a, b). Also D_m is nilpotent whenever m is a power of 2; indeed, if $m = 2^k$ then D_m has nilpotency class k. On the other hand, for all $m \geq 2$ the subgroup generated by a is normal, and hence not core-free in D_m.

Example 3.3 For any $n \geq 1$, let $m = 2^n$ and let P be the wreath product $C_m \wr S_2$, which is generated by two commuting elements u and v of order m and an involution t that interchanges u and v by conjugation. Now $u^t = v$, so $\langle u, t \rangle = P$, and u and t have orders m and 2, so P is a quotient of Γ. Next, $[u, t] = u^{-1} u^t = u^{-1} v = w$, say, and this has order m, and is centralised by u and inverted by (conjugation by) t, so w generates a cyclic normal subgroup of P. It follows that $P_1 = [P, P]$ is generated by w, and an easy induction shows that P_j is the subgroup generated by $w^{2^{j-1}}$, for all j. The smallest j for which P_j is trivial is $n+1$, since w has order $m = 2^n$, and so P has nilpotency class $n+1$. Also t conjugates each non-trivial power of u to the same non-trivial power of v, and in particular, t does not centralise the involution $u^{m/2} = u^{2^{n-1}}$. Hence the core K of the cyclic subgroup generated by u is trivial.

Note The regular map associated with Example 3.2 is the dual of the standard embedding of the m-cycle on the sphere. In Example 3.3 the underlying graph of the associated regular map is the complete bipartite graph $K_{m,m}$ and the map is known as the *standard embedding* of $K_{m,m}$. The associated surface is the Fermat curve $x^m + y^m = 1$; see [14].

We now proceed with some theory of nilpotent quotients of the group $\Gamma = \mathbb{Z} * C_2$.

Lemma 3.4 *The factor group Γ_n / Γ_{n+1} is non-trivial and abelian, for all $n \geq 0$.*

Proof First, the derived subgroup of Γ_n is contained in $[\Gamma, \Gamma_n] = \Gamma_{n+1}$, and so Γ_n / Γ_{n+1} is abelian, for all $n \geq 0$. To see that Γ_n / Γ_{n+1} is non-trivial, we note that the dihedral group $D_{2^{n+1}} = \langle a, b \mid a^{2^{n+1}} = b^2 = (ab)^2 = 1 \rangle$ is a quotient of Γ, and is nilpotent of class $n + 1$. It follows that $D_{2^{n+1}}$ is a quotient of Γ / Γ_{n+1} but not of Γ / Γ_n, so $\Gamma_n \neq \Gamma_{n+1}$, and Γ_n / Γ_{n+1} is non-trivial. $\qquad \square$

Even more can be said about the factors Γ_n / Γ_{n+1} when $n \geq 1$.

Proposition 3.5 *For each $n \geq 1$, the factor Γ_n / Γ_{n+1} of the lower central series of Γ is a finite elementary abelian 2-group.*

Proof First, we know that Γ_n / Γ_{n+1} is abelian (by Lemma 3.4), and also that Γ_n / Γ_{n+1} is finitely generated, since Γ_n is generated by Γ_{n+1} and commutators $[x_1, x_2, \ldots, x_{n+1}]$ with $x_i \in X = \{x, y\}$ for $1 \leq i \leq n+1$ (by [10, Hilfsatz III.1.11]). Hence all we have to do is to show that Γ_n / Γ_{n+1} has exponent 2. On the other hand, Γ_n is generated by conjugates of elements of the form $[a, b]$, where a and b are elements of Γ and Γ_{n-1}, respectively, and so this reduces to showing that the

square of each such $[a, b]$ lies in Γ_{n+1}. To do this, we use the easily proved fact that $[a, b]^2 = [a, b^2][b, [a, b]]$ for all a and b (in a given group).

Now Γ_1 is generated by all conjugates of $z = [x, y]$, since adjoining the relation $[x, y] = 1$ to the presentation for the 2-generator group Γ gives the abelianisation $\Gamma/[\Gamma, \Gamma] = \Gamma/\Gamma_1$. Also if $w \in \Gamma$ then $w^{-1}zw = z[z, w] = z[[x, y], w] \in z[\Gamma_1, \Gamma] = z\Gamma_2$, and therefore every conjugate of z lies in the coset $z\Gamma_2$. Thus Γ_1/Γ_2 is cyclic, generated by $z\Gamma_2$. Moreover, if we take $a = x$ and $b = y$ in the commutator identity $[a, b]^2 = [a, b^2][b, [a, b]]$, we find that $z^2 = [x, y]^2 = [x, y^2][y, [x, y]] = [y, [x, y]] \in [\Gamma, \Gamma_1] = \Gamma_2$. Thus Γ_1/Γ_2 has order 2.

Next, suppose $n > 1$, and let a and b be elements of Γ and Γ_{n-1}, respectively. Then by induction, we may assume that Γ_{n-1}/Γ_n has exponent 2, and hence that $b^2 \in \Gamma_n$. It follows that $[a, b^2] \in [\Gamma, \Gamma_n] = \Gamma_{n+1}$, and then since also $[b, [a, b]] \in [\Gamma, [\Gamma, \Gamma_{n-1}]] = [\Gamma, \Gamma_n] = \Gamma_{n+1}$, the above commutator identity gives $[a, b]^2$ as the product of a pair of elements of Γ_{n+1}, so $[a, b]^2 \in \Gamma_{n+1}$. Hence the square of every generator of Γ_n lies in Γ_{n+1}, and so Γ_n/Γ_{n+1} has exponent 2, as required. $\qquad\square$

Note that Proposition 3.5 does not hold for $n = 0$, because $\Gamma_0/\Gamma_1 = \Gamma/[\Gamma, \Gamma]$ is the abelianisation of Γ, which is isomorphic to $\mathbb{Z} \oplus C_2$ (since x has infinite order).

Corollary 3.6 $(xy)^{2^n} \in x^{2^n}\Gamma_n$ for all $n \geq 1$.

Proof We prove this by induction on n. First $(xy)^2 = x^2x^{-1}yxy = x^2[x, y] \in x^2\Gamma_1$. Next, suppose $(xy)^{2^n} = x^{2^n}z$ where $z \in \Gamma_n$. Then

$$(xy)^{2^{n+1}} = (x^{2^n}z)^2 = (x^{2^n})^2x^{-2^n}zx^{2^n}z^{-1}z^2 = x^{2^{n+1}}[x^{2^n}, z^{-1}]z^2,$$

and as $[x^{2^n}, z^{-1}] \in [\Gamma, \Gamma_n] = \Gamma_{n+1}$ and also $z^2 \in \Gamma_{n+1}$ (because Γ_j/Γ_{j+1} has exponent 2), it follows that $(xy)^{2^{n+1}} \in x^{2^{n+1}}\Gamma_{n+1}$. $\qquad\square$

The first factor Γ/Γ_1 of the lower central series of Γ is the abelianisation of Γ, which is isomorphic to $\mathbb{Z} \oplus C_2$ since x has infinite order while y has order 2. The second factor Γ_1/Γ_2 is cyclic of order 2, as shown in the proof of Proposition 3.5.

The ranks of the factor groups Γ_n/Γ_{n+1} for $n > 1$ are given by some work by Labute [15] on the associated Lie algebra. These ranks are determined by a sequence of integers and two related sequences of partial sums, defined as follows:

Set $h_1 = 0$, and then for $n \geq 2$ take

$$h_n = \frac{1}{n} \sum_{d|n} \mu(n/d) \left(\sum_{0 \leq i \leq d/3} (-1)^i \frac{d}{d-2i} \binom{d-2i}{i} 2^{d-3i} \right),$$

$$g_n = h_1 + h_2 + \cdots + h_n = \sum_{j=1}^{n} h_n,$$

$$f_n = g_1 + g_2 + \cdots + g_n = \sum_{i=1}^{n} g_n,$$

where μ denotes the Möbius function. The first few terms of these three sequences are given below, each starting from $n = 1$:

h_n : 0, 1, 1, 1, 2, 2, 4, 5, 8, 11, 18, 25, 40, 58, 90, 135, 210, 316, …
g_n : 0, 1, 2, 3, 5, 7, 11, 16, 24, 35, 53, 78, 118, 176, 266, 401, 611, 927, …
f_n : 0, 1, 3, 6, 11, 18, 29, 45, 69, 104, 157, 235, 353, 529, 795, 1196, 1807, 2734, ….

We note that the sequence h_n above agrees in all but the first term with the sequence A006206 in Neil Sloane's On-line Encyclopedia of Integer Sequences [20], which counts (among other things) the number of aperiodic binary necklaces of length n with no subsequence 00, excluding the single-term necklace '0'.

The main theorem we use from the work of Labute is the following:

Theorem 3.7 (Labute [15]) *The rank of the factor group* Γ_{n-1}/Γ_n *is* g_n, *for all* $n \geq 2$.

An immediate consequence is the following:

Corollary 3.8 *The order of the quotient* Γ_1/Γ_n *is* 2^{f_n}, *for all* $n \geq 2$.

Proof $|\Gamma_1 : \Gamma_n| = |\Gamma_1 : \Gamma_2||\Gamma_2 : \Gamma_3| \dots |\Gamma_{n-1} : \Gamma_n| = 2^{g_2+g_3+\dots+g_n} = 2^{f_n-f_1} = 2^{f_n}$. □

The above information will help us find the orders of the largest nilpotent simple maps of given class.

Before continuing further, we define $H^{(n)}$ to be the class n nilpotent quotient Γ/Γ_n of Γ. Also we need the following lemma, concerning the generators x and y of Γ:

Lemma 3.9 $[x^{2k}, y] = [x^k, [y, x^k]][[y, x^k], y]$ *for all* k.

Proof Since $y^2 = 1$, we have

$$[x^k, [y, x^k]][[y, x^k], y] = x^{-k}[x^k, y]x^k[y, x^k][x^k, y]y[y, x^k]y$$
$$= x^{-k}[x^k, y]x^k y[y, x^k]y$$
$$= x^{-k}x^{-k}yx^k yx^k yyx^{-k}yx^k y = x^{-2k}yx^{2k}y = [x^{2k}, y].$$

□

Proposition 3.10 *For* $n \geq 0$, *the subgroup of* $H^{(n+1)} = \Gamma/\Gamma_{n+1}$ *generated by the image of* x^{2^n} *is normal, but for* $n > 0$, *the subgroup generated by the image of* $x^{2^{n-1}}$ *is not.*

Proof First, we show that $[x^{2^i}, y]$ lies in Γ_{i+1}, for all $i \geq 0$. This is obvious for $i = 0$, and we can prove it easily by induction for all $i > 0$: if $k = 2^{i-1}$ and $[x^k, y] \in \Gamma_i$, then $[x^k, [y, x^k]] \in [\Gamma, \Gamma_i] = \Gamma_{i+1}$ and $[[y, x^k], y] \in [\Gamma_i, \Gamma] = \Gamma_{i+1}$, and so Lemma 3.9 implies that $[x^{2^i}, y] = [x^{2k}, y]$ is a product of a pair of elements of Γ_{i+1}.

It follows that $y^{-1}(x^{2^n})y = x^{2^n}[x^{2^n}, y]$ lies in the coset $x^{2^n}\Gamma_{n+1}$, for all n. Hence in the quotient Γ/Γ_{n+1}, the image of y normalises the cyclic subgroup X_n/Γ_{n+1} generated by $x^{2^n}\Gamma_{n+1}$. Since x^{2^n} is centralised by x, the latter subgroup is also centralised by the image of x, and so we find that $X_n/\Gamma_{n+1} \lhd \Gamma/\Gamma_{n+1}$.

On the other hand, suppose that the subgroup generated by the image of $x^{2^{n-1}}$ in Γ/Γ_{n+1} were normal. Then in the quotient $C_{2^n} \wr S_2$ of Γ considered in Example 3.3, the subgroup generated by $u^{2^{n-1}}$ would be normal, but it is not—contradiction. □

For a quotient of Γ to be the group of a regular map with simple underlying graph, we need the stabiliser of a vertex to be core-free. It follows that the largest nilpotent quotient of class c that is admissible is the quotient $U^{(c)} = H^{(c)}/K$, where K is the core of the subgroup generated by the image of x in $H^{(c)} = \Gamma/\Gamma_c$, and every other admissible quotient that is nilpotent of class at most c with a core-free vertex-stabiliser is a quotient of $U^{(c)}$.

Theorem 3.11 *For each $c \geq 1$, the group $U^{(c)} = H^{(c)}/K$ has order 2^{c+f_c}, where f_c is as given just before Theorem 3.7. The corresponding regular map has type $\{2^c, 2^{c-1}\}_{2^c}$, with valency 2^{c-1}, face-size 2^c, and Petrie polygons of length 2^c.*

Proof By Proposition 3.10 we know that the core K of the cyclic subgroup X_c/Γ_c generated by the image of x in $H^{(c)} = \Gamma/\Gamma_c$ is generated by the image of $x^{2^{c-1}}$. It follows that in the quotient $U^{(c)} = H^{(c)}/K$, the order of the image of x is 2^{c-1}. Also because the subgroup generated by x has trivial intersection with both Γ_1 and the subgroup generated by y, the abelianisation of $U^{(c)}$ has order $2^{c-1} \cdot 2 = 2^c$. Next, by Corollary 3.8 the factor group Γ_1/Γ_c has order 2^{f_c}, and hence so does its image in the quotient $(\Gamma/\Gamma_c)/K = U^{(c)}$, which is the derived subgroup $(U^{(c)})_1$. Thus $|U^{(c)}| = |U^{(c)} : (U^{(c)})_1||(U^{(c)})_1| = 2^c \cdot 2^{f_c} = 2^{c+f_c}$.

We now calculate the orders of the images of xy and $[x, y]$ in $U^{(c)}$, which determine the sizes of faces and Petrie polygons in the resulting map. By Corollary 3.6, we know that $(xy)^{2^c}$ lies in Γ_c, and so the order of the image of xy in $U^{(c)}$ divides 2^c. Similarly, the commutator $[x, y]$ lies in Γ_1, and the exponent of Γ_1/Γ_c divides 2^{c-1} (since Γ_j/Γ_{j+1} has exponent 2 for $1 \leq j < c$), and so the order of the image of $[x, y]$ in $U^{(c)}$ divides 2^{c-1}. On the other hand, consider the class c nilpotent quotient $C_{2^{c-1}} \wr S_2$ of Γ in Example 3.3 (with $n = c-1$). In this group, the images of xy and $[x, y]$ are the elements ut and $[u, t]$, and these have orders $2m = 2^c$ and $m = 2^{c-1}$, respectively (since $(ut)^2 = uu^t = uv$ and $[u, t] = u^{-1}u^t = u^{-1}v$). It follows that the orders of the images of xy and $[x, y]$ in $U^{(c)}$ are exactly 2^c and 2^{c-1}, and hence in the corresponding map, the face-size is 2^c, and the length of Petrie polygons is $2 \cdot 2^{c-1} = 2^c$ as well. $\square$

Corollary 3.12 *For every integer $c \geq 1$ there exists a unique nilpotent regular map $\mathcal{M}_c$ of class c with simple underlying graph, having 2^{1+f_c} vertices, and type $\{2^c, 2^{c-1}\}_{2^c}$. Furthermore, this map is 'universal', in the sense that every nilpotent regular map of class at most c with simple underlying graph is a quotient of $\mathcal{M}_c$.*

We note that by its universality, the map $\mathcal{M}_c$ is reflexible, and also self-Petrie (isomorphic to its Petrie dual—see [5] or [21]). In fact, universality of $\mathcal{M}_c$ implies also that for every odd j the map $\mathcal{M}_c$ is invariant under the Coxeter/Wilson 'hole' operator H_j, which replaces the image $\bar{x}$ of the generator x by its j-th power $\bar{x}^j$ (see [21]). Similarly, $\mathcal{M}_c$ is invariant under every 'e-switch', which replaces the local rotations at all vertices in one part of a bipartite map by their e-th powers (see [18]). The cases $j = -1$ and $e = -1$ correspond to reflection and the Petrie dual operator, respectively.

4 Nilpotent simple regular maps of small class

We list below all of the nilpotent maps of class 2 to 4, having simple underlying graph. There are 26 such maps, and for each one, we give its genus g, its type $\{p, q\}_r$ and the order of its orientation-preserving automorphism group $G = \mathrm{Aut}^+(\mathcal{M})$, as well as defining relators for G (as words in the given generators x and y of $\Gamma = \mathbb{Z} * C_2 = \langle x, y \mid y^2 = 1 \rangle$). These were found with the help of the MAGMA system [2]. The last one for each class c is the universal map $\mathcal{M}_c$. All of these maps are reflexible.

c	g	$\{p, q\}_r$	$\lvert G \rvert$	Defining relators for G
2	0	$\{4, 2\}_4$	8	$x^2, y^2, (xy)^4$
3	0	$\{8, 2\}_8$	16	$x^2, y^2, (xy)^8$
3	1	$\{4, 4\}_4$	32	$x^4, y^2, (xy)^4, [x, y]^2$
3	3	$\{8, 4\}_8$	32	$x^4, y^2, (xy)^2(x^{-1}y)^2$
3	5	$\{8, 4\}_8$	64	$x^4, y^2, xyx^2yx^{-1}yx^{-2}y, (xy)^4(x^{-1}y)^4$
4	0	$\{16, 2\}_{16}$	32	$x^2, y^2, (xy)^{16}$
4	1	$\{4, 4\}_8$	64	$x^4, y^2, (xy)^4, (x^2y)^2(x^{-2}y)^2$
4	5	$\{8, 4\}_4$	64	$x^4, y^2, [x, y]^2, (x^2y)^2(x^{-2}y)^2$
4	7	$\{16, 4\}_{16}$	64	$x^4, y^2, xyx^2yx^{-1}yx^{-2}y, (xy)^6(x^{-1}y)^2$
4	9	$\{8, 4\}_8$	128	$x^4, y^2, (x^2y)^2(x^{-2}y)^2, xyx^2yxyx^{-1}yx^{-2}yx^{-1}y, (xy)^4(x^{-1}y)^4$
4	13	$\{16, 4\}_{16}$	128	$x^4, y^2, xyx^2yx^{-1}yx^{-2}y, (xy)^8(x^{-1}y)^8$
4	13	$\{16, 4\}_{16}$	128	$x^4, y^2, (x^2y)^2(x^{-2}y)^2, xyx^2yxyx^{-1}yx^{-2}yx^{-1}y, (xy)^5x^{-1}yxyx^{-1}y$
4	21	$\{16, 8\}_{16}$	128	$x^8, y^2, (xy)^2(x^{-1}y)^2$
4	21	$\{16, 8\}_{16}$	128	$x^8, y^2, xyx^2yx^{-1}yx^{-2}y, x^3yx^{-1}yxyx^{-3}y$
4	25	$\{16, 4\}_{16}$	256	$x^4, y^2, (x^2y)^2(x^{-2}y)^2, xyx^2yxyx^{-1}yx^{-2}yx^{-1}y,$ $(xy)^4(x^{-1}y)^2(xy)^2(x^{-1}y)^2(xy)^2(x^{-1}y)^4$
4	33	$\{8, 8\}_8$	256	$x^8, y^2, xyx^2yx^{-1}yx^{-2}y, (xy)^8$
4	33	$\{8, 8\}_8$	256	$x^8, y^2, x^3yx^{-2}yxyx^{-2}y, (xy)^8$
4	41	$\{16, 8\}_{16}$	256	$x^8, y^2, xyx^2yx^{-1}yx^{-2}y, (xy)^4(x^{-1}y)^4$
4	41	$\{16, 8\}_{16}$	256	$x^8, y^2, x^3yx^{-2}yxyx^{-2}y, (xy)^4(x^{-1}y)^4$
4	65	$\{8, 8\}_8$	512	$x^8, y^2, (x^2y)^2(x^{-2}y)^2, xyx^4yx^{-1}yx^{-4}y, (xyxyx^{-2}y)^2, (xy)^8$
4	81	$\{16, 8\}_{16}$	512	$x^8, y^2, xyx^2yx^{-1}yx^{-2}y, (xy)^5(x^{-1}y)^4(xy)^3(x^{-1}y)^4$
4	81	$\{16, 8\}_{16}$	512	$x^8, y^2, (x^2y)^2(x^{-2}y)^2, xyx^4yx^{-1}yx^{-4}y, (xyxyx^{-2}y)^2, (xy)^4(x^{-1}y)^4$
4	81	$\{16, 8\}_{16}$	512	$x^8, y^2, x^3yx^{-2}yxyx^{-2}y,$ $(xy)^3(x^{-1}y)^2xy(x^{-1}y)^3(xy)^2x^{-1}y(xy)^2(x^{-1}y)^2$
4	81	$\{16, 8\}_{16}$	512	$x^8, y^2, (x^2y)^2(x^{-2}y)^2, xyx^4yx^{-1}yx^{-4}y, (xyxyx^{-2}y)^2,$ $(xy)^3(x^{-1}y)^2xy(x^{-1}y)^2$
4	81	$\{16, 8\}_{16}$	512	$x^8, y^2, (x^2y)^2(x^{-2}y)^2, xyx^4yx^{-1}yx^{-4}y, (xyxyx^{-2}y)^2,$ $x^3yx^{-1}yxyx^{-1}yxy(x^{-1}y)^3$
4	161	$\{16, 8\}_{16}$	1024	$x^8, y^2, (x^2y)^2(x^{-2}y)^2, xyx^4yx^{-1}yx^{-4}y, (xyxyx^{-2}y)^2,$ $(xy)^5(x^{-1}y)^2xy(x^{-1}y)^3(xy)^2(x^{-1}y)^3$

For class 5, there are 681 simple maps, of which 663 are reflexible (with genera ranging from 0 to 13313), while the remaining 18 such maps fall into nine chiral pairs (of genera 45, 105, 177, 417 (two pairs) and 833 (four pairs)). The smallest chiral examples are a pair of genus 25 and type $\{16, 4\}_8$, with $G = \mathrm{Aut}^+(\mathcal{M})$ having order 256 and nilpotency class 6; these are the dual and mirror-dual of the chiral map labelled

C25.1 in the first author's list of all arc-transitive chiral maps of genus 2 to 101 (see [4]).

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