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Hecke algebras with independent parameters

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Abstract We study the Hecke algebra $\mathcal{H}(\mathbf{q})$ over an arbitrary field $\mathbb{F}$ of a Coxeter system (W, S) with independent parameters $\mathbf{q} = (q_s \in \mathbb{F} : s \in S)$ for all generators. This algebra always has a spanning set indexed by the Coxeter group W, which is indeed a basis if and only if every pair of generators joined by an odd edge in the Coxeter diagram receives the same parameter. In general, the dimension of $\mathcal{H}(\mathbf{q})$ could be as small as 1. We construct a basis for $\mathcal{H}(\mathbf{q})$ when (W, S) is simply laced. We also characterize when $\mathcal{H}(\mathbf{q})$ is commutative, which happens only if the Coxeter diagram of (W, S) is simply laced and bipartite. In particular, for type A, we obtain a tower of semisimple commutative algebras whose dimensions are the Fibonacci numbers. We show that the representation theory of these algebras has some features in analogy/connection with the representation theory of the symmetric groups and the 0-Hecke algebras.

Keywords Hecke algebra · Independent parameters · Fibonacci number · Independent set · Grothendieck group

1 Introduction

Let $W := \langle S : (st)^{m_{st}} = 1,\ \forall s, t \in S\rangle$ be a Coxeter group. The *(Iwahori-)Hecke algebra* of the Coxeter system (W, S) is a one-parameter deformation of the group

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algebra of W, which has significance in many areas, such as algebraic combinatorics, knot theory, quantum groups, representation theory of p-adic groups, and so on. We generalize the definition of the Hecke algebra of (W, S) from a single parameter to multiple independent parameters.

Definition 1.1 Let $\mathbb{F}$ be an arbitrary field. The *Hecke algebra* $\mathcal{H}(\mathbf{q}) = \mathcal{H}_S(\mathbf{q})$ *of the Coxeter system* (W, S) *with independent parameters* $\mathbf{q} = (q_s \in \mathbb{F} : s \in S)$ is the (associative) $\mathbb{F}$-algebra generated by $\{T_s : s \in S\}$ with

- quadratic relations $(T_s - 1)(T_s + q_s) = 0$ for all $s \in S$,
- braid relations $(T_s T_t T_s \cdots)_{m_{st}} = (T_t T_s T_t \cdots)_{m_{st}}$ for all $s, t \in S$.

Here $(aba \cdots)_m$ is an alternating product of m terms.

The algebra $\mathcal{H}(\mathbf{q})$ can be represented by the Coxeter diagram of (W, S) with extra labels q_s for all vertices $s \in S$. For simplicity, we only draw the labels of the vertices but not the vertices themselves. For example, we draw

$$1 = 0 - 1 - 0 - 1 - 0 - 1 - 0$$

for the usual Coxeter system of type B_8 whose Coxeter diagram is

$$s_1 = s_2 - s_3 - s_4 - s_5 - s_6 - s_7 - s_8$$

with independent parameters $\mathbf{q} = (q_{s_i} : 1 \le i \le 8) = (1, 0, 1, 0, 1, 0, 1, 0)$.

The quadratic relations for $\mathcal{H}(\mathbf{q})$ can be rewritten as $T_s^2 = (1 - q_s)T_s + q_s$ for all $s \in S$. If $q_s \neq 0$, then T_s is invertible and $T_s^{-1} = q_s^{-1} T_s + 1 - q_s^{-1}$. For any $w \in W$ with a reduced expression $w = st \cdots r$ where $s, t, \ldots, r \in S$, the element $T_w := T_s T_t \cdots T_r$ is well defined thanks to the word property of W (see, e.g., [3, Theorem 3.3.1]).

If $q_s = q$ for all $s \in S$, then $\mathcal{H}(\mathbf{q})$ is the usual Hecke algebra of (W, S) with parameter q. If one only insists $q_s = q_t$ whenever m_{st} is odd, then $\mathcal{H}(\mathbf{q})$ is the *Hecke algebra with unequal parameters* in the sense of Lusztig [7]. Now, we allow $\mathbf{q} = (q_s \in \mathbb{F} : s \in S)$ to be arbitrary. The following result may be well known to the experts, and we include a proof for it in the end of Sect. 3 for completeness.

Theorem 1.2 *The algebra* $\mathcal{H}(\mathbf{q})$ *is always spanned by* $\{T_w : w \in W\}$*, which is indeed a basis if and only if* $\mathcal{H}(\mathbf{q})$ *is a Hecke algebra with unequal parameters, i.e.,* $q_s = q_t$ *whenever* m_{st} *is odd.*

In general, we show that the algebra $\mathcal{H}(\mathbf{q})$ could be much smaller than the group algebra $\mathbb{F}W$.

Theorem 1.3 *If there exist* $s, t \in S$ *with* m_{st} *odd such that* q_s *and* q_t *are distinct nonzero parameters, then one has* $\mathcal{H}_S(\mathbf{q}) \cong \mathcal{H}_{S \setminus R}(\mathbf{q})$ *where* R *consists of all elements* $r \in S$ *connected to* s *via some path with odd edge weights and nonzero vertex labels in the Coxeter diagram of* (W, S).

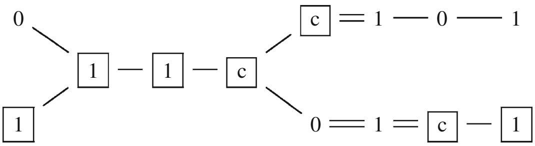
Thus, we always assume without loss of generality that $\mathcal{H}(\mathbf{q})$ is *collapse free*, i.e., if m_{st} is odd and $q_s \neq q_t$, then at least one of q_s and q_t is 0. We next characterize when $\mathcal{H}(\mathbf{q})$ is commutative.

Theorem 1.4 *The algebra $\mathcal{H}(\mathbf{q})$ is collapse free and commutative if and only if (W, S) is simply laced and exactly one of q_s and q_t is 0 for any pair of elements $s, t \in S$ with $m_{st} = 3$.*

We construct a basis for $\mathcal{H}(\mathbf{q})$ (not necessarily commutative) when (W, S) is simply laced (Theorem 4.3). It implies the dimension of a commutative $\mathcal{H}(\mathbf{q})$, giving one motivation for our study of the commutative case.

Corollary 1.5 *Let G be the underlying graph of the Coxeter diagram of (W, S), and let $\mathcal{I}(G)$ be the set of all independent sets in G. If $\mathcal{H}(\mathbf{q})$ is collapse free and commutative, then its dimension is $|\mathcal{I}(G)|$ (the Merrifield-Simmons index of the graph G). In particular, if (W, S) is of type A_n, then the dimension of $\mathcal{H}(\mathbf{q})$ is the Fibonacci number F_{n+2}.*

Example 1.6 Let $\mathbb{F}$ be a field with at least 3 distinct elements 0, 1, and c. Let $\mathcal{H}(\mathbf{q})$ be given by the diagram below.



Removing the boxed elements gives 3 connected components 0, $0 = 1$, and $1 - 0 - 1$. Thus, the dimension of $\mathcal{H}(\mathbf{q})$ is $2 \cdot 8 \cdot 5 = 80$ by Theorems 1.2, 1.3, 1.4, and Corollary 1.5.

Theorem 1.4 shows that if $\mathcal{H}(\mathbf{q})$ is collapse free and commutative, then the Coxeter diagram of (W, S) must be a simply laced bipartite graph. Here a *bipartite graph* is a graph whose vertices can be partitioned into two disjoint sets U and V such that every edge connects one vertex in U and one in V. Computations in Magma suggest the following conjecture, which is verified for type A (Theorem 5.4). This gives another motivation for our study of the commutative case.

Conjecture 1.7 *If the Coxeter diagram of (W, S) is a simply laced bipartite graph G, then a collapse-free $\mathcal{H}(\mathbf{q})$ has minimum dimension equal to $|\mathcal{I}(G)|$, which is attained when $\mathcal{H}(\mathbf{q})$ is commutative.*

For the irreducible simply laced Coxeter systems of type A, D, $\widetilde{A}$, and $\widetilde{D}$, the dimensions of collapse free and commutative Hecke algebras $H(\mathbf{q})$ are given below, which all happen to satisfy the Fibonacci recurrence.

Note that the Coxeter diagram of $\widetilde{A}_n$ is a cycle of length n, which is bipartite if and only if n if even. However, the dimensions given above for $\widetilde{A}_n$ make sense for all integers $n \geq 1$. This is because we can define a commutative algebra $\mathcal{H}(G, R)$ whose

Coxeter diagram	Dimensions	Known as	OEIS entry
A_n $(n \geq 1)$	2,3,5,8,13,…	Fibonacci numbers F_{n+2}	A000045
D_n $(n \geq 2)$	4,5,9,14,23,…	?	A000285
$\widetilde{A}_n$ $(n \geq 3)$	4,7,11,18,29,…	Lucas numbers L_n	A000032
$\widetilde{D}_n$ $(n \geq 5)$	17, 24, 41,65,106,…	?	A190996

dimension is $|\mathcal{I}(G)|$ for any (unweighted) simple graph G with vertex set $V(G)$ and edge set $E(G)$ and for any $R \subseteq V(G)$, such that a collapse-free and commutative Hecke algebra $\mathcal{H}(\mathbf{q})$ is isomorphic to $\mathcal{H}(G, R)$ where G is the Coxeter diagram of the simply laced (W, S) and $R = \{s \in S : q_s = -1\}$. This algebra $\mathcal{H}(G, R)$ is defined as the quotient of the polynomial algebra $\mathbb{F}[x_v : v \in V(G)]$ by its ideal generated by

$$\{x_r^2 : r \in R\} \cup \{x_v^2 - x_v : v \in V(G) \setminus R\} \cup \{x_u x_v : uv \in E(G)\}.$$

It is also a quotient of the *Stanley-Reisner ring of the independence complex of* G [5].

We show the following results on the representation theory of $\mathcal{H}(G, R)$. The projective indecomposable $\mathcal{H}(G, R)$-modules are indexed by $\mathcal{I}(G - R)$, where $G - R$ is the graph obtained from G by deleting R and all edges incident to R. The simple $\mathcal{H}(G, R)$-modules are all one dimensional and also indexed by $\mathcal{I}(G - R)$. The Cartan matrix of $\mathcal{H}(G, R)$ is a diagonal matrix. The algebra $\mathcal{H}(G, R)$ is semisimple if and only if $R = \emptyset$.

We next apply the above results to type A. Let $G = P_{n-1}$ be a path with $n - 1$ vertices. One sees that the dimension of the algebra $\mathcal{H}(P_{n-1}, R)$ is equal to the Fibonacci number F_{n+1}. We further assume that this algebra is semisimple, i.e., $R = \emptyset$, and write $\mathcal{H}_n := \mathcal{H}(P_{n-1}, \emptyset)$. If char $(\mathbb{F}) \neq 2$ then $\mathcal{H}_n$ is isomorphic to the Hecke algebra $\mathcal{H}(\mathbf{q})$ of the Coxeter system of type A_{n-1} with independent parameters $\mathbf{q} = (0, 1, 0, 1, \ldots)$ or $\mathbf{q} = (1, 0, 1, 0, \ldots)$. We summarize our results on the algebra $\mathcal{H}_n$ below. The reader who is familiar with the representation theory of the symmetric group $\mathfrak{S}_n$ and/or the 0-Hecke algebra $\mathcal{H}_n(0)$ can see certain features of our results in analogy with $\mathfrak{S}_n$ and/or $H_n(0)$.

The semisimple commutative algebra $\mathcal{H}_n$ has F_{n+1} many nonisomorphic simple modules, which are all one dimensional and indexed by compositions of n with internal parts larger than 1. The *Grothendieck group* $G_0(\mathcal{H}_n)$ of finite-dimensional representations of $\mathcal{H}_n$ is a free abelian group on these simple $\mathcal{H}_n$-modules. The tower of algebras $\mathcal{H}_\bullet : \mathcal{H}_0 \hookrightarrow \mathcal{H}_1 \hookrightarrow \mathcal{H}_2 \hookrightarrow \cdots$ has a *Grothendieck group*

$$G_0(\mathcal{H}_\bullet) := \bigoplus_{n \geq 0} G_0(\mathcal{H}_n)$$

with a product and a coproduct given by the induction and restriction along the embeddings $\mathcal{H}_m \otimes \mathcal{H}_n \hookrightarrow \mathcal{H}_{m+n}$.

Although *not* a bialgebra, $G_0(\mathcal{H}_\bullet)$ has a self-dual basis consisting of simple $\mathcal{H}_n$-modules for all $n \geq 0$. We provide explicit formulas for the structure constants of the product and coproduct of $G_0(\mathcal{H}_\bullet)$ in terms of this self-dual basis, which are naturally all positive. This result connects $G_0(\mathcal{H}_\bullet)$ to the Grothendieck groups of the finite-dimensional (projective) representations of the 0-Hecke algebras $\mathcal{H}_n(0)$, or equivalently, the dual Hopf algebras **NSym** of *noncommutative symmetric functions* and QSym of *quasisymmetric functions*. It turns out that $G_0(\mathcal{H}_\bullet)$ is a quotient algebra of **NSym** and a subcoalgebra of QSym, but its antipode satisfies a different rule than the antipodes of QSym and **NSym**. The *Bratteli diagram* of the tower $\mathcal{H}_\bullet$ is a binary tree on compositions with internal parts larger than 1.

This paper is structured as follows. We first provide preliminaries in Sect. 2. Then, we discuss when $\mathcal{H}(\mathbf{q})$ collapses or becomes commutative in Sect. 3. We study the algebra $\mathcal{H}(\mathbf{q})$ of a simply laced Coxeter system in Sect. 4 and investigate the simply laced bipartite case in Sect. 5. We provide more results on the commutative case in Sect. 6 and give the type A specialization in Sect. 7. Finally, we give remarks and questions in Sect. 8.

2 Preliminaries

2.1 Coxeter groups and Hecke algebras

A *Coxeter group* is a group with the following presentation

$$W := \langle S : s^2 = 1,\ (sts\cdots)_{m_{st}} = (tst\cdots)_{m_{st}},\ \forall s, t \in S,\ s \neq t \rangle$$

where the generating set S is finite, $m_{st} = m_{ts} \in \{2, 3, \ldots\} \cup \{\infty\}$, and $(aba\cdots)_m$ is an alternating product of m terms. By convention, no relation is imposed between s and t if $m_{st} = \infty$. The pair (W, S) is called a *Coxeter system*.

The Coxeter diagram of (W, S) is an edge-weighted graph whose vertices are the elements in S and whose edges are the unordered pairs $\{s, t\}$ with weight m_{st} for all $s, t \in S$ such that $m_{st} \geq 3$, $s \neq t$. An edge with weight $m_{st} \leq 5$ is often drawn as $m_{st} - 2$ many multiple edges between s and t. An edge is *simply laced* if its weight is 3. If every edge is simply laced, then the Coxeter system (W, S) and its Coxeter diagram are both called *simply laced*.

An element w in W can be written as a product of elements in S. Among all such expressions, the shortest ones are called *reduced*, and the length of a reduced expression of w is called the *length* of w and denoted by $\ell(w)$. A *nil-move* deletes s^2, and a *braid-move* replaces $(sts\cdots)_{m_{st}}$ with $(tst\cdots)_{m_{st}}$ in the expressions of $w \in W$ as products of elements in S. By [3, Theorem 3.3.1], W satisfies the following word property.

Word property *Any expression of $w \in W$ as a product of elements in S can be transformed into a reduced expression of w by braid-moves and nil-moves, and every pair of reduced expressions for w can be connected via braid-moves.*

A subset $I \subseteq S$ generates a *parabolic subgroup* $W_I := \langle I \rangle$ of W. The pair (W_I, I) is a Coxeter system whose Coxeter diagram is the edge-weighted subgraph of the Coxeter diagram of (W, S) induced by the vertex subset $I \subseteq S$. If $S_1, \ldots, S_k$ are the vertex sets of the connected components of the Coxeter diagram of (W, S), then $W = W_{S_1} \times \cdots \times W_{S_k}$. Thus, (W, S) is *irreducible* if its Coxeter diagram is connected.

There is a well-known classification for finite irreducible Coxeter groups, among which type A is of particular interest. The symmetric group $\mathfrak{S}_n$ is the Coxeter group of type A_{n-1} with generating set S consisting of the adjacent transpositions $s_i := (i, i+1)$ for $i = 1, \dots, n-1$. The Coxeter diagram of $\mathfrak{S}_n$ is the path $s_1 — s_2 — \cdots — s_{n-1}$.

The *(Iwahori-)Hecke algebra* $\mathcal{H}_S(q)$ of a Coxeter system (W, S) is a one-parameter deformation of the group algebra of W. Let $\mathbb{F}$ be a field, and let $q \in \mathbb{F}$. Then, $\mathcal{H}_S(q)$ is defined as the $\mathbb{F}$-algebra generated by $\{T_s : s \in S\}$ with

- quadratic relations: $(T_s - 1)(T_s + q) = 1,\ \forall s \in S$,
- braid relations: $(T_s T_t T_s \cdots)_{m_{st}} = (T_t T_s T_t \cdots)_{m_{st}},\ \forall s, t \in S,\ s \neq t.$

The specialization of the Hecke algebra $\mathcal{H}_S(q)$ at $q = 1$ gives the group algebra $\mathbb{F}W$, and the specialization at $q = 0$ gives the *0-Hecke algebra* $\mathcal{H}_S(0)$. If (W, S) is of type A_{n-1}, then we write $\mathcal{H}_n(q) := \mathcal{H}_S(q)$ and $\mathcal{H}_n(0) := \mathcal{H}_S(0)$.

If $w \in W$ has a reduced expression $w = st \cdots r$, where $s, t, \dots, r \in S$, then $T_w := T_s T_t \cdots T_r$ is well defined thanks to the word property of W. It is well known that $\{T_w : w \in W\}$ is a basis for $\mathcal{H}_S(q)$. One has

$$T_s T_w = \begin{cases} (1-q)T_w + qT_{sw}, & \ell(sw) < \ell(w), \\ T_{sw}, & \ell(sw) > \ell(w), \end{cases} \tag{2.1}$$

for all $s \in S$ and $w \in W$. This gives the *regular representation* of $\mathcal{H}_S(q)$.

2.2 Representation theory of associative algebras

We review some general results on the representation theory of associative algebras (see, e.g., [2, § I]). Let $\mathbb{F}$ be a field, and let A be a finite-dimensional (unital associative) $\mathbb{F}$-algebra. Let M be a (left) A-module. If M has no submodules except 0 and itself, then M is *simple*. If M is a direct sum of simple A-modules, then M is *semisimple*. The algebra A is *semisimple* if it is semisimple as an A-module. Every module over a semisimple algebra is also semisimple. If M cannot be written as a direct sum of two nonzero A-submodules, then M is *indecomposable*. If M is a direct summand of a free A-module, then M is *projective*.

The *(Jacobson) radical* $\mathrm{rad}(M)$ of M is the intersection of all maximal A-submodules of M, which turns out to be the smallest submodule N of M such that M/N is semisimple. One has $\mathrm{rad}(M_1 \oplus M_2) = \mathrm{rad}(M_1) \oplus \mathrm{rad}(M_2)$ if M_1 and M_2 are two A-modules. The radical of the algebra A is defined as $\mathrm{rad}(A)$ with A itself viewed as an A-module. If A happens to be commutative, then all nilpotent elements in A form an ideal of A, called the *nilradical* of A, which is always contained in $\mathrm{rad}(A)$. The *top* of M is the quotient module $\mathrm{top}(M) := M/\mathrm{rad}(M)$. The *socle* $\mathrm{soc}(M)$ of M is the sum of all minimal submodules of M, which is the largest semisimple submodule of M.

Every A-module can be written as a direct sum of indecomposable A-submodules. Let A itself as an A-module be a direct sum of indecomposable A-modules $\mathbf{P}_1, \dots, \mathbf{P}_k$. Although $\mathbf{P}_i$ is not simple in general, its top $\mathbf{C}_i$ is. Moreover, every projective indecomposable A-module is isomorphic to some $\mathbf{P}_i$, and every simple A-module is isomorphic to some $\mathbf{C}_i$. Suppose without loss of generality that $\{\mathbf{P}_1, \ldots, \mathbf{P}_\ell\}$ and $\{\mathbf{C}_1, \ldots, \mathbf{C}_\ell\}$ are complete lists of nonisomorphic projective indecomposable A-modules and simple A-modules, respectively, where $\ell \le k$. Then, the *Cartan matrix* of A is $[a_{ij}]_{i,j\in[\ell]}$ where a_{ij} is the multiplicity of $\mathbf{C}_j$ among the composition factors of $\mathbf{P}_i$.

The *Grothendieck group* $G_0(A)$ *of the category of finitely generated* A*-modules* is defined as the abelian group F/R, where F is the free abelian group on the isomorphism classes $[M]$ of finitely generated A-modules M, and R is the subgroup of F generated by the elements $[M] - [L] - [N]$ corresponding to all exact sequences $0 \to L \to M \to N \to 0$ of finitely generated A-modules. The *Grothendieck group* $K_0(A)$ *of the category of finitely generated projective* A*-modules* is defined similarly. We often identify a finitely generated (projective) A-module with the corresponding element in the Grothendieck group $G_0(A)$ ($K_0(A)$). It turns out that $G_0(A)$ and $K_0(A)$ are free abelian groups with bases $\{\mathbf{C}_1, \ldots, \mathbf{C}_\ell\}$ and $\{\mathbf{P}_1, \ldots, \mathbf{P}_\ell\}$, respectively. If L, M, N are all projective A-modules, then the exact sequence $0 \to L \to M \to N \to 0$ is equivalent to the direct sum decomposition $M \cong L \oplus N$. If A is semisimple, then $G_0(A) = K_0(A)$ since $\mathbf{P}_i = \mathbf{C}_i$ for all i.

Let B be a subalgebra of A. The *induction* $N \uparrow_B^A$ of a B-module N from B to A is the A-module $A \otimes_B N$. The *restriction* $M \downarrow_B^A$ of an A-module M from A to B is M itself viewed as a B-module. The induction and restriction are well defined for isomorphic classes of modules.

2.3 Representation theory of symmetric groups and 0-Hecke algebras

The (complex) representation theory of the symmetric group is fascinating and has rich connections with symmetric function theory. The simple $\mathbb{C}\mathfrak{S}_n$-modules S_λ are indexed by partitions λ of n, and every $\mathbb{C}\mathfrak{S}_n$-module is a direct sum of simple $\mathbb{C}\mathfrak{S}_n$-modules, i.e., $\mathbb{C}\mathfrak{S}_n$ is semisimple. Thus, the Grothendieck group $G_0(\mathbb{C}\mathfrak{S}_n) = K_0(\mathbb{C}\mathfrak{S}_n)$ is a free abelian group on the isomorphism classes $[S_\lambda]$ for all partitions λ of n. The tower of groups $\mathfrak{S}_\bullet : \mathfrak{S}_0 \hookrightarrow \mathfrak{S}_1 \hookrightarrow \mathfrak{S}_2 \hookrightarrow \cdots$ has a Grothendieck group

$$G_0(\mathbb{C}\mathfrak{S}_\bullet) := \bigoplus_{n\ge 0} G_0(\mathbb{C}\mathfrak{S}_n).$$

Using the natural embedding $\mathfrak{S}_m \times \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{m+n}$, one can define the product of S_μ and S_ν as the induction of $S_\mu \otimes S_\nu$ from $\mathfrak{S}_m \times \mathfrak{S}_n$ to $\mathfrak{S}_{m+n}$ for all partitions $\mu \vdash m$ and $\nu \vdash n$, and define the coproduct of S_λ as the sum of its restriction to $\mathfrak{S}_i \times \mathfrak{S}_{n-i}$ for $i = 0, 1, \ldots, n$, for all partitions $\lambda \vdash n$. This gives $G_0(\mathbb{C}\mathfrak{S}_\bullet)$ a self-dual graded Hopf algebra structure, as the product and coproduct share the same structure constants, namely the *Littlewood-Richardson coefficients*.

The *Frobenius characteristic map* ch sends a simple S_λ to the Schur function s_λ, giving a Hopf algebra isomorphism between the Grothendieck group $G_0(\mathbb{C}\mathfrak{S}_\bullet)$ and Sym, the *ring of symmetric functions* (see Stanley [11, Chapter 7]).

The 0-Hecke algebra $\mathcal{H}_n(0)$ has analogous representation theory as the symmetric group $\mathfrak{S}_n$. We first review some notation. A *composition* is a sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers. Let $\sigma_i := \alpha_1 + \cdots + \alpha_i$ for $i = 1, \ldots, \ell$.

The *size* $|\alpha|$ of the composition α is the sum of all its *parts* $\alpha_1, \ldots, \alpha_\ell$, i.e., $|\alpha| = \sigma_\ell$. If $|\alpha| = n$ then we say that α is a composition of n and write $\alpha \models n$. The *descent set* of α is $D(\alpha) := \{\sigma_1, \ldots, \sigma_{\ell-1}\}$. Sending α to $D(\alpha)$ gives a bijection between compositions of n and subsets of $[n-1]$.

Now recall from Norton [8] that the 0-Hecke algebra $\mathcal{H}_n(0)$ has the following decomposition

$$\mathcal{H}_n(0) = \bigoplus_{\alpha \models n} \mathbf{P}_\alpha(0)$$

where the $\mathbf{P}_\alpha(0)$ is pairwise nonisomorphic indecomposable $\mathcal{H}_n(0)$-modules. The top of $\mathbf{P}_\alpha(0)$ is one dimensional and denoted by $\mathbf{C}_\alpha(0)$. Thus, the two Grothendieck groups $G_0(\mathcal{H}_n(0))$ and $K_0(\mathcal{H}_n(0))$ are free abelian groups on the isomorphism classes of $\mathbf{C}_\alpha(0)$ and $\mathbf{P}_\alpha(0)$, respectively, for all compositions α. Associated with the tower of algebras $\mathcal{H}_\bullet(0) : \mathcal{H}_0(0) \hookrightarrow \mathcal{H}_1(0) \hookrightarrow \mathcal{H}_2(0) \hookrightarrow \cdots$ are two Grothendieck groups

$$G_0(\mathcal{H}_\bullet(0)) := \bigoplus_{n\geq 0} G_0(\mathcal{H}_n(0)) \quad \text{and} \quad K_0(\mathcal{H}_\bullet(0)) := \bigoplus_{n\geq 0} K_0(\mathcal{H}_n(0)).$$

They are dual graded Hopf algebras with product and coproduct again given by induction and restriction of representations along the natural embeddings $\mathcal{H}_m(0) \otimes \mathcal{H}_n(0) \hookrightarrow \mathcal{H}_{m+n}(0)$ of algebras. The duality is given by the pairing $\langle \mathbf{P}_\alpha(0), \mathbf{C}_\beta(0)\rangle := \delta_{\alpha,\beta}$ for all compositions α and β.

For later use, we review the explicit formulas for the product of $K_0(\mathcal{H}_\bullet(0))$ and the coproduct of $G_0(\mathcal{H}_\bullet(0))$. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \ldots, \beta_k)$ be compositions of m and n, respectively. We write

$$\alpha\beta := (\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_k) \quad \text{and} \quad \alpha \rhd \beta := (\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \ldots, \beta_k).$$

For any $i \in \{0, 1, \ldots, m\}$, let r be the largest integer such that $\sigma_r := \alpha_1 + \cdots + \alpha_r$ is no more than i, and write

$$\alpha_{\leq i} := (\alpha_1, \ldots, \alpha_r, i - \sigma_r) \quad \text{and} \quad \alpha_{>i} := (\sigma_{r+1} - i, \alpha_{r+2}, \ldots, \alpha_\ell)$$

where we ignore $i - \sigma_r$ if it happens to be 0.

Proposition 2.1 (Krob and Thibon [6]) *For any $\alpha \models m$ and $\beta \models n$, one has*

$$\mathbf{P}_\alpha(0) \hat{\otimes} \mathbf{P}_\beta(0) := \big(\mathbf{P}_\alpha(0) \otimes \mathbf{P}_\beta(0)\big) \uparrow\,^{\mathcal{H}_{m+n}(0)}_{\mathcal{H}_m(0)\otimes\mathcal{H}_n(0)} = \mathbf{P}_{\alpha\beta}(0) \oplus \mathbf{P}_{\alpha\rhd\beta}(0),$$
$$\Delta(\mathbf{C}_\alpha(0)) := \sum_{i=0}^{m} \mathbf{C}_\alpha(0) \downarrow\,^{\mathcal{H}_m(0)}_{\mathcal{H}_i(0)\otimes\mathcal{H}_{m-i}(0)} = \sum_{i=0}^{m} \mathbf{C}_{\alpha_{\leq i}}(0) \otimes \mathbf{C}_{\alpha_{>i}}(0).$$

For example, one has $\mathbf{P}_{213}(0)\,\hat{\otimes}\,\mathbf{P}_{223}(0) = \mathbf{P}_{213223}(0) \oplus \mathbf{P}_{21523}(0)$. Let $\emptyset$ be the empty composition of $n = 0$. Then

$$\Delta(\mathbf{C}_{121}(0)) = \mathbf{C}_{\emptyset}(0) \otimes \mathbf{C}_{121}(0) + \mathbf{C}_{1}(0) \otimes \mathbf{C}_{21}(0) + \mathbf{C}_{11}(0) \otimes \mathbf{C}_{11}(0) + \mathbf{C}_{12}(0) \otimes \mathbf{C}_{1}(0) + \mathbf{C}_{121}(0) \otimes \mathbf{C}_{\emptyset}(0).$$

The representation theory of the 0-Hecke algebras is connected with the dual graded Hopf algebras QSym of *quasisymmetric functions* and **NSym** of *noncommutative symmetric functions*. There are dual bases for QSym and **NSym** consisting of the *fundamental quasisymmetric functions* F_α and the *noncommutative ribbon Schur functions* $\mathbf{s}_\alpha$ for all compositions α. Krob and Thibon [6] introduced two Hopf algebra isomorphisms

$$\mathrm{Ch} : G_0(\mathcal{H}_\bullet(0)) \cong \mathrm{QSym} \quad \text{and} \quad \mathbf{ch} : K_0(\mathcal{H}_\bullet(0)) \cong \mathbf{NSym}$$

defined by $\mathrm{Ch}(\mathbf{C}_\alpha(0)) = F_\alpha$ and $\mathbf{ch}(\mathbf{P}_\alpha(0)) = \mathbf{s}_\alpha$ for all compositions α. There is an injection Sym $\hookrightarrow$ QSym of Hopf algebras given by inclusion, as well as a surjection **NSym** $\twoheadrightarrow$ Sym of Hopf algebras by taking commutative image.

3 Collapse and commutativity

Let (W, S) be a Coxeter system and let $\mathbb{F}$ be a field. Recall from Definition 1.1 that the *Hecke algebra* $\mathcal{H}(\mathbf{q}) = \mathcal{H}_S(\mathbf{q})$ *of the Coxeter system* (W, S) *with independent parameters* $\mathbf{q} = (q_s \in \mathbb{F} : s \in S)$ is the (associative) $\mathbb{F}$-algebra generated by $\{T_s : s \in S\}$ with

- quadratic relations $(T_s - 1)(T_s + q_s) = 0$ for all $s \in S$,
- braid relations $(T_s T_t T_s \cdots)_{m_{st}} = (T_t T_s T_t \cdots)_{m_{st}}$ for all $s, t \in S$.

In this section, we study when the algebra $\mathcal{H}(\mathbf{q})$ collapses or becomes commutative.

We first study the *parabolic subalgebras* of $\mathcal{H}(\mathbf{q})$. We know that any subset $R \subseteq S$ generates a Coxeter subsystem (W_R, R) of (W, S). However, the subalgebra of $\mathcal{H}(\mathbf{q})$ generated by $\{T_r : r \in R\}$ is not necessarily isomorphic to the Hecke algebra $\mathcal{H}_R(\mathbf{q})$ of the Coxeter system (W_R, R) with independent parameters $(q_r : r \in R)$. For example, if there exist two elements s and t in S such that q_s and q_t are distinct nonzero parameters and m_{st} is odd, then the algebra $\mathcal{H}_{\{s\}}(\mathbf{q})$ is two dimensional, but Theorem 3.2 below gives $T_s = 1$ in $\mathcal{H}(\mathbf{q})$. To guarantee an isomorphism between these two algebras, we assume that $R \subseteq S$ is *admissible*, i.e., if m_{st} is odd for $s \in R$ and $t \in S \setminus R$ then either $q_s = 0$ or $q_t = 0$. If R is admissible, then one sees that $S \setminus R$ is also admissible. We denote the generating set of $\mathcal{H}_R(\mathbf{q})$ by $\{T'_r : r \in R\}$, which satisfies the relations $(T'_r - 1)(T'_r + q_r) = 0$ and $(T'_r T'_t T'_r \cdots)_{m_{rt}} = (T'_t T'_r T'_t \cdots)_{m_{rt}}$ for all $r, t \in R$.

Proposition 3.1 *For any $R \subseteq S$, there is an algebra surjection from $\mathcal{H}_R(\mathbf{q})$ to the subalgebra of $\mathcal{H}(\mathbf{q})$ generated by $\{T_r : r \in R\}$ by sending T'_r to T_r for all $r \in R$, which is an isomorphism when R is admissible.*

Proof Sending T_r' to T_r for all $r \in R$ gives an algebra map $\phi : \mathcal{H}_R(\mathbf{q}) \to \mathcal{H}(\mathbf{q})$ whose image is the subalgebra of $\mathcal{H}(\mathbf{q})$ generated by $\{T_r : r \in R\}$. Suppose that R is admissible and define

$$\psi(T_s) = \begin{cases} T_s', & \text{if } s \in R, \\ 1, & \text{if } s \in S \setminus R,\ q_s \neq 0, \\ 0, & \text{if } s \in S \setminus R,\ q_s = 0. \end{cases}$$

One sees that the quadratic relations are preserved by ψ. We next check the braid relations. Let $s, t \in S$ with $m_{st} = m$.

If s and t are both in R then $\psi(T_s) = T_s'$ and $\psi(T_t) = T_t'$ satisfy the same braid relation as T_s and T_t.

If $s \in R$ and $t \in S \setminus R$, then $\psi(T_t) \in \{0, 1\}$. When m is even, one has

$$(\psi(T_s)\psi(T_t)\psi(T_s)\cdots)_m = (\psi(T_t)\psi(T_s)\psi(T_t)\cdots)_m.$$

When m is odd and $q_t = 0$, one has $\psi(T_t) = 0$ and the above quality still holds. When m is odd and $q_t \neq 0$, one has $\psi(T_t) = 1$ and the admissibility of R implies $q_s = 0$. Thus,

$$(\psi(T_s)\psi(T_t)\psi(T_s)\cdots)_m = (T_s')^{(m+1)/2} = (T_s')^{(m-1)/2} = (\psi(T_t)\psi(T_s)\psi(T_t)\cdots)_m.$$

It follows that ψ is a well-defined algebra map. Restricted to the image of ϕ, the map ψ is nothing but the inverse of ϕ. Thus, the result holds. □

We say that a path in the Coxeter diagram of (W, S) is *odd* if all its edges have odd weights, and *nonzero* if all its vertices, including the two end vertices, correspond to nonzero parameters. The *collapsed subset* of S consists of all elements $r \in S$ that are connected to some other vertex s (depend on r) with $q_s \neq q_r$ via an odd nonzero path.

Theorem 3.2 *If R is the collapsed subset of S, then (i) $T_r = 1$, $\forall r \in R$, (ii) $T_s \notin \mathbb{F}$, $\forall s \in S \setminus R$, and (iii) $\mathcal{H}(\mathbf{q}) \cong \mathcal{H}_{S \setminus R}(\mathbf{q})$.*

Proof By definition, for any $r \in R$, there exists an odd nonzero path $(r, s, \ldots, t)$ from r to some $t \in S$ such that $q_r \neq q_t$. We show (i) by induction on the length of the path. First assume that the length is 1, i.e., there is an edge between r and t with an odd weight $m := m_{rt}$. The braid relation between T_r and T_t implies that

$$T_r(T_rT_tT_r\cdots T_r)_m = (T_rT_tT_r\cdots T_t)_{m+1} = (T_tT_rT_t\cdots T_t)_m T_t.$$

Using the quadratic relations for T_r and T_t, one obtains

$$q_r(T_tT_rT_t\cdots)_{m-1} + (1-q_r)(T_rT_tT_r\cdots)_m = q_t(T_tT_rT_t\cdots)_{m-1}$$
$$+(1-q_t)(T_tT_rT_t\cdots)_m.$$

Hence,

$$(q_r-q_t)(T_tT_rT_t\cdots T_r)_{m-1}=(q_r-q_t)(T_rT_tT_r\cdots T_r)_m=(q_r-q_t)(T_tT_rT_t\cdots T_t)_m.$$

Since $q_r \neq 0$, $q_t \neq 0$, and $q_r \neq q_t$, one can apply the inverses of T_r, T_t, and $(q_r - q_t)$ to get $T_r = T_t = 1$.

Now suppose that the path $(r, s, \ldots, t)$ has length at least two. If $q_r \neq q_s$ then $T_r = 1$ by the above argument. Otherwise, $q_r = q_s \neq q_t$ and one has $T_s = 1$ by induction, since $(s, \ldots, t)$ is an odd nonzero path of smaller length. Then, applying T_r^{-1} to the braid relation between T_r and T_s gives $T_r = 1$. This proves (i).

To show (ii), we assume $T_s \in \mathbb{F}$ for some $s \in S$. If $q_s = 0$, then $\{s\}$ is admissible, and thus, the subalgebra of $\mathcal{H}(\mathbf{q})$ generated by T_s is two dimensional by Proposition 3.1, which is absurd. Therefore, $q_s \neq 0$. Let U be the set of all elements in S that are connected to s via odd nonzero paths, including s itself. Then, $q_u \neq 0$ for all $u \in U$. One sees that U is admissible, and hence, the subalgebra of $\mathcal{H}(\mathbf{q})$ generated by $\{T_u : u \in U\}$ is isomorphic to the algebra $\mathcal{H}_U(\mathbf{q})$ by Proposition 3.1. If $|\{q_u : u \in U\}| = 1$, then $\mathcal{H}_R(\mathbf{q})$ has a basis indexed by W_U, and hence, $T_s \notin \mathbb{F}$, a contradiction. Therefore, $|\{q_u : u \in U\}| \geq 2$. This forces $s \in R$ and establishes (ii).

Finally, one sees that $S \setminus R$ is admissible. By Proposition 3.1, $\mathcal{H}_{S\setminus R}(\mathbf{q})$ is isomorphic to the subalgebra of $\mathcal{H}(\mathbf{q})$ generated by $\{T_s : s \in S \setminus R\}$. Hence, (iii) follows from (i). □

By Theorem 3.2, we may always assume without loss of generality that $\mathcal{H}(\mathbf{q})$ is *collapse free*, i.e., if m_{st} is odd and $q_s \neq q_t$, then either q_s or q_t is 0. We next develop some lemmas in order to characterize when $\mathcal{H}(\mathbf{q})$ is commutative.

Lemma 3.3 *If $S = \{s, t\}$, $q_s = 0 \neq q_t$, and $m := m_{st}$ is odd, then $\mathcal{H}(\mathbf{q})$ has dimension $2m - 3$ and a basis*

$$\{(T_sT_tT_s\cdots)_k,\ (T_tT_sT_t\cdots)_k : k = 0, 1, 2, \ldots, m-2\}.$$

Proof Since $q_s = 0 \neq q_t$ and m is odd, it follows from the defining relations for $\mathcal{H}(\mathbf{q})$ that

$$\begin{aligned}(T_sT_tT_s\cdots T_s)_m &= (T_sT_tT_s\cdots T_t)_{m+1} = (T_tT_sT_t\cdots T_t)_m T_t\\ &= q_t(T_tT_sT_t\cdots)_{m-1} + (1-q_t)(T_tT_sT_t\cdots)_m\end{aligned}$$

which implies $(T_tT_sT_t\cdots)_{m-1} = (T_tT_sT_t\cdots)_m$ and thus $(T_sT_tT_s\cdots)_{m-2} = (T_sT_tT_s\cdots)_{m-1}$. Similarly,

$$\begin{aligned}(T_sT_tT_s\cdots)_m &= (T_tT_sT_t\cdots T_s)_{m+1} = T_t(T_tT_sT_t\cdots)_m\\ &= q_t(T_sT_tT_s\cdots)_{m-1} + (1-q_t)(T_tT_sT_t\cdots)_m.\end{aligned}$$

Thus, $(T_sT_tT_s\cdots T_t)_{m-1}=(T_tT_sT_t\cdots T_t)_m$ and $(T_sT_tT_s\cdots)_{m-2}=(T_tT_sT_t\cdots)_{m-1}$. It follows that $\mathcal{H}(\mathbf{q})$ is spanned by the desired basis. Then, it remains to show that the dimension of $\mathcal{H}(\mathbf{q})$ is at least $2m - 3$.

To achieve this, we define an $\mathcal{H}(\mathbf{q})$-action on the $\mathbb{F}$-span of $Z := \{(sts\cdots)_k, (tst\cdots)_k : k = 0, 1, 2, \ldots, m-2\}$ where $(sts\cdots)_0 = (tst\cdots)_0 = 1$ by convention.

The dimension of $\mathbb{F}Z$ is by definition $|Z| = 2m - 3$. Define

$$\begin{cases} T_s(tst\cdots)_k = (sts\cdots)_{k+1}, & 0 \le k \le m-3, \\ T_t(sts\cdots)_k = (tst\cdots)_{k+1}, & 0 \le k \le m-3, \\ T_s(sts\cdots)_k = (sts\cdots)_k, & 1 \le k \le m-2, \\ T_t(tst\cdots)_k = q_t(sts\cdots)_{k-1} + (1-q_t)(tst\cdots)_k, & 1 \le k \le m-2, \\ T_s(tst\cdots)_{m-2} = T_t(sts\cdots)_{m-2} = (sts\cdots)_{m-2}. \end{cases}$$

One sees that the quadratic relations for T_s and T_t are both satisfied by this action, and so is the braid relation because

$$(T_sT_tT_s\cdots)_m(z) = (T_tT_sT_t\cdots)_m(z) = (sts\cdots)_{m-2}, \quad \forall z \in Z.$$

Hence, $\mathbb{F}Z$ becomes a cyclic $\mathcal{H}(\mathbf{q})$-module generated by 1. This forces the dimension of $\mathcal{H}(\mathbf{q})$ to be at least $2m - 3$. □

Lemma 3.4 *Suppose that there exists a path* $(s = s_0, s_1, s_2, \ldots, s_k = t)$ *consisting of simply laced edges in the Coxeter diagram of* (W, S), *where* $k \ge 1$. *If* $q_{s_i} \ne 0$ *and* $m_{ss_i} \le 3$ *for all* $i \in [k]$, *and* $q_s = 0$, *then* $T_sT_t = T_tT_s = T_s$.

Proof We show $T_sT_t = T_tT_s = T_s$ by induction on k. One has $T_s^2 = T_s$ since $q_s = 0$. One also sees that T_{s_i} is invertible and $T_{s_{i-1}}T_{s_i}T_{s_{i-1}} = T_{s_i}T_{s_{i-1}}T_{s_i}$ for each $i \in [k]$. If $k = 1$, then

$$T_sT_tT_s = (T_sT_tT_s)T_s = T_t(T_sT_tT_s) = T_t^2T_sT_t = q_tT_sT_t + (1-q_t)T_tT_sT_t.$$

Since $q_t \ne 0$, one has $T_sT_t = T_tT_sT_t$, and thus, $T_s = T_tT_s$ using T_t^{-1}. Then, $T_sT_t = T_tT_sT_t = T_sT_tT_s = T_s^2 = T_s$.

Now assume $k \ge 2$. If $m_{st} = 3$, then $T_sT_t = T_tT_s = T_s$ by the above argument. Assume $m_{st} = 2$, i.e., $T_sT_t = T_tT_s$. Let $r = s_{k-1}$. Then, $T_rT_s = T_sT_r = T_s$ by induction hypothesis. Thus,

$$T_tT_s = T_tT_sT_r = T_sT_tT_r = T_sT_rT_tT_r = T_sT_tT_rT_t = T_t^2T_s = q_tT_s + (1-q_t)T_tT_s.$$

This implies $T_sT_t = T_tT_s = T_s$, which completes the proof. □

Now, we provide a characterization for when $\mathcal{H}(\mathbf{q})$ is commutative. It implies that there exists $\mathbf{q} \in \mathbb{F}^S$ such that $\mathcal{H}(\mathbf{q})$ is collapse free and commutative if and only if the Coxeter diagram of (W, S) is simply laced and bipartite.

Theorem 3.5 *Suppose that* $\mathcal{H}(\mathbf{q})$ *is collapse free. Then,* $\mathcal{H}(\mathbf{q})$ *is commutative if and only if the Coxeter diagram of* (W, S) *is simply laced and exactly one of* q_s, q_t *is* 0 *for any pair of elements* $s, t \in S$ *with* $m_{st} = 3$.

Proof We first assume that $\mathcal{H}(\mathbf{q})$ is commutative. Let $s, t \in S$ with $m_{st} \ge 3$. We need to show that $m_{st} = 3$ and exactly one of q_s and q_t is 0. To attain this, we first show that $\{s, t\}$ is admissible. By symmetry, it suffices to show that $q_r q_s = 0$ for any $r \in S \setminus \{s, t\}$ with m_{rs} odd. Suppose to the contrary that $q_r q_s \neq 0$. Then, $q_r = q_s$ since $\mathcal{H}(\mathbf{q})$ is collapse free. Let R be a maximal subset of S containing s such that $q_a = q_b$ whenever $a, b \in R$ and m_{ab} is odd. Then, $r \in R$. The maximality forces R to be admissible. By Proposition 3.1, $\mathcal{H}_R(\mathbf{q})$ is isomorphic to a subalgebra of $\mathcal{H}(\mathbf{q})$ and thus commutative. It also has a basis $\{T_w : w \in W_R\}$ by Theorem 1.2. Hence, $m_{rs} \leq 2$, a contradiction.

Therefore, $\{s, t\}$ is admissible. Then, $\mathcal{H}_{\{s,t\}}(\mathbf{q})$ is isomorphic to a subalgebra of $\mathcal{H}(\mathbf{q})$ and hence commutative. Since $m_{st} \geq 3$, Theorem 1.2 implies that m_{st} is odd and $q_s \neq q_t$. Then, exactly one of q_s and q_t must be 0 since $\mathcal{H}(\mathbf{q})$ is collapse free. By Lemma 3.3, the dimension of $\mathcal{H}_{\{s,t\}}(\mathbf{q})$ is $2m - 3$, and hence, $m_{st} = 3$. This proves one direction of the theorem. The other direction follows from Lemma 3.4. □

Finally, using the results in this section, we obtain a proof for Theorem 1.2. One can check that $\{T_w : w \in W\}$ spans $\mathcal{H}(\mathbf{q})$ using the word property of W and the defining relations of $\mathcal{H}(\mathbf{q})$. If $q_s = q_t$ whenever m_{st} is odd, then $\{T_w : w \in W\}$ is a basis for $\mathcal{H}(\mathbf{q})$ by Lusztig [7, Proposition 3.3]. Conversely, suppose that $\{T_w : w \in W\}$ is a basis for $\mathcal{H}(\mathbf{q})$. Let $s, t \in S$ with $m := m_{st}$ odd. The dimension d of the subalgebra of $\mathcal{H}(\mathbf{q})$ generated by T_s and T_t equals the cardinality of the subgroup $\langle s, t\rangle$ of W, which is $2m$ by the word property of W. On the other hand, if $q_s \neq q_t$, then either $d = 1 < 2m$ when $q_s q_t \neq 0$ by Theorem 3.2, or $d \leq 2m - 3 < 2m$ when $q_s q_t = 0$ by Proposition 3.1 and Lemma 3.3. Hence $q_s = q_t$.

4 The simply laced case

In this section, we study a collapse-free Hecke algebra $\mathcal{H}(\mathbf{q})$ with independent parameters $\mathbf{q} = (q_s \in \mathbb{F} : s \in S)$ of a simply laced Coxeter system (W, S). We first give some lemmas in order to construct a basis for $\mathcal{H}(\mathbf{q})$.

Lemma 4.1 *If (W, S) is simply laced, then S decomposes into a disjoint union of $S_1, \ldots, S_k$ such that*

(i) *the elements of each S_i receive the same parameters and are connected in the Coxeter diagram of (W, S),*
(ii) *if $s \in S_i$, $t \in S_j$, $i \neq j$, then either $m_{st} = 2$ or exactly one of q_s and q_t is 0.*

Proof We remove from the Coxeter diagram of (W, S) all the edges whose two end vertices correspond to distinct parameters. Let $S_1, \ldots, S_k$ be the vertex sets of the connected components of the resulting graph.

If $s, t \in S_i$, then there exists a path from s to t, whose vertices have the same parameter. Thus, (i) holds.

If $s \in S_i$, $t \in S_j$, $i \neq j$, and $m_{st} = 3$, then one has $q_s \neq q_t$, and thus, exactly one of q_s and q_t is 0 since $\mathcal{H}(\mathbf{q})$ is collapse free. Hence, (ii) holds. □

Let $W_i := \langle S_i\rangle$, where S_i is as in Lemma 4.1, for all $i = 1, \ldots, k$. We say an element $w_i \in W_i$ *dominates* S_j if $i \neq j$ and there exist $s \in S_i$ and $t \in S_j$ such that $q_s = 0$, $m_{st} = 3$, and s occurs in some reduced expression of w_i. Let $W(\mathbf{q})$ be the set of all elements $(w_1, \dots, w_k) \in W_1 \times \cdots \times W_k$ such that $w_j = 1$ whenever some w_i dominates S_j. We need to define an $\mathcal{H}(\mathbf{q})$-action on $\mathbb{F}W(\mathbf{q})$. Let s be an arbitrary element in S. Then, $s \in S_i$ for some $i \in [k]$. Let $\mathbf{w} = (w_1, \dots, w_k) \in W(\mathbf{q})$. We define $T_s(\mathbf{w}) := (T_s(\mathbf{w})_1, \dots, T_s(\mathbf{w})_k) \in \mathbb{F}W(\mathbf{q})$ as follows.

If S_i is dominated by some w_j, then T_s acts *trivially* on $\mathbf{w}$, meaning that $T_s(\mathbf{w}) := \mathbf{w}$. Otherwise, T_s acts *nontrivially* on $\mathbf{w}$: if $\ell(sw_i) < \ell(w_i)$ then $T_s(\mathbf{w})_i = (1-q)w_i + qsw_i$ and $T_s(\mathbf{w})_j = w_j$ for all $j \neq i$; if $\ell(sw_i) > \ell(w_i)$, then $T_s(\mathbf{w})_i = sw_i$, $T_s(\mathbf{w})_j = 1$ for all $j \neq i$ such that s dominates S_j, and $T_s(\mathbf{w})_j = w_j$ for all $j \neq i$ such that s does not dominates S_j. In other words, if S_i is not dominated by w_j for all $j \neq i$, then T_s acts on the ith component of $\mathbf{w}$ in the same way as the regular representation of the Hecke algebra $\mathcal{H}_{S_i}(q_s)$ (see (2.1)), and for all $j \neq i$, one has

$$T_s(\mathbf{w})_j = \begin{cases} w_j, & \text{if } s \text{ does not dominate } S_j, \\ 1, & \text{if } s \text{ dominates } S_j. \end{cases}$$

Lemma 4.2 *One has a well defined $\mathcal{H}(\mathbf{q})$-action on $\mathbb{F}W(\mathbf{q})$ such that every element $(w_1, \dots, w_k)$ in $W(\mathbf{q})$ is equal to $T_{w_1} \cdots T_{w_k}(1)$.*

Proof Let $s \in S_i$ and let $\mathbf{w} = (w_1, \dots, w_k) \in W(\mathbf{q})$. We first show that $T_s(\mathbf{w}) \in \mathbb{F}W(\mathbf{q})$. We may assume that T_s acts nontrivially on $\mathbf{w}$, i.e., S_i is not dominated by w_j for all $j \neq i$. If $\ell(sw_i) < \ell(w_i)$, then $\mathbf{w} \in W(\mathbf{q})$ implies

$$T_s(\mathbf{w}) = (1-q)\mathbf{w} + q(w_1, \dots, w_{i-1}, sw_i, w_{i+1}, \dots, w_k) \in W(\mathbf{q}).$$

If $\ell(sw_i) > \ell(w_i)$, then $T_s(\mathbf{w}) \in W(\mathbf{q})$ since $T_s(\mathbf{w})_i = sw_i$ and $T_s(\mathbf{w})_j = 1$ whenever s dominates S_j.

Next, we verify the quadratic relation for the action of T_s. If T_s acts trivially on $\mathbf{w}$, then $T_s^2 = (1-q_s)T_s + q_s$ clearly holds. Assume that T_s acts nontrivially on $\mathbf{w}$ and apply T_s again to $T_s(\mathbf{w})$. For the i-th component, this is the same as the regular representation of $\mathcal{H}_{S_i}(q_s)$ (see 2.1). Hence, $T_s^2 = (1-q_s)T_s + q_s$ holds for the i-th component. Let $j \neq i$. If s does not dominate S_j, then $T_s(\mathbf{w})_j = w_j$ is fixed by T_s. If s dominates S_j, then $T_s(w_j) = 1$ is also fixed by T_s, and $q_s = 0$. Hence, $T_s^2 = (1-q_s)T_s + q_s$ also holds for the j-th component for all $j \neq i$.

Next, we verify the braid relation between T_s and T_t for any $t \in S_i \setminus \{s\}$. If one of T_s and T_t acts trivially on $\mathbf{w}$, then so does the other. Thus, we may assume that T_s and T_t both act nontrivially on $\mathbf{w}$. Then, they both act on the i-th component of $\mathbf{w}$ by the regular representation of $\mathcal{H}_{S_i}(q_s)$, and hence, the braid relation holds for this component. Let $j \neq i$ and let $T(s,t)$ be any product of T_s and T_t that contains both of them. If either s or t dominates S_i, then $T(s,t)$ sends w_j to 1. If neither of s and t dominates S_j, then $T(s,t)$ fixes w_j. Hence, the braid relation between T_s and T_t also holds for the j-th component for all $j \neq i$.

Next, assume that $t \in S_j$ and $i \neq j$. First consider the case when s dominates S_j. Since $q_s = 0$, one has $T_s(\mathbf{w})_i = w_i$ if $\ell(sw_i) < \ell(w_i)$ and $T_s(\mathbf{w})_i = sw_i$ if $\ell(sw_i) > \ell(w_i)$. In either case, T_t acts trivially on $T_s(\mathbf{w})$, i.e., $T_t(T_s(\mathbf{w})) = T_s(\mathbf{w})$. On the other hand, since $q_t \neq 0$, one sees that T_t dominates nothing and thus fixes all components of

$\mathbf{w}$ except the j-th one. Since s dominates S_j, one also has $T_s(T_t(\mathbf{w}))_j = T_s(\mathbf{w})_j = 1$. Hence, $T_s(T_t(\mathbf{w})) = T_s(\mathbf{w})$.

Similarly, if t dominates S_i, then one has $T_sT_t(\mathbf{w}) = T_t(\mathbf{w}) = T_tT_s(\mathbf{w})$. For the remaining case, that is, when s does not dominate S_j and t does not dominates S_i, one has $m_{st} = 2$ by Lemma 4.1 (ii). We need to show that both actions of T_sT_t and T_tT_s on $\mathbf{w}$ are the same. One sees for both actions that T_s and T_t act separately on w_i and w_j by the regular representations of $\mathcal{H}_{S_i}(q_s)$ and $\mathcal{H}_{S_j}(q_t)$, respectively. Let $h \in [k] \setminus \{i, j\}$. If S_h is dominated by either s or t, then both T_sT_t and T_tT_s send w_h to 1. Otherwise, both T_sT_t and T_tT_s fix w_j. Hence, $T_sT_t(\mathbf{w}) = T_tT_s(\mathbf{w})$.

Therefore, one has a well-defined action of $\mathcal{H}(\mathbf{q})$ on $\mathbb{F}W(\mathbf{q})$. One sees that every element $(w_1, \ldots, w_k)$ in $W(\mathbf{q})$ is equal to $T_{w_1}\cdots T_{w_k}(1)$ by induction on $\ell(w_1) + \cdots + \ell(w_k)$. This completes the proof. □

Theorem 4.3 *Assume that* (W, S) *is simply laced and* $\mathcal{H}(\mathbf{q})$ *is collapse free. Then,* $\mathcal{H}(\mathbf{q})$ *has a basis*

$$B(\mathbf{q}) := \{T_{w_1}\cdots T_{w_k} : (w_1, \ldots, w_k) \in W(\mathbf{q})\}.$$

Proof Theorem 1.2 shows that $\mathcal{H}(\mathbf{q})$ is spanned by $\{T_w : w \in W\}$. Let $s \in S_i$, $t \in S_j$, and $i \neq j$. If $m_{st} = 2$, then $T_sT_t = T_tT_s$. If $m_{st} = 3$, then we may assume $0 = q_s \neq q_t$ by Lemma 4.1 and it follows from Lemma 3.4 that $T_sT_r = T_s = T_rT_s$ for all $r \in S_j$. Hence, for any $w \in W$, one can write $T_w = T_{w_1}\cdots T_{w_k}$ where $\mathbf{w} = (w_1, \ldots, w_k) \in W(\mathbf{q})$. This shows that $B(\mathbf{q})$ is a spanning set for $\mathcal{H}(\mathbf{q})$. On the other hand, it follows from Lemma 4.2 that $B(\mathbf{q})$ is also linearly independent. Thus, $B(\mathbf{q})$ is a basis for $\mathcal{H}(\mathbf{q})$. □

Corollary 4.4 *Suppose that* (W, S) *is simply laced and let* $S_1, \ldots, S_k$ *be given by Lemma 4.1.*

(i) *A collapse-free* $\mathcal{H}(\mathbf{q})$ *is finite dimensional if and only if* $W_i := \langle S_i \rangle$ *is finite for all* $i \in [k]$.
(ii) *There exists* $\mathbf{q} \in \mathbb{F}^S$ *such that* $\mathcal{H}(\mathbf{q})$ *is collapse free and finite dimensional if and only if there exists* $R \subseteq S$ *such that the parabolic subgroups* $\langle R \rangle$ *and* $\langle S \setminus R \rangle$ *are finite.*

Proof (i) By Theorem 4.3, a collapse-free $\mathcal{H}(\mathbf{q})$ is finite dimensional if and only if $W(\mathbf{q})$ is finite. For any $i \in [k]$, there are injections $W_i \hookrightarrow W(\mathbf{q}) \hookrightarrow W_1 \times \cdots \times W_k$. Hence, $W(\mathbf{q})$ is finite if and only if W_i is finite for all $i \in [k]$.

(ii) Suppose that $\mathcal{H}(\mathbf{q})$ is collapse free and finite dimensional. Let $R := \{s \in S : q_s = 0\}$. By Lemma 4.1, we may assume $R = S_1 \cup \cdots \cup S_j$. Then, $\langle R \rangle = \langle S_1 \rangle \times \cdots \times \langle S_j \rangle$ and $\langle S \setminus R \rangle = \langle S_{j+1} \rangle \times \cdots \times \langle S_k \rangle$ are both finite groups by (i). Conversely, if there exists a subset $R \subseteq S$ such that $\langle R \rangle$ and $\langle S \setminus R \rangle$ are both finite groups, then $\mathcal{H}(\mathbf{q})$ is finite dimensional by (i), where $\mathbf{q}$ is defined by $q_s = 0$ for all $s \in R$ and $q_s = 1$ for all $s \notin R$. □

Example 4.5 (i) It is well known that the Coxeter group of affine type A is infinite and so is the associated Hecke algebra with a single parameter. However, if one takes some parameters to be 0 and others to be 1, the resulting algebra is finite dimensional, since all the W_is given in the above theorem are of finite type A.

(ii) Let the Coxeter diagram of (W, S) be the complete graph K_5 with 5 vertices. Assume that $\mathcal{H}(\mathbf{q})$ is collapse free. There can be at most two different parameters 0 and $q \neq 0$. Both $R := \{s \in S : q_s = 0\}$ and its complement $S \setminus R = \{s \in S : q_s = q\}$ are admissible subsets of S, the larger one of which contains at least 3 elements and thus gives a copy of the infinite-dimensional Hecke algebra of affine type A_3 with a single parameter as a subalgebra of $\mathcal{H}(\mathbf{q})$. Therefore, $\mathcal{H}(\mathbf{q})$ is never finite dimensional in such cases.

5 The simply laced bipartite case

By Theorem 3.5, there exists $\mathbf{q} \in \mathbb{F}^S$ such that $\mathcal{H}(\mathbf{q})$ is collapse free and commutative if and only if the Coxeter diagram of (W, S) is simply laced and bipartite. We give more results for such case in this section. Recall from graph theory that an *independent set* of a graph is a set of vertices of which no two are adjacent. Let $T_I := \prod_{i \in I} T_i$ for all $I \in \mathcal{I}(G)$, where $\mathcal{I}(G)$ consists of independent sets in the underlying graph G of the Coxeter diagram of (W, S).

Corollary 5.1 *A collapse-free and commutative $\mathcal{H}(\mathbf{q})$ has a basis $\{T_I : I \in \mathcal{I}(G)\}$. In particular, if (W, S) is of type A_n, then the dimension of $\mathcal{H}(\mathbf{q})$ equals the Fibonacci number F_{n+2}.*

Proof By Theorem 3.5, the Coxeter diagram of (W, S) is a simply laced and bipartite graph G with all edges between the two subsets $\{s \in S : q_s = 0\}$ and $\{t \in S : q_t \neq 0\}$. Hence, the subsets $S_1, \ldots, S_k$ given by Lemma 4.1 are all singleton sets. Then, the basis $B(\mathbf{q})$ for $\mathcal{H}(\mathbf{q})$ given in Theorem 4.3 consists of the elements T_I for all $I \in \mathcal{I}(G)$.

Now suppose that (W, S) is of type A_n, i.e., its Coxeter diagram is isomorphic to the path P_n with n vertices. If an independent set I in P_n contains one end vertex of P_n, then removing this end point from I gives an independent set of P_{n-2}; otherwise, I is an independent set of P_{n-1}. Thus, $|\mathcal{I}(P_n)| = |\mathcal{I}(P_{n-1})| + |\mathcal{I}(P_{n-2})|$. One also sees that $|\mathcal{I}(P_i)| = i + 1$ if $i = 0, 1$. Thus $|\mathcal{I}(P_n)| = F_{n+2}$ for all $n \geq 0$. □

Computations in Magma suggest the following conjecture.

Conjecture 5.2 *Suppose that the Coxeter diagram of (W, S) is a simply laced and bipartite graph G. The minimum dimension of a collapse-free $\mathcal{H}(\mathbf{q})$ is $|\mathcal{I}(G)|$, which is attained when it is commutative.*

We will verify this conjecture for type A_n. We first need a lemma on the *Fibonacci numbers*, which are defined as $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

Lemma 5.3 *If $k \geq 4$, then $k! \geq F_{k+3} + 2$. Also, if $a \geq 1$ and $b \geq 0$, then $F_{a+b} = F_a F_{b+1} + F_{a-1} F_b \leq F_a F_{b+2}$.*

Proof The first result follows easily by induction. It is well known that $F_{a+b} = F_a F_{b+1} + F_{a-1} F_b$ (see Example 7.2). Hence, $F_{a+b} \leq F_a(F_{b+1} + F_b) = F_a F_{b+2}$. □

Theorem 5.4 *Let $\mathcal{H}(\mathbf{q})$ be a collapse-free Hecke algebra of type A_n with independent parameters. Then, its dimension is at least the Fibonacci number F_{n+2}, and the equality holds if and only if $\mathcal{H}(\mathbf{q})$ is commutative.*

Proof We prove the result by induction on n. The Coxeter diagram for type A_n is the path $s_1 — s_2 — \cdots — s_n$. We write $q_i := q_{s_i}$ for all $i \in [n]$. Let $S_1, \dots, S_k$ be the subsets of S given by Lemma 4.1. Then, S_j is a path of length $n_j \geq 1$ for every $j \in [k]$. We may assume, without loss of generality, that

$$S_j = \{s_i : n_1 + \cdots + n_{j-1} < i \leq n_1 + \cdots + n_j\}, \quad \forall j \in [k].$$

If all parameters in $\mathbf{q}$ are the same, then $\mathcal{H}(\mathbf{q})$ has dimension $(n+1)! \geq F_{n+2}$. Thus, we may assume that there exists $j \in [k]$ such that $q_s = q \neq 0$ for all $s \in S_j$. Let $a = n_1 + \cdots + n_{j-1}$, $b = n_j$, and $c = n_{j+1} + \cdots + n_k$. By convention, $a = 0$ if $j = 1$, and $c = 0$ if $j = k$. One sees that s_a and s_{a+b+1} both dominate S_j.

By Theorem 4.3, $\mathcal{H}(\mathbf{q})$ has dimension $|W(\mathbf{q})|$. We need to count the elements $(w_1, \dots, w_k)$ in $W(\mathbf{q})$. If $w_j \neq 1$, then any reduced word of w_{j-1} cannot contain s_a and any reduced word of w_{j+1} cannot contain s_{a+b+1}. It follows that $(w_1, \dots, w_{j-1})$ and $(w_{j+1}, \dots, w_k)$ are arbitrary elements in $W(q_i : 1 \leq i \leq a-1)$ and $W(q_i : a+b+2 \leq i \leq n)$, respectively. Then, the number of choices for $(w_1, \dots, w_k)$ in this case is at least $F_{a+1}((b+1)!-1)F_{c+1}$, by induction hypothesis. Note that this still holds even if $a = 0$ or $c = 0$, since $F_1 = 1$.

Similarly, if $w_j = 1$, the number of choices for $(w_1, \dots, w_k)$ is at least $F_{a+2}F_{c+2}$ by induction hypothesis.

Thus, the dimension of $\mathcal{H}(\mathbf{q})$ is at least $f(a,b,c) := F_{a+1}((b+1)!-1)F_{c+1} + F_{a+2}F_{c+2}$. By Lemma 5.3,

$$f(a,b,c) = F_{a+1}((b+1)!-2)F_{c+1} + F_{a+c+3}.$$

If $b = 1$, then this becomes $f(a,b,c) = F_{a+c+3} = F_{n+2}$. If $b = 2$, then Lemma 5.3 implies that

$$f(a,b,c) > 3F_{a+1}F_{c+1} + F_{a+c+3} \geq F_4F_{a+c} + F_{n+1} \geq F_n + F_{n+1} = F_{n+2}.$$

If $b \geq 3$ then Lemma 5.3 implies that

$$f(a,b,c) > F_{a+1}F_{b+4}F_{c+1} \geq F_{a+b+3}F_{c+1} \geq F_{n+2}.$$

Therefore, $f(a,b,c) \geq F_{n+2}$ always holds.

Finally, assume $f(a,b,c) = F_{n+2}$. By the above argument, this equality is possible only if $b = 1$ and the dimensions of $\mathcal{H}(q_1, \dots, q_a)$ and $\mathcal{H}(q_{a+2}, \dots, q_n)$ are F_{a+2} and F_{c+2}, respectively. Then, $\mathcal{H}(q_1, \dots, q_a)$ and $\mathcal{H}(q_{a+2}, \dots, q_n)$ are commutative by induction hypothesis. The definition for a, b, and c implies $q_a = 0$, $q_{a+1} \neq 0$, and $q_{a+2} = 0$. It follows from Theorem 3.5 that $q_i = 0$ when $i \equiv a \bmod 2$ and $q_i \neq 0$ otherwise. Hence, $\mathcal{H}(\mathbf{q})$ must be commutative. On the other hand, if $\mathcal{H}(\mathbf{q})$ is commutative, then its dimension is F_{n+2} by Corollary 5.1. This completes the proof. □

Next, we explain the connection between a collapse-free and commutative $\mathcal{H}(\mathbf{q})$ and the *Möbius algebra* $A(L)$ of a finite lattice L. According to Stanley [10, § 3.9], the Möbius algebra $A(L)$ is the monoid algebra of L over $\mathbb{F}$ with the meet operation, and it is a direct sum of $|L|$ many one-dimensional subalgebras.

Now let Z be a finite rank two poset. Set $X := \{x \in Z : x > y \text{ for some } y \in Z\}$ and $Y = Z \setminus X$. By abuse of notation, we denote by Z the underlying graph of Z. Let L be the distribute lattice $J(Z)$ of the order ideals of Z ordered by reverse inclusion (so that the meet operation is the union of ideals). Suppose that (W, S) is a Coxeter system whose Coxeter diagram coincides with Z. Denote by $\mathcal{H}(Z)$ the Hecke algebra $\mathcal{H}(\mathbf{q})$ of (W, S) with parameters $\mathbf{q} = (q_s : s \in S)$ given by $q_s = 0$ for all $s \in X$ and $q_s = 1$ for all $s \in Y$.

Proposition 5.5 *When* char $(\mathbb{F}) \neq 2$ *the algebra* $\mathcal{H}(Z)$ *is isomorphic the Möbius algebra of* $J(Z)$.

Proof By definition, the algebra $\mathcal{H}(Z)$ is generated by $\{T_x : x \in X\} \cup \{T_y : y \in Y\}$ with relations

$$\begin{cases} T_x^2 = T_x,\ T_y^2 = 1, & \forall x \in X,\ \forall y \in Y, \\ T_z T_{z'} = T_{z'} T_z, & \forall z, z' \in Z, \\ T_x T_y = T_x, & \text{if } x > y \text{ in } Z \text{ (by Lemma 3.4).} \end{cases}$$

One has a basis $\{T_I : I \in \mathcal{I}(Z)\}$ for $\mathcal{H}(Z)$ by Corollary 5.1.

When char $(\mathbb{F}) \neq 2$, one can replace the generator T_y with $T'_y := (T_y + 1)/2$, which is now an idempotent, for every $y \in Y$. One checks that all other relations given above remain same. Write $T'_x = T_x$ for all $x \in X$. Then, the algebra $H(Z)$ is generated by $\{T'_x : x \in X\} \cup \{T'_y : y \in Y\}$ and has a basis $\{T'_I : I \in \mathcal{I}(Z)\}$ where $T'(I) := \prod_{z \in I} T'_z$.

Any independent set I in $\mathcal{I}(Z)$ is an antichain in Z, generating an order ideal $J(I)$ consisting of all elements weakly below some element of I. Conversely, an order ideal of Z corresponds to an independent set $I \in \mathcal{I}(Z)$ consisting of all maximal elements in this order ideal. Hence, sending $T'(I)$ to the order ideal $J(I)$ for all $I \in \mathcal{I}(Z)$ gives a vector space isomorphism $H(Z) \cong A(J(Z))$. To see this isomorphism preserves multiplications, let I_1 and I_2 be two elements in $\mathcal{I}(Z)$. Then, $T'(I_1)T'(I_2) = T'(I_1 \circ I_2)$ where $I_1 \circ I_2$ is obtained from $I_1 \cup I_2$ by removing all the elements that are less than some element of $I_1 \cup I_2$. On the other hand, the order ideal $J(I_1) \cup J(I_2)$ has maximal elements given by $I_1 \circ I_2$ and thus equals $J(I_1 \circ I_2)$. This completes the proof. □

6 The commutative case

By Theorem 3.5 and Corollary 5.1, if $\mathcal{H}(\mathbf{q})$ is collapse free and commutative, then the Coxeter diagram of (W, S) is simply laced with a bipartite underlying graph G, and the dimension of $\mathcal{H}(\mathbf{q})$ is $|\mathcal{I}(G)|$. In this section, we define and study a more general commutative algebra for any (unweighted) simple graph G, whose dimension is still $|\mathcal{I}(G)|$.

6.1 Basic results

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $R \subseteq V(G)$. We define an algebra $\mathcal{H}(G, R)$ to be the quotient of the polynomial algebra $\mathbb{F}[x_v : v \in V(G)]$ by the ideal generated by

$$\{x_r^2 : r \in R\} \cup \{x_v^2 - x_v : v \in V(G) \setminus R\} \cup \{x_u x_v : uv \in E(G)\}.$$

The image of x_v in the quotient algebra $\mathcal{H}(G, R)$ is still denoted by x_v for all $v \in V$. This algebra $\mathcal{H}(G, R)$ generalizes the commutative algebra $\mathcal{H}(\mathbf{q})$ by the following result.

Proposition 6.1 *If $\mathcal{H}(\mathbf{q})$ is collapse free and commutative, then it is isomorphic to $\mathcal{H}(G, R)$ as an algebra, where G is the underlying graph of the Coxeter diagram of (W, S) and $R := \{s \in S : q_s = -1\}$.*

Proof The algebra $\mathcal{H}(\mathbf{q})$ has another generating set $\{x_s : s \in S\}$ given by

$$x_s := \begin{cases} T_s, & q_s = 0, \\ T_s - 1, & q_s = -1, \\ (1 - T_s)/(1 + q_s), & \text{otherwise}. \end{cases}$$

If $\mathcal{H}(\mathbf{q})$ is collapse free and commutative, then one can check that the relations for $\{T_s : s \in S\}$ are equivalent to the relations for $\{x_s : s \in S\}$ in the definition of $\mathcal{H}(G, R)$ using Lemma 3.4. Thus, the result holds. □

Remark 6.2 (i) The set $R = \{s \in S : q_s = -1\}$ associated with $\mathcal{H}(\mathbf{q})$ depends on char $(\mathbb{F})$. For example, an element $s \in S$ with $q_s = 1$ belongs to R if and only if char $(\mathbb{F}) = 2$. However, once R is chosen for the algebra $\mathcal{H}(G, R)$, our results on $\mathcal{H}(G, R)$ do not depend on char $(\mathbb{F})$ any more.

(ii) By Theorem 3.5, if $\mathcal{H}(\mathbf{q})$ is collapse free and commutative, then $R = \{s \in S : q_s = -1\}$ must be an independent set of G. But the commutative algebra $\mathcal{H}(G, R)$ is well defined for any simple graph G and any subset $R \subseteq V(G)$.

(iii) The *Stanley-Reisner ring of the independence complex of* G is defined as the quotient of the polynomial algebra $\mathbb{F}[y_v : v \in V(G)]$ by the *edge ideal* generated by $(y_u y_v : uv \in E(G))$ (see, e.g., [5]). The algebra $\mathcal{H}(G, R)$ is a further quotient of the Stanley-Reisner ring of the independence complex of G.

Now, we study the algebra $\mathcal{H}(G, R)$ and our results will naturally apply to the commutative algebra $\mathcal{H}(\mathbf{q})$ by Proposition 6.1. We first need some notation. For any $U \subseteq V(G)$, we write

$$X_U := \prod_{u \in U} x_u \quad \text{and} \quad X_U^- := \prod_{u \in U} x_u^-$$

where $x_v^- := 1 - x_v$ for all $v \in V(G)$. One sees that $X_U \neq 0$ if and only if U belongs to $\mathcal{I}(G)$, the set of all independent sets in G. We define the *length* of a nonzero monomial X_I to be the cardinality $|I|$ of the independent set I. We partially order the nonzero monomials by their lengths. We denote by $N(U)$ the set of all vertices that are adjacent to some vertex $u \in U$ in G. We will often identify a subset U of $V(G)$ with the subgraph of G induced by U, whose vertex set is U and whose edge set is $\{\{u, v\} \in E(G) : u, v \in U\}$. We will also write "+" and "−" for set union and difference. For example, we write $G - R$ for the subgraph of G induced by $V(G) - R$, and hence, $\mathcal{I}(G - R)$ consists of all independent sets of $G - R$. We give two bases for $\mathcal{H}(G, R)$ in the following proposition, which generalizes Corollary 5.1.

Proposition 6.3 *The algebra $\mathcal{H}(G, R)$ has dimension $|\mathcal{I}(G)|$ and two bases $\{X_I : I \in \mathcal{I}(G)\}$ and*

$$\left\{X_{I+J} X^{-}_{G-R-I} : I \in \mathcal{I}(G - R),\ J \in \mathcal{I}(R - N(I))\right\}. \tag{6.1}$$

Proof The defining relations for $\mathcal{H}(G, R)$ immediately imply that it is spanned by $\{X_I : I \subseteq I(G)\}$. Let $\mathbb{F}\mathcal{I}(G)$ be the vector space over $\mathbb{F}$ with a basis $\mathcal{I}(G)$. We define an action of $\mathcal{H}(G, R)$ on $\mathbb{F}\mathcal{I}(G)$ by

$$x_v(I) = \begin{cases} 0, & \text{if } v \in I \cap R \text{ or } I \cup \{v\} \notin \mathcal{I}(G), \\ I \cup \{v\}, & \text{otherwise.} \end{cases}$$

It is not hard to check that this action satisfies the defining relations for $\mathcal{H}(G, R)$. For any $I \in \mathcal{I}(G)$, one has $X_I(\emptyset) = I$. This forces the spanning set $\{X_I : I \subseteq I(G)\}$ to be a basis for $\mathcal{H}(G, R)$.

One sees that any independent set of G can be written uniquely as $I + J$ for some $I \in \mathcal{I}(G - R)$ and $J \in \mathcal{I}(R - N(I))$, and the shortest term in $X_{I+J} X^{-}_{G-R-I}$ is X_{I+J}. Thus, (6.1) is also a basis for $\mathcal{H}(G)$. □

Let G' be a subgraph of G induced by $V' \subseteq V(G)$, and let $R' = V' \cap R$. The following corollary allows us to study the induction of $\mathcal{H}(G', R')$-modules to $\mathcal{H}(G, R)$ and the restriction of $\mathcal{H}(G, R)$-modules to $\mathcal{H}(G', R')$.

Corollary 6.4 *The subalgebra of $\mathcal{H}(G, R)$ generated by $\{x_v : v \in V'\}$ is isomorphic to $\mathcal{H}(G', R')$.*

Proof There is an injection $\phi : \mathcal{H}(G', R') \hookrightarrow \mathcal{H}(G, R)$ of algebras defined by sending the generators x'_v for $\mathcal{H}(G', R')$ to the generators x_v for $\mathcal{H}(G, R)$ for all $v \in V'$. By Proposition 6.3, the algebra $\mathcal{H}(G', R')$ admits a basis consisting of the elements $X'_I := \prod_{v \in I} x'_v$ for all $I \in \mathcal{I}(G')$. The map ϕ sends this basis to the basis $\{X_I : I \in \mathcal{I}(G')\}$ for the subalgebra of $\mathcal{H}(G, R)$ generated by $\{x_v : v \in V'\}$, giving the desired isomorphism. □

6.2 Projective indecomposable modules and simple modules

We first decompose the algebra $\mathcal{H}(G, R)$ into a direct sum of indecomposable submodules.

Theorem 6.5 *There is an $\mathcal{H}(G, R)$-module decomposition*

$$\mathcal{H}(G, R) = \bigoplus_{I \subseteq \mathcal{I}(G-R)} \mathbf{P}_I(G, R) \tag{6.2}$$

where each $\mathbf{P}_I(G, R) := \mathcal{H}(G, R) X_I X^-_{G-R-I}$ is an indecomposable $\mathcal{H}(G, R)$-module with a basis

$$\left\{ X_{I+J} X^-_{G-R-I} : J \in \mathcal{I}(R - N(I)) \right\} \tag{6.3}$$

and hence has dimension $|\mathcal{I}(R - N(I))|$. The top of $\mathbf{P}_I(G, R)$, denoted by $\mathbf{C}_I(G, R)$, is one dimensional and admits an $\mathcal{H}(G, R)$-action by

$$x_v = \begin{cases} 1, & \text{if } v \in I, \\ 0, & \text{if } v \in G - I. \end{cases}$$

Proof Let $I \in \mathcal{I}(G - R)$. Since $x_v x_v^- = 0$ for any $v \in G - R - I$, and $x_u x_v = 0$ whenever $v \in I$ and $u \in N(v)$, one has

$$X_J(X_I X^-_{G-R-I}) = \begin{cases} X_{I+J} X^-_{G-R-I}, & \text{if } J - I \in \mathcal{I}(R - N(I)), \\ 0, & \text{otherwise} \end{cases} \tag{6.4}$$

for any $J \in \mathcal{I}(G)$. Hence, (6.3) spans $\mathbf{P}_I(G, R)$. By Proposition 6.3, $\mathcal{H}(G, R)$ has a basis (6.1) which is the union of the spanning sets (6.3) for all $I \in \mathcal{I}(G - R)$. This implies the direct sum decomposition (6.2) of $\mathcal{H}(G, R)$ and forces the spanning set (6.3) to be a basis for $\mathbf{P}_I(G, R)$. The dimension of $\mathbf{P}_I(G, R)$ is then clear.

Now, we prove that $\mathbf{P}_I(G, R)$ is indecomposable and find its top. Since $x_r^2 = 0$ for any $r \in R$, the elements in (6.3) are all nilpotent except $X_I X^-_{G-R-I}$. The span $\mathbf{N}_I$ of these nilpotent elements is contained in the nilradical of $\mathcal{H}(G, R)$ and hence in the radical of $\mathbf{P}_I(G, R)$. By (6.4), the quotient $\mathbf{P}_I(G, R)/\mathbf{N}_I$ is isomorphic to the one-dimensional $\mathcal{H}(G, R)$-module $\mathbf{C}_I(G, R)$. It follows that the radical of $\mathbf{P}_I(G, R)$ equals $\mathbf{N}_I$, and the top of $\mathbf{P}_I(G, R)$ is isomorphic to $\mathbf{C}_I(G, R)$. Then, $\mathbf{P}_I(G, R)$ must be indecomposable as its top is simple. □

By Theorem 6.5, $\{\mathbf{P}_I(G, R) : I \in \mathcal{I}(G - R)\}$ and $\{\mathbf{C}_I(G, R) : I \in \mathcal{I}(G - R)\}$ are complete lists of pairwise nonisomorphic projective indecomposable $\mathcal{H}(G, R)$-modules and simple $\mathcal{H}(G, R)$-modules, respectively. The proof of Theorem 6.5 shows that the radical of $\mathbf{P}_I(G, R)$ is spanned by $\{X_{I+J} X^-_{G-R-I} : \emptyset \neq J \in \mathcal{I}(R - N(I))\}$, and hence, the radical of $\mathcal{H}(G, R)$ is the ideal generated by $\{x_r : r \in R\}$. This ideal coincides with the nilradical of $\mathcal{H}(G, R)$, showing that $\mathcal{H}(G, R)$ is a *Jacobson ring*. Some other consequences of Theorem 6.5 are listed below.

Corollary 6.6 *Theorem 6.5 implies the following results.*

(i) *The algebra $\mathcal{H}(G, R)$ is semisimple if and only if $R = \emptyset$.*
(ii) *For any $I \in \mathcal{I}(G - R)$ one has $\mathbf{P}_I(G, R) \cong \mathcal{H}(G, R) \otimes_{\mathcal{H}(G-R,\emptyset)} \mathbf{C}_I(G - R, \emptyset)$.*

(iii) *The socle of* $\mathbf{P}_I(G,R)$ *is the direct sum of* $\mathbb{F}X_{I+J}X^-_{G-R-I} \cong \mathbf{C}_I(G,R)$ *for all maximal* J *in* $\mathcal{I}(R-N(I))$.

(iv) *The Cartan matrix of* $\mathcal{H}(G,R)$ *is the diagonal matrix* $\operatorname{diag}\{|\mathcal{I}(R-N(I))| : I \in \mathcal{I}(G-R)\}$.

(v) *A complete set of primitive orthogonal idempotents of* $H(G)$ *is given by* $\{X_I X^-_{G-R-I} : I \in \mathcal{I}(G-R)\}$.

Proof (i) An algebra is semisimple if and only if its radical is 0. The radical of $\mathcal{H}(G,R)$ is generated by $\{x_r : r \in R\}$, which is 0 if and only if $R = \emptyset$.

(ii) There is a bilinear map $\mathcal{H}(G,R) \times \mathbf{C}_I(G-R,\emptyset) \to \mathbf{P}_I(G,R)$ defined by sending (X_J, z_I) to $X_J X_I X^-_{G-R-I}$ for all $J \in \mathcal{I}(G)$, where z_I is an element spanning $\mathbf{C}_I(G-R,\emptyset)$. This induces an algebra surjection

$$\phi : \mathcal{H}(G,R) \otimes_{\mathcal{H}(G-R,\emptyset)} \mathbf{C}_I(G-R,\emptyset) \twoheadrightarrow \mathbf{P}_I(G,R)$$

which sends $X_J \otimes_{\mathcal{H}(G-R,\emptyset)} z_I$ to $X_J X_I X^-_{G-R-I}$ for all $J \in \mathcal{I}(G)$. One sees that $\mathcal{H}(G,R) \otimes_{\mathcal{H}(G-R,\emptyset)} \mathbf{C}_I(G-R,\emptyset)$ is spanned by $\{X_J \otimes_{\mathcal{H}(G-R,\emptyset)} z_I : J \in \mathcal{I}(R-N(I))\}$, which is sent by ϕ to the basis (6.3) for $\mathbf{P}_I(G,R)$. Hence, ϕ must be an isomorphism.

(iii) If J is maximal in $\mathcal{I}(R-N(I))$, then $\mathbb{F}X_{I+J}X^-_{G-R-I}$ admits the same action of $\mathcal{H}(G,R)$ as $\mathbf{C}_I(G,R)$. Thus, $\mathbb{F}X_{I+J}X^-_{G-R-I}$ is a simple submodule of $\mathbf{P}_I(G,R)$ and must be contained in the socle of $\mathbf{P}_I(G,R)$. Conversely, we need to show that any simple submodule M of $\mathbf{P}_I(G,R)$ is contained in the direct sum of $\mathbb{F}X_{I+J}X^-_{G-R-I}$ for all maximal $J \in \mathcal{I}(R-N(I))$. Using the basis (6.3) for $\mathbf{P}_I(G,R)$, one writes an arbitrary element of M as

$$z = \sum_{J \in \mathcal{I}(R-N(I))} c_J X_{I+J} X^-_{G-R-I}, \quad c_J \in \mathbb{F}.$$

Let K be a minimal independent set in $\mathcal{I}(R-N(I))$ such that $c_K \neq 0$. It suffices to show that K is also maximal in $\mathcal{I}(R-N(I))$. If not, then there exists $r \in R-K$ such that $K+r \in \mathcal{I}(R-N(I))$. For any $J \in \mathcal{I}(R-N(I))$, one sees that

$$x_r X_{I+J} X^-_{G-R-I} = \begin{cases} 0, & \text{if } r \in J \cup N(I \cup J), \\ X_{I+J+r} X^-_{G-R-I} \neq 0, & \text{otherwise.} \end{cases}$$

Thus, in the expansion of $x_r z$ in terms of the basis (6.3), the coefficients of $X_{I+K}X^-_{G-R-I}$ and $X_{I+K+r}X^-_{G-R-I}$ are 0 and $c_K \neq 0$, respectively. It follows that $x_r z \notin \mathbb{F}z$ and M is at least two dimensional. This contradicts the simplicity of M.

(iv) Let $I \in \mathcal{I}(G-R)$. We order the elements $X_{I+J}X^-_{G-R-I}$ by $|J|$ for all $J \in \mathcal{I}(R-N(I))$. This induces a filtration for $\mathbf{P}_I(G,R)$, under which

$$x_v X_{I+J} X^-_{G-R-I} \equiv \begin{cases} X_{I+J} X^-_{G-R-I}, & v \in I, \\ 0, & v \notin I. \end{cases}$$

Hence, every simple composition factor of $\mathbf{P}_I(G, R)$ is isomorphic to $\mathbf{C}_I(G, R)$. The Cartan matrix follows.

(v) This follows from the decomposition of $\mathcal{H}(G, R)$ given in Theorem 6.5 and the equality

$$\sum_{I\in\mathcal{I}(G-R)} X_I X^-_{G-R-I} = \sum_{J\in\mathcal{I}(G-R)} \sum_{I\subseteq J} (-1)^{|J\setminus I|} X_J = 1.$$

The reader who is not familiar with primitive orthogonal idempotents can find more details in [2, § I.4]. □

6.3 Induction and restriction

Let G' be an induced subgraph of G and let $R' = G' \cap R$. By Corollary 6.4, the following induction and restriction are well defined for isomorphism classes of modules:

- the induction $M \uparrow^{G,R}_{G',R'} := \mathcal{H}(G, R) \otimes_{\mathcal{H}(G',R')} M$ of an $\mathcal{H}(G', R')$-module M to $\mathcal{H}(G, R)$,
- the restriction $N \downarrow^{G,R}_{G',R'}$ of an $\mathcal{H}(G, R)$-module N to $\mathcal{H}(G', R')$.

Proposition 6.7 *Assume $R = \emptyset$, and hence, $R' = \emptyset$. Write $(G, R) = (G)$ and $(G', R') = (G')$. Then, for any $I' \in \mathcal{I}(G')$,*

$$\mathbf{C}_{I'}(G') \uparrow^{G}_{G'} \cong \bigoplus_{I\in\mathcal{I}(G):I\cap G'=I'} \mathbf{C}_I(G).$$

Proof Suppose that $\mathbf{C}_{I'}(G') = \mathbb{F}z$. Using the universal property of the tensor product, one obtains an algebra surjection

$$\phi : \mathcal{H}(G) \otimes_{\mathcal{H}(G')} \mathbb{F}z \twoheadrightarrow \mathcal{H}(G) X_{I'} X^-_{G'-I'}$$

which sends $X_J \otimes_{\mathcal{H}(G')} z$ to $X_J X_{I'} X^-_{G'-I'}$ for all $J \in \mathcal{I}(G)$. One sees that $\mathcal{H}(G) \otimes_{\mathcal{H}(G')} \mathbb{F}z$ is spanned by

$$\{X_I \otimes_{\mathcal{H}(G')} z : I \in \mathcal{I}(G),\ I \cap G' = I'\}$$

since $x_v z = 0$ for all $v \in G' - I'$. This spanning set is sent by ϕ to

$$\{X_I X^-_{G'-I'} : I \in \mathcal{I}(G),\ I \cap G' = I'\}$$

which is a basis for $\mathcal{H}(G) X_{I'} X^-_{G'-I'}$ since it is a spanning set triangularly related to $\{X_I : I \in \mathcal{I}(G),\ I \cap G' = I'\}$, a linearly independent set in $\mathcal{H}(G)$. Thus, ϕ is an isomorphism. Using the length filtration induced by $|I|$ for all I appearing in the above basis, one sees that the composition factors of $\mathcal{H}(G) X_{I'} X^-_{G'-I'}$ are $\mathbf{C}_I(G)$ for all $I \in \mathcal{I}(G)$ with $I \cap G' = I'$, each appearing exactly once. This completes the proof as $\mathcal{H}(G)$ is semisimple by Corollary 6.6 (i). □

Proposition 6.8 *Let $I \in \mathcal{I}(G-R)$ and $J \in \mathcal{I}(G'-R')$. Then,* $\mathbf{C}_I(G,R)\downarrow^{G,R}_{G',R'} \cong \mathbf{C}_{I\cap G'}(G',R')$ *and*

$$\mathbf{P}_J(G',R')\uparrow^{G,R}_{G',R'} \cong \bigoplus_{K\in\mathcal{I}(G-R):K\cap G'=J} \mathbf{P}_K(G,R).$$

Proof The restriction of $\mathbf{C}_I(G,R)$ follows easily from the definition. By Corollary 6.6 (ii) and Proposition 6.7,

$$\begin{aligned}
\mathbf{P}_J(G',R')\uparrow^{G,R}_{G',R'} &\cong \mathbf{C}_J(G'-R',\emptyset)\uparrow^{G',R'}_{G'-R',\emptyset}\uparrow^{G,R}_{G',R'}\\
&\cong \mathbf{C}_J(G'-R',\emptyset)\uparrow^{G,R}_{G'-R',\emptyset}\\
&\cong \mathbf{C}_J(G'-R',\emptyset)\uparrow^{G-R,\emptyset}_{G'-R',\emptyset}\uparrow^{G,R}_{G-R,\emptyset}\\
&\cong \bigoplus_{K\in\mathcal{I}(G-R),\ K\cap G'=J} \mathbf{C}_K(G-R,\emptyset)\uparrow^{G,R}_{G-R,\emptyset}\\
&\cong \bigoplus_{K\in\mathcal{I}(G-R),\ K\cap G'=J} \mathbf{P}_K(G,R).
\end{aligned}$$

This completes the proof. □

Remark 6.9 It is not hard to obtain the simple composition factors of the induction of a simple $\mathcal{H}(G',R')$-module to $\mathcal{H}(G,R)$. But the restriction of a projective indecomposable $\mathcal{H}(G,R)$-module to $\mathcal{H}(G',R')$ is not always projective.

7 Commutative Hecke algebras of type A

We apply the previous results to commutative Hecke algebras of type A with independent parameters.

7.1 Decomposition of Fibonacci numbers

Let (W,S) be the Coxeter system of type A_n whose Coxeter diagram is the path $s_1 — s_2 — \cdots — s_n$. We often identify s_i with i and write $\mathbf{q} := (q_1,\ldots,q_n) \in \mathbb{F}^n$. Let $\mathcal{H}(\mathbf{q})$ be a collapse-free and commutative Hecke algebra of (W,S) with independent parameters $\mathbf{q}$. Then Theorem 3.5 implies that either $q_i = 0$ for all odd $i \in [n]$ and $q_i \neq 0$ for all even $i \in [n]$, or the other way around. Proposition 6.1 provides an algebra isomorphism $H(\mathbf{q}) \cong \mathcal{H}(P_n,R)$, where $R := \{i \in [n] : q_i = -1\}$. Note that the set R obtained from $\mathcal{H}(\mathbf{q})$ depends on char $(\mathbb{F})$. For example, if $\mathbf{q} = (1,0,1,0,1,\ldots)$, then $R = \emptyset$ and $\mathcal{H}(P_n,R)$ is semisimple if char $\mathbb{F} \neq 2$, but $R = \{1,3,5,\ldots\}$ and $\mathcal{H}(P_n,R)$ is not semisimple if char $(\mathbb{F}) = 2$. However, the algebra $\mathcal{H}(P_n,R)$ is defined for any subset $R \subseteq [n]$, and our results do not depend on char $(\mathbb{F})$. We first give decompositions of the Fibonacci numbers.

Proposition 7.1 *Let $R \subseteq [n]$. Then,*

$$F_{n+2} = \sum_{I \in \mathcal{I}(P_n - R)} |\mathcal{I}(R - N(I))|.$$

Proof Let G be a simple graph, and let $R \subseteq V(G)$. By Proposition 6.3, the dimension of $\mathcal{H}(G, R)$ is $|\mathcal{I}(G)|$. By Theorem 6.5, $\mathcal{H}(G, R)$ is the direct sum of $\mathbf{P}_I(G, R)$ for all $I \in \mathcal{I}(G - R)$, and the dimension of each $\mathbf{P}_I(G, R)$ is $|\mathcal{I}(R - N(I))|$. Hence,

$$|\mathcal{I}(G)| = \sum_{I \in \mathcal{I}(G - R)} |\mathcal{I}(R - N(I))|.$$

Now, take $G = P_n$. We know that $|\mathcal{I}(P_n)| = F_{n+2}$ by Corollary 5.1. Thus, the result holds.

Example 7.2 Let $R := [m]$ for some $m \in [n-1]$. Then, the subgraph of P_n induced by R is the path P_m. If $I \in \mathcal{I}(P_n - [m+1])$, then $\mathcal{I}(R - N(I)) = \mathcal{I}(R)$. If $I \in \mathcal{I}(P_n - R)$ contains $m + 1$, then $I - \{m + 1\} \in \mathcal{I}(P_n - [m + 2])$ and $\mathcal{I}(R - N(I) = \mathcal{I}([m - 1])$. Thus, we recover a well-known identity $F_{n+2} = F_{m+2} F_{n-m+1} + F_{m+1} F_{n-m}$.

Example 7.3 Let X and Y be the subsets of odd and even numbers in $[n]$, respectively. Then,

$$F_{n+2} = \sum_{I \subseteq X} 2^{|Y - N(I)|} = \sum_{J \subseteq Y} 2^{|X - N(J)|}.$$

This writes a Fibonacci number as a sum of $2^{|X|}$ or $2^{|Y|}$ many powers of 2. Some small examples are provided below.

$n = 1$	$2 = 1 + 1 = 2$	$n = 2$	$3 = 2 + 1$
$n = 3$	$5 = 2 + 1 + 1 + 1 = 4 + 1$	$n = 4$	$8 = 4 + 2 + 1 + 1$
$n = 5$	$13 = 4+2+2+1+1+1+1+1 = 8+2+2+1$	$n = 6$	$21 = 8+4+2+2+2+1+1+1$

7.2 The semisimple commutative case

Now, we study the representation theory of the semisimple commutative algebra $\mathcal{H}_n := \mathcal{H}(P_{n-1}, \emptyset)$, where $\mathcal{H}_0 := \mathbb{F}$ by convention. We write $\alpha \propto n$ if $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ is a composition of n with all internal parts larger than 1, i.e., $\alpha_i > 1$ whenever $1 < i < \ell$.

Proposition 7.4 *The algebra $\mathcal{H}_n$ decomposes into a direct sum of F_{n+1} many one-dimensional simple submodules $\mathbf{C}_\alpha$ indexed by $\alpha \propto n$, with the $\mathcal{H}_n$-action on $\mathbf{C}_\alpha$ given by $x_i = 1$ if $i \in D(\alpha)$ or $x_i = 0$ otherwise.*

Proof For any composition α of n, one sees that $D(\alpha)$ is an independent set of P_{n-1} if and only if α has no internal parts equal to 1. Thus, the result follows from Theorem 6.5.

Since $\mathcal{H}_n$ is semisimple, its two Grothendieck groups $G_0(\mathcal{H}_n)$ and $K_0(\mathcal{H}_n)$ are the same. Given nonnegative integers m and n, the subalgebra of $\mathcal{H}_{m+n}$ generated by $x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_{m+n-1}$ is isomorphic to $\mathcal{H}_m \otimes \mathcal{H}_n$, giving a natural embedding $\mathcal{H}_m \otimes \mathcal{H}_n \hookrightarrow \mathcal{H}_{m+n}$. Thus, there is a tower $\mathcal{H}_\bullet : \mathcal{H}_0 \hookrightarrow \mathcal{H}_1 \hookrightarrow \mathcal{H}_2 \hookrightarrow \cdots$ of algebras, whose Grothendieck group $G_0(\mathcal{H}_\bullet) := \bigoplus_{n\ge 0} G_0(\mathcal{H}_n)$ has a product and a coproduct defined by

$$\mathbf{C}_\alpha \hat{\otimes} \mathbf{C}_\beta := \left(\mathbf{C}_\alpha \otimes \mathbf{C}_\beta\right) \uparrow_{\mathcal{H}_m\otimes\mathcal{H}_n}^{\mathcal{H}_{m+n}} \quad \text{and} \quad \Delta(\mathbf{C}_\alpha) := \sum_{0\le i\le m} \mathbf{C}_\alpha \downarrow_{\mathcal{H}_i\otimes\mathcal{H}_{m-i}}^{\mathcal{H}_m}$$

for all $\alpha \propto m$ and $\beta \propto n$. One sees that the product $\hat{\otimes}$ and the coproduct Δ are well defined, with unit u sending 1 to $C_\emptyset$, and counit ϵ sending $C_\emptyset$ to 1 and C_α to 0 for all $\alpha \propto n$, $n \ge 1$. Applying Proposition 6.8 immediately gives the following explicit formulas for the product and coproduct below. See §2.3 for the notation $\alpha\beta$, $\alpha \rhd \beta$, $\alpha_{\le i}$, and $\alpha_{>i}$.

Proposition 7.5 *For any $\alpha \propto m$ and $\beta \propto n$, one has*

$$\mathbf{C}_\alpha \hat{\otimes} \mathbf{C}_\beta = \begin{cases} \mathbf{C}_{\alpha\beta} \oplus \mathbf{C}_{\alpha\rhd\beta}, & \text{if } \alpha\beta \propto m+n, \\ \mathbf{C}_{\alpha\rhd\beta}, & \text{otherwise,} \end{cases} \quad \textit{and} \quad \Delta(\mathbf{C}_\alpha) = \sum_{0\le i\le m} \mathbf{C}_{\alpha_{\le i}} \otimes \mathbf{C}_{\alpha_{>i}}.$$

For example, one has $\mathbf{C}_{132} \hat{\otimes} \mathbf{C}_{41} = \mathbf{C}_{13241} \oplus \mathbf{C}_{1361}$, $\mathbf{C}_{121} \hat{\otimes} \mathbf{C}_{32} = \mathbf{C}_{1242}$, and

$$\Delta(\mathbf{C}_{122}) = \mathbf{C}_\emptyset \otimes \mathbf{C}_{122} + \mathbf{C}_1 \otimes \mathbf{C}_{22} + \mathbf{C}_{11} \otimes \mathbf{C}_{12} + \mathbf{C}_{12} \otimes C_2 + \mathbf{C}_{121} \otimes \mathbf{C}_1 + \mathbf{C}_{122} \otimes \mathbf{C}_\emptyset.$$

Corollary 7.6 (i) *The graded algebra and coalgebra structures of $G_0(\mathcal{H}_\bullet)$ are dual to each other via the pairing defined by $\langle \mathbf{C}_\alpha, \mathbf{C}_\beta \rangle := \delta_{\alpha,\beta}$ for all $\alpha \propto m$ and $\beta \propto n$, with a self-dual basis $\{\mathbf{C}_\alpha : \alpha \propto n,\ \forall n \ge 0\}$.*

(ii) *There is a surjection $\sigma : K_0(\mathcal{H}_\bullet(0)) \twoheadrightarrow G_0(\mathcal{H}_\bullet)$ of graded algebras and an injection $\iota : G_0(\mathcal{H}_\bullet) \hookrightarrow G_0(\mathcal{H}_\bullet(0))$ of graded coalgebras such that the two maps are dual to each other.*

Proof The first assertion holds since it follows from Proposition 7.5 that

$$\langle \mathbf{C}_\alpha \hat{\otimes} \mathbf{C}_\beta, \mathbf{C}_\gamma \rangle = \langle \mathbf{C}_\alpha \otimes \mathbf{C}_\beta, \Delta(\mathbf{C}_\gamma) \rangle, \quad \langle \mathbf{C}_\emptyset, \mathbf{C}_\alpha \rangle = \epsilon(\mathbf{C}_\alpha). \tag{7.1}$$

For the second assertion, first recall the representation theory of the 0-Hecke algebra $H_n(0)$ from §2.3. We define the surjection σ by

$$\sigma(\mathbf{P}_\alpha(0)) = \begin{cases} \mathbf{C}_\alpha, & \text{if } \alpha \propto n, \\ 0, & \text{otherwise.} \end{cases} \tag{7.2}$$

We define the injection ι by sending $\mathbf{C}_\alpha$ to $\mathbf{C}_\alpha(0)$ for all $\alpha \propto n$. One sees that σ and ι are maps of graded algebras and coalgebras, respectively, by comparing Proposition 7.5 with Proposition 2.1. It is not hard to check that

$$\langle \sigma(\mathbf{P}_\alpha(0)), \mathbf{C}_\beta \rangle = \langle \mathbf{P}_\alpha(0), \iota(\mathbf{C}_\beta) \rangle = \delta_{\alpha,\beta}, \quad \forall \alpha \models m,\ \forall \beta \propto n.$$

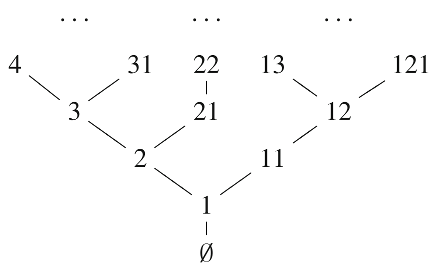
This shows that σ and ι are dual maps. Hence, (ii) holds. □

Remark 7.7 (i) Comparing the definitions for $\mathcal{H}_n$ and $\mathcal{H}_n(0)$, one sees that the former is a quotient of the latter by the relations $T_i T_{i+1} = 0$ for all $i = 1, \ldots, n-2$. Thus, any $\mathcal{H}_n$-module is automatically an $\mathcal{H}_n(0)$-module. This induces the injection $\iota : G_0(\mathcal{H}_\bullet) \hookrightarrow G_0(\mathcal{H}_\bullet(0))$ given in the previous proposition. On the other hand, $\mathbf{C}_\alpha(0) = \mathrm{top}(\mathbf{P}_\alpha(0))$ admits an $\mathcal{H}_n$-action and is hence isomorphic to $\mathbf{C}_\alpha$ if and only if the composition α has all internal parts larger than 1. This induces the surjection $\sigma : K_0(\mathcal{H}_\bullet(0)) \twoheadrightarrow G_0(\mathcal{H}_\bullet)$ defined in (7.2).

(ii) It is well known that the number of partitions of n is no more than the Fibonacci number F_{n+1}. One may suspect that the surjection $K_0(\mathcal{H}_\bullet(0)) \cong \mathbf{NSym} \twoheadrightarrow \mathrm{Sym} \cong G_0(\mathbb{C}\mathfrak{S}_\bullet)$ factors through the surjection $\sigma : K_0(\mathcal{H}_\bullet(0)) \twoheadrightarrow G_0(\mathcal{H}_\bullet)$. This is *not* true since the commutative image of the noncommutative ribbon Schur function $\mathbf{s}_\alpha$ is the ribbon schur function s_α, but $f(\mathbf{P}_\alpha(0)) = 0$ if α is a composition with an internal part equal to 1. Similarly, one sees that the injection $G_0(\mathbb{C}\mathfrak{S}_\bullet) \cong \mathrm{Sym} \hookrightarrow \mathrm{QSym} \cong G_0(\mathcal{H}_\bullet(0))$ does not factor through the injection $\iota : G_0(\mathcal{H}_\bullet) \hookrightarrow G_0(\mathcal{H}_\bullet(0))$, since the image of the injection i is spanned by $\mathbf{C}_\alpha(0)$ for all $\alpha \propto n$, $n \geq 0$, but $F_\alpha \in \mathrm{Sym}$ when $\alpha = 1^n$, $n \geq 3$.

(iii) Unfortunately, $G_0(\mathcal{H}_\bullet)$ is not a bialgebra: one checks that $\Delta(\mathbf{C}_{11} \hat{\otimes} \mathbf{C}_1) \neq \Delta(\mathbf{C}_{11}) \hat{\otimes} \Delta(\mathbf{C}_1)$ where the product on the right- hand side is tensor-componentwise. Thus, it does not fit into Zelevinsky's theory on *positive self-dual Hopf algebras* [12]. One also checks that $G_0(\mathcal{H}_\bullet)$ is not a *weak bialgebra* (c.f. [4]), nor an *infinitesimal bialgebra* (c.f. [1]).

Next, we consider the *Bratteli diagram* of the tower of algebras $\mathcal{H}_0 \hookrightarrow \mathcal{H}_1 \hookrightarrow \mathcal{H}_2 \hookrightarrow \cdots$. It has vertices at level n indexed by $\alpha \propto n$, for $n = 0, 1, 2, \ldots$, and it has an edge between $\alpha \propto n$ and $\beta \propto n-1$ if and only if $\mathbf{C}_\alpha \downarrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}} \cong \mathbf{C}_\beta$. One can draw this diagram using Proposition 7.5. The first 5 levels are illustrated below.



7.3 Antipode

We consider the antipode of $G_0(\mathcal{H}_\bullet)$. In general, let A be an algebra with product μ and unit u, and let C be a coalgebra with coproduct Δ and counit ϵ. The *convolution product* of two maps $f, g \in \mathrm{Hom}_{\mathbb{F}}(C, A)$ is defined as $f \star g := \mu \circ (f \otimes g) \circ \Delta$. One can check that $u \circ \epsilon$ is the two-sided identity element for this convolution product.

Let (A', μ', u') be another algebra and (C', Δ', ϵ') be another coalgebra such that there exists an algebra surjection $\sigma : A \twoheadrightarrow A'$ and a coalgebra injection $\iota : C' \hookrightarrow C$. Then, $u' = \sigma \circ u$, $\epsilon' = \epsilon \circ \iota$, and the following diagram is commutative, where $f' := \sigma \circ f \circ \iota$ and $g' := \sigma \circ g \circ \iota$.

$$\begin{array}{ccccccc} C & \xrightarrow{\Delta} & C \otimes C & \xrightarrow{f\otimes g} & A \otimes A & \xrightarrow{\mu} & A \\ {\scriptstyle \iota}\uparrow & & {\scriptstyle \iota\otimes\iota}\uparrow & & \downarrow{\scriptstyle \sigma\otimes\sigma} & & \downarrow{\scriptstyle \sigma} \\ C' & \xrightarrow{\Delta'} & C' \otimes C' & \xrightarrow{f'\otimes g'} & A' \otimes A' & \xrightarrow{\mu'} & A' \end{array} \tag{7.3}$$

The *antipode* S of a Hopf algebra H is nothing but the two-sided inverse of the identity map 1_H under the convolution product for the endomorphism algebra $\mathrm{End}_{\mathbb{F}}(H)$. In other words, S is defined by the commutative diagram below.

$$\begin{array}{ccccccc} & \overset{\Delta}{\nearrow} & H \otimes H & \xrightarrow{S\otimes 1_H} & H \otimes H & \overset{\mu}{\searrow} & \\ H & & \xrightarrow{\epsilon} & \mathbb{F} & \xrightarrow{u} & & H \\ & \underset{\Delta}{\searrow} & H \otimes H & \xrightarrow[1_H\otimes S]{} & H \otimes H & \underset{\mu}{\nearrow} & \end{array}$$

Note that the definition for the antipode S only requires H to be simultaneously an algebra and a coalgebra. Moreover, if the antipode S of H exists, and if there is an algebra surjection $\sigma : H \twoheadrightarrow H'$ and a coalgebra injection $\iota : H' \hookrightarrow H$, then one sees from (7.3) that $S' := \sigma \circ S \circ \iota$ is the antipode of H'.

The antipodes of the dual graded Hopf algebras QSym and **NSym** are well known to the experts. If $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ is a composition of n, then its *reverse* is the composition $\mathrm{rev}(\alpha) := (\alpha_\ell, \ldots, \alpha_1)$, its *complement* is the unique composition α^c of n with $D(\alpha^c) = [n-1] \setminus D(\alpha)$, and its *conjugate* is the composition $\omega(\alpha) := (\mathrm{rev}(\alpha))^c = \mathrm{rev}(\alpha^c)$. For example, if $\alpha = 21321$, then $\mathrm{rev}(\alpha) = 12312$, $\alpha^c = 13122$, and $\omega(\alpha) = 22131$. The antipodes of QSym and **NSym** are defined by $S(F_\alpha) = (-1)^n F_{\omega(\alpha)}$ and $S(\mathbf{s}_\alpha) = (-1)^n \mathbf{s}_{\omega(\alpha)}$ for all $\alpha \models n$, $n \geq 0$, where $\{F_\alpha\}$ and $\{\mathbf{s}_\alpha\}$ are dual bases for QSym and **NSym**.

However, the same rule does not work for $G_0(\mathcal{H}_\bullet)$. To give the antipodes of $G_0(\mathcal{H}_\bullet)$, we introduce a free $\mathbb{Z}$-module $\mathfrak{Comp}$ with a basis consisting of all compositions. By Proposition 2.1, we can define a product $\alpha \hat{\otimes} \beta := \alpha\beta + \alpha \triangleright \beta$ and a coproduct $\Delta(\alpha) := \sum_{0 \leq i \leq |\alpha|} \alpha_{\leq i} \otimes \alpha_{>i}$ for all compositions α and β, such that there is an algebra isomorphism $\mathfrak{Comp} \cong K_0(\mathcal{H}_\bullet(0))$ and a coalgebra isomorphism $\mathfrak{Comp} \cong G_0(\mathcal{H}_\bullet(0))$. The basis of all compositions for $\mathfrak{Comp}$ is self-dual under the pairing $\langle \alpha, \beta \rangle := \delta_{\alpha,\beta}$.

There is an algebra surjection $\sigma : \mathfrak{Comp} \twoheadrightarrow G_0(\mathcal{H}_\bullet)$ defined by

$$\sigma(\alpha) = \begin{cases} \mathbf{C}_\alpha, & \alpha \propto n, \\ 0, & \text{otherwise}, \end{cases} \quad \forall \alpha \models n, \quad \forall n \geq 0$$

and a coalgebra injection $\iota : G_0(\mathcal{H}_\bullet) \hookrightarrow \mathfrak{Comp}$ sending $\mathbf{C}_\alpha$ to α for all $\alpha \propto n, n \geq 0$. They are dual to each other by Corollary 7.6 (ii). One can check that $\mathfrak{Comp}$ is not a bialgebra, but its antipode exists, giving the antipode of $G_0(\mathcal{H}_\bullet)$.

Proposition 7.8 *The map S sending α to $(-1)^n\alpha^c$ for all $\alpha \models n$, $n \geq 0$, is the antipode of $\mathfrak{Comp}$. Consequently, the antipode of $G_0(\mathcal{H}_\bullet)$ is $\sigma \circ S \circ \iota$, which sends $\mathbf{C}_\alpha$ to $(-1)^n \mathbf{C}_{\alpha^c}$ if both $\alpha \propto n$ and $\alpha^c \propto n$ hold for some $n \geq 0$, that is, if $\alpha \in \{22\cdots 2, 122\cdots 2, 22\cdots 21, 122\cdots 21\}$, or sends $\mathbf{C}_\alpha$ to 0 otherwise.*

Proof If S is the antipode of $\mathfrak{Comp}$ then $\sigma \circ S \circ \iota$ is the antipode of $G_0(\mathcal{H}_\bullet)$. Thus, it suffices to show that

$$\sum_{i=0}^{n} S(\alpha_{\leq i}) \hat{\otimes} \alpha_{>i} = u \circ \epsilon(\alpha) = \sum_{i=0}^{n} \alpha_{\leq i} \hat{\otimes} S(\alpha_{>i}), \quad \forall \alpha \models n.$$

We only show the first equality and one can check that the same argument works for the second equality. It is trivial when $\alpha = \emptyset$. Assume $n \geq 1$ below. Then, $u \circ \epsilon(\alpha) = 0$. For any $\beta \propto n$, it follows the self-duality of $\mathfrak{Comp}$ that

$$\left\langle \sum_{i=0}^{n} S(\alpha_{\leq i}) \hat{\otimes} \alpha_{>i}, \beta \right\rangle = \sum_{i=0}^{n} \big\langle S(\alpha_{\leq i}) \otimes \alpha_{>i}, \Delta(\beta) \big\rangle = \sum_{i=0}^{n} \langle S(\alpha_{\leq i}), \beta_{\leq i}\rangle \cdot \langle \alpha_{>i}, \beta_{>i} \rangle. \tag{7.4}$$

Thus, it suffices to show that the sum of $L_i := \langle S(\alpha_{\leq i}), \beta_{\leq i}\rangle \cdot \langle \alpha_{>i}, \beta_{>i}\rangle$ for $i = 0, 1, \ldots, n$ equals 0. One sees that

$$L_i = \begin{cases} (-1)^i, & \text{if } (\alpha_{\leq i})^c = \beta_{\leq i},\ \alpha_{>i} = \beta_{>i}, \\ 0, & \text{otherwise}. \end{cases}$$

Let N be the set of all $i \in \{0, 1, \ldots, n\}$ such that $L_i \neq 0$. It is trivial if $N = \emptyset$.

Suppose that $i \in N$. One sees that $D(\alpha_{\leq j}) = D(\alpha) \cap [j-1]$ and $D(\alpha_{>j}) = D(\alpha) \cap \{j+1, \ldots, n-1\}$ for any j; similarly for β. Hence, $(\alpha_{\leq i})^c = \beta_{\leq i}$ implies $(\alpha_{\leq j}) = \beta_{\leq j}$ for all $j < i$, and $\alpha_{>i} = \beta_{>i}$ implies $\alpha_{>j} = \beta_{>j}$ for all $j > i$.

Since $(\alpha_{\leq i})^c = \beta_{\leq i}$, the number $i-1$ must belong to exactly one of $D(\alpha)$ and $D(\beta)$. This forces $\alpha_{>j} \neq \beta_{>j}$ for all $j < i-1$. Similarly, since $\alpha_{>i} = \beta_{>i}$, the number $i+1$ belongs to both or neither of $D(\alpha)$ and $D(\beta)$. This forces $(\alpha_{\leq j})^c \neq \beta_{\leq j}$ for all $j > i+1$. Hence, $N \subseteq \{i-1, i, i+1\}$.

If i belongs to exactly one of $D(\alpha)$ and $D(\beta)$, then $N = \{i, i+1\}$ since $(\alpha_{\leq i+1})^c = \beta_{\leq i+1}$ and $\alpha_{>i-1} \neq \beta_{i-1}$.

If i belongs to both or neither of $D(\alpha)$ and $D(\beta)$, then $N = \{i-1, i\}$ since $(\alpha_{\leq i+1})^c \neq \beta_{\leq i+1}$ and $\alpha_{>i-1} = \beta_{i-1}$.

In either case, above equation (7.4) equals $1 - 1 = 0$. This completes the proof. □

8 Questions and remarks

8.1 Dimension

If the Coxeter system (W, S) is simply laced, then using the basis for $\mathcal{H}(\mathbf{q})$ provided in Theorem 4.3, one can obtain recursive formulas for the dimension of $\mathcal{H}(\mathbf{q})$. Is there anything else (e.g., closed formula and combinatorial interpretation) one can say about this dimension? More generally, how to write down a basis for $\mathcal{H}(\mathbf{q})$ of an arbitrary Coxeter system?

8.2 Type A

In type A, we know that the dimension of a collapse free and commutative $\mathcal{H}(\mathbf{q})$ is a Fibonacci number; for example, one can take $\mathbf{q} = (0, 1, 0, 1, \ldots)$ or $\mathbf{q} = (1, 0, 1, 0, \ldots)$. What if $\mathcal{H}(\mathbf{q})$ is not commutative?

For instance, let $\mathbf{q}$ be a sequence of $m-1$ zeros followed by $n-1$ ones. Then, $\mathcal{H}(\mathbf{q})$ is a quotient of $\mathcal{H}_m(0) \otimes \mathbb{F}\mathfrak{S}_n$ and has dimension $(m-1)!(n!+m-1)$, by Theorem 4.3. How does the representation theory of this algebra connect to the representation theory of $\mathcal{H}_m(0)$ and $\mathfrak{S}_n$?

Here is another example. If $\mathbf{q}$ consists of a many copies of 0 followed by b many copies of $q \neq 0$ and then c many copies of 0, one can use Theorem 4.3 to show that

$$\dim \mathcal{H}(\mathbf{q}) = c!(a!((b+1)! + a) + (a+1)!c).$$

If $\mathbf{q}$ consists of a many copies of $q \neq 0$ followed by b many copies of 0 and then c many copies of $q' \neq 0$, then

$$\dim \mathcal{H}(\mathbf{q}) = b!((a+1)! + b) + (b-1)!((a+1)! + b - 1)((c+1)! - 1).$$

What is the representation theory of $\mathcal{H}(\mathbf{q})$ in these two cases?

A final remark for type A: The tower of algebras $\mathcal{H}_0 \hookrightarrow \mathcal{H}_1 \hookrightarrow \mathcal{H}_2 \hookrightarrow \cdots$ is different from the tower of algebras defined by Okada [9], whose dimensions are $n!$ and whose Bratteli diagram is the Young-Fibonacci poset.

8.3 Other types

Our results on the commutative algebra $\mathcal{H}(G, R)$ applies to affine type A. Let G be the cycle C_n with vertices $1, \ldots, n$ and edges $\{1, 2\}, \ldots, \{n-1, n\}, \{n, 1\}$. We know that $\mathcal{H}(C_n, R)$ has a basis indexed by $\mathcal{I}(C_n)$. One checks that if $n \geq 3$, then $\mathcal{I}(C_n) = \mathcal{I}(P_{n-1}) \sqcup \mathcal{I}(P_{n-3})$, which is the shadow of the decomposition

$$\mathcal{H}(C_n, R) \cong \mathcal{H}(P_{n-1}, R \cap [n-1]) \oplus \mathcal{H}(P_{n-1}, R \cap [n-1])x_n.$$

Hence, for $n \geq 3$, one has $|\mathcal{I}(C_n)| = F_{n+1} + F_{n-1} = L_n$, where L_n is the n-th *Lucas number*. When $R = \emptyset$ the algebra $\mathcal{H}(C_n, \emptyset)$ is semisimple and has all simple modules one dimensional. Unfortunately, we do not have a tower of algebras $\mathcal{H}(C_n, \emptyset)$, since there is no natural embedding $C_n \hookrightarrow C_{n+1}$, and thus have no further result in this direction.

One can also take G to be the Coxeter diagram of finite type D_n ($n \geq 2$) or affine type $\widetilde{D}_n$ ($n \geq 5$). The dimension of $\mathcal{H}(G, R)$ is 4, 5, 9, 14, 23, ... (OEIS entry A000285) or 17, 24, 41, 65, 106, ... (OEIS entry A190996) in these cases.

8.4 Power series realization

In Sect. 7, we defined an algebra and coalgebra structure for the Grothendieck group $G_0(\mathcal{H}_\bullet)$ of the tower of algebras $\mathcal{H}_\bullet : \mathcal{H}_0 \hookrightarrow \mathcal{H}_1 \hookrightarrow \mathcal{H}_2 \hookrightarrow \cdots$, with a self-dual basis consisting of the simple modules, which are indexed by compositions with internal parts larger than 1. This is further extended to $\mathfrak{Comp}$ with a basis indexed by all compositions. Is there a Frobenius type of characteristic map for $G_0(\mathcal{H}_\bullet)$, or in other words, is there a power series realization of $G_0(\mathcal{H}_\bullet)$ as both an algebra and a coalgebra, similarly to $G_0(\mathbb{C}\mathfrak{S}_\bullet) \cong \mathrm{Sym}$, $G_0(\mathcal{H}_\bullet(0)) \cong \mathrm{QSym}$, and $K_0(\mathcal{H}_\bullet(0)) \cong \mathbf{NSym}$? And how about $\mathfrak{Comp}$?

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