J Algebr Comb (2015) 42:1027–1058
DOI 10.1007/s10801-015-0615-0

Combinatorics of tropical Hurwitz cycles

Simon Hampe[1]

Received: 22 July 2014 / Accepted: 24 June 2015 / Published online: 8 July 2015

Abstract We study properties of the tropical double Hurwitz loci defined by Bertram, Cavalieri and Markwig. We show that all such loci are connected in codimension one. If we mark preimages of simple ramification points, then for a generic choice of such points, the resulting cycles are weakly irreducible, i.e. an integer multiple of an irreducible cycle. We study how Hurwitz cycles can be written as divisors of rational functions and show that they are numerically equivalent to a tropical version of a representation as a sum of boundary divisors. The results and counterexamples in this paper were obtained with the help of `a-tint`, an extension for `polymake` for tropical intersection theory.

Keywords Hurwitz theory · Tropical geometry · Computational geometry

1 Introduction

Roughly speaking, Hurwitz numbers count covers of $\mathbb{P}^1$ by complex curves C of some genus g—but with a given degree and some special ramification profile over a certain number of points.[1] For example, *single* Hurwitz numbers require the cover $C \to \mathbb{P}^1$ to have a specific ramification profile over some special point (usually ∞) and only simple ramification elsewhere. These numbers have played a significant role in the study of the intersection theory of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of curves.

[1] In fact, one can consider this problem in even greater generality by counting covers $C \to C'$, where C and C' are curves of prescribed genera g and g'.

✉ Simon Hampe
simon.hampe@googlemail.com; hampe@math.tu-berlin.de

1 Institut für Mathematik, Technische Universität Berlin, Sekretariat MA 6-2, Strasse des 17. Juni 136, 10623 Berlin, Germany

The ELSV formula [11] relates Hurwitz numbers to certain intersection products of tautological classes on $\overline{\mathcal{M}}_{g,n}$. This was then used by Okounkov and Pandharipande to prove Witten's conjecture [32]—though the first proof of this is of course due to Kontsevich [26].

To obtain *double* Hurwitz numbers, we fix the ramification over two points in $\mathbb{P}^1$, usually 0 and ∞. These numbers occur not only in algebraic geometry, but also in representation theory and combinatorics—thus providing a strong connection between a wide variety of disciplines. An overview over the different definitions of double Hurwitz numbers can, for example, be found in [24]. An ELSV-type formula has been conjectured by Goulden et al. in [18], where it is also shown that these numbers are piecewise polynomial in terms of the ramification profile. By convention, one writes the profile as $x \in \mathbb{Z}^n$ with $\sum x_i = 0$. The interpretation of this is that the positive part x^+ gives the ramification profile over 0 and the negative part x^- gives the ramification profile over ∞. A special feature of double Hurwitz numbers is the fact that the number of simple ramification points only depends on the length of the ramification profile, not on the multiplicities. The number of additional simple ramification points is then $n - 2 + 2g$. This fact will be very helpful in defining higher-dimensional cycles.

The generalization to Hurwitz cycles is achieved by letting one or more of the images of simple ramification points "move around" in $\mathbb{P}^1$. In the general case, these loci were defined and studied by Graber and Vakil in [21]. In the genus 0 case, Bertram, Cavalieri and Markwig proved that these cycles are linear combinations of cycles with coefficients that are piecewise polynomial in the entries of the ramification profile [4]. They also considered tropical versions $\mathbb{H}_k^{\text{trop}}(x, p)$ and $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$, respectively, of double Hurwitz loci and showed that their combinatorics relate very nicely to the combinatorics of the different strata of the algebraic loci via dualizing of graphs. Here, $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ differs from $\mathbb{H}_k^{\text{trop}}(x, p)$ in that the preimages of the simple ramification points p_i are also marked.

Higher-dimensional Hurwitz loci were a key ingredient in the study of tautological classes of $\overline{M}_{g,n}$ in [21]. For tropical geometers, they are also of particular interest in the search for a more conceptual approach to enumerative geometry. So far, tropical enumerative results could only be translated into results in algebraic geometry by using correspondence theorems (e.g. [3,8,28,31]). These theorems only apply to very specific enumerative problems. A more general result which could, for example, relate intersection rings of algebraic and tropical moduli spaces, would make tropical enumerative geometry much more powerful. The fact that Hurwitz numbers (and possibly, Hurwitz cycles) are so closely related to intersection theory on $\overline{M}_{g,n}$ makes them a good starting point for this approach. A natural question to ask in this context is whether the algebraic Hurwitz cycle somehow tropicalizes onto the tropical one. In [4], the tropical Hurwitz cycles are obtained by translating a Gromov–Witten-type formula to its tropical analogue. While the definition is rather simple and involves only the well-known tropical moduli space of rational curves, the cycles itself are rather large (in terms of ambient dimension and number of polyhedral cells) even for small examples and difficult to study "by hand". This makes it very hard to prove a more concrete tropicalization result. We will therefore start by studying the tropical Hurwitz cycles and their properties to make them more accessible.

There are two main properties we want to consider in this paper: connectedness in codimension one and irreducibility. The first is relevant for computational purposes, as well as a necessary condition for the second property. Irreducibility itself is important if one wants to prove equality of tropical cycles—thus providing an important step towards a potential tropicalization statement relating classical and tropical Hurwitz cycles. We will also consider how Hurwitz cycles can be written as divisors of rational functions and how they relate to tropical translations of other representations of algebraic Hurwitz cycles.

Classically, questions about irreducibility and connectedness of Hurwitz spaces have been considered for a long time. Hurwitz [23] showed that the space of simple branched covers of $\mathbb{P}^1$ is connected, using results of Clebsch and Lüroth [7]. Severi [34] used this to show that $\mathcal{M}_g$ is irreducible. These questions become much more difficult, however, if one allows target curves of higher genus or more complicated ramification and monodromy—this is a very actively researched topic, see for example [5,16,25,37].

A very helpful tool in the study of Hurwitz cycles is `a-tint`[2] [22], an extension for `polymake`[3] [17] for tropical intersection theory. With its focus on moduli of curves, it provides an easy way to compute examples and a quick method for testing conjectures.

In Sect. 2.1, we review the basic definitions of tropical geometry. We define tropical varieties and the basic notions of tropical intersection theory. We give a definition of connectedness and irreducibility and discuss their relevance in more detail. We conclude this section with a short introduction to moduli of rational curves and stable maps. In 2.2, we define algebraic and tropical Hurwitz cycles. We then look at the latter in more detail, i.e. we describe the tropical covers that they parametrize and how a tropical Hurwitz cycle can be computed. In Sect. 3.1, we study whether tropical Hurwitz cycles are connected in codimension one. We give a combinatorial proof of the following result:

Theorem (Theorem 3.9) *For all k, p and x, the cycles $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ and $\mathbb{H}_k^{\text{trop}}(x, p)$ are connected in codimension one.*

In 3.2, we use this to show that all marked Hurwitz cycles are weakly irreducible for a generic choice of simple ramification points:

Corollary (Corollary 3.11) *For any x and any pairwise different p_j, $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ is weakly irreducible.*

We conclude that section with computational examples showing that this is the strongest possible statement.

In 3.3, we study how Hurwitz cycles can be cut out by rational functions on $\mathcal{M}_{0,n}^{\text{trop}}$. We know from [12] that each subcycle of a matroidal fan (such as $\mathcal{M}_{0,n}^{\text{trop}}$) can be written as the sum of products of rational functions, but the result is non-constructive. We show

2 See also https://github.com/simonhampe/atint.

3 See also www.polymake.org.

that $\mathbb{H}_{n-4}^{\text{trop}}(x, 0)$ can be cut out by the rational function that adds up distances of vertex images of covers. To prove this, we define the *push-forward* of a rational function under a morphism of equidimensional tropical varieties whose target is smooth.

Finally, in 3.4 we consider an alternative representation of the algebraic Hurwitz cycle given in [4] and its "tropicalization". We show that this new tropical cycle is *numerically equivalent* to $\mathbb{H}_k^{\text{trop}}(x)$, thus obtaining a strong indicator that our notion of naively tropicalizing is the correct one.

Remark A paper by Cavalieri, Markwig and Ranganathan [9], which appeared shortly after the first submission of this paper, proves that indeed tropical Hurwitz cycles are tropicalizations of algebraic Hurwitz cycles. As a corollary, they obtain the connectedness in codimension one for unmarked Hurwitz cycles (Theorem 3 and Corollary 4).

2 Preliminaries

2.1 Tropical geometry

2.1.1 Weighted polyhedral complexes

Notation 2.1 Let Λ be a lattice (i.e. a finitely generated free Abelian group) and $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ the associated vector space. We assume all polyhedra in V to be rational, i.e. defined by inequalities $g(x) \geq \alpha$ with $g \in \Lambda^\vee$. For a polyhedron σ, we write $V_\sigma := \langle a - b; a, b \in \sigma \rangle_{\mathbb{R}}$ for the linear part of its affine space and $\Lambda_\sigma := V_\sigma \cap \Lambda$ for its associated lattice.

Definition 2.2 A *weighted polyhedral complex* (Σ, ω) is a pure, rational, polyhedral complex Σ in $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ together with a weight function ω on its maximal cells, taking values in $\mathbb{Z}$. We write $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ for the *support* of Σ.

Let σ be a rational d-dimensional polyhedron and τ a face of σ of dimension $d-1$. The *lattice normal vector* of τ with respect to σ, denoted by $u_{\sigma/\tau}$, is the unique generator of $\Lambda_\sigma / \Lambda_\tau \cong \mathbb{Z}$, such that $g(u_{\sigma/\tau}) > 0$ for all $g \in \Lambda_\sigma^\vee$ with $g_{|\tau} = 0$ and $g_{|\sigma} \geq 0$. By abuse of notation, we also write any representative of $u_{\sigma/\tau}$ in V with the same letter.

We call a weighted complex (Σ, ω) *balanced*, if for all codimension one cells τ the following holds:

$$\sum_{\sigma > \tau} \omega(\sigma) u_{\sigma/\tau} \in V_\tau.$$

A *tropical cycle* is the equivalence class of a balanced weighted complex modulo refinement, i.e. we consider two balanced complexes to be the same, if they have a common refinement respecting the weights. By abuse of notation, we will often use the same letter for a tropical cycle and its polyhedral structure.

A *tropical variety* is a tropical cycle whose weights are greater than zero.

Let (Σ, ω) be a weighted complex and τ any cell in Σ. We define the local fan at τ to be the weighted fan

$$\mathrm{Star}_\Sigma(\tau) := (\{\Pi(\sigma - \tau); \tau \le \sigma\}, \omega_{\mathrm{Star}}),$$

where $\Pi : \mathbb{R}^n \to \mathbb{R}^n / V_\tau$ is the residue map, $\sigma - \tau$ denotes the pointwise difference and the weight function is defined by $\omega_{\mathrm{Star}} : \Pi(\sigma - \tau) \mapsto \omega(\sigma)$.

The *recession cone* of a polyhedron $\sigma \subseteq V$ is the set

$$\mathrm{rec}(\sigma) := \{v \in V; \exists x \in \sigma \text{ such that } x + \mathbb{R}_{\ge 0} v \subseteq \sigma\}.$$

If X is a tropical cycle, then by [33, Lemma 1.4.10], there exists a refinement $\mathcal{X}$ of its polyhedral structure such that $\delta(X) := \{\mathrm{rec}(\sigma); \sigma \in \mathcal{X}\}$ is a polyhedral fan (one can use a construction similar to the one used for defining push-forwards). If we define a weight function

$$\omega_\delta(\mathrm{rec}(\sigma)) := \sum_{\sigma' : \mathrm{rec}(\sigma') = \mathrm{rec}(\sigma)} \omega_X(\sigma'),$$

then $(\delta(X), \omega_\delta)$ is a tropical cycle by [33, Theorem 1.4.12].

We call two tropical cycles *rationally equivalent* if $\delta(X) = \delta(Y)$ (up to refinement, of course).

Let (X, ω_X) be a tropical cycle. A *rational function* on X is a function $\varphi : X \to \mathbb{R}$ that is piecewise affine linear with integer slopes with respect to some polyhedral structure $\mathcal{X}_\varphi$ of X.

The *divisor* of φ is the tropical cycle $\varphi \cdot X := (\mathcal{Y}, \omega_\varphi)$, with $\mathcal{Y}$ the codimension one skeleton of $\mathcal{X}_\varphi$ and

$$\omega_\varphi(\tau) := \sum_{\sigma > \tau} \omega_X(\sigma) \varphi_\sigma(u_{\sigma/\tau}) - \varphi_\tau \left(\sum_{\sigma > \tau} \omega_X(\sigma) u_{\sigma/\tau} \right),$$

where $\varphi_\sigma, \varphi_\tau$ denote the linear part of the function restricted to the corresponding cell.

A *morphism* of tropical cycles $f : X \to Y$ is a map from $|X|$ to $|Y|$ which is locally a linear map and respects the underlying lattice, i.e. maps Λ_X to Λ_Y.

The *push-forward* of X is defined as follows: by [19, Construction 2.24], there exists a refinement $\mathcal{X}$ of the polyhedral structure on X such that $\{f(\sigma); \sigma \in \mathcal{X}\}$ is a polyhedral complex. We then set

$$f_*(X) = \{f(\sigma); \sigma \in \mathcal{X}; f \text{ injective on } \sigma\}$$

with weights

$$\omega_{f_*(X)}(f(\sigma)) = \sum_{\sigma' : f(\sigma') = f(\sigma)} \left| \Lambda_{f(\sigma)} / f(\Lambda_{\sigma'}) \right| \omega_X(\sigma').$$

It is shown in [19, Proposition 2.25] that this yields a tropical cycle and does not depend on the choice of $\mathcal{X}$.

If $f : X \to Y$ is a morphism of tropical cycles and φ is a rational function on Y, then $f^*\varphi = \varphi \circ f$ is the *pullback* of φ via f.

2.1.2 Connectedness and irreducibility

Definition 2.3 A tropical cycle X is *connected in codimension one*, if for any two maximal cells σ, σ' there exists a sequence of maximal cells $\sigma = \sigma_0, \ldots, \sigma_r = \sigma'$, such that two subsequent cells σ_i, σ_{i+1} intersect in codimension one (it is easy to see that this does not depend on the actual choice of polyhedral structure).

We call X *irreducible*, if any $(\dim X)$-dimensional subcycle Y (i.e. a tropical cycle with $|Y| \subseteq |X|$) is an integer multiple of X.

We call X *weakly irreducible* if X is an integer multiple of an irreducible cycle.

Remark 2.4 We can measure irreducibility of a tropical cycle X by computing its *weight lattice* Ω_X: this is the lattice of weight functions making it balanced. It has been shown in [22] that this does not depend on the choice of polyhedral structure and that (X, ω) is irreducible if and only if the rank of Ω_X and the greatest common divisor of all weights $\omega(\sigma)$ are both 1. Ω_X can be computed as the common solutions of all local balancing equations, which in turn can be interpreted as linear equations in the space of weight functions.

Somewhat contrary to the terminology, connectedness should probably be considered the “tropicalization” of irreducibility in the algebraic setting. It was shown in [10] that the tropicalization of any irreducible variety over an algebraically closed field is connected in codimension one. This property is also interesting from a computational point of view: roughly speaking, a connected complex can be computed by starting with a single maximal cell and recursively computing maximal cells that are attached to codimension one faces. This often provides a more efficient approach (see [6] for an example).

It is not as easy to find an analogue for tropical irreducibility. By [30, Theorem 6.7.5], the weight lattice of a d-dimensional complex Σ in $\mathbb{R}^n$ is in bijection to $A_{n-d}(X_\Sigma)$. From a purely tropical point of view, irreducibility is a helpful property if you want to show equality of cycles, as one then only needs to prove one inclusion.

Connectedness in codimension one is clearly a necessary condition for irreducibility. Together with local irreducibility, we obtain a sufficient criterion:

Proposition 2.5 (This is an easy generalization of [33, Lemma 1.2.29].) *Let X be a tropical cycle. If X is locally (weakly) irreducible (i.e.* $\mathrm{Star}_X(\tau)$ *is (weakly) irreducible for each codimension one face τ) and X is connected in codimension one, then X is (weakly) irreducible.*

2.1.3 Tropical rational curves, moduli spaces and Psi classes

We only present the basic notations and definitions related to tropical moduli spaces. For more detailed information, see for example [19].

Definition 2.6 An *n-marked rational tropical curve* is a metric tree with n unbounded edges, labelled with numbers $\{1, \ldots, n\}$, such that all vertices of the graph are at least trivalent. We can associate with each such curve C its metric vector $(d(C)_{i,j})_{i<j} \in \mathbb{R}^{\binom{n}{2}}$, where $d(C)_{i,j}$ is the distance between the unbounded edges (called *leaves*) marked i and j determined by the metric on C.

Define $\Phi_n : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{2}}, a \mapsto (a_i + a_j)_{i<j}$. Then,

$$\mathcal{M}_{0,n}^{\text{trop}} := \{d(C); C\ n - \text{marked curve}\} \subseteq \mathbb{R}^{\binom{n}{2}} / \Phi_n(\mathbb{R}^n)$$

is the *moduli space* of n-marked rational tropical curves.

Remark 2.7 The space $\mathcal{M}_{0,n}^{\text{trop}}$ is also known as the *space of phylogenetic trees* [36]. It is shown (e.g. in [19]) that $\mathcal{M}_{0,n}^{\text{trop}}$ is a pure $(n-3)$-dimensional fan and if we assign weight 1 to each maximal cone, it is balanced (though [19] does not use the standard lattice, as we will see below). Points in the interior of the same cone correspond to curves with the same *combinatorial type*: the combinatorial type of a curve is its equivalence class modulo homeomorphisms respecting the labelings of the leaves, i.e. morally we forget the metric on each graph. In particular, maximal cones correspond to curves where each vertex is exactly trivalent. We call this particular polyhedral structure on $\mathcal{M}_{0,n}^{\text{trop}}$ the *combinatorial subdivision*.

The lattice for $\mathcal{M}_{0,n}^{\text{trop}}$ under the embedding defined above is generated by the rays of the fan. These correspond to curves with exactly one bounded edge. Hence, each such curve defines a partition or *split* $I|I^c$ on $\{1, \ldots, n\}$ by dividing the set of leaves into those lying on the "same side" of e. We denote the resulting ray by v_I (note that $v_I = v_{I^c}$). Similarly, given any rational n-marked curve, each bounded edge E_i of length α_i induces some split $I_i|I_i^c, i = 1, \ldots, d$ on the leaves. In the moduli space, this curve is then contained in the cone spanned by the v_{I_i} and can be written as $\sum \alpha_i v_{I_i}$. In particular, $\mathcal{M}_{0,n}^{\text{trop}}$ is a simplicial fan.

There are several reasons why $\mathcal{M}_{0,n}^{\text{trop}}$ should be considered the tropical analogue of $M_{0,n}$, the algebraic space of rational n-marked curves. Perhaps easiest to see is the fact that there is a one-to-one, dimension-reversing relation between combinatorial types of tropical rational curves and boundary strata of $\overline{M}_{0,n}$. Each boundary stratum corresponds to a nodal curve X, to which we can assign a *dual graph*. This is a graph which has a vertex for each component of X, a bounded edge for each node and an unbounded leaf for each marked point.

A much stronger relation was proven in [20], where it is shown that (for the right embedding) the tropicalization of $M_{0,n}$ is $\mathcal{M}_{0,n}^{\text{trop}}$ and the closure of $M_{0,n}$ in the toric variety $X(\mathcal{M}_{0,n}^{\text{trop}})$ is $\overline{M}_{0,n}$ (i.e. $\overline{M}_{0,n}$ is a *tropical compactification*).

Definition 2.8 Let $n \geq 3$ and $i \in \{1, \ldots, n\}$. The ith Psi class is the subset ψ_i of $\mathcal{M}_{0,n}$, consisting of the locus of all n-marked curves such that the ith leaf is attached to a vertex that is at least four-valent.

Remark 2.9 In the combinatorial subdivision of $\mathcal{M}_{0,n}$, ψ_i is actually a codimension one sub*fan* and assigning weight 1 to each maximal cone produces a tropical variety.

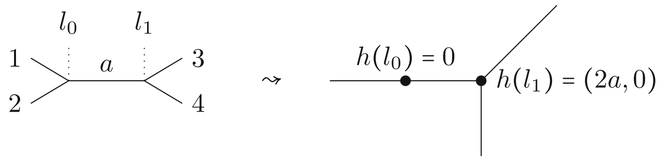


Fig. 1 On the *left*, the abstract 6-marked curve $\Gamma = a \cdot v_{\{1,2,l_0\}}$. If we pick $\Delta = ((-1,0),(-1,0),(2,2),(0,-2))$ and fix $h(l_0) = 0$ in $\mathbb{R}^2$, we obtain the curve on the *right-hand side* as $h(\Gamma)$

Tropical Psi classes were first defined by Mikhalkin in [29], as a direct translation of the classical definition. In [27], the authors define Psi classes as divisors of rational functions on $\mathcal{M}_{0,n}$ and give a complete combinatorial description of all products of Psi classes.

2.1.4 Tropical stable maps

To study covers of $\mathbb{R}$ in tropical geometry, we will need a tropical space of stable maps. A precise definition can be found in [19, Section 4]. For shortness, we will use their result from Proposition 4.7 as definition and explain the geometric interpretation behind it afterwards.

Definition 2.10 Let $m \geq 4$, $r \geq 1$. For any $\Delta = (v_1, \ldots, v_n)$, $v_i \in \mathbb{R}^r$ with $\sum v_i = 0$ we denote by

$$\mathcal{M}_{0,m}^{\text{trop}}(\mathbb{R}^r, \Delta) := \mathcal{M}_{0,n+m}^{\text{trop}} \times \mathbb{R}^r$$

the *space of stable m-pointed maps of degree Δ.*

Remark 2.11 An element of $\mathcal{M}_{0,m}^{\text{trop}}(\mathbb{R}^r, \Delta)$ represents an $(n+m)$-marked abstract curve C together with a continuous, piecewise integer affine linear (with respect to the metric on C) map $h : C \to \mathbb{R}^r$. We label the first n leaves by $\{1, \ldots, n\}$ and require h to have slope $v_1, \ldots, v_n$ on them. We denote the last m leaves by $l_0, \ldots, l_{m-1}$. These are contracted to a point under h. Since we want the image curve to be a tropical curve in $\mathbb{R}^r$, the slope on the bounded edges is already uniquely defined by the condition that the outgoing slopes of h at each vertex have to add up to 0. This defines the map h up to a translation in $\mathbb{R}^r$. The translation is fixed by the $\mathbb{R}^r$-coordinate, which can, for example, be interpreted as the image of the first contracted end l_0 under h (see Fig. 1 for an example). There are obvious evaluation maps $\text{ev}_i : \mathcal{M}_{0,m}^{\text{trop}}(\mathbb{R}^r, \Delta) \to \mathbb{R}^r$, $i = 0, \ldots, m-1$, mapping a stable map to $h(l_i)$. [19, Proposition 4.8] shows that these are morphisms. Similarly, there is a *forgetful morphism* $\text{ft} : \mathcal{M}_{0,m}^{\text{trop}}(\mathbb{R}^r, \Delta) \to \mathcal{M}_{0,n}^{\text{trop}}$, forgetting the contracted ends and the map h.

2.2 Hurwitz cycles

2.2.1 Algebraic Hurwitz cycles

We will only briefly cover algebraic Hurwitz cycles, as we will be working exclusively on the tropical side. For a more in-depth discussion of its definition and properties, see, for example, [4,21].

Let $n \geq 4$. We define

$$\mathcal{H}_n := \left\{ x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0 \right\} \setminus \{0\}.$$

Let $x \in \mathcal{H}_n$ and choose distinct points $p_0, \ldots, p_{n-3-k} \in \mathbb{P}^1 \setminus \{0, \infty\}$. The *double Hurwitz cycle* $\mathbb{H}_k(x)$ is a k-dimensional cycle in the moduli space of rational n-marked curves $\overline{M}_{0,n}$. It parametrizes curves C that *allow* covers $C \xrightarrow{\pi} \mathbb{P}^1$ with the following properties:

- C is a smooth connected rational curve.
- π has ramification profile $x^+ := (x_i; x_i > 0)$ over 0 and ramification profile $x^- := (x_i; x_i < 0)$ over ∞. The corresponding ramification points are the marked points of C.
- π has simple ramification over the p_i and at most simple ramification elsewhere.

The precise definition [4, Section 3] actually involves some moduli spaces. For the sake of simplicity, we will just cite the following result that can be taken as a definition throughout this paper.

Lemma 2.12 ([4, Lemma 3.2])

$$\mathbb{H}_k(x) = st_* \left(\prod_{i=1}^{n-2-k} \psi_i \mathrm{ev}_i^*([pt]) \right),$$

where

- *the intersection product is taken in* $\overline{M}_{0,n-2-k}(x)$*, the space of relative stable maps to* $\mathbb{P}^1$ *with ramification profile* x^+, x^- *over* 0 *and* ∞ *(see also* [21] *for a definition. In their language, this is the space of maps to a rigid target).*
- $st : \overline{M}_{0,n-2-k}(x) \to \overline{M}_{0,n}$ *is the morphism forgetting the map and all marked points but the ramification points over* 0 *and* ∞ *(and stabilizing the result by contracting components that become unstable, i.e. contain less than three special points).*

2.2.2 Tropical Hurwitz cycles

We already have all ingredients at hand to "tropicalize" Lemma 2.12. Note that a point $q \in \mathbb{R}$ can be considered as the divisor of the tropical polynomial $\max\{x, q\}$, so it can be pulled back along a morphism to $\mathbb{R}$. Also, as $\mathcal{M}_{0,n+m}^{\text{trop}}$ is a subcycle of $\mathcal{M}_{0,m}^{\text{trop}}(\mathbb{R}^r, \Delta) = \mathcal{M}_{0,n+m}^{\text{trop}} \times \mathbb{R}^r$, we can define Psi classes on the latter: for $i = 0, \dots, m-1$, we define

$$\Psi_i := \psi(l_i) \times \mathbb{R}^r,$$

where $\psi(l_i)$ is the Psi class of $\mathcal{M}_{0,n+m}^{\text{trop}}$ associated with the leaf l_i we defined in 2.8.

Definition 2.13 ([4, Definition 6]) Let $x \in \mathbb{Z}^n \setminus \{0\}$ with $\sum x_i = 0, k \geq 0$ and $N := n - 2 - k$. Choose $p := (p_0, \dots, p_{N-1}), p_i \in \mathbb{R}$. We define the *tropical marked Hurwitz cycle*

$$\tilde{\mathbb{H}}_k^{\text{trop}}(x, p) := \left(\prod_{i=0}^{N-1} (\Psi_i \text{ev}_i^*(p_i)) \right) \cdot \mathcal{M}_{0,N}^{\text{trop}}(\mathbb{R}, x).$$

We then define the *tropical Hurwitz cycle*

$$\mathbb{H}_k^{\text{trop}}(x, p) := \text{ft}_*(\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)) \subseteq \mathcal{M}_{0,n}^{\text{trop}}.$$

Remark 2.14 In [8], the authors show that Hurwitz numbers can be considered as a weighted count of tropical covers of $\mathbb{R}$, which are monodromy graphs of algebraic covers. In particular, the ramification profile over 0 and ∞ appears on the tropical side as the slopes of the ends going to $\pm\infty$. Thus, a tropical analogue of a cover with prescribed ramification profile x is an element of $\mathcal{M}_{0,N}^{\text{trop}}(\mathbb{R}, x)$. Hence, the above definition becomes the exact analogue of Lemma 2.12 and gives us k-dimensional tropical cycles $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p), \mathbb{H}_k^{\text{trop}}(x, p)$. While it formally depends on the choice of the p_j, two different choices p, p' lead to rationally equivalent cycles $\mathbb{H}_k^{\text{trop}}(x, p) \sim \mathbb{H}_k^{\text{trop}}(x, p')$. The reason for this is that any two points in $\mathbb{R}$ are rationally equivalent and this is compatible with pullbacks and taking intersection products. In particular, if we choose all p_i to be equal (e.g. equal to 0), we obtain fans, which we denote by $\tilde{\mathbb{H}}_k^{\text{trop}}(x)$ and $\mathbb{H}_k^{\text{trop}}(x)$. They are obviously the recession fans of $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p), \mathbb{H}_k^{\text{trop}}(x, p)$ for any p.

Example 2.15 Let us now see what kind of object these Hurwitz cycles represent. As discussed in Remark 2.11, for any fixed x and any n-marked curve C, we obtain a map $h : C \to \mathbb{R}$ up to translation. To determine such a map, we have to fix an orientation of each edge and leaf of C and an integer slope along this orientation. In informal terms, the orientation determines how we position an edge or leaf on $\mathbb{R}$ (the "tip" of the arrow points towards $+\infty$). The slope can then be seen as a stretching factor.

The orientation of each leaf i is chosen so that it "points away" from its vertex if and only if $x_i > 0$. We define its slope to be $|x_i|$. Any bounded edge e induces a split I_e. Its slope is $|x_{I_e}|$, where $x_{I_e} = \sum_{i \in I_e} x_i$. We pick the orientation such that at each vertex, the sum of slopes of incoming edges is the sum of slopes of outgoing edges (it is not hard to see that such an orientation exists and must be unique).

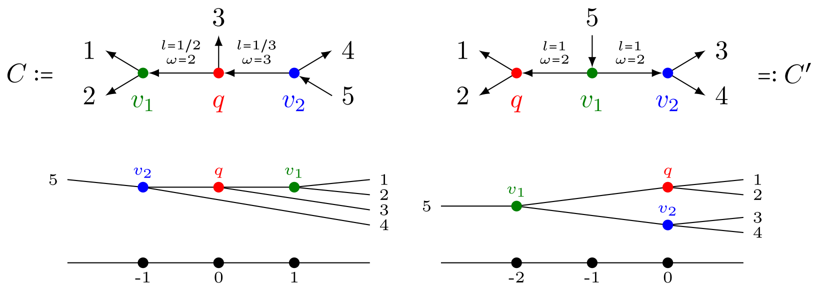


Fig. 2 Covers defined by two 5-marked rational curves after fixing the image of a vertex q to be $\alpha = 0$. We chose $x = (1, 1, 1, 1, -4)$ and denoted edge lengths by l, edge slopes by ω

As we discussed before, we can fix the translation of h by requiring the image of any of its vertices q to be some $\alpha \in \mathbb{R}$. Denote by $h(C, q, \alpha) : C \to \mathbb{R}$ the corresponding map. Figure 2 gives two examples of this construction.

Now choose $p_0, \ldots, p_{N-1} \in \mathbb{R}$. Then, $\mathbb{H}_k^{\text{trop}}(x, p)$ is (set theoretically) the set of all curves C, where we can find vertices $q_0, \ldots, q_{N-1}$ (each vertex q can be picked a number of times equal to $\text{val}(q) - 2$), such that $h(C, q_0, p_0)(q_l) = p_l$ for all l, i.e. all curves that *allow* a cover with fixed images for some of its vertices. For example in Fig. 2, we have

- $C \in \mathbb{H}_1^{\text{trop}}(x, p = (0, 1))$, but $C \notin \mathbb{H}_1^{\text{trop}}(x, p = (0, 0))$.
- $C' \in \mathbb{H}_1^{\text{trop}}(x, p = (0, 0))$, but $C' \notin \mathbb{H}_1^{\text{trop}}(x, p = (0, 1))$.

In particular, if we choose $p_i = 0$ for all i, $\mathbb{H}_k^{\text{trop}}(x, p)$ is the set of all curves, such that $n - 2 - k$ of its vertices have the same image (again, counting higher-valent vertices v with multiplicity $\text{val}(v) - 2$).

Of course there may be several possible choices of vertices that are compatible with p. In $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$, we fix a choice by attaching the contracted end l_i to the vertex we wish to be mapped to p_i; i.e. $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ is the set of all curves C, such that $l_0, \ldots, l_{N-1}$ are attached to vertices and such that in the corresponding cover the vertex with leaf l_i is mapped to p_i. For example, in Fig. 2 on the left-hand side, there are two possible choices of vertices that are compatible with $p = (0, 1)$. Hence, there are two preimages in $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ corresponding to attaching the contracted leaves l_0, l_1 either to q and v_1 or to v_2 and q.

Remark 2.16 Let us see how the weight of a cell of $\mathbb{H}_k^{\text{trop}}(x)$ is computed if we choose the p_i to be generic, i.e. pairwise different. Let τ be a maximal cell of $\mathbb{H}_k^{\text{trop}}(x)$ and C the curve corresponding to an interior point of τ. Then, τ must lie in the interior of a maximal cell σ of $\mathcal{M}_{0,n}^{\text{trop}}$, and for a generic choice of C, there is a unique choice of vertices $q_0, \ldots, q_{N-1}$ compatible with the p_i (which fixes a cover). Marking these vertices accordingly, we can consider σ as a cone in $\mathcal{M}_{0,N}^{\text{trop}}(\mathbb{R}, x)$. We thus obtain well-defined and linear evaluation maps $\text{ev}_i : \sigma \to \mathbb{R}$, mapping each curve in σ to the image of the vertex q_i. Assume σ is spanned by the rays $v_{I_1}, \ldots, v_{I_{n-3}}$, and then, we can write ev_i in the coordinates of these rays as $(a_1^i, \ldots, a_{n-3}^i)$, where $a_k^i = \text{ev}_i(v_{I_k})$.

It is shown in [4, Lemma 4.4] that the weight of τ is then the greatest common divisor of the maximal minors of the matrix $(a_k^i)_{k,i}$.

In the case that all p_i are 0, we use the fact that $\mathbb{H}_k^{\text{trop}}(x, p)$ is the recession fan of the Hurwitz cycle obtained for a generic choice of p_i. By its definition, this means that the total weight of a cell τ is obtained as

$$\omega(\tau) = \sum_{\tau \subseteq \sigma} \sum_{q_i} g_{\sigma, q_i},$$

where the first sum runs over all maximal cones σ of $\mathcal{M}_{0,n}^{\text{trop}}$ containing τ, the second sum runs over all vertex choices $q_0, \dots, q_{N-1}$ that are compatible in σ with a *generic* choice of p_i and g_{σ,q_i} is the gcd we obtained in the previous construction. In fact, one can easily see that the same method can be used for computing weights if only some of the p_i are equal.

2.3 Computation

If we approach this naively, we already have everything at hand to compute at least marked Hurwitz cycles with `a-tint`: [22] tells us how to compute a product of Psi classes (without having to compute the ambient moduli space, which will be huge!), and then, we only have to compute divisors of tropical polynomials on this product. However, this only works for small k, i.e. large codimension. Otherwise, the Psi class product will already be too large to make this computation feasible.

Also, we will mostly be interested in unmarked Hurwitz cycles and computing push-forwards is, computationally speaking, not desirable. One has to produce a very fine polyhedral structure to make sure that the images of the cones form a fan. The following approach to compute unmarked cycles directly proves to be more suitable:

Assume we want to compute $\mathbb{H}_k^{\text{trop}}(x, p = (p_0, \dots, p_{N-1}))$ for $x \in \mathbb{Z}^n$. Fix a combinatorial type C of a trivalent rational n-marked curve, i.e. a maximal cone σ of $\mathcal{M}_{0,n}^{\text{trop}}$. For each choice of distinct vertices $q_0, \dots, q_{N-1}$ of C, we obtain linear evaluation maps on σ, by considering it as a cone of stable maps, where the additional marked ends are attached to the q_i. We can now refine σ by intersecting it with the fan F_i, whose maximal cones are

$$F_i^+ := \{x \in \sigma : \text{ev}_i(x) \geq p_i\}, \; F_i^- := \{x \in \sigma : \text{ev}_i(x) \leq p_i\}.$$

Iterating over all possible choices of q_i, this will finally give us a subdivision σ' of σ. The part of $\mathbb{H}_k^{\text{trop}}(x, p)$ that lives in σ is now a subcomplex of the k-skeleton of σ': it consists of all k-dimensional cells τ of σ' such that there exists a choice of vertices q_i with the property that the corresponding evaluation maps fulfil $\text{ev}_i(x) = p_i$ for all $x \in \tau$. The weight of such a τ can then be computed using the method described in Remark 2.16.

The full Hurwitz cycle can now be computed by iterating over all maximal cones of $\mathcal{M}_{0,n}^{\text{trop}}$. This gives a feasible algorithm at least for $n \leq 8$—after that, the moduli space itself becomes too large.

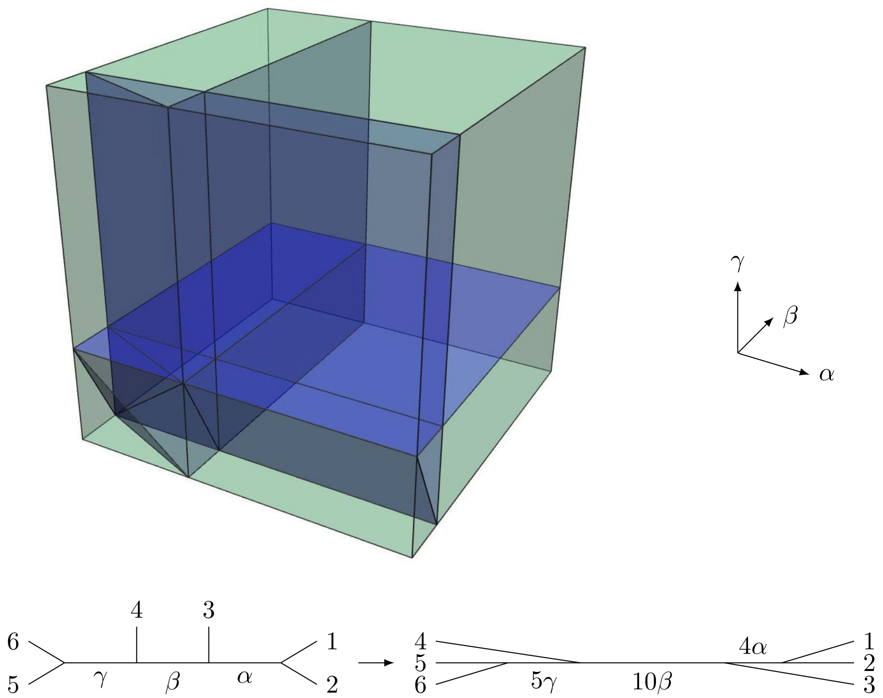


Fig. 3 *Cube* represents the three-dimensional cone in $\mathcal{M}_{0,6}^{\text{trop}}$ that corresponds to the combinatorial type $v_{\{1,2\}}+v_{\{4,5,6\}}+v_{\{5,6\}}$ drawn on the *bottom left part* of the picture. We denote the length of the interior edges by α, β, γ as indicated. The *blue cells* represent the Hurwitz cycle living in this cone. The *bottom right* figure indicates the corresponding cover. The parameters we chose here are $k = 2$, $x = (2, 2, 6, -5, -4, -1)$ and $(p_0, p_1) = (0, 1)$ (Color figure online)

Example 2.17 We want to compute (part of) a Hurwitz cycle: we choose $k = 2$, $x = (2, 2, 6, -5, -4, -1)$ and $(p_0, p_1) = (0, 1)$. Since the complete cycle would be rather large and difficult to visualize (3755 maximal cells living in $\mathbb{R}^9$), we only consider the part of $\mathbb{H}_2^{\text{trop}}(x, p)$ lying in the three-dimensional cone of $\mathcal{M}_{0,6}^{\text{trop}}$ corresponding to the combinatorial type

$$C = v_{\{1,2\}} + v_{\{4,5,6\}} + v_{\{5,6\}}.$$

Figure 3 shows the corresponding cover, together with the part of the Hurwitz cycle we computed using the method described above. Each cell of the cycle is obtained by choosing specific vertices of C for the additional marked points p_0 and p_1. The correspondence between these choices and the actual cells, together with the corresponding equation, is laid out in Fig. 4. While there are of course in theory $4 \times 4 = 16$ possible choices, not all of them produce a cell: we only display choices of distinct vertices, such that the image of the vertex for $p_1 = 1$ is larger than the image of the vertex for $p_0 = 0$. This gives $\binom{4}{2} = 6$ valid choices.

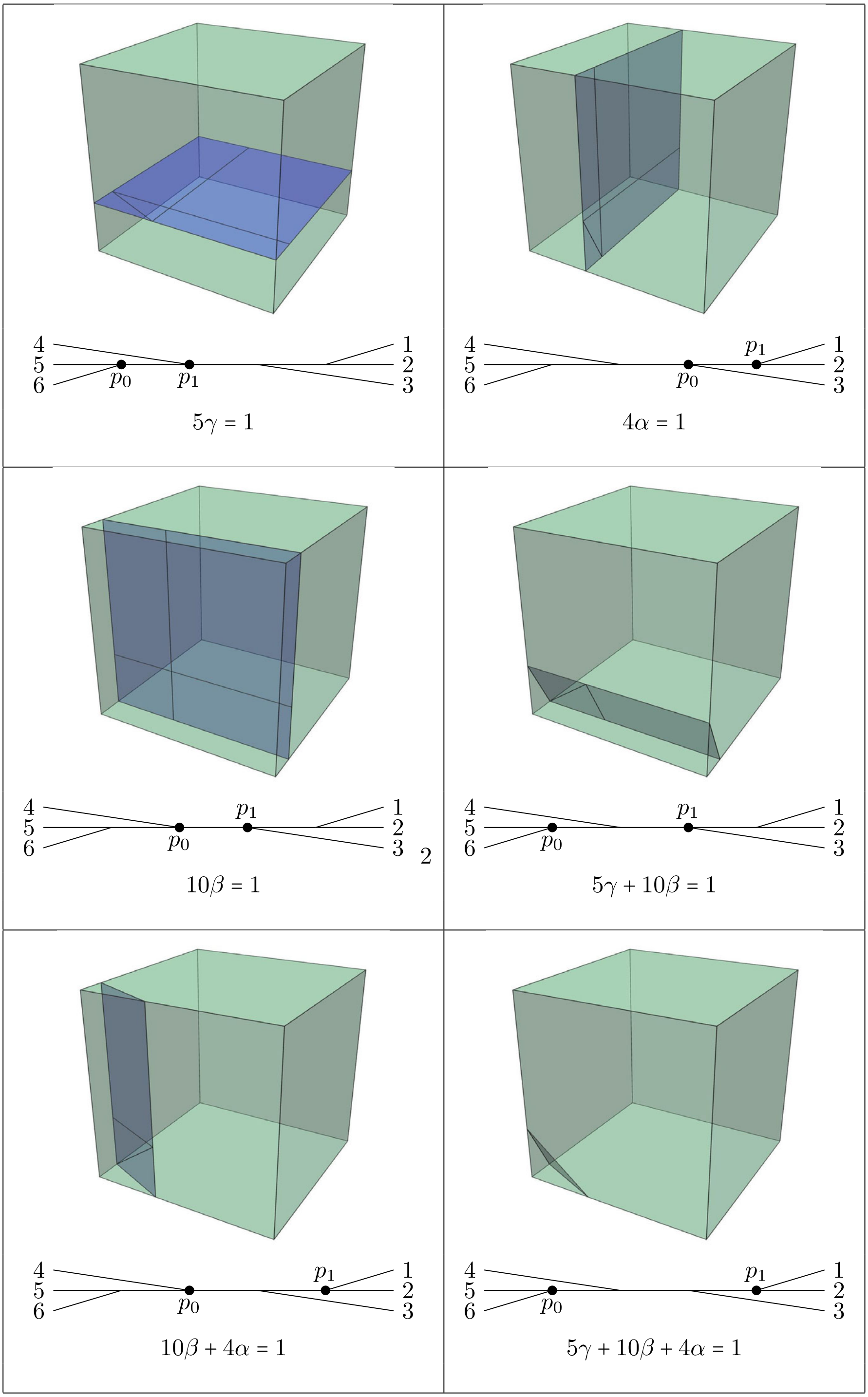


Fig. 4 Different choices of vertices yield different cells of $\mathbb{H}_2^{\text{trop}}(x, p)$

3 Properties of Hurwitz cycles

In the first two parts of this section, we want to study whether tropical Hurwitz cycles are irreducible. For this purpose, we will first prove that all (marked and unmarked) Hurwitz cycles are connected in codimension one. We will go on to show that for a generic choice of p_j, all marked cycles $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ are locally and globally a multiple of an irreducible cycle. Finally, we will see that $\mathbb{H}_k^{\text{trop}}(x, p)$ is in general not irreducible.

3.1 Connectedness in codimension one

It is well known that $\mathcal{M}_{0,n}^{\text{trop}}$ is connected in codimension one. In this particular case, the property has a very nice combinatorial description: maximal cones correspond to rational curves with $n - 3$ bounded edges. A codimension one face of a maximal cone is attained by shrinking any of these edges to length 0, thus obtaining a single four-valent vertex. This vertex can then be "drawn apart" or *resolved* in three different ways, thus moving into a maximal cone again. Saying that $\mathcal{M}_{0,n}^{\text{trop}}$ is connected in codimension one means that we can transform any trivalent curve into another by alternatingly contracting edges and resolving four-valent vertices.

A similar correspondence holds for Hurwitz covers. An element of a maximal cone of $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p) \subseteq \mathcal{M}_{0,N}^{\text{trop}}(\mathbb{R}, x)$ can be considered as an n-marked rational curve C with $N = n - 2 - k$ additional leaves attached to vertices of C. By abuse of notation, throughout this chapter we will also label these additional leaves by $p_0, \ldots, p_{N-1}$. By the *valence* of a vertex of an element of $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$, we will mean the valence of the vertex in the underlying n-marked curve.

For a generic choice of p, maximal cells of $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ will also correspond to curves with $n - 3$ bounded edges and codimension one cells are obtained by shrinking an edge. Hence, the problem of connectedness can be formulated in the same manner as for $\mathcal{M}_{0,n}^{\text{trop}}$. However, the requirement that the contracted leaves be mapped to specific points excludes certain combinatorial "moves", as we will shortly see.

Also note that the problem of connectedness does not really change if we allow non-generic points: the combinatorial problem remains essentially the same, we just allow some edge lengths to be 0. Hence, we will assume throughout this section that $p_0 < p_1 < \cdots < p_{N-1}$.

We will first show connectedness in the case $k = 1$. In this case, the Hurwitz cycle is a tropical curve, so saying that $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ is connected in codimension one is the same as requiring that it is path-connected. So we will prove that for any two vertices q, q' of $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$, there exists a sequence of edges connecting them.

We will prove this by induction on n, the length of x. For the case $n = 5$, we will simply go through all possible cases explicitly. For $n > 5$, we will first show that any two covers of a special type, called *chain covers*, are connected. Having shown this, we will then introduce a construction that allows us to connect any cover to a chain cover.

The general case is then an easy corollary, since we mark fewer vertices in higher-dimensional Hurwitz cycles, thus obtaining more degrees of freedom.

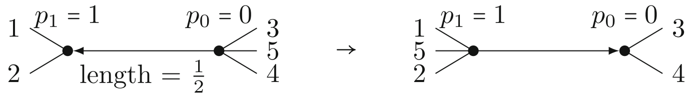


Fig. 5 *Curve on the left* is a vertex of $\tilde{\mathbb{H}}_1^{\text{trop}}(1, 1, 1, 1, -4)$. In $\mathcal{M}_{0,7}$ it corresponds to a ray spanning a cone with the *curve on the right*. However, the *right curve* is not an element of $\tilde{\mathbb{H}}_1^{\text{trop}}(1, 1, 1, 1, -4)$ (for any edge length), since the edge direction is not compatible with the vertex ordering

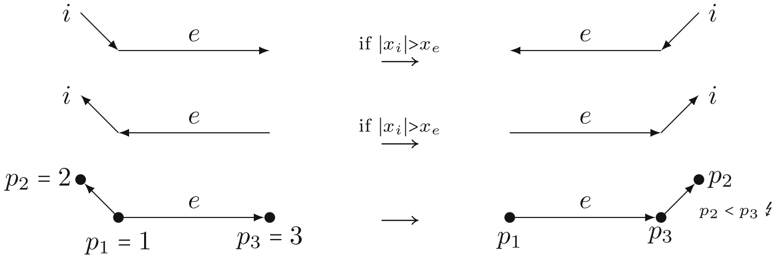


Fig. 6 Invalid moves on a Hurwitz cover: in the first two cases, when moving the leaf/edge i along the bounded edge e, the direction of e changes. In the third case, the edge direction of e remains the same, but the direction is not compatible with the order of the p_i

Remark 3.1 Before we start, we want to discuss why this problem is so difficult. Since $\mathcal{M}_{0,N}^{\text{trop}}$ is connected in codimension one, one would expect to be able to move from one combinatorial type to another without problems. However, the intermediate types need not be valid covers: a vertex of $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ can be considered as a point in a codimension one cone of $\mathcal{M}_{0,n}^{\text{trop}}$, i.e. a curve with one four-valent vertex and only trivalent vertices besides, with an additional marked end attached to every vertex. Moving along an edge of $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ means moving an edge or leaf of that codimension one type along a bounded edge. However, this cannot be done in an arbitrary manner, since not all of these movements will produce valid covers (see Fig. 5 for an example). Note that the p_j already fix the length of all bounded edges of a vertex curve in $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ uniquely. So, we will usually identify each vertex of $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ with the combinatorial type of the corresponding curve.

Recall that the *weight* or slope of an edge e is $x_e := \left|\sum_{i\in I} x_i\right|$, where I is the split on $[n]$ induced by e. The *orientation* of e is chosen as in Example 2.15: e "points towards I" if and only if $\sum_{i\in I} x_i > 0$.

Now, when moving some leaf along a bounded edge, that edge might change direction. But the direction of the edges is dictated by the order of the p_i, so this is not a valid move. One can easily see the following (see Fig. 6 for an illustration): moving an edge/leaf i to the other side of a bounded edge e changes the direction of that edge if and only if one of them is incoming and one is outgoing (recall that we consider leaves as incoming if they have negative weight) and $|x_i| > x_e$. Note that, even if the direction of an edge does not change, moving an edge might be illegal (see the last diagram in Fig. 6), if the resulting edge configuration does not agree with the order on the p_j.

Definition 3.2 A *vertex-type cover* is any cover corresponding to a vertex of $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$.

Lemma 3.3 *For $n = 5$, the cycle $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ is connected in codimension one for any p and x.*

Proof Let q, q' be two vertices of $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ and C, C' the corresponding rational curves. Both curves consist of a single bounded edge connecting contracted ends $p_0 < p_1$ with three leaves on one side and two on the other. We distinguish different cases, depending on how many leaves have to switch sides to go from C to C'.

Assume first that both curves only differ by the placement of one leaf, i.e. we want to move one leaf i from the four-valent vertex in C to the other side of the bounded edge. We can assume without restriction that the four-valent vertex in C is at p_0. Assume that moving i to the other side is an invalid move. Then, the direction of the bounded edge would be inverted in C', which is a contradiction to the fact that $p_0 < p_1$.

Now assume that both curves differ by an exchange of two leaves. Again we assume that the four-valent vertex in C (and hence also in C') is at p_0. Denote the leaves in C at p_0 by i, a, b and the remaining two at p_1 by j, c and assume that C' is obtained by exchanging i and j. If we can move either i in C or j in C', then we are in the case where only one leaf needs to be moved, which we already studied. So assume that i and j cannot be moved in C and C', respectively. By Remark 3.1, this means that $x_i < -x_e < 0$, where x_e is the weight of the bounded edge in C. Furthermore, $x_i + x_a + x_b = -x_e$, so $x_a + x_b > 0$. We assume without restriction that $x_a > 0$. Hence, we can move a along the bounded edge to obtain a valid cover C_1, whose four-valent vertex is at p_1. Since we assumed that we cannot move j in C', we must have $x_j < 0$ (it must be an incoming edge). This implies that we can move it to the left in C_1 to obtain a cover C_2. We now have i, j, b at p_0 and c, a at p_1. We want to show that we can move i to the other side. Assume this is not possible. Then, $-x_i > x_e'$, where x_e' is the weight of the bounded edge in C_2. But $x_e' = -x_i - x_j - x_b$. This implies $0 > -x_j - x_b$. Again, since j cannot be moved in C', we have $-x_j > x_a + x_b$. Finally, we obtain that $0 > x_a + x_b - x_b = x_a > 0$, which is a contradiction. Thus, we can move i to the right side to obtain a cover C_3. This cover now only differs from C' by the placement of leaf a, so we are again in the first case (see Fig. 7 for an illustration).

Now assume we have to move three leaves (see Fig. 8). That means we have to exchange two leaves i, j from the four-valent vertex in C (again assume it is at p_0) for one leaf k at p_1. Assume we cannot move i in C. In particular, $x_i < 0$. But that means

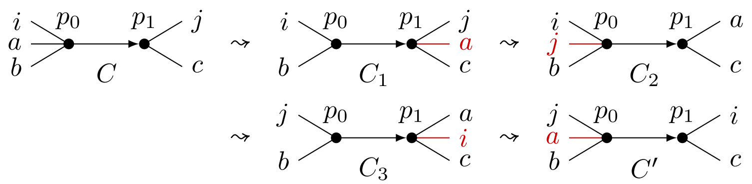
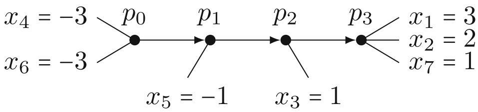


Fig. 7 Connecting two *curves* differing by an exchange of leaves. The leaf we moved in each step is marked by a *red line* (Color figure online)

Fig. 8 Two vertex types differing by a movement of three leaves. Depending on the direction of i, we can move it either in C or in C'

Fig. 9 Standard cover for $x = (3, 2, 1, -3, -1, -3, 1)$ at p_3



we can move i in C' to obtain a cover C_1. This cover differs from C by the exchange of j and k, so we already know they are connected.

Finally, assume that four leaves have to switch sides, i.e. we exchange two leaves i, j at the four-valent at p_0 for the two leaves k, l at p_1. Assume we can move neither i nor j. This means that $x_i, x_j < 0$. But then $x_i + x_j < 0$ as well, so the edge direction would be inverted in C', which is a contradiction. Hence, we can move i or j and reduced the problem to the case where only three leaves need to be moved.

It is easy to see that these are all possible cases. In particular, it is impossible to let all five leaves switch sides, since this would automatically invert the direction of the bounded edge. □

As mentioned above, we want to show that for $n > 5$, we can connect each vertex type to a vertex corresponding to a *standard cover*. Let us define this:

Definition 3.4 Let $x \in \mathcal{H}_n$. We define an order $<_x$ on $[n]$ by:

$$i <_x j :\iff x_i < x_j \text{ or } (x_i = x_j \text{ and } i < j).$$

A *chain cover for x* is a vertex-type cover with the additional property that the vertex marked with p_i is connected to the vertex marked with p_j, if and only if $|i - j| = 1$ (i.e. the p_j are arranged as a single chain in order of their size). Fix an $s \in \{0, \dots, n-4\}$. The *standard cover for x at p_s* is the unique chain cover, where the leaves are attached to the p_j according to their size (defined by $<_x$) and p_s is at the four-valent vertex. More precisely, if leaf i is attached to p_k and leaf j is attached to p_l, then $i <_x j \iff p_k < p_l$ (see Fig. 9 for an example of this construction).

Lemma 3.5 *Each standard cover is a valid Hurwitz cover.*

Proof We have to show that the edge connecting p_j and p_{j+1} points towards p_{j+1} for all j. Note that the weight and direction of an edge only depend on the split defined by it.

We will say that a leaf *lies behind* p_k, if it is attached to some $p_{k'}, k' \geq k$. Denote the leaves lying behind p_{j+1} by $i_1, \dots, i_l$. Their weights are by construction larger than or equal to all weights of remaining leaves. Considering that the sum over all leaves is 0, this implies that $\sum_{s=1}^{l} x_{i_s} > 0$ (if it was 0, then all x_i would have to be 0). Hence, the bounded edge points towards p_{j+1}. □

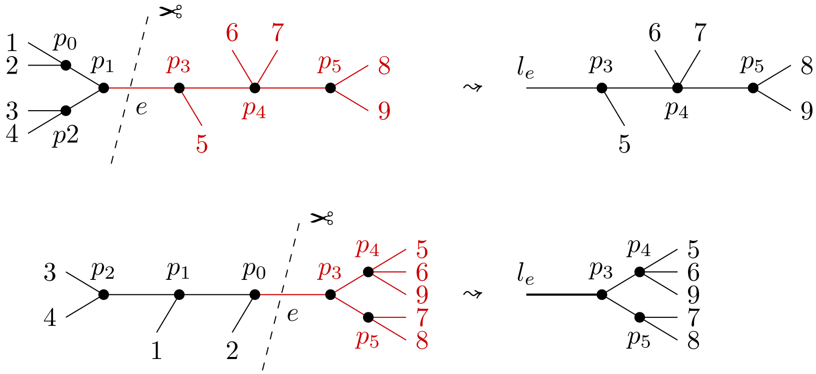


Fig. 10 Two Hurwitz covers for $n = 9$. In each case, the split cover at the edge marked by e is a cover for $n = 6$ (the labels at the leaves are just indices in this case, not weights)

We will also need another construction in our proofs:

Definition 3.6 Let C be a vertex-type cover and e any bounded edge in C connecting the contracted ends p and q. Removing e, we obtain two path-connected components. For any contracted end r, we write $C_e(r)$ for the component containing r.

Now assume $C_e(p)$ contains the four-valent vertex and at least one other bounded edge. The *split cover at e* is a cover C' obtained in the following way: remove the edge e and keep only $C_e(p)$. Then attach a leaf to p whose weight is the original weight of e (or its negative, if e pointed towards p). This is obviously a vertex-type cover for some $x' = (x'_1, \ldots, x'_m)$, where $m < n$ (see Fig. 10 for an example). We denote the leaf replacing e by l_e and call it the *splitting leaf.*

We now want to see that all chain covers are connected:

Lemma 3.7 *Let $x \in \mathcal{H}_n$ and let $p_0, \ldots, p_{n-4} \in \mathbb{R}$ with $p_j \le p_{j+1}$ for all j. Then, all chain covers for x are connected to each other.*

Proof We will show that all chain covers are connected to a standard cover at some p_s. We prove this by induction on n. For $n = 5$, all covers are chain covers and our claim follows from Lemma 3.3.

So let $n > 5$ and C be any chain cover. We can assume without restriction that the vertices at p_0 and p_{n-4} are trivalent (if they are not, one can easily see that at least one leaf can be moved away). Take any bounded edge e connecting some p_j and p_{j+1}. Suppose there is a leaf k at p_j and a leaf l at p_{j+1}, such that $k >_x l$. This means that exchanging k and l still gives a valid cover. We can assume without restriction that $j > 0$, i.e. e is not the first edge (if $j = 0$, we can use a similar argument using a split cover at the edge connecting p_{n-5} and p_{n-4}).

Let C' be the split cover at the edge connecting p_0 and p_1. This is a cover on $n-1$ leaves. By induction we know that C' is connected to the cover which only differs from C' by exchanging k and l. Let C'' be any vertex-type cover occurring along that path. Since p_0 is smaller than all p_j, we can lift C'' to a cover on n leaves: simply re-attach the splitting leaf to p_0. (see Fig. 11 for an illustration of the split-and-lift construction in a different case).

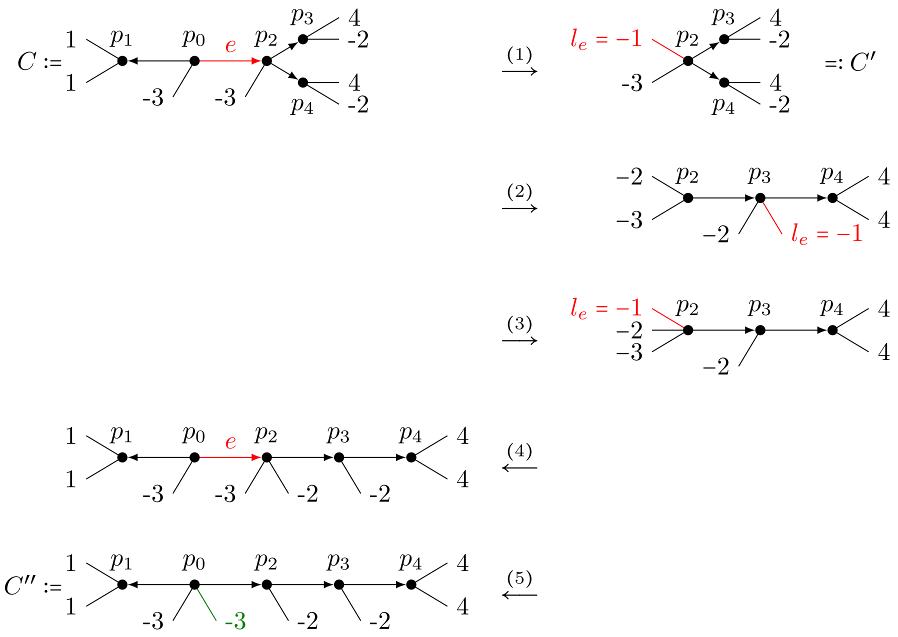


Fig. 11 Branch sorting construction: (*1*) Take the split cover C' at e. (*2*) Move that split cover to a standard cover using induction. (*3*) Move the splitting leaf to the smallest p_j. (*4*) Consider the lift of this cover. (*5*) Move the smallest leaf at $p_1' = p_2$ to p_0 to obtain C''

Hence, we obtain a path between C and the cover $\tilde{C}$, where k and l have been exchanged. We can apply this procedure iteratively to sort all leaves to obtain a standard cover at some p_s.

Finally, note that all standard covers are connected: one can always move the smallest leaf at the four-valent vertex to the left (except of course at p_0) and the largest leaf to the right. This way the four-valent vertex can be placed at any contracted end. ☐

Lemma 3.8 *Let $x \in \mathcal{H}_n$. Then, $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ is connected in codimension one.*

Proof We prove this by induction on n. The case $n = 5$ was already covered in lemma 3.3. Also note that for $n = 4$, the Hurwitz cycle $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ is by definition equal to a Psi class and hence a fan curve.

So assume $n > 5$ and let q be a vertex of $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ with corresponding rational curve C. We want to show that it is connected to the standard cover on p_0. First, we prove the following technical statement:

1) Let e be a bounded edge connecting p_0 and some p_j, such that $C_e(p_j)$ contains the four-valent vertex. Let C' be the split cover at e with degree $x' = (x_1', \ldots, x_m')$. Let $P = \{p_1', \ldots, p_m'\}$ be the set of contracted ends in C' and assume $p_1' < \ldots < p_m'$. Then, C is connected to the cover C'', obtained in the following way: first, remove all leaves and contracted ends contained in C' from C together with any bounded edges that are attached to them. Then, attach all $p \in P$ as an ordered chain to p_0, i.e. p_1' to

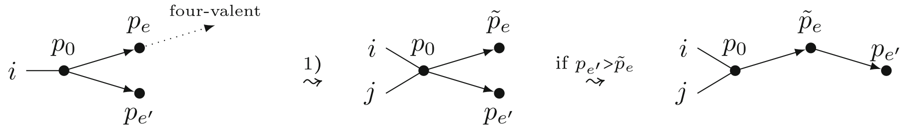


Fig. 12 How to reduce the number of bounded edges at p_0: first move the four-valent vertex to p_0 using the construction from *1)*. Then, move one bounded edge along another according to the size of the p_e

p_0, p'_2 to p'_1, ... etc. Assume the leaves in C' have weights $x_{i_1} \leq \cdots \leq x_{i_{m-1}}$. Attach leaf i_1 to p_0, i_2 to p'_1 and so on (see Fig. 11).

We know by induction that C' is connected to the standard cover for x' at any $p \in P$. Choose p, such that the standard cover at p has the splitting leaf attached to the four-valent vertex. Since the splitting leaf has negative weight, we can move it to the smallest contracted end. This gives us a chain cover C_2 connected to C'. As in the proof of Lemma 3.7, we can lift the connecting path to a path of covers with degree x by attaching p_0 to the splitting leaf. Denote the lift of C_2 by C'_2. This cover has its four-valent vertex at p'_1. Denote by k the smallest leaf at p'_1 with respect to $<_x$ and let ω be the weight of the edge connecting p_0 and p'_1. By definition $\omega = \sum_{i \in I} x_i$, where I is the set of all leaves contained in C'. By construction, k is the minimal element of I with respect to $<_x$. Hence, $\omega > k$ and we can move k to p_0 to obtain C''.

We can now use this to prove the following:

2) If p_0 has only one bounded edge attached to it, then C is connected to the standard cover at p_0.

We can assume without restriction that p_0 is not at the four-valent vertex (otherwise, we can move at least one leaf). We now apply the construction described in 1) to the single bounded edge at p_0. This gives us a chain cover for x, which by Lemma 3.7 is connected to the standard cover.

It remains to prove the following statement, which implies our theorem:

3) C is always connected to a cover C', in which p_0 has only one bounded edge attached to it.

As any vertex is at most four-valent, p_0 can have at most four bounded edges attached to it. First, assume that only two bounded edges e, e' are attached to p_0 and their other vertices are attached to contracted ends $p_e \leq p_{e'}$. If p_0 is four-valent, we can move e' along e to obtain a valid cover in which p_0 has a single bounded edge attached to it. If the four-valent vertex lies behind one of the edges, say e, we apply the construction of 1) to this edge. This way we obtain a cover in which p_0 is still attached to two bounded edges and is also four-valent.

Assume p_0 has three bounded edges e, e', e'' attached to it, connecting it to contracted ends $p_e \leq p_{e'} \leq p_{e''}$. With the same argument as in the case of two bounded edges, we can assume that the vertex at p_0 is four-valent. Now we can move e' along e. Thus, we obtain a cover where p_0 has only two bounded edges attached to it. A similar argument covers the case of four bounded edges (see also Fig. 12 for an illustration in the case of two edges). □

Theorem 3.9 *For all k, p and x, the cycles $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ and $\mathbb{H}_k^{\text{trop}}(x, p)$ are connected in codimension one.*

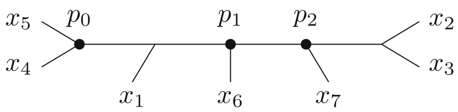


Fig. 13 Sorted maximal cover in $\mathbb{H}_2^{\text{trop}}(x)$ for $x = (1, 2, 3, -3, -5, 1, 1)$ on $S = \{1, 3, 4\}$

Proof Note that it suffices to show the statement for $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$, since $\mathbb{H}_k^{\text{trop}}(x, p) = \text{ft}(\tilde{\mathbb{H}}_k^{\text{trop}}(x, p))$ and connectedness in codimension one is independent of the chosen polyhedral structure.

The general idea of the proof is that for larger k, we mark fewer vertices with contracted ends and thus have more degrees of freedom to "move around", so we can apply induction on k.

Similar to Definition 3.4, we define a *sorted maximal cover* for x on S, with $S \subseteq [n-2]$, $|S| = n-2-k$: we obtain a trivalent curve by attaching the leaves to a chain of $n-3$ bounded edges sorted according to the ordering $<_x$. We number the vertices $\{1, \ldots, n-2\}$ (from lowest leaf to highest). We then attach the contracted ends p_j, in order of their size, to the vertices with numbers in S (see Fig. 13 for an example).

It is easy to see that all sorted maximal covers are connected in codimension one: assume that $(j-1) \notin S \ni j$ (i.e. there is a contracted end at vertex j but none at vertex $(j-1)$). Then, the sorted cover on $(S \setminus \{j\}) \cup \{j-1\}$ shares a codimension one face with this cover, obtained by shrinking the edge between the two vertices $(j-1), j$ to length 0. In this manner, we see that every sorted cover is connected to the sorted maximal cover on $S = \{1, \ldots, n-2-k\}$.

Now, we want to see that every maximal cell σ is connected to the maximal cone of a sorted cover. The cell σ corresponds to a trivalent curve, with some of the vertices marked with contracted ends $p_0, \ldots, p_{n-3-k}$. We now add a further marking $q \in \mathbb{R}$ on an arbitrary vertex such that it is compatible with the edge directions. This gives us an element of $\tilde{\mathbb{H}}_{k-1}^{\text{trop}}(x, p)$. By induction, the corresponding cell is connected to a sorted cover on S', with $\left|S'\right| = n-3-k$. We can "lift" each intermediate step in the connecting path to a valid cover in $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ simply by forgetting the mark q. Thus, we have connected σ to a sorted maximal cover. □

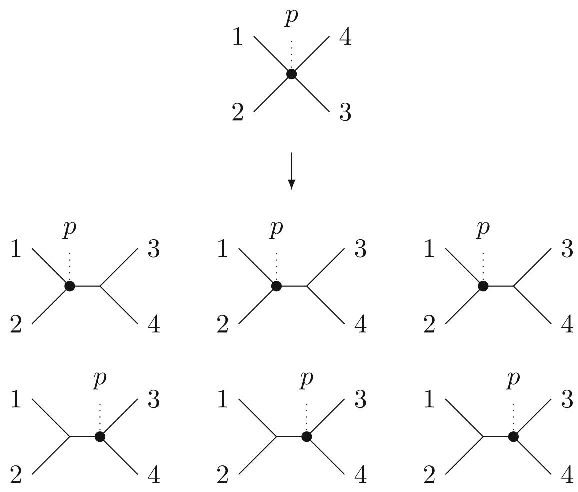
3.2 Irreducibility

We now want to see when a Hurwitz cycle is irreducible. We just proved that it is connected in codimension one, so we can try to apply Proposition 2.5. To see whether a Hurwitz cycle is locally irreducible, we will make use of our knowledge of the local structure of $\mathcal{M}_{0,N}^{\text{trop}}$ (Corollary 6.18 in [22]): if τ is a cone of the combinatorial subdivision of $\mathcal{M}_{0,N}^{\text{trop}}$, corresponding to a curve C with vertices $q_1, \ldots, q_k$, then

$$\text{Star}_{\mathcal{M}_{0,N}^{\text{trop}}}(\tau) = \mathcal{M}_{0,\text{val}(q_1)}^{\text{trop}} \times \cdots \times \mathcal{M}_{0,\text{val}(q_k)}^{\text{trop}}.$$

Lemma 3.10 *For any $x \in \mathcal{H}_n$ and pairwise different p_j, the cycle $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ is locally at each codimension one face weakly irreducible.*

Fig. 14 Six possible resolutions of a four-valent vertex with a contracted end



Proof Let τ be a codimension one cell of $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ and C_τ the corresponding combinatorial type. Since we chose the p_j to be pairwise different, C_τ has exactly one vertex v adjacent to four bounded edges or non-contracted leaves. Depending on whether a contracted end is also attached, the vertex is either four- or five-valent, corresponding to an $\mathcal{M}_4$- or $\mathcal{M}_5$-coordinate.

Denote by $S := \text{Star}_{\tilde{\mathbb{H}}_k^{\text{trop}}(x)}(\tau)$. First let us assume that no contracted end is attached to v. Then, there are three maximal cones adjacent to τ, corresponding to the three different possible resolutions of v. The projections of the normal vectors to the $\mathcal{M}_4$-coordinate of v are (multiples of) the three rays of $\mathcal{M}_4$. In particular, there is only one possible way to assign weights to these rays so that they add up to 0. Hence, the rank of Ω_S is 1, showing that S is a multiple of an irreducible cycle.

Now assume there is a contracted end p at v and four edges/non-contracted ends. Then, there are six maximal cones adjacent to τ: consider v as a four-valent vertex with an additional point for the contracted end. Then, we still have three possibilities to resolve v, but in each case, we have two possibilities to place the additional point (see Fig. 14). Now label the four ends and p with numbers $1, \ldots, 5$ and assume p is labelled with 5. Then, the projections of the normal vectors are multiples of the vectors $v_{\{i,j\}} \in \mathcal{M}_5$ with $i, j \neq 5$. The set of these vectors has been studied in [27], and it is shown there that there is only one way to assign weights to these rays such that they add up to 0. □

Corollary 3.11 *For any $x \in \mathcal{H}_n$ and any pairwise different p_j, $\tilde{\mathbb{H}}_k^{\text{trop}}(x, p)$ is weakly irreducible.*

Example 3.12 We now want to see that this is the strongest possible statement (see also the subsequent `polymake` example).

- **Non-generic points:** Let $n = 5, k = 1, x = (1, 1, 1, 1, -4)$. If we choose $p_0 = p_1 = 0$, then $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ is not irreducible: one can use `a-tint` to compute that the rank of $\Omega_{\tilde{\mathbb{H}}_1^{\text{trop}}(x)}$ is 3.

- **Strict irreducibility:** Let $x' = (1, 1, 1, 1, 1, -5)$, $k' = 1$. For $(p_0, p_1, p_2) = (0, 1, 2)$, we obviously obtain a cycle with weight lattice $\Omega_{\tilde{\mathbb{H}}_1^{\text{trop}}(x',p)}$ of rank 1. However, the gcd of all weights in this cycle is 2, so it is not irreducible in the strict sense.
- **Unmarked cycles:** Again, choose $x = (1, 1, 1, 1, -4)$, $k' = 1$ and generic points $p_0 = 0$, $p_1 = 1$. Passing to $\mathbb{H}_1^{\text{trop}}(x, p)$, the rank of $\Lambda_{\mathbb{H}_1^{\text{trop}}}(x, p)$ is 18. One can also see that $\mathbb{H}_1^{\text{trop}}(x, p)$ is not locally irreducible: it contains the two lines $\{\frac{1}{2}v_{\{1,2\}} + \mathbb{R}_{\geq 0}v_{\{3,4\}}\}$, $\{\frac{1}{2}v_{\{3,4\}} + \mathbb{R}_{\geq 0}v_{\{1,2\}}\}$, which intersect transversely in the vertex $\frac{1}{2}v_{\{1,2\}} + \frac{1}{2}v_{\{3,4\}}$. Locally at this vertex, the curve is just the union of two lines, which is of course not irreducible. However, one can again use the computer to see that there are also vertices of $\tilde{\mathbb{H}}_1^{\text{trop}}(x, p)$ such that the map induced locally by ft is injective, but such that the image of the local variety at that vertex is not irreducible.

polymake example: computing Hurwitz cycles.
We compute the Hurwitz cycles from Example 3.12. First, we compute the cycle $\tilde{\mathbb{H}}_1^{\text{trop}}((1, 1, 1, 1, -4), p)$ for $p_0 = p_1 = 0$ (If no points are given, they are set to 0). A basis for its weight space is given as row vectors of a matrix. We then compute $\tilde{\mathbb{H}}_1^{\text{trop}}((1, 1, 1, 1, 1, -5), q)$ for generic points $q = (0, 1, 2)$ (the first point is always zero in a-tint) and display its weight space dimension. Finally, we compute $\mathbb{H}_1^{\text{trop}}(1, 1, 1, 1, -4)$ for generic points $(0, 1)$ and the dimension of its weight space.

```
atint > $h1 = hurwitz_marked_cycle(1,(new Vector<Int>(1,1,1,1,-4)));
atint > print $h1->WEIGHT_SPACE->rows();
3
atint > $h2 = hurwitz_marked_cycle(1,(new Vector<Int>(1,1,1,1,1,-5)),
        (new Vector<Rational>(1,2)));
atint > print $h2->WEIGHT_SPACE->rows();
1
atint > print gcd($h2->TROPICAL_WEIGHTS);
2
atint > $h3 = hurwitz_cycle(1,(new Vector<Int>(1,1,1,1,-4)),
        (new Vector<Rational>([1])));
atint > print $h3->WEIGHT_SPACE->rows();
18
```

Remark 3.13 So far, we have not found a single example of an irreducible Hurwitz cycle $\mathbb{H}_k^{\text{trop}}(x, p)$. If we pick $p = 0$, it is actually obvious that the cycle must be reducible: for any $i = 1, \ldots, n$ it contains the Psi class product $\psi_i^{(n-3-k)}$ as a non-trivial k-dimensional subcycle. In fact, finding a canonical decomposition, e.g. in terms of Psi class products, would be a large step towards finding a higher-dimensional ELSV formula. However, while possible decompositions can be found with a-tint, the problem proves computationally infeasible in all but the smallest cases.

3.3 Cutting out Hurwitz cycles

For intersection-theoretic purposes, it is very tedious to have a representation of the cycle $\mathbb{H}_k^{\text{trop}}(x)$ only as a push-forward. We would like to find rational functions that successively cut out (recession fans of) Hurwitz cycles directly in the moduli space $\mathcal{M}_{0,n}^{\text{trop}}$. It turns out that there is a very intuitive rational function cutting out the codimension one Hurwitz cycle $\mathbb{H}_{n-4}^{\text{trop}}(x)$ in $\mathcal{M}_{0,n}^{\text{trop}}$. Alas, this seems to be the strongest possible statement already that we can make in this generality. For $n \geq 7$, we can find examples where there is *no rational function at all* that cuts out $\mathbb{H}_{n-5}^{\text{trop}}$ from $\mathbb{H}_{n-4}^{\text{trop}}(x)$. It remains to be seen whether there might be other rational functions or piecewise polynomials cutting out lower-dimensional Hurwitz cycles from $\mathcal{M}_{0,n}^{\text{trop}}$.

Throughout this section, we assume $p_i = 0$ for all i, i.e. $\mathbb{H}_k^{\text{trop}}$ is a fan in $\mathcal{M}_{0,n}^{\text{trop}}$.

3.3.1 Push-forwards of rational functions

We already know that $\tilde{\mathbb{H}}_{n-4}^{\text{trop}}(x)$ can by definition be cut out from

$$\text{ev}_0^*(0) \cdot \Psi_0 \cdot \Psi_1 \cdot \mathcal{M}_{0,2}^{\text{trop}}(\mathbb{R}, x) =: \mathcal{M}_x$$

by the rational function $\text{ev}_1^*(0)$ (note that there is an obvious isomorphism $\mathcal{M}_x \cong \psi_{n+1} \cdot \psi_{n+2} \cdot \mathcal{M}_{0,n+2}^{\text{trop}}$). The forgetful map $\text{ft} : \mathcal{M}_{0,2}^{\text{trop}}(\mathbb{R}, x) \to \mathcal{M}_{0,n}^{\text{trop}}$ now induces a (surjective) morphism of equidimensional tropical varieties (by abuse of notation we also denote it by ft)

$$\text{ft} : \mathcal{M}_x \to \mathcal{M}_{0,n}^{\text{trop}},$$

which is injective on each cone of $\mathcal{M}_x$. We will see that under these conditions, we can actually define the *push-forward* of a rational function. Note that we call a tropical variety X *smooth*, if it is locally at each point isomorphic to a matroidal fan (modulo some linear space). We will not go into the details of matroids and matroidal fans, which can, for example, be found in [2,14,35].

Definition 3.14 Let X, Y be d-dimensional tropical cycles and assume Y is smooth. Let $x \in X$. If $f : X \to Y$ is a morphism, we denote by f_x the induced local map

$$f_x : \text{Star}_X(x) \to \text{Star}_Y(f(x)) =: V_x.$$

We define the *mapping multiplicity* of x to be

$$m_x := f_x^*(f(x)).$$

Note that, since V_x is a smooth fan, any two points in it are rationally equivalent by [13, Theorem 9.5], so $\deg f_x^*(\cdot)$ is constant on V_x. In particular, to compute m_x, we can replace $f(x)$ by any point y in a sufficiently small neighbourhood.

Now let $g : X \to \mathbb{R}$ be a rational function. We define the *push-forward* of g under f to be the function

$$f_*g : Y \to \mathbb{R}, y \mapsto \sum_{x : f(x)=y} m_x g(x).$$

Proposition 3.15 *Under the assumptions above, $f_* g$ is a rational function on Y.*

Proof We can assume without restriction that X and Y have been refined in such a manner that f maps cells of X to cells of Y and g is affine linear on each cell of X. Let us first see that $f_* g$ is well defined:

Let $y \in Y$ and denote by τ the minimal cell containing it. We want to see that y has only finitely many preimages $x \in X$ with $m_x \neq 0$. Assume there is a cell ρ in X such that $f(\rho) = \tau$, but $\dim(\rho) > \dim(\tau)$, so $f_{|\rho}$ is not injective. In particular, all maximal cells $\xi > \rho$ map to a cell of dimension strictly less than d. Now let $x \in \mathrm{relint}(\rho)$ with $f(x) = y$. If we pick a point $q \in V_y$ that lies in a maximal cone adjacent to τ, it has no preimage under f_x: all maximal cones in $\mathrm{Star}_X(x)$ are mapped to a lower-dimensional cone. It follows that $m_x = 0$.

We now have to show that $f_* g$ is continuous. Let σ be a maximal cell of Y. Denote by

$$C_\sigma = \{\xi \in X^{(d)}, f(\xi) = \sigma\}.$$

Then, for each $y \in \mathrm{relint}(\sigma)$, we have

$$f_* g(y) = \sum_{\xi \in C_\sigma} \omega_X(\xi) \mathrm{ind}(\xi) g(f_{|\xi}^{-1}(y)),$$

where $\mathrm{ind}(\xi) := \left|\Lambda_\sigma / f(\Lambda_\xi)\right|$ is the index of f on ξ. Since $f_{|\xi}$ is a homeomorphism, this is just a sum of continuous maps, so $(f_* g)_{|\mathrm{relint}(\sigma)}$ is continuous.

Assume τ is a cell of Y of dimension strictly less than d and contained in some maximal cell σ. Let $s : [0, 1] \to \sigma$ be a continuous path with:

- $s([0, 1)) \subseteq \mathrm{relint}(\sigma)$
- $s(1) \in \mathrm{relint}(\tau)$

We write $y_t := s(t)$ for $t \in [0, 1]$. Then, we have to show that $\lim_{t \to 1} f_* g(y_t) = f_* g(y_1)$. If we denote by $s_\xi = (f_{|\xi}^{-1} \circ s)$ the unique lift of s to any $\xi \in C_\sigma$, we have

$$\begin{aligned} \lim_{t \to 1} f_* g(y_t) &= \lim_{t \to 1} \sum_{\xi \in C_\sigma} \omega_X(\xi) \mathrm{ind}(\xi) g(s_\xi(t)) \\ &= \sum_{\xi \in C_\sigma} \omega_X(\xi) \mathrm{ind}(\xi) \lim_{t \to 1} g(s_\xi(t)) \\ &= \sum_{\xi \in C_\sigma} \omega_X(\xi) \mathrm{ind}(\xi) g(\underbrace{\lim_{t \to 1} s_\xi(t)}_{=: x_\xi}), \end{aligned}$$

where the last equality is due to the continuity of g. Note that x_ξ lies in the unique face $\rho_\xi < \xi$ such that $f(\rho_\xi) = \tau$.

Conversely, let ρ be any cell of X with $\dim(\rho) = \dim(\tau)$ and $f(\rho) = \tau$. Assume ρ has no adjacent maximal cell mapping to σ. Then, if we let $x := f_{|\rho}^{-1}(y)$, we must again have $m_x = 0$. We define

$$C_\tau := \{\rho \in X^{(\dim \tau)};\ f(\rho) = \tau \text{ and there exists } \xi > \rho \text{ with } f(\xi) = \sigma\}.$$

Then, we have

$$\begin{aligned} f_* g(y_1) &= \sum_{\substack{\rho \in X^{(\dim \tau)} \\ f(\rho)=\tau}} m_{f_{|\rho}^{-1}(y_1)} g(f_{|\rho}^{-1}(y_1)) \\ &= \sum_{\rho \in C_\tau} m_{f_{|\rho}^{-1}(y_1)} g(f_{|\rho}^{-1}(y_1)) \end{aligned} \tag{1}$$

If $x_\rho := f_{|\rho}^{-1}(y_1)$, then for small ϵ, we have

$$m_{f_{|\rho}^{-1}(y_1)} = \deg f_{x_\rho}^* y_{1-\epsilon} = \sum_{\substack{\xi > \rho \\ f(\xi)=\sigma}} \omega_X(\xi) \mathrm{ind}(\xi).$$

If we plug this into (1), we see that each $\xi \in C_\sigma$ occurs exactly once (since ξ cannot have two faces ρ mapping to τ due to injectivity), so finally we have $\lim_{t\to 1} f_* g(y_t) = f_* g(y_1)$. □

Proposition 3.16 *Let $f : X \to Y$ be a morphism of d-dimensional tropical cycles. Assume Y is smooth and f is injective on each cell of X. Then,*

$$f_* g \cdot Y = f_*(g \cdot X).$$

Proof By studying this identity locally and dividing out lineality spaces we can assume that:

- Y is a smooth one-dimensional tropical fan.
- $X = \coprod_{i=1}^r X_i$ is a disjoint union of one-dimensional tropical fan cycles.
- $f_{|X_i} : X_i \to Y$ is a linear map.
- g is affine linear on each ray of X_i.

We write $Z := f_* g \cdot Y$ and $Z' := f_*(g \cdot X)$. We have to show that $\omega_Z(0) = \omega_{Z'}(0)$. We know that

$$\omega_{Z'}(0) = \sum_{i=1}^r \omega_{g \cdot X_i}(0) = \sum_{i=1}^r \sum_{\rho \in X_i^{(1)}} \omega_{X_i}(\rho) g(u_\rho),$$

where u_ρ is the integer primitive generator of ρ. On the other hand, we have

$$\begin{aligned} \omega_Z(0) &= \sum_{\sigma \in Y^{(1)}} f_* g(u_\sigma) \\ &= \sum_{\sigma \in Y^{(1)}} \sum_{i=1}^r \sum_{\substack{\rho \in X_i^{(1)} \\ f(\rho)=\sigma}} \omega_{X_i}(\rho) \mathrm{ind}(\rho) g\left(\frac{u_\rho}{\mathrm{ind}(\rho)}\right). \end{aligned}$$

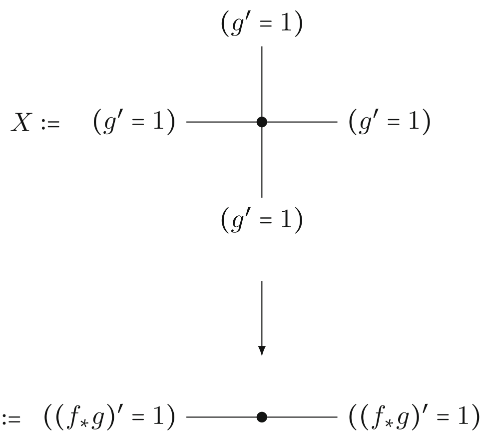


Fig. 15 A morphism where the push-forward of a function does not give the same divisor as the push-forward of the divisor of this function. All weights are 1, and the function slopes of g and f_*g are given in *brackets*

Obviously, each ray ρ can occur at most once in this sum, and by assumption, it occurs at least once. Hence, we see that $\omega_Z(0) = \omega_{Z'}(0)$.

Example 3.17 Note that the assumption that f is injective on each cone is necessary. Consider the morphism depicted in Fig. 15: in this case, we get that $f_*(g \cdot X) = 4$ and $f_*g \cdot Y = 2$.

3.3.2 Cutting out the codimension one cycle

By definition, we have

$$\mathbb{H}^{\text{trop}}_{n-4}(x) = \text{ft}_*(\tilde{\mathbb{H}}^{\text{trop}}_{n-4}(x)) = \text{ft}_*(\text{ev}_1^*(0) \cdot \mathcal{M}_x)$$

and we already discussed that $\text{ft} : \mathcal{M}_x \to \mathcal{M}^{\text{trop}}_{0,n}$ is a morphism of $(n-3)$-dimensional tropical varieties which is injective on each cone of $\mathcal{M}_x$. Since $\mathcal{M}^{\text{trop}}_{0,n}$ is smooth, we immediately obtain the following result:

Corollary 3.18 *The codimension one Hurwitz cycle can be cut out as*

$$\mathbb{H}^{\text{trop}}_{n-4} = (\text{ft}_*(\text{ev}_1^*(0))) \cdot \mathcal{M}^{\text{trop}}_{0,n}.$$

We now want to describe the rational function $(\text{ft}_*(\text{ev}_1^*(0)))$ in more intuitive and geometric terms:

Lemma 3.19 *Let C be any curve in $\mathcal{M}^{\text{trop}}_{0,n}$. Given $x \in \mathcal{H}_n$, this defines a cover of $\mathbb{R}$ up to translation. Pick any such cover $h : C \to \mathbb{R}$. Let $v_1, \ldots, v_r$ be the vertices of C. Then,*

$$(\text{ft}_*(\text{ev}_1^*(0)))(C) = \sum_{i \neq j} (\text{val}(v_i) - 2)(\text{val}(v_j) - 2) \left| h(v_i) - h(v_j) \right|.$$

Proof It suffices to show this for curves in maximal cones. Since $(\text{ft}_*(\text{ev}_1^*(0)))$ is continuous by Proposition 3.15, the claim follows for all other cones.

So let C be an n-marked trivalent curve with vertices $v_1, \ldots, v_{n-2}$. We obtain all preimages in $\mathcal{M}_x$ by going over all possible choices of vertices v_i, v_j and attaching the additional leaves l_0 to v_i and l_1 to v_j. We denote the corresponding $n+2$-marked curve by $C(i, j)$. Note that ev_1 maps $C(i, j)$ to the image of l_1 under the cover obtained by fixing the image of l_0 to be 0. We immediately see the following:

- $\text{ev}_1(C(i, i)) = 0$.
- $\text{ev}_1(C(i, j)) = -\text{ev}_1(C(j, i))$.
- $|\text{ev}_1(C(i, j))| = \left|h(v_i) - h(v_j)\right|$

Since $\text{ev}_1^*(0)(x) = \max\{0, \text{ev}_1(x)\}$ and the forgetful map has index 1, the claim follows. □

3.4 Hurwitz cycles as linear combinations of boundary divisors

In [4], the authors present several different representations of $\mathbb{H}_k(x)$. One is given in

Lemma 3.20 ([4, Lemma 3.6])

$$\mathbb{H}_k(x) = \sum_{\Gamma \in \mathcal{T}_{n-3-k}} \left(m(\Gamma)\varphi(\Gamma) \prod_{v \in \Gamma^{(0)}} (\text{val}(v) - 2)\Delta_\Gamma \right),$$

where Γ runs over $\mathcal{T}_{n-3-k}$, the set of all combinatorial types of rational n-marked curves with $n - 3 - k$ bounded edges and Δ_Γ is the stratum of all covers with dual graph Γ. Furthermore, $m(\Gamma)$ is the number of total orderings on the vertices of Γ compatible with edge directions and $\varphi(\Gamma)$ is the product over all edge weights.

There is an obvious, "naive" tropicalization of this: $\mathcal{T}_{n-3-k}$ corresponds to the codimension k skeleton of $\mathcal{M}_{0,n}^{\text{trop}}$. We will write $m(\tau) := m(\Gamma_\tau)$, $x_\tau := \varphi(\Gamma_\tau)$ for any codimension k cone τ and its corresponding combinatorial type Γ_τ. The boundary stratum Δ_Γ we translate like this:

Definition 3.21 Let $(\mathcal{X}, w)$ be a simplicial tropical fan. For a d-dimensional cone τ generated by rays $v_1, \ldots, v_d$ we define rational functions φ_{v_i} on $\mathcal{X}$ by fixing its value on all rays:

$$\varphi_{v_i}(r) = \begin{cases} 1, & \text{if } r = v_i \\ 0, & \text{otherwise} \end{cases}$$

for all $r \in \mathcal{X}^{(1)}$. We then write $\varphi_\tau := \varphi_{v_1} \ldots \varphi_{v_d}$ for subsequently applying these d functions. In the case of $\mathcal{X} = \mathcal{M}_{0,n}^{\text{trop}}$ and $v_i = v_I$, we will also write φ_I instead of φ_{v_i}.

As a shorthand notation, we will write $\mathcal{C}_k$ for all dimension k cells of $\mathcal{M}_{0,n}^{\text{trop}}$ and $\mathcal{C}^k$ for all codimension k cells (in its combinatorial subdivision) (Fig. 16).

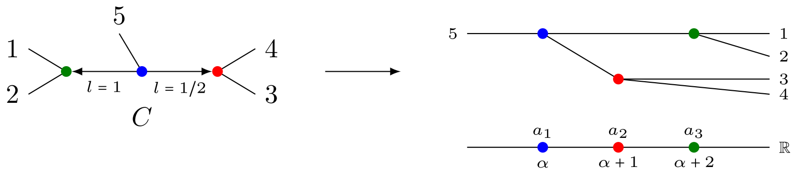


Fig. 16 We compute $(\mathrm{ft}_*(\mathrm{ev}_1^*(0)))(C)$ for an example. We choose parameters $x = (1, 1, 1, 1, -4)$ and $C = v_{\{1,2\}} + \frac{1}{2} v_{\{3,4\}}$. In this case, Lemma 3.19 tells us that the value of the function at C is $|a_2 - a_1| + |a_3 - a_1| + |a_3 - a_2| = 1 + 2 + 1 = 4$

Now, we define the following divisor of a piecewise polynomial (see, for example, [12] for a treatment of piecewise polynomials. For now it suffices if we define them as sums of products of rational functions):

$$D_k(x) := \sum_{\tau \in \mathcal{C}^k} m(\tau) \cdot x_\tau \cdot \left(\prod_{v \in \Gamma_\tau^{(0)}} (\mathrm{val}(v) - 2) \right) \cdot \varphi_\tau \cdot \mathcal{M}_{0,n}^{\mathrm{trop}},$$

where $\varphi_\tau = \prod_{v_I \in \tau^{(1)}} \varphi_I$ and the sum is to be understood as a sum of tropical cycles.

We can now ask ourselves, what the relation between $D_k(x)$ and $\mathbb{H}_k^{\mathrm{trop}}(x)$ is. They are obviously not equal: $D_k(x)$ is a subfan of $\mathcal{M}_{0,n}^{\mathrm{trop}}$ (in its coarse subdivision), but even if we choose all p_i to be equal to make $\mathbb{H}_k^{\mathrm{trop}}(x)$ a fan, it will still contain rays in the interior of higher-dimensional cones of $\mathcal{M}_{0,n}^{\mathrm{trop}}$.

This also rules out *rational equivalence* (as defined in [1]): two cycles are equivalent, if and only if their recession fans are equal.

But there is another, coarser equivalence on $\mathcal{M}_{0,n}^{\mathrm{trop}}$ that comes from toric geometry. As was shown in [20], the classical $M_{0,n}$ can be embedded in the toric variety $X(\mathcal{M}_{0,n}^{\mathrm{trop}})$ and we have

$$\mathrm{Cl}(X(\mathcal{M}_{0,n}^{\mathrm{trop}})) \cong \mathrm{Pic}(\overline{M}_{0,n})$$
$$D_I \mapsto \delta_I,$$

where D_I is the divisor associated with the ray v_I and δ_I is the boundary stratum of curves consisting of two components, each containing the marked points in I and I^c, respectively. By [15], D_I corresponds to some tropical cycle of codimension one in $\mathcal{M}_{0,n}^{\mathrm{trop}}$ and [33, Corollary 1.2.19] shows that this is precisely $\varphi_I \cdot \mathcal{M}_{0,n}^{\mathrm{trop}}$. Hence, the following is a direct translation of numerical equivalence in $\overline{M}_{0,n}$.

Definition 3.22 Two k-dimensional cycles $C, D \subseteq \mathcal{M}_{0,n}^{\mathrm{trop}}$ are *numerically equivalent*, if for all k-dimensional cones $\rho \in \mathcal{C}_k$, we have

$$\varphi_\rho \cdot C = \varphi_\rho \cdot D \in \mathbb{Z}.$$

Theorem 3.23 $\mathbb{H}_k^{\text{trop}}(x)$ *is numerically equivalent to* $D_k(x)$.

Proof Note that for a generic choice of p_i, the cycle $\mathbb{H}_k^{\text{trop}}(x)$ does not intersect any cones of codimension larger than k and intersects all codimension k cones transversely. For the proof, we will need the following result from [4, Proposition 5.4], describing the intersection multiplicity of $\mathbb{H}_k^{\text{trop}}(x)$ with a codimension k-cell τ:

$$\tau \cdot \mathbb{H}_k^{\text{trop}}(x) = m(\tau) \cdot x_\tau \cdot \prod_{v \in C_\tau^{(0)}} (\text{val}(v) - 2).$$

This implies that for any $\rho \in \mathcal{C}_k$, we have

$$\begin{aligned}
\varphi_\rho \cdot \mathbb{H}_k^{\text{trop}}(x) &= \sum_{\tau \in \mathcal{C}^k} (\tau \cdot \mathbb{H}_k^{\text{trop}}(x)) \cdot \omega_{\varphi_\rho \cdot \mathcal{M}_{0,n}^{\text{trop}}}(\tau) \\
&= \sum_{\tau \in \mathcal{C}^k} m(\tau) \cdot x_\tau \cdot \prod_{v \in C_\tau^{(0)}} (\text{val}(v) - 2) \cdot \omega_{\varphi_\rho \cdot \mathcal{M}_{0,n}^{\text{trop}}}(\tau) \\
&= \sum_{\tau \in \mathcal{C}^k} m(\tau) \cdot x_\tau \cdot \prod_{v \in C_\tau^{(0)}} (\text{val}(v) - 2) \cdot (\varphi_\tau \cdot \varphi_\rho \cdot \mathcal{M}_{0,n}^{\text{trop}}) \\
&= \varphi_\rho \cdot D_k(x),
\end{aligned}$$

where $\omega_{\varphi_\rho \cdot \mathcal{M}_{0,n}^{\text{trop}}}(\tau) = \varphi_\tau \cdot \varphi_\rho \cdot \mathcal{M}_{0,n}^{\text{trop}}$ by [12, Lemma 4.7]. □

Acknowledgments I would like to thank Hannah Markwig for many inspiring discussions and the anonymous referees for their helpful suggestions. I was supported by DFG Grants MA 4797/3-1 and MA 4797/1-2.

References

1. Allermann, L., Hampe, S., Rau, J.: On rational equivalence in tropical geometry (2014). arXiv:1408.1537 (preprint)
2. Ardila, F., Klivans, C.: The Bergman complex of a matroid and phylogenetic trees. J. Comb. Theory Ser. B. **96**, 38–49 (2006). arXiv:math/0311370v2
3. Bertrand, B., Brugallée, E., Mikhalkin, G.: Tropical open Hurwitz numbers. Rend. Semin. Mat. Univ. Padova. **125**, 157–171 (2011). arXiv:1005.4628
4. Bertram, A., Cavalieri, R., Markwig, H.: Polynomiality, wall crossings and tropical geometry of rational double hurwitz cycles. JCTA **120**, 1604–1631 (2013)
5. Berstein, I., Edmonds, A.L.: On the classification of generic branched coverings of surfaces. Ill. J. Math. **28**(1), 64–82 (1984)
6. Bogart, T., Jensen, A.N., Speyer, D., Sturmfels, B., Thomas, R.R.: Computing tropical varieties. J. Symb. Comput. **42**(1–2), 54–73 (2007)
7. Clebsch, A.: Zur Theorie der Riemann'schen Fläche. Math. Ann. **6**(2), 216–230 (1873)
8. Cavalieri, R., Johnson, P., Markwig, H.: Tropical Hurwitz numbers, J. Algebr. Comb. **32**(2), 241–265 (2010). arXiv:0804.0579
9. Cavalieri, R., Markwig, H., Ranganathan, D.: Tropical compactification and the gromov-witten theory of $\mathbb{P}^1$ (2014). arXiv:1410.2837 (preprint)
10. Cartwright, D., Payne, S.: Connectivity of tropicalizations. Math. Res. Lett. **19**(5), 1089–1095 (2012). arXiv:1204.6589
11. Ekedahl, T., Lando, S., Shapiro, M., Vainshtein, A.: Hurwitz numbers and intersections on moduli spaces of curves. Invent. Math. **146**(2), 297–327 (2001)

12. Francois, G.: Cocycles on tropical varieties via piecewise polynomials. Proc. Am. Math. Soc. **141**, 481–497 (2013)
13. Francois, G., Rau, J.: The diagonal of tropical matroid varieties and cycle intersections. Collect. Math. **64**, 185–210 (2013)
14. Feichtner, E.M., Sturmfels, B.: Matroid polytopes, nested sets and Bergman fans. Port. Math. Nova Série **62**, 437–468 (2005)
15. Fulton, W., Sturmfels, B.: Intersection theory on toric varieties. Topology **36**(2), 335–353 (1997). (English)
16. Graber, T., Harris, J., Starr, J.: A note on Hurwitz schemes of covers of a positive genus curve (2002). arXiv:math/0205056 (preprint)
17. Gawrilow, E., Joswig, M.: Polymake: a framework for analyzing convex polytopes. In: Polytopes: Combinatorics and Computation (Oberwolfach, 1997), pp. 43-73. (2000)
18. Goulden, I., Jackson, D., Vakil, R.: Towards the geometry of double Hurwitz numbers. Adv. Math. **198**(1), 43–92 (2005). arXiv:math/0309440
19. Gathmann, A., Kerber, M., Markwig, H.: Tropical fans and the moduli spaces of tropical curves. Compos. Math. **145**, 173–195 (2009). arXiv:0708.2268
20. Gibney, A., Maclagan, D.: Equations for Chow and Hilbert quotients. Algebra Number Theory. **7**, 855–885 (2010). arXiv:0707.1801
21. Graber, T., Vakil, R.: Relative virtual localization and vanishing of tautological classes on moduli spaces of curves. Duke Math. J. **130**(1), 1–37 (2005). arXiv:math/0309227
22. Hampe, S.: A-tint: a polymake extension for algorithmic tropical intersection theory. Eur. J. Comb. **36**, 579–607 (2014). arXiv:1208.4248
23. Hurwitz, A.: Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten. Math. Ann. **39**, 1–61 (1891)
24. Johnson, P.: Hurwitz numbers, ribbon graphs and tropicalizations. Contemp. Math. **580**, 55–72 (2012). arXiv:1303.1543
25. Kanev, V.: Irreducible components of Hurwitz spaces parameterizing Galois coverings of curves of positive genus. Pure Appl. Math. Q. **10**(2), 193–222 (2014)
26. Kontsevich, M.: Intersection theory on the moduli space of curves and the matrix Airy function. Commun. Math. Phys. **147**(1), 1–23 (1992)
27. Kerber, M., Markwig, H.: Intersecting psi-classes on tropical $\mathcal{M}_{\prime,\backslash}$. Int. Math. Res. Not. (2008). arXiv:0709.3953
28. Mikhalkin, G.: Enumerative tropical algebraic geometry in $\mathbb{R}^2$. J. Am. Math. Soc. **18**(2), 313–377 (2005). arXiv:math/0312530
29. Mikhalkin, G.: Moduli spaces of rational tropical curves. In: Proceedings of Gökova Geometry: Topology Conference 2006, pp. 39–51. (2007)
30. Maclagan, D., Sturmfels, B.: Introduction to Tropical Geometry, Graduate Studies in Mathematics, vol. 161. American Mathematical Society, Providence, RI (2015)
31. Nishinou, T., Siebert, B.: Toric degenerations of toric varieties and tropical curves. Duke Math. J. **135**(1), 1–51 (2006). arXiv:math/0409060
32. Okounkov, A., Pandharipande, R.: Gromov-Witten theory, Hurwitz numbers and matrix models. In: I (2001). arXiv:math/0101147
33. Rau, J.: Tropical intersection theory and gravitational descendants. Ph.D. Thesis (2009)
34. Severi, F.: Vorlesungen über algebraische Geometrie. Teubner-Verlag, Berlin (1921)
35. Speyer, D.: Tropical linear spaces. SIAM J. Discrete Math. **22**, 1527–1558 (2008). arXiv:math/0410455
36. Speyer, D., Sturmfels, B.: The tropical Grassmannian. Adv. Geom. **4**(3), 389–411 (2004). arXiv:math/0304218
37. Vetro, F.: Irreducibility of Hurwitz spaces of coverings with one special fiber and monodromy group a Weyl group of type D_d. Manuscr. Math. **125**(3), 353–368 (2008)