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Construction of chiral 4-polytopes with alternating or symmetric automorphism group

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Abstract In this paper we describe a construction for finite abstract chiral 4-polytopes with Schläfli type $\{3, 3, k\}$ (with tetrahedral facets) and with an alternating or symmetric group as automorphism group. We use it to prove that for all but finitely many n, both A_n and S_n are the automorphism groups of such a polytope. We also show that the vertex-figures of the polytopes obtained from our construction are chiral.

Keywords Abstract polytopes · Chiral polytopes · Symmetric groups · Alternating groups

1 Introduction

Abstract polytopes generalise the classical notion of convex geometric polytopes to more general structures. Highly symmetric examples include not only classical regular polytopes such as the Platonic solids and more exotic structures such as the 120-cell

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and 600-cell, but also non-degenerate regular maps on surfaces (such as Klein's quartic, of genus 3).

Roughly speaking, an abstract polytope $\mathcal{P}$ is a partially ordered set endowed with a rank function, satisfying certain conditions that arise naturally from a geometric setting. Such objects were proposed by Grünbaum in the 1970s, and their definition (initially as 'incidence polytopes') and theory were developed by Danzer and Schulte.

Every automorphism of an abstract polytope is uniquely determined by its effect on any flag, which is a maximal chain in $\mathcal{P}$ (when this is regarded as a poset). The most symmetric examples are *regular*, with all flags lying in a single orbit, and a comprehensive description of these is given in a book on the subject by McMullen and Schulte [6]. These objects are also known as 'thin residually connected geometries with a linear diagram'.

An interesting class of examples which are not quite regular are the *chiral* polytopes, for which the automorphism group has two orbits on flags, with any two flags that differ in a single element lying in different orbits. The study of chiral abstract polytopes was pioneered by Schulte and Weiss (see [10,11] for example). Chiral polytopes of rank 3 are essentially the same as chiral maps on surfaces, with some modest extra geometric conditions.

For quite some time, the only known finite examples of chiral polytopes had ranks 3 and 4, but then some finite examples of rank 5 were constructed by Conder et al. [3], and now quite a few such examples are known. Many small examples of regular or chiral polytopes have been assembled in collections, as in [4,5], for example.

In early 2009, the first author and Alice Devillers devised a construction for chiral polytopes whose facets are simplices, and used this to construct examples of finite chiral polytopes of ranks 6, 7 and 8 (unpublished). At about the same time, the fourth author of this paper devised a quite different method for constructing finite chiral polytopes with given regular facets, and used this construction to prove the existence of finite chiral polytopes of every rank $d \geq 3$; see [7]. The latter polytopes are enormous, however, and not easy to describe. It is still an open problem to find alternative constructions for families of chiral polytopes of relatively small order, or which have more easily described automorphism groups. A large number of other open questions about chiral polytopes are given by the fourth author in [8].

In this paper, we make a contribution towards producing infinite families of chiral polytopes with well-known groups. Specifically, we describe a construction for chiral 4-polytopes of type $\{3, 3, k\}$, with tetrahedral facets, using a way of combining together permutation representations of the tetrahedral group A_4 into the automorphism group.

Our main result is the following:

Theorem 1.1 *For all but finitely many positive integers n, both A_n and S_n are the automorphism groups of chiral 4-polytopes of type $\{3, 3, k\}$ for some k.*

In fact our construction proves this theorem for all $n \geq 50$, but thanks to an easy computation with MAGMA [1], we know it is also true for $20 \leq n \leq 49$ and hence for all $n \geq 20$. In addition, we know that the only smaller values of n for which A_n is the automorphism group of such a chiral 4-polytope are 9, 13, 14, 15, 17 and 18, while the only such values of n for S_n are 12, 16, 17, 18 and 19. Examples of generating permutations for A_n and S_n in the cases not covered by our construction are given in [2].

In a planned sequel, we will extend the ideas presented here to the construction of infinite families of chiral polytopes of larger rank d, using permutation representations of the alternating group A_d (as the rotation group of the regular $(d-1)$ simplex) to build their automorphism groups.

Here, we give some further background on regular and chiral polytopes in Sect. 2, and then in Sect. 3 we set up some of the building blocks and other things needed for our construction. We describe our construction and prove Theorem 1.1 in Sect. 4. Finally, in Sect. 5 we show that the vertex-figures of the chiral 4-polytopes resulting from our construction are all chiral.

2 Abstract polytopes and chirality

An *abstract d-polytope* (or *abstract polytope of rank d*) is a partially ordered set $\mathcal{P}$, the elements and maximal totally ordered subsets of which are called *faces* and *flags* respectively, such that certain properties are satisfied, which we explain below.

2.1 Definition of abstract polytopes

First, $\mathcal{P}$ contains a minimum face F_{-1} and a maximum face F_d, and there is a rank function from $\mathcal{P}$ to the set $\{-1, 0, \ldots, d\}$ such that $\text{rank}(F_{-1}) = -1$ and $\text{rank}(F_d) = d$. Every flag of $\mathcal{P}$ contains precisely $d+2$ elements, including F_{-1} and F_d. The faces of rank i are called *i-faces*, the 0-faces are called *vertices*, the 1-faces are called *edges*, and the $(d-1)$-faces are called *facets*. If F and G are faces of ranks r and s with $F \leq G$, then we say that F and G are *incident*, we define $G/F := \{H \mid F \leq H \leq G\}$, and call this a *section* of $\mathcal{P}$, of rank $s-r-1$. When convenient, we identify the section G/F_{-1} with the face G itself in $\mathcal{P}$, and if $v = F_0$ is a vertex, then the rank $d-1$ section $F_d/F_0 := \{H \mid F_0 \leq H\}$ is called the *vertex-figure* of $\mathcal{P}$ at v.

Whenever G/F is a rank 1 section (with $\text{rank}(G) - \text{rank}(F) = 2$), there are precisely two faces H_1 and H_2 such that $F < H_i < G$. This property is called the *diamond condition*. It implies that for any flag Φ and for every $i \in \{0, \ldots, d-1\}$, there is a unique flag Φ^i differing from Φ in precisely the i-face. We call Φ^i the *i-adjacent flag* for Φ.

Finally, for any two flags Φ and Φ' of $\mathcal{P}$, there exists a sequence $\Psi_0, \Psi_1, \ldots, \Psi_m$ of flags of $\mathcal{P}$ from $\Psi_0 = \Phi$ to $\Psi_m = \Phi'$ such that Ψ_{k-1} is adjacent to Ψ_k, and $\Phi \cap \Phi' \subseteq \Psi_k$, for $1 \leq k \leq m$. The last condition is known as *strong flag-connectivity* and completes the definition of an abstract d-polytope.

In this paper, we will deal with finite polytopes (namely those with finite rank and only finitely many faces of each rank).

Every rank 2 section G/F between an $(i-2)$-face F and an incident $(i+1)$-face G of a finite abstract polytope $\mathcal{P}$ is isomorphic to the face lattice of a polygon, and by convention, we assume that each such polygon is non-degenerate (having at least 3 sides). If the number of sides of each such polygon depends only on i, and not on F or G, then we say that $\mathcal{P}$ is *equivelar*. Regular and chiral polytopes (defined below) are examples of equivelar polytopes. We define the *Schläfli type* of an equivelar d-polytope $\mathcal{P}$ as $\{p_1, \ldots, p_{d-1}\}$, when each section between an $(i-2)$-face and an $(i+1)$-face is an abstract p_i-gon. By finiteness, $p_i < \infty$ for all i, and by our non-degeneracy assumption, $p_i > 2$ for all i.

2.2 Automorphisms and regular polytopes

An *automorphism* of an abstract polytope $\mathcal{P}$ is an order-preserving permutation of its faces. We denote the group of automorphisms of $\mathcal{P}$ by $\Gamma(\mathcal{P})$. By the diamond condition and strong flag-connectivity, every automorphism is uniquely determined by its effect on any flag, and it follows that the number of automorphisms of $\mathcal{P}$ is bounded above by the number of flags of $\mathcal{P}$.

A d-polytope $\mathcal{P}$ is said to be *regular* whenever $\Gamma(\mathcal{P})$ acts transitively (and therefore regularly) on the set of all flags of $\mathcal{P}$. When that happens, the automorphism group $\Gamma(\mathcal{P})$ is generated by involutions $\rho_0, \ldots, \rho_{d-1}$, where ρ_i is the unique automorphism mapping a given *base flag* Φ to its i-adjacent flag Φ^i. Moreover, the generators $\rho_0, \ldots, \rho_{d-1}$ satisfy

$$\rho_i^2 = 1 \quad \text{for all } i, \tag{1}$$
$$(\rho_i \rho_j)^2 = 1 \quad \text{whenever } |i-j| \geq 2. \tag{2}$$

These generators also satisfy the following *intersection condition*:

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_k \mid k \in I \cap J \rangle \quad \text{for all } I, J \subseteq \{0, 1, \ldots, d-1\}. \tag{3}$$

The stabiliser in $\Gamma(\mathcal{P})$ of the i-face of the base flag Φ is generated by $\{\rho_0, \ldots, \rho_{d-1}\} \setminus \{\rho_i\}$, for $0 \leq i < d$, and the order of the element $\rho_{i-1}\rho_i$ coincides with the i-th term p_i of the Schläfli type $\{p_1, \ldots, p_{d-1}\}$, for $1 \leq i < d$.

These properties of the automorphism group of a regular polytope can be exploited to construct examples from particular groups, called *string C-groups*. A string C-group of rank d is a finite group Γ and an associated set $\{\rho_0, \ldots, \rho_{d-1}\}$ of d generators for Γ which satisfy (1) and (2), as well as the intersection condition (3). For any such Γ, we may construct a regular d-polytope $\mathcal{P}$ with $\Gamma = \Gamma(\mathcal{P})$, by taking as its i-faces the (right) cosets of the subgroup generated by $\{\rho_0, \ldots, \rho_{d-1}\} \setminus \{\rho_i\}$, for $0 \leq i < d$, and defining incidence by non-empty intersection; see [6, Theorem 2E11].

Hence up to isomorphism, regular d-polytopes are in one-to-one correspondence with string C-groups.

Next, we define the *rotation group* $\Gamma^+(\mathcal{P})$ of a regular d-polytope $\mathcal{P}$ as the subgroup of $\Gamma(\mathcal{P})$ consisting of words of even length in the generators $\rho_0, \ldots, \rho_{d-1}$, or equivalently, the subgroup generated by the *abstract rotations* $\sigma_i = \rho_{i-1}\rho_i$ for $1 \leq i < d$. The index of $\Gamma^+(\mathcal{P})$ in the full automorphism group $\Gamma(\mathcal{P})$ is at most 2. Motivated by what happens for maps (in rank 3), we say that $\mathcal{P}$ is *orientably regular* whenever this index is 2, and otherwise we say that $\mathcal{P}$ is *non-orientably regular*.

Note that $\sigma_i = \rho_{i-1}\rho_i$ has order p_i for all i. Moreover, these generators satisfy the relations

$$(\sigma_i \sigma_{i+1} \cdots \sigma_j)^2 = 1 \quad \text{for } 1 \le i < j < d. \tag{4}$$

The involutory element $\tau_{i,j} = \sigma_i \sigma_{i+1} \cdots \sigma_j$ is called an *abstract half-turn*, for $1 \le i < j < d$. If we extend this definition of $\tau_{i,j}$ by setting $\tau_{0,i} = \tau_{i,d} = 1$ for $0 \le i \le d$, and $\tau_{i,i} = \sigma_i$ for $0 < i < d$, so that $\tau_{i,j}$ is defined whenever $0 \le i \le j \le d$, and we define the subgroup $H_I = \langle\, \tau_{i+1,j} \mid i, j \in I,\ i < j \,\rangle$ for every $I \subseteq \{-1, 0, \ldots, d\}$, then these subgroups satisfy the intersection condition

$$H_I \cap H_J = H_{I \cap J} \quad \text{for all } I, J \subseteq \{-1, 0, \ldots, d\}. \tag{5}$$

2.3 Chiral polytopes

The abstract d-polytope $\mathcal{P}$ is said to be *chiral* if its automorphism group $\Gamma(\mathcal{P})$ has two orbits on flags, with every two adjacent flags lying in different orbits. The reason for this terminology is that any such $\mathcal{P}$ has maximum possible 'rotational' symmetry (admitting analogues of the abstract rotations $\sigma_i = \rho_{i-1}\rho_i$), without admitting the 'reflections' ρ_i.

The rank d of a chiral polytope is at least 3, since every abstract 2-polytope is combinatorially isomorphic to a regular convex polygon with at least 3 sides (by our non-degeneracy assumption). The facets and vertex-figures of a chiral d-polytope $\mathcal{P}$ may be regular or chiral, but the $(d-2)$-faces (and dually the co-edges) are always regular (by a nice argument given in [10, Proposition 9]).

The structure of the automorphism group of a chiral polytope $\mathcal{P}$ closely resembles that of the rotation group of a regular polytope. In particular, $\Gamma(\mathcal{P})$ is generated by elements $\sigma_1, \ldots, \sigma_{d-1}$, where σ_i maps a given base flag Φ to the flag $(\Phi^i)^{i-1}$ which differs from Φ in its $(i-1)$- and i-faces. The rank 2 section of $\mathcal{P}$ between the $(i-2)$- and $(i+1)$-faces of Φ is then isomorphic to a regular p_i-gon for some p_i, and the automorphism σ_i permutes the $(i-1)$- and i-faces of this section in two cycles of length p_i.

Moreover, the generators σ_i also satisfy (4), and if we define elements $\tau_{i,j} = \sigma_i \sigma_{i+1} \cdots \sigma_j$ for $1 \le i < j < d$, and exactly as in the previous subsection for other values of i and j, then the subgroups $H_I = \langle\, \tau_{i+1,j} \mid i, j \in I \,\rangle$ also satisfy the intersection condition (5).

For simplicity and consistency, we still refer to these generators σ_i of $\Gamma(\mathcal{P})$ as *abstract rotations*, and the products $\tau_{i,j}$ for $1 \le i < j < d$ as *abstract half-turns*. Also we often refer to the automorphism group of the chiral polytope $\mathcal{P}$ as its *rotation group* and sometimes denote it by $\Gamma^+(\mathcal{P})$.

Conversely, any finite group Γ generated by $d-1$ elements $\sigma_1, \sigma_2, \ldots, \sigma_{d-1}$ satisfying (4) and the intersection condition (5) is the rotation subgroup of an abstract d-polytope $\mathcal{P}$ that is either (orientably) regular or chiral; see [10, Theorem 1]. Indeed $\mathcal{P}$ is regular if and only if there is a group automorphism ρ of Γ of order 2 such that

$$\sigma_i^{\rho} = \begin{cases} \sigma_i^{-1} & \text{when } i = 1, \\ {\sigma_1}^2\sigma_i & \text{when } i = 2, \\ \sigma_i & \text{when } 2 < i < d. \end{cases} \tag{6}$$

Note (for later use) that for rank 3, the automorphism ρ has to invert σ_1 and take σ_2 to ${\sigma_1}^2\sigma_2 = \sigma_1\sigma_2^{-1}\sigma_1^{-1}$, so the composite of ρ with conjugation by σ_1 inverts both σ_1 and σ_2; the existence of such an automorphism is the more customary test for chirality of maps.

Each chiral d-polytope $\mathcal{P}$ occurs in two *enantiomorphic forms*, which may be understood as $\mathcal{P}$ and its 'mirror image' (and hence as a right- and left-handed version of $\mathcal{P}$). The group of the mirror image of $\mathcal{P}$ is also $\Gamma(\mathcal{P})$, but with respect to the generators $\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \ldots, \sigma_{d-1}$. Further information can be found in [11].

2.4 Chiral 4-polytopes

In this paper we concentrate on chiral polytopes of rank $d = 4$.

By [10, Lemma 11], the intersection condition for a chiral 4-polytope $\mathcal{P}$ can be reduced to just three cases, as follows:

$$\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = \{1\}, \quad \langle\sigma_2\rangle \cap \langle\sigma_3\rangle = \{1\} \quad \text{and} \quad \langle\sigma_1, \sigma_2\rangle \cap \langle\sigma_2, \sigma_3\rangle = \langle\sigma_2\rangle. \tag{7}$$

We will also make use of an alternative generating set for $\Gamma^+(\mathcal{P})$, namely $\{\tau_1, \tau_2, \tau_3\}$, where $\tau_i = \tau_{1,i} = \sigma_1\sigma_2\cdots\sigma_i$ for $1 \le i \le 3$. In terms of these three generators, the relations $(\sigma_i\sigma_{i+1}\cdots\sigma_j)^2 = 1$ in (4) are equivalent to

$$(\tau_1\tau_3)^2 = \tau_2^2 = \tau_3^2 = 1. \tag{8}$$

Furthermore, the test in (6) for regularity of $\mathcal{P}$ simplifies to the existence of a group automorphism ρ of $\Gamma^+(\mathcal{P})$ such that

$$\tau_i^{\rho} = \tau_i^{-1} \text{ for } 1 \le i \le 3. \tag{9}$$

Finally we note that $\langle\tau_1\rangle = \langle\sigma_1\rangle$ and $\langle\tau_1, \tau_2\rangle = \langle\sigma_1, \sigma_2\rangle$, but a comparison of orders shows that $\langle\tau_2\rangle \ne \langle\sigma_2\rangle$, and similarly it need not be true that $\langle\tau_2, \tau_3\rangle = \langle\sigma_2, \sigma_3\rangle$.

3 Actions of A_4

In Sect. 4 we will construct families of chiral 4-polytopes whose facets are tetrahedra. The construction involves extending an intransitive action of the rotation group A_4 of the tetrahedron on a set with n elements, to the standard action of A_n or S_n on the same set, by adjoining a new permutation that represents a generator of the automorphism group of the 4-polytope.

In this section we create some building blocks for the construction, via transitive permutation representations of A_4. We will be particularly interested in the permutations

τ_1 and τ_2 representing the generators of A_4 as the rotation group of the tetrahedron. These permutations satisfy the relations $\tau_1^3 = \tau_2^2 = (\tau_1^{-1}\tau_2)^3 = 1$.

3.1 Building blocks

The transitive representations of A_4 that we use as building blocks are those on 1, 4, 6 and 12 points, as follows:

Representation A: the trivial representation of A_4, of degree 1;
Representation B: the standard representation of A_4 on 4 points, with

$$\tau_1 = (1, 3, 2)(4) \quad \text{and} \quad \tau_2 = (1, 2)(3, 4);$$

Representation C: the transitive representation of A_4 on 6 points, given by

$$\tau_1 = (1, 2, 3)(4, 5, 6) \quad \text{and} \quad \tau_2 = (1, 4)(2, 5);$$

Representation D: the regular representation of A_4 on 12 points, given by

$$\begin{aligned} \tau_1 &= (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12) \quad \text{and} \\ \tau_2 &= (1, 4)(2, 7)(3, 10)(5, 12)(6, 8)(9, 11). \end{aligned}$$

Note that these transitive representations are unique up to re-labelling points, because A_4 has a single conjugacy class of subgroups of each of the orders 12, 3, 2 and 1.

We will also be interested in the orbits of the subgroup $\langle \tau_1 \rangle$. In Representation B there are two orbits, of lengths 3 and 1, respectively, while in Representations C and D there are two of length 3 and four of length 3, respectively.

For later use, we illustrate these representations in Fig. 1 by subdivided boxes, with each subdivision giving the length of an orbit of $\langle \tau_1 \rangle$.

3.2 Extending the action of A_4

Our construction involves extending an intransitive action of $A_4 = \langle \tau_1, \tau_2 \rangle$ to a transitive action of $\langle \tau_1, \tau_2, \tau_3 \rangle$, by a suitable definition of the third generator τ_3.

The first and third of the relations $(\tau_1\tau_3)^2 = \tau_2^2 = \tau_3^2 = 1$ given in (8) imply that τ_3 must be an involution which conjugates the generator τ_1 to its inverse. For this reason, τ_3 must permute the fixed points of $\langle \tau_1 \rangle$ among themselves, and permute the orbits of length 3 among themselves. To make the resulting action of $\langle \tau_1, \tau_2, \tau_3 \rangle$ transitive, we must link together the orbits of $A_4 = \langle \tau_1, \tau_2 \rangle$, and this can be achieved by defining τ_3

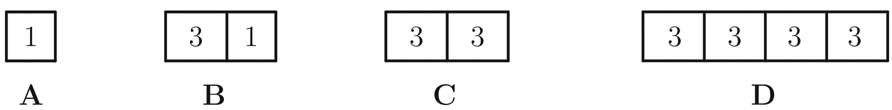

Fig. 1 Transitive permutation representations of $A_4 = \langle \tau_1, \tau_2 \rangle$ on 1, 4, 6 and 12 points

in such a way as to link together the orbits of $\langle \tau_1 \rangle$, perhaps sometimes linking an orbit to itself.

There is just one way of linking together two orbits of $\langle \tau_1 \rangle$ of length 1, namely by making τ_3 interchange the single points from the orbits. On the other hand, linking together two different orbits of $\langle \tau_1 \rangle$ of length 3 can be done in three ways. If τ_1 acts on one orbit as the 3-cycle (y_1, y_2, y_3), where $y_1 = \min\{y_1, y_2, y_3\}$, and on the other as the 3-cycle (z_1, z_2, z_3), where $z_1 = \min\{z_1, z_2, z_3\}$, then we have these three possibilities for the effect of τ_3 on the set $\{y_1, y_2, y_3, z_1, z_2, z_3\}$:

$$\begin{array}{ll} (y_1, z_1)(y_2, z_3)(y_3, z_2) & \ldots \text{ type I} \\ (y_1, z_2)(y_2, z_1)(y_3, z_3) & \ldots \text{ type II} \\ (y_1, z_3)(y_2, z_2)(y_3, z_1) & \ldots \text{ type III.} \end{array}$$

In the special case where these orbits are the same (so that $(y_1, y_2, y_3) = (z_1, z_2, z_3)$), the element τ_3 induces (y_2, y_3), (y_1, y_2) and (y_1, y_3) for types I, II and III, respectively.

Also at this stage, we note that for an orientably regular or chiral 4-polytope $\mathcal{P}$ of type $\{3, 3, k\}$, whose facets are tetrahedra, the reduced intersection condition (7) can be simplified even further.

Lemma 3.1 *Let Γ be a transitive permutation group of degree n generated by three elements σ_1, σ_2 and σ_3 satisfying*

$$\sigma_1^3 = \sigma_2^3 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1$$

with $\langle \sigma_1, \sigma_2 \rangle \cong A_4$. If $\langle \sigma_2, \sigma_3 \rangle$ is intransitive and σ_2 is not a power of σ_3, then the intersection condition (7) holds.

Proof First $\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \{1\}$, since σ_1 and σ_2 are two elements of order 3 generating A_4. Next, observe that $\langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle$ is a subgroup of $\langle \sigma_1, \sigma_2 \rangle$, containing $\langle \sigma_2 \rangle$ and that $\langle \sigma_2 \rangle$ is maximal in $\langle \sigma_1, \sigma_2 \rangle$, since every cyclic subgroup of order 3 in A_4 is maximal in A_4. It follows that if $\langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle \neq \langle \sigma_2 \rangle$, then $\langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle = \langle \sigma_1, \sigma_2 \rangle$, and therefore $\sigma_1 \in \langle \sigma_2, \sigma_3 \rangle$, which gives $\Gamma = \langle \sigma_1, \sigma_2, \sigma_3 \rangle = \langle \sigma_2, \sigma_3 \rangle$. But that is clearly impossible, because Γ is transitive while $\langle \sigma_2, \sigma_3 \rangle$ is not. Thus $\langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle = \langle \sigma_2 \rangle$. Finally, $\langle \sigma_2 \rangle \cap \langle \sigma_3 \rangle = \{1\}$, since the element σ_2 of order 3 does not lie in $\langle \sigma_3 \rangle$. □

3.3 Other facts needed

To conclude this section we mention some results from group theory that we need for the construction presented in Sect. 4, specifically for recognising when a transitive subgroup of S_n is either A_n or S_n, and also about the automorphism groups of A_n and S_n.

Theorem 3.2 (Jordan, 1873) *Let G be a primitive group of permutations on a set X of degree n, and suppose G contains an element that acts as a p-cycle, fixing the other $n - p$ points, where p is a prime such that $p \leq n - 3$. Then G is isomorphic to A_n or S_n.*

For a proof, see [12, Theorem 13.9]. The next theorem is well known; proofs can be found in [9, Corollary 7.7] for $\mathrm{Aut}(S_n)$, and [13, Theorem 2.3] for $\mathrm{Aut}(A_n)$, for example.

Theorem 3.3 *For every $n \geq 7$, every automorphism of A_n and every automorphism of S_n is induced by conjugation by an element of S_n. In particular,* $\mathrm{Aut}(A_n) \cong \mathrm{Aut}(S_n) \cong S_n$ *for every $n \geq 7$.*

4 Construction of chiral 4-polytopes

In this section we use the building blocks given earlier to construct two families of chiral 4-polytopes, with automorphism groups S_n and A_n respectively, for all $n \geq 46$.

4.1 General approach

We let X be the set $\{1, 2, \ldots, n\}$, and define permutations τ_1, τ_2, and $\tau_3 \in S_n$ such that $\langle \tau_1, \tau_2 \rangle = A_4$ and τ_1, τ_2, and τ_3 satisfy (8). In order to prove that the construction actually gives a chiral 4-polytope, we need to do three things:

Step (a): *Show that $\Gamma = \langle \tau_1, \tau_2, \tau_3 \rangle$ is A_n or S_n.*

Our construction ensures that the action of Γ is transitive on X. We exhibit an element of Γ that acts as a cycle of prime length p, fixing at least 3 points, and then use this to prove that Γ is primitive on X and apply Theorem 3.2 to give $\Gamma \cong A_n$ or $\Gamma \cong S_n$.

Step (b): *Show that Γ is the rotation subgroup of an orientably regular polytope or the automorphism group of a chiral polytope.*

For this step, all we need to do is prove that the permutations $\sigma_1 = \tau_1$, $\sigma_2 = \tau_1^{-1}\tau_2$ and $\sigma_3 = \tau_2^{-1}\tau_3$ satisfy the reduced form of the intersection condition given in (7). By Lemma 3.1, it is sufficient to show that $\langle \sigma_2, \sigma_3 \rangle$ is intransitive on X and that $\sigma_2 \notin \langle \sigma_3 \rangle$.

Step (c): *Verify chirality, by ruling out the existence of a permutation $\rho \in S_n$ such that $\tau_i^{\rho} = \tau_i^{-1}$ for all $i \in \{1, 2, 3\}$.*

Note the permutations τ_1 and τ_2 are always even, since they come from permutation representations of A_4. It follows that once we have completed step (a), we can decide whether Γ is A_n or S_n by simply checking whether τ_3 is even or odd. In some cases we will make an adjustment to τ_3 that will still ensure that $\Gamma = \langle \tau_1, \tau_2, \tau_3 \rangle$ is the automorphism group of some chiral 4-polytope of type $\{3, 3, k\}$ for some k, but has a different parity, in which case we change Γ from an alternating group to a symmetric group, or vice versa.

We will consider a number of cases, based on the residue class of n mod 6. Before that, we give a concrete example (for $n = 46$), which will show how most of the construction works. This can then be adapted in a number of ways for other values of the degree n.

4.2 Example: degree $n = 46$

Consider the following three permutations on 46 points:

$$\tau_1 = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)(16, 17, 18)(19, 20, 21)(22, 23, 24)(25, 26, 27)(28, 29, 30)(31, 32, 33)(34, 35, 36)(37, 38, 39)(40, 41, 42)(43, 44, 45),$$

$$\tau_2 = (1, 4)(2, 7)(3, 10)(5, 12)(6, 8)(9, 11)(13, 16)(14, 17)(19, 22)(20, 23)(25, 28)(26, 29)(31, 34)(32, 35)(37, 40)(38, 41)(43, 44)(45, 46),$$

$$\tau_3 = (1, 2)(4, 7)(5, 9)(6, 8)(10, 13)(11, 15)(12, 14)(16, 19)(17, 21)(18, 20)(22, 25)(23, 27)(24, 26)(28, 31)(29, 33)(30, 32)(34, 39)(35, 38)(36, 37)(40, 43)(41, 45)(42, 44).$$

These satisfy the required relations, and generate a transitive subgroup of S_{46}. The orbits of $\langle \tau_1, \tau_2 \rangle$ are the sets $\{1, 2, \ldots, 12\}$, $\{13, 14, \ldots, 18\}$, $\{19, 20, \ldots, 24\}$, $\{25, 26, \ldots, 30\}$, $\{31, 32, \ldots, 36\}$, $\{37, 38, \ldots, 42\}$ and $\{43, 44, 45, 46\}$, of lengths 12, 6, 6, 6, 6, 6 and 4. The way in which the orbits of $\langle \tau_1 \rangle$ are linked together by τ_3 is illustrated in Fig. 2, where the Roman numerals indicate the type of link.

In this representation, observe that the elements $\sigma_2 = \tau_1^{-1}\tau_2$ and $\sigma_3 = \tau_2^{-1}\tau_3 = \tau_2\tau_3$ are as follows:

$$\sigma_2 = (1, 10, 5)(2, 4, 8)(3, 7, 11)(6, 12, 9)(13, 15, 17)(14, 16, 18)(19, 21, 23)(20, 22, 24)(25, 27, 29)(26, 28, 30)(31, 33, 35)(32, 34, 36)(37, 39, 41)(38, 40, 42)(43, 46, 45);$$

$$\sigma_3 = (1, 7)(2, 4)(3, 13, 19, 25, 31, 39, 34, 28, 22, 16, 10)(5, 14, 21, 17, 12, 9, 15, 11)(18, 20, 27, 23)(24, 26, 33, 29)(30, 32, 38, 45, 46, 41, 35)(36, 37, 43, 42, 44, 40).$$

In particular, the cycle structure of σ_3 is $1^2\, 2^2\, 4^2\, 6^1\, 7^1\, 8^1\, 11^1$, and so its order is 1848. Also ${\sigma_3}^{168}$ is an 11-cycle, namely (3, 25, 34, 16, 13, 31, 28, 10, 19, 39, 22).

We claim that the action of $\Gamma = \langle \tau_1, \tau_2, \tau_3 \rangle$ is primitive on $\{1, \ldots, 46\}$. To verify this, we assume the contrary (but we will ignore the fact that 3 and 11 do not divide 46, just to exhibit a more general argument). All the 11 points moved by ${\sigma_3}^{168}$ would have to belong to the same block of imprimitivity, say U, since 11 is prime and every block containing a fixed point of ${\sigma_3}^{168}$ would be fixed by ${\sigma_3}^{168}$. Next, τ_2 preserves U since it interchanges the points 3 and 10 of U, and similarly, τ_3 preserves U, since it fixes the point 3. It follows that τ_1 cannot preserve U, and so the images of U under τ_1 and its inverse

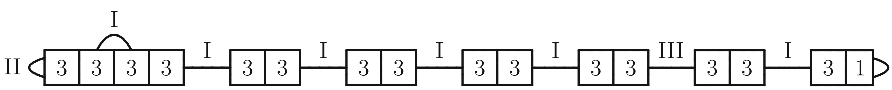

Fig. 2 Orbit links for a permutation representation of degree 46

τ_1^2 must be new blocks V and W, containing $\{1, 26, 35, 17, 14, 32, 29, 11, 20, 37, 23\}$ and $\{2, 27, 36, 18, 15, 33, 30, 12, 21, 38, 24\}$, respectively. Now τ_2 preserves both V and W, since it interchanges the points 26 and 29 and fixes the point 24, and similarly, τ_3 interchanges V with W since it interchanges the points 1 and 2. By transitivity, it follows that there are just three blocks, with τ_1, τ_2 and τ_3 inducing the permutations (U, V, W), $(U)(V)(W)$ and $(U)(V, W)$ on them. In particular, τ_2 preserves every block, while τ_1 preserves no block. But that is impossible, since τ_1 fixes the point 46.

By Theorem 3.2, we find that $\Gamma = A_{46}$ or S_{46}, and since τ_3 is even, we have $\Gamma = A_{46}$.

Next, we verify the intersection condition. First, $\langle \sigma_2, \sigma_3 \rangle = \langle \tau_1^{-1}\tau_2, \tau_2\tau_3 \rangle$ is intransitive, since it has $\{2, 4, 8\}$ as an orbit, and second, $\sigma_2 \notin \langle \sigma_3 \rangle$, since $\sigma_2 = \tau_1^{-1}\tau_2$ induces the 3-cycle $(1, 10, 5)$, while σ_3 interchanges the points 1 and 7. Hence by Lemma 3.1, the intersection condition (7) holds.

Thus A_{46} is the rotation group of a regular or chiral 4-polytope $\mathcal{P}$ (of type $\{3, 3, 1848\}$).

Next, suppose $\mathcal{P}$ is regular. Then there must exist an involutory group automorphism ρ of Γ inverting each of τ_1, τ_2 and τ_3. By Theorem 3.3, this automorphism ρ can be taken as a permutation in S_{46}. In particular, since ρ inverts τ_1, it must permute the orbits of $\langle \tau_1 \rangle$ among themselves, and hence must fix the point 46. Then since ρ inverts τ_2, it follows that ρ preserves the orbit $\{45, 46\}$ of $\langle \tau_2 \rangle$ and hence fixes the point 45. In turn, since ρ inverts τ_1, it must interchange the other two points 43 and 44 of the 3-cycle $(43, 44, 45)$ of τ_1, and then must interchange the orbits $\{40, 43\}$ and $\{42, 44\}$ of $\langle \tau_3 \rangle$, and hence must interchange the points 40 and 42. But this is impossible, since 42 is fixed by τ_2, while 40 is not. Thus $\mathcal{P}$ is a chiral 4-polytope, of type $\{3, 3, 1848\}$, with automorphism group A_{46}.

To do the same for S_{46}, we define τ_1 and τ_2 exactly as above, but now take

$$\begin{aligned}\tau_3 = {} & (1, 2)(4, 6)(7, 8)(10, 13)(11, 15)(12, 14)(16, 19)(17, 21)(18, 20)(22, 25) \\ & (23, 27)(24, 26)(28, 31)(29, 33)(30, 32)(34, 39)(35, 38)(36, 37)(40, 43) \\ & (41, 45)(42, 44).\end{aligned}$$

This is almost the same as the permutation taken for τ_3 above, but with the three transpositions $(4, 7)$, $(5, 9)$ and $(6, 8)$ replaced by the two transpositions $(4, 6)$ and $(7, 8)$, and two fixed points 5 and 9. With regard to Fig. 2, we have replaced the τ_3-link between the first two orbits of $\langle \tau_1 \rangle$ by self-links for those two orbits, of types III and II respectively. This time we have

$$\begin{aligned}\sigma_3 = {} & (1, 6, 7)(2, 8, 4)(3, 13, 19, 25, 31, 39, 34, 28, 22, 16, 10)(5, 14, 21, 17, 12) \\ & (9, 15, 11)(18, 20, 27, 23)(24, 26, 33, 29)(30, 32, 38, 45, 46, 41, 35) \\ & (36, 37, 43, 42, 44, 40),\end{aligned}$$

which has cycle structure $3^3\, 4^2\, 5^1\, 6^1\, 7^1\, 11^1$, so its order is 4620.

Now σ_3^{420} is an 11-cycle, in fact the 8th power of the one found before, and it can be used to prove that the action of $\Gamma = \langle \tau_1, \tau_2, \tau_3 \rangle$ is primitive, and hence that $\Gamma = A_{46}$ or S_{46}. This time τ_3 is odd, so $\Gamma = S_{46}$. Again the intersection condition is satisfied (noting that {2, 4, 8} is still an orbit of $\langle \sigma_2, \sigma_3 \rangle = \langle \tau_1^{-1}\tau_2, \tau_2\tau_3 \rangle$ and that (1, 10, 5) is still a cycle of σ_2), and the same argument as before proves chirality. Thus we have another chiral 4-polytope, but now of type {3, 3, 4620}, and with automorphism group S_{46}.

4.3 Adding extra orbits of A_4 of length 6

Now take the above example (for $n = 46$) and insert an additional orbit of length 6 for $\langle \tau_1, \tau_2 \rangle \cong A_4$ between the last two on the right, with a τ_3-link of type I to the previous final orbit of length 6 and a τ_3-link of type III to the orbit of length 4, as in Fig. 3.

This gives a transitive permutation representation on $46 + 6 = 52$ points, with the following changes made to the three permutations τ_i used to generate A_{46}:

τ_1: adjoin the two 3-cycles (46, 47, 48) and (49, 50, 51), and the fixed point 52,
τ_2: replace (43, 44)(45, 46) by (43, 46)(44, 47)(49, 50)(51, 52), fixing points 45 and 48,
τ_3: replace the fixed point 46 by (46, 51)(47, 50)(48, 49), fixing the point 52.

With these changes, the only effect on the permutation $\sigma_3 = \tau_2\tau_3$ is to alter cycles containing any of the points numbered greater than 42, and in fact, it is easy to see that the two cycles (30, 32, 38, 45, 46, 41, 35) and (36, 37, 43, 42, 44, 40) of lengths 7 and 6 are replaced by (30, 32, 38, 45, 41, 35), (36, 37, 43, 51, 52, 46, 40) and (42, 44, 50, 48, 49, 47), of lengths 6, 7 and 6. In particular, the cycle structure of σ_3 remains the same except for the addition of one further cycle of length 6, and σ_3^{168} is still the same 11-cycle, namely (3, 25, 34, 16, 13, 31, 28, 10, 19, 39, 22).

Again this 11-cycle and the existence of a fixed point of τ_1 can be used to prove that the group $\Gamma = \langle \tau_1, \tau_2, \tau_3 \rangle$ is primitive, and then since the parity of τ_3 has changed from even to odd, we have $\Gamma = S_{52}$. The intersection condition (7) holds for exactly the same reasons as for degree 46, and the proof of chirality is entirely similar: any involutory group automorphism ρ inverting each of τ_1, τ_2 and τ_3 would have to fix the points 52 and 51, and swap the points 49 and 50, and then swap the points 47 and 48, which is impossible.

Thus S_{52} is the automorphism group of a chiral 4-polytope $\mathcal{P}$ (of type {3, 3, 1848}).

Furthermore, we can now make the same change to the effect of τ_3 on the first orbit of $\langle \tau_1, \tau_2 \rangle$ (of length 12) as we did for degree 46, with a change in parity of τ_3, and the same arguments work again, to prove that A_{52} is the automorphism group of a chiral 4-polytope of type {3, 3, 4620}.

In summary, inserting the extra orbit of A_4 of length 6 increased the degree n by 6, but retained the properties of the permutations τ_1, τ_2 and τ_3 needed to prove that A_n and S_n are the automorphism groups of chiral 4-polytopes of type {3, 3, k} for some k.

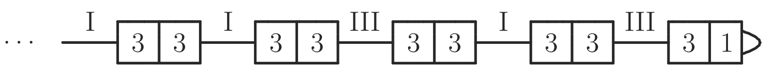

Fig. 3 Inserting an extra orbit of $\langle \tau_1, \tau_2 \rangle \cong A_4$ of length 6

But clearly we can do the same kind of thing again. Suppose we insert another new orbit of A_4 of length 6 between the last one and the orbit of length 4, with a τ_3-link of type III to the previous final orbit of length 6 and a τ_3-link of type I to the orbit of length 4. Then the degree n increases by 6, and we return to a situation similar to that for degree 46. With the obvious re-numbering of points in the last two orbits of A_4, the cycles $(36, 37, 43, 51, 52, 46, 40)$ and $(42, 44, 50, 48, 49, 47)$ of lengths 7 and 6 for σ_3 in the case of degree 52 are replaced by $(36, 37, 43, 51, 46, 40)$, $(42, 44, 50, 57, 58, 53, 47)$ and $(48, 49, 55, 54, 56, 52)$, of lengths 6, 7 and 6. Hence the cycle structure of σ_3 is again changed only by the addition of another cycle of length 6. All the previous arguments work in the same way, to prove that A_{58} and S_{58} are the automorphism groups of chiral 4-polytopes of types $\{3, 3, 1848\}$ and $\{3, 3, 4620\}$, respectively.

These insertions can be repeated over and over again, increasing the degree by 6 through insertion of a new orbit of length 6 for A_4 each time. Provided that the types of the τ_3-links joining successive new orbits of A_4 are chosen to alternate between types I and III, the important properties of the the permutations τ_1, τ_2 and τ_3 will be retained, and all our arguments will go through in the same way as for degrees 46 and 52.

Thus we have the following: *for every integer* $n \geq 46$ *such that* $n \equiv 4$ *mod* 6, *both* A_n *and* S_n *are the automorphism groups of chiral* 4*-polytopes of type* $\{3, 3, k\}$ *for some* k.

In fact, k can be taken as 1848 or 4620, depending on the residue class of n mod 12, and in particular, our construction shows there are infinitely many chiral 4-polytopes of type $\{3, 3, k\}$ for each of these two values of k.

4.4 Adding an extra point fixed by A_4

In all of the cases considered so far in this section, with degree $n \equiv 4$ mod 6, the subgroup $\langle \tau_1, \tau_2 \rangle \cong A_4$ had single orbits of lengths 12 and 4, and $\frac{n-16}{6}$ orbits of length 6, and the permutation τ_1 had a single fixed point (which we chose to be n) and $\frac{n-1}{3}$ cycles of length 3. We will now consider what happens when we adjoin a single orbit of length 1.

Necessarily, the permutations τ_1 and τ_2 will fix this point, while τ_3 must interchange it with the only other fixed point of τ_1. The only change to the permutation σ_3 is to enlarge its unique 7-cycle (containing the original fixed point of τ_1) to an 8-cycle. For example, when $n = 46$, the cycle $(30, 32, 38, 45, 46, 41, 35)$ becomes $(30, 32, 38, 45, 47, 46, 41, 35)$.

The order of σ_3 changes from 1848 to $1848/7 = 264$, or from 4620 to $2 \cdot 4620/7 = 1320$, and in those two cases respectively, the permutations σ_3^{24} and σ_3^{120} are 11-cycles, namely $\xi = (3, 19, 31, 34, 22, 10, 13, 25, 39, 28, 16)$ and $\xi^5 = (3, 10, 16, 22, 28, 34, 39, 31, 25, 19, 13)$.

In each case, the 11-cycle and the existence of a fixed point of τ_1 can be used to prove that the resulting permutations τ_1, τ_2 and τ_3 generate a primitive group, and hence an alternating or symmetric group. Also the intersection condition holds for exactly the same reasons as before. On the other hand, the proof of chirality needs a small variation.

Take n to be the resulting degree, and $n-1$ and n as the fixed points of τ_1, and $n-2$ as the image of $n-1$ under τ_2 in the orbit of A_4 of length 4. Now suppose there exists an involutory group automorphism ρ of $\Gamma = \langle \tau_1, \tau_2, \tau_3 \rangle$ inverting each τ_i. By Theorem 3.3, this automorphism ρ can be taken as an element of S_n, and since ρ inverts τ_1, it must fix or interchange the points $n-1$ and n. If it fixes both, then the same argument as before gives a contradiction, and so it must interchange them. But that is impossible, since $n-1$ and n lie in cycles of τ_2 of different lengths (namely 2 and 1). Hence there is no such automorphism ρ, and we have a chiral 4-polytope.

Thus we have the following: *for every integer* $n \geq 47$ *such that* $n \equiv 5$ *mod* 6, *both* A_n *and* S_n *are the automorphism groups of chiral* 4*-polytopes of type* $\{3, 3, k\}$ *for some* k.

In fact, k can be taken as 264 or 1320, depending on the residue class of n mod 12, and in particular, our construction shows there are infinitely many chiral 4-polytopes of type $\{3, 3, k\}$ for each of these two values of k.

4.5 Adding a second orbit of A_4 of length 4

Next, we consider what happens when we add a second orbit of length 4 for A_4 to the permutations given earlier for A_{46}, but at the 'first end', linked to the orbit of length 12 for A_4 by a τ_3 link of type II, as in Fig. 4.

Specifically (and to avoid altering the numbering too much), we introduce four new points, labelled v, x, y and z, with the assumption that $v < x < y < z$ and make the following changes made to the three permutations τ_i used to generate A_{46}:

τ_1: adjoin the 3-cycle (x, y, z) and the fixed point v,
τ_2: adjoin the transpositions (v, z) and (x, y),
τ_3: replace the transposition $(1, 2)$ and fixed point 3 by $(x, 2)(y, 1)(z, 3)$, fixing v.

With these changes, the only effect on the permutation $\sigma_3 = \tau_2\tau_3$ is to alter the cycles containing any of the points numbered 1, 2 and 3, namely the transpositions $(1, 7)$ and $(2, 4)$ and the 11-cycle $(3, 13, 19, 25, 31, 39, 34, 28, 22, 16, 10)$. These cycles are replaced by $(x, 1, 7)$, $(y, 2, 4)$ and $(v, 3, 13, 19, 25, 31, 39, 34, 28, 22, 16, 10, z)$, of lengths 3, 3 and 13, respectively.

In particular, the cycle structure of σ_3 becomes $1^2\, 3^2\, 4^2\, 6^1\, 7^1\, 8^1\, 13^1$, and so σ_3 now has order 2184. Also ${\sigma_3}^{168}$ is a 13-cycle, namely $(3, v, z, 10, 16, 22, 28, 34, 39, 31, 25, 19, 13)$.

We claim that the action of $\Gamma = \langle \tau_1, \tau_2, \tau_3 \rangle$ is primitive on $\{1, \ldots, 46\} \cup \{v, x, y, z\}$. If not, then the 13 points moved by ${\sigma_3}^{168}$ would belong to the same block U, and U would be preserved by τ_1 and τ_3, since the point v is fixed by both τ_1 and τ_3, and U would be preserved by τ_2, since τ_2 swaps v with z. But then U would be preserved

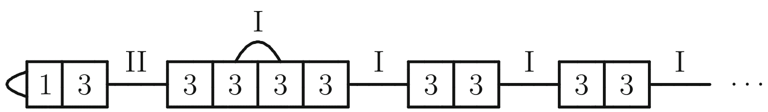

Fig. 4 Adding a second orbit of length 4 for A_4

by $\langle \tau_1, \tau_2, \tau_3 \rangle = \Gamma$ and so could not be a block of imprimitivity. Since τ_3 is even, it follows that $\Gamma \cong A_{50}$.

Also the subgroup generated by σ_2 and σ_3 is intransitive, because it has $\{y, 2, 4, 8\}$ as an orbit, and σ_2 does not lie in $\langle \sigma_3 \rangle$, because σ_2 induces the 3-cycle $(2, 4, 8)$, while σ_3 induces the 3-cycle $(y, 2, 4)$ on this orbit. By Lemma 3.1, the intersection condition holds.

We still need to confirm chirality. Suppose there is an involution ρ in S_{50} which conjugates each of τ_1, τ_2 and τ_3 to its inverse. Then ρ fixes or interchanges the two fixed points of τ_1, namely v and 46, and if it fixes 46, then the same argument as before gives a contradiction, so it must interchange them. It follows that ρ swaps $v^{\tau_2} = z$ with $46^{\tau_2} = 45$, and also $z^{\tau_3} = 3$ with $45^{\tau_3} = 41$. But that is impossible, since 3 and 41 lie in cycles of τ_2 of different lengths (namely 2 and 1).

Thus A_{50} is the automorphism group of a chiral 4-polytope of type $\{3, 3, 2184\}$.

Next, if we make the same change to the effect of τ_3 on the orbit $\{1, 2, \ldots, 12\}$ of $\langle \tau_1, \tau_2 \rangle$ as we did for degree 46, then we find that the cycles of σ_3 containing the points of $\{v, x, y, z, 1, 2, \ldots, 12\}$ are $(v, 3, 13, 19, 25, 31, 39, 34, 28, 22, 16, 10, z)$, $(x, 1, 6, 7)$, $(y, 2, 8, 4)$, $(5, 14, 21, 17, 12)$ and $(9, 15, 11)$. In this case, σ_3 has cycle structure $3^1\, 4^4\, 5^1\, 6^1\, 7^1\, 13^1$ and hence order 5460. Again the existence of the 13-cycle and the effect of τ_1, τ_2 and τ_3 on the points v and z imply that $\Gamma = \langle \tau_1, \tau_2, \tau_3 \rangle$ is primitive, and this time the change in parity of τ_3 gives $\Gamma \cong S_{50}$. Also $\{y, 2, 4, 8\}$ is an orbit of $\langle \sigma_2, \sigma_3 \rangle$, on which σ_2 induces the 3-cycle $(2, 4, 8)$ and σ_3 induces the 4-cycle $(y, 2, 8, 4)$, and hence the intersection condition holds, again by Lemma 3.1. Chirality follows from the same argument as for A_{50} above.

Thus S_{50} is the automorphism group of a chiral 4-polytope of type $\{3, 3, 5460\}$.

Now we can repeat the process begun in Sect. 4.3 and introduce further orbits of length 6 for A_4 near the 'other end'. As before, this adds extra 6-cycles to the cycle structure for σ_3, but does not affect the proof of primitivity, and therefore still gives the group $\Gamma = \langle \tau_1, \tau_2, \tau_3 \rangle$ as either A_n or S_n each time. Also verification of the intersection condition and proof of chirality are entirely analogous to those for the A_{50} case, above.

Thus we have the following: *for every integer* $n \geq 50$ *such that* $n \equiv 2$ *mod* 6, *both* A_n *and* S_n *are the automorphism groups of chiral* 4*-polytopes of type* $\{3, 3, k\}$ *for some* k.

In fact, k can be taken as 2184 or 5460, depending on the residue class of n mod 12, and in particular, our construction shows there are infinitely many chiral 4-polytopes of type $\{3, 3, k\}$ for each of these two values of k.

Moreover, we can make the same adjustment as in Sect. 4.4, by adding an extra fixed point of $\langle \tau_1, \tau_2 \rangle \cong A_4$ at the 'other end'. In this case, the order of σ_3 changes from 2184 to $2184/7 = 312$, or from 5460 to $2 \cdot 5460/7 = 1560$, respectively, and the permutations σ_3^{24} and σ_3^{120} are 13-cycles, namely $\zeta = (3, z, 16, 28, 39, 25, 13, v, 10, 22, 34, 31, 19)$ and $\zeta^5 = (3, 25, 34, 16, v, 19, 39, 22, z, 13, 31, 28, 10)$. Again it is easy to verify primitivity, and deduce that $\Gamma = \langle \tau_1, \tau_2, \tau_3 \rangle$ is isomorphic to A_n or S_n. Also the intersection condition holds for exactly the same reasons as before, but again, the proof of chirality needs a small variation. This time there are three fixed points of τ_1, two of which are interchanged by τ_3. If there exists an involutory permutation ρ of the points that inverts each τ_i, then it must fix or interchange those two, and then the argument follows in the same way as in Sect. 4.4.

Thus we have the following: *for every integer $n \geq 51$ such that $n \equiv 3$ mod 6, both A_n and S_n are the automorphism groups of chiral 4-polytopes of type $\{3, 3, k\}$ for some k.*

In fact, k can be taken as 312 or 1560, depending on the residue class of n mod 12, and in particular, our construction shows there are infinitely many chiral 4-polytopes of type $\{3, 3, k\}$ for each of these two values of k.

4.6 Adding a third orbit of A_4 of length 4

We are left with the cases of degree $n \equiv 0$ or 1 mod 6. For these, we start with our constructions from the previous subsection for degrees congruent to 2 or 3 mod 6 (beginning with 50 and 51), and adjoin a third orbit of length 4 for A_4, at the same end as the second such orbit, with a τ_3-link of type III to itself. This is illustrated in Fig. 5.

Specifically, we introduce another four new points, labelled p, q, r and s, with the assumption that $p < q < r < s$ and make the following changes to the three permutations τ_i used for generating A_{n-4} or S_{n-4}:

τ_1: adjoin the 3-cycle (p, q, r) and the fixed point s,
τ_2: adjoin the transpositions (p, q) and (r, s),
τ_3: replace the fixed point v by $(p, r)(s, v)$, fixing q.

Obviously this increases the degree by 4, from $n - 4$ to n, and in all cases the only effect on the permutation σ_3 is to replace the 13-cycle $(v, 3, 13, 19, 25, 31, 39, 34, 28, 22, 16, 10, z)$ by the cycle $(v, 3, 13, 19, 25, 31, 39, 34, 28, 22, 16, 10, z, s, p, q, r)$, which has length 17.

In particular, the order of σ_3 changes from 2184 or 5460 to 2856 or 7140 when $n - 4 \equiv 4$ mod 6, and from 312 or 1560 to 408 or 2040 when $n - 4 \equiv 5$ mod 6.

In all cases, some power of σ_3 is a single 17-cycle containing all the points p, q and r, and this can be used to prove that $\Gamma = \langle\tau_1, \tau_2, \tau_3\rangle$ is primitive, since it contains the point p and its images under each of the generators of Γ. It follows that $\Gamma = A_n$ or S_n, again depending on the parity of τ_3.

The intersection condition holds for the same reasons as in the previous subsection, but again a little more care is needed to prove chirality. When $n \equiv 0$ mod 6, there are three fixed points of τ_1, and two of them (namely s and v) are interchanged by τ_3, while the third one (at the 'other end') is fixed by τ_3. Hence any permutation ρ in S_n that conjugates each τ_i to its inverse must fix the third one, and then chirality follows from the same argument as for degree 46. On the other hand, when $n \equiv 1$ mod 6, there are four fixed points of τ_1, with two at each end, both interchanged by τ_3. Just

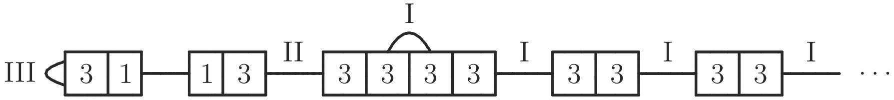

Fig. 5 Adding a third orbit of length 4 for A_4

one of those, however, is a fixed point of τ_2, and so it is fixed by any such ρ, and then chirality follows from the same argument as for degree 47.

Thus we have the following: *for every integer* $n \geq 54$ *such that* $n \equiv 0$ *or* 1 *mod* 6, *both* A_n *and* S_n *are the automorphism groups of chiral* 4*-polytopes of type* $\{3, 3, k\}$ *for some* k.

In fact, k can be taken as 2856 or 7140 when $n \equiv 0$ mod 6, and as 408 or 2040 when $n \equiv 1$ mod 6, in both cases depending on the residue class of n mod 12, and in particular, our construction shows there are infinitely many chiral 4-polytopes of type $\{3, 3, k\}$ for each of these four values of k.

5 Vertex-figures

In this section we prove the following:

Theorem 5.1 *The vertex-figures of the polytopes constructed in Sect. 4 are all chiral.*

Again there is some variation in the argument of different residue classes of n mod 6, but the approach is much the same in all cases.

Proof Let τ_1, τ_2 and τ_3 be the generators of $\Gamma(\mathcal{P})$ as given, and take $\sigma_1 = \tau_1$, $\sigma_2 = \tau_1^{-1}\tau_2$ and $\sigma_3 = \tau_2^{-1}\tau_3 = \tau_2\tau_3$ as before. Then the subgroup Γ_0 generated by σ_2 and σ_3 is the rotation group of a vertex-figure of $\mathcal{P}$.

It is easy to verify that the group $\Gamma_0 = \langle\sigma_2, \sigma_3\rangle$ always has two orbits on the n-point set X, one of which has length 3 or 4, with the other having length $n-3$ or $n-4$. Indeed if $n \equiv 4$ or 5 mod 6, the small orbit Y is $\{2, 4, 8\}$, while otherwise Y is $\{2, 4, 8, y\}$, where y is the middle point of the 3-cycle (x, y, z) of the 'second' A_4-orbit of length 4, which is linked by τ_3 to the A_4-orbit of length 12 as in Sects. 4.5 and 4.6.

Also some power ξ of σ_3 is either the 11-cycle (3, 25, 34, 16, 13, 31, 28, 10, 19, 39, 22), or the 13-cycle $(v, 3, 25, 34, 16, 13, 31, 28, 10, 19, 39, 22, z)$, where v and z are another two of the four points of the second A_4-orbit of length 4 introduced in 4.5, or the 17-cycle $(v, 3, 25, 34, 16, 13, 31, 28, 10, 19, 39, 22, z, s, p, q, r)$, where p, q, r and s are the four points of the third A_4-orbit of length 4 introduced in 4.6.

We will first show that Γ_0 acts on the set $Z = X \setminus Y$ as an alternating or symmetric group of degree $|Z| = n-3$ or $n-4$ and then show that Γ_0 admits no automorphism that inverts both σ_2 and σ_3, which is enough to prove chirality of the vertex-figures.

Suppose Γ_0 is imprimitive on Z. Then all the points of the cycle ξ of prime length (obtained as a power of σ_3) lie in the same block of imprimitivity, say U. Now U is preserved by σ_3 and so cannot be preserved by σ_2, and furthermore, since σ_2 has order 3, the images of U under σ_2 and its inverse σ_2^2 must be new blocks V and W. Next, in all cases, σ_2 takes 10–5, 19–21 and 14–16, while σ_3 takes 5–14 and 14–21. It follows that V contains $10^{\sigma_2} = 5$ and $19^{\sigma_2} = 21$, while W contains $16^{\sigma_2^2} = 14$, and therefore σ_3 interchanges V and W. Hence there are just three blocks, cyclically permuted by σ_2. But also σ_2 fixes at least one point, namely one of the points of the first A_4-orbit of length 4, and so σ_2 preserves at least one block, a contradiction.

Thus Γ_0 is primitive on $Z = X \setminus Y$. Moreover, the existence of the prime cycle ξ shows that Γ_0 is alternating or symmetric on Z (by Jordan's theorem).

On the other hand, σ_2 induces (2, 4, 8) on Y, and σ_3 induces either (2, 4) or (2, 8, 4) on Y when $|Y| = 3$, or $(2, 4, y)$ or $(2, 8, 4, y)$ on Y when $|Y| = 4$, so $\Gamma_0 = \langle\sigma_2, \sigma_3\rangle$ acts on Y as S_3, A_3, A_4 or S_4. It follows that Γ_0 is isomorphic to a sub-direct product $G_1 \times G_2$ where $G_1 = A_{n-3}, S_{n-3}, A_{n-4}$ or S_{n-4}, and $G_2 = S_3, A_3, A_4$ or S_4. (Recall that a sub-direct product of groups G_1 and G_2 is a subgroup G of $G_1 \times G_2$ with the property that the restrictions to G of the projections $\pi_i : G_1 \times G_2 \to G_i$ are both surjective.)

Now each of $A_{n-3}, S_{n-3}, A_{n-4}$ and S_{n-4} is insoluble, with no non-trivial abelian normal subgroup, while A_3, S_3, A_4 and S_4 are soluble, and so the kernel K of the action of Γ_0 on $Z = X \setminus Y$ is the largest soluble normal subgroup of Γ_0 and is therefore characteristic in Γ_0 (that is, invariant under all automorphisms of Γ_0). Thus every automorphism of Γ_0 induces an automorphism of the group $\Pi_0 \cong \Gamma_0/K$ induced by Γ_0 on Z, which of course is $A_{n-3}, S_{n-3}, A_{n-4}$ or S_{n-4}.

Next, suppose the vertex-figures of $\mathcal{P}$ are regular, so that Γ_0 has an automorphism that inverts both σ_2 and σ_3. Then by the above argument, this automorphism induces an automorphism of Π_0 which inverts the permutations induced by σ_2 and σ_3 on Z. Also by Theorem 3.3, we know that the latter can be viewed as a permutation on Z. We can therefore complete the proof of chirality by showing that there is no permutation ρ in $\mathrm{Sym}(Z)$ that conjugates each of σ_2 and σ_3 to its inverse.

In exactly half of the cases we have considered, the permutation σ_3 has exactly two fixed points, namely 6 and 8. These are the cases where τ_3 links the second and third orbits of $\langle\tau_1\rangle$ in the A_4-orbit of length 12, or equivalently, where τ_3 contains the transpositions (4, 7), (5, 9) and (6, 8). In all these cases, (1, 10, 5), (6, 12, 9) and (14, 16, 18) are 3-cycles of σ_2, and (5, 14, 21, 17, 12, 9, 15, 11) and (18, 20, 27, 23) are an 8-cycle and a 4-cycle of σ_3, and the point 1 lies in a 2-cycle or 3-cycle of σ_3.

Now ρ must fix the unique fixed point of σ_3 on $Z = X \setminus Y$, namely 6, and therefore ρ interchanges the other two points 9 and 12 of the 3-cycle (6, 12, 9) of σ_2. It follows that conjugation by ρ inverts the 8-cycle (5, 14, 21, 17, 12, 9, 15, 11) of σ_3, and hence interchanges the points 5 and 14, and must then conjugate the 3-cycle (1, 10, 5) of σ_2 to the inverse of the 3-cycle (14, 16, 18) of σ_2. Hence ρ interchanges the points 1 and 18. But that is impossible, since 18 lies in a 4-cycle of σ_3, while 1 lies in a 2-cycle or 3-cycle of σ_3.

In the other half of all cases, σ_3 has no fixed points, but has a unique 5-cycle, namely (5, 14, 21, 17, 12), and this must be inverted by ρ, and the same is true for the prime cycle ξ of length 11, 13 or 17. Now each of the four points 5, 14, 21 and 17 of the 5-cycle (5, 14, 21, 17, 12) of σ_3 lies in a 3-cycle of σ_2 that has a point in common with the prime cycle ξ, but the fifth point 12 does not have this property. Hence ρ fixes the point 12 and therefore must interchange the other two points 6 and 9 of the 3-cycle (6, 12, 9) of σ_2.

In all these remaining cases, the point 9 lies in a 3-cycle of σ_3, namely (9, 15, 11), and it follows that the point 6 must also lie in a 3-cycle of σ_3. In the cases where there are two or more A_4-orbits of length 4 (and σ_3 has no fixed points), the point 6 lies in the 4-cycle $(1, 6, 7, x)$ of σ_3, and so we can ignore those. This leaves only the cases where there is just one A_4-orbit of length 4, namely those with $n \equiv 4$ or 5 mod 6. For these, we consider what happens locally around the single A_4-orbit of length 4.

When $n \equiv 4 \bmod 6$ (as in the case $n = 46$ and its extensions considered in Sects. 4.2 and 4.3), we may label the points of X such that the generators τ_i of Γ have the following forms:

$$\tau_1 = \ldots(n-12, n-11, n-10)(n-9, n-8, n-7)(n-6, n-5, n-4)(n-3, n-2, n-1),$$
$$\tau_2 = \ldots(n-15, n-12)(n-14, n-11)(n-9, n-6)(n-8, n-5)(n-3, n-2)(n-1, n),$$
$$\tau_3 = \ldots(n-12, n-7)(n-11, n-8)(n-10, n-9)(n-6, n-3)(n-5, n-1)(n-4, n-2)$$
$$\text{or } \ldots(n-12, n-9)(n-11, n-7)(n-10, n-8)(n-6, n-1)(n-5, n-2)(n-4, n-3).$$

With this labelling, $n-2$ is the only fixed point of σ_2, and this lies in a 6-cycle of σ_3, which is $(n-10, n-9, n-3, n-4, n-2, n-6)$ when $n \equiv 10 \bmod 12$ (such as when $n = 46$), or $(n-10, n-8, n-2, n-4, n-3, n-5)$ when $n \equiv 4 \bmod 12$ (such as when $n = 52$). Also the unique 7-cycle of σ_3 is $(n-16, n-14, n-8, n-1, n, n-5, n-11)$ when $n \equiv 10 \bmod 12$, or $(n-16, n-15, n-9, n-1, n, n-6, n-12)$ when $n \equiv 4 \bmod 12$.

In both cases ρ must fix the only fixed point of σ_2, namely $n-2$, and so the 6-cycle of σ_3 containing $n-2$ must be inverted by ρ. When $n \equiv 10 \bmod 12$, this implies that ρ fixes $n-9$ and hence interchanges the two other points $n-7$ and $n-5$ of the 3-cycle $(n-9, n-7, n-5)$ of σ_2. But that is impossible, since $n-5$ lies in a 7-cycle of σ_3, while $n-7$ does not. Similarly, when $n \equiv 4 \bmod 12$, we find that ρ fixes $n-5$, and hence swaps $n-7$ and $n-9$, which is impossible since $n-9$ lies in a 7-cycle of σ_3, while $n-7$ does not.

A similar approach works when $n \equiv 5 \bmod 6$. In this case σ_2 has two fixed points, one being the (unique) point fixed by $\langle \tau_1, \tau_2 \rangle \cong A_4$. This lies in an 8-cycle of σ_3, while the other lies in a 6-cycle of σ_3, and hence both points must be fixed by ρ. Then the same argument as in the case $n \equiv 4 \bmod 6$ shows that two points from a 3-cycle of σ_2 are interchanged by ρ, but one of them lies in the 8-cycle of σ_3 while the other does not. Hence no such ρ exists, and this completes the proof.

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References

1. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system I: the user language. J. Symb. Comput. **24**, 235–265 (1997)
2. Conder, M.D.E.: Chiral 4-polytopes of type {3, 3, k} with automorphism group A_n and S_n for small n. www.math.auckland.ac.nz/~conder/chiral4-polytopes/.
3. Conder, M., Hubard, I., Pisanski, T.: Constructions for chiral polytopes. J. Lond. Math. Soc. **77**, 115–129 (2008).

4. Hartley, M., Hubard, I., Leemans, D.: Two atlases of abstract chiral polytopes for small groups. Math. Contemp. **5**, 371–382 (2012)
5. Leemans, D., Vauthier, L.: An atlas of abstract regular polytopes for small groups. Aequationes Math. **72**, 313–320, (2006). See also www.math.auckland.ac.nz/~dleemans/polytopes/.
6. McMullen, P., Schulte, E.: Abstract Regular Polytopes, Encyclopedia of Math. and its Applications, vol. 92. Cambridge University Press, Cambridge (2002)
7. Pellicer, D.: A construction of higher rank chiral polytopes. Discrete Math. **310**, 1222–1237 (2010)
8. Pellicer, D.: Developments and open problems on chiral polytopes. Ars Math. Contemp. **5**, 333–354 (2012)
9. Rotman, J.J.: An Introduction to the Theory of Groups, 4th ed., Graduate Texts in Mathematics, vol. **48**, 4th edn. Springer, New York (1995)
10. Schulte, E., Weiss, A.I.: Chiral polytopes. In: Gritzmann, P., Sturmfels, B. (eds) Applied Geometry and Discrete Mathematics (The "Victor Klee Festschrift"), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., Vol. **4**, Amer. Math. Soc., pp. 493–516. Providence, RI, (1991)
11. Schulte, E., Weiss, A.I.: Chirality and projective linear groups. Discrete Math. **131**, 221–261 (1994)
12. Wielandt, H.: Finite Permutation Groups. Academic Press, New York (1964)
13. Wilson, R.: The Finite Simple Groups, Graduate Texts in Mathematics, vol. 251. Springer, London (2009)