Maximum scattered linear sets of pseudoregulus type and the Segre variety $\mathcal{S}_{n,n}$

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Abstract In this paper we study a family of scattered $\mathbb{F}_q$-linear sets of rank tn of the projective space $PG(2n − 1, q^t)$ ($n \geq 1$, $t \geq 3$), called of *pseudoregulus type*, generalizing results contained in Lavrauw and van de Voorde, Des. Codes Crypt. **20**(1) (2013) and in Marino et al. J. Combin. Theory, Ser. A 114:769–788 (2007). As an application, we characterize, in terms of the associated linear sets, some classical families of semifields: the Generalized Twisted Fields and the 2-dimensional Knuth semifields.

1 Introduction

In recent years the theory of linear sets has constantly increased its importance mainly because of its connection with other geometric objects such as blocking sets, translation ovoids and semifield planes (for an overview see [26]).

In this paper we study a family of maximum scattered $\mathbb{F}_q$-linear sets of the projective space $\Lambda = PG(2n − 1, q^t)$ ($n \geq 1$, $t \geq 3$). They were first introduced in [25] for

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$n = 2$ and $t = 3$, and further generalized in [18] for $n \geq 2$ and $t = 3$. If Λ is not a line, it is possible to associate with any such linear set a family of $(q^{nt} - 1)/(q^t - 1)$ pairwise disjoint lines admitting exactly two $(n-1)$-dimensional transversal spaces. Such a set of lines is called *pseudoregulus*, in analogy to the pseudoregulus of $PG(3, q^2)$ introduced by Freeman in [8]. For this reason, we refer to the relevant family of linear sets as *linear sets of pseudoregulus type*.

All maximum scattered $\mathbb{F}_q$-linear sets of $\Lambda = PG(2n - 1, q^3)$ ($n \geq 2$) are of pseudoregulus type and they are all equivalent under the action of the collineation group of Λ (see [25, Propositions 2.7 and 2.8] for $n = 2$ and [18, Sect. 3 and Theorem 4] for $n \geq 3$). In this paper, we characterize $\mathbb{F}_q$-linear sets of $PG(2n - 1, q^t)$ ($n \geq 1$, $t \geq 3$) of pseudoregulus type in terms of the associated projected subgeometry and we prove that, when $n > 1$ there are $\varphi(t)/2$ (where φ denotes Euler's phi function) orbits of such $\mathbb{F}_q$-linear sets under the action of the collineation group of $PG(2n - 1, q^t)$ (Theorems 3.11, 3.13). Also, we show that, when $t \geq 4$ and $q > 3$, there exist examples of maximum scattered $\mathbb{F}_q$-linear sets of $PG(2n - 1, q^t)$ ($n \geq 1$) which are not of pseudoregulus type (Example 4.6).

Finally, in Sect. 5 we first prove some geometric properties of the Segre Variety $\mathcal{S}_{n,n}$ of the projective space $\mathbb{P} = PG(n^2 - 1, q)$. These properties, together with the results contained in Sects. 3 and 4, allow us to describe and characterize the linear sets associated with some classical semifields: the Generalized Twisted Fields and the Knuth semifields 2-dimensional over their left nucleus (Propositions 5.5 and 5.9, Theorems 5.6 and 5.10).

2 Preliminary results

A $(t-1)$-spread of a projective space $PG(nt - 1, q)$ is a family $\mathcal{S}$ of mutually disjoint subspaces of dimension $t - 1$ such that each point of $PG(nt - 1, q)$ belongs to an element of $\mathcal{S}$. A first example of spread can be obtained in the following way. Let $PG(n - 1, q^t) = PG(V, \mathbb{F}_{q^t})$. Any point P of $PG(n - 1, q^t)$ defines a $(t-1)$-dimensional subspace $X(P)$ of the projective space $PG(nt - 1, q) = PG(V, \mathbb{F}_q)$ and $\mathcal{D} = \{X(P) : P \in PG(n - 1, q^t)\}$ is a spread of $PG(nt - 1, q)$, called a *Desarguesian spread* (see [27], Sect. 25).[1] If $n > 2$, the incidence structure $\Pi_{n-1}(\mathcal{D})$, whose points are the elements of $\mathcal{D}$ and whose lines are the $(2t - 1)$-dimensional subspaces of $PG(nt - 1, q)$ joining two distinct elements of $\mathcal{D}$, is isomorphic to $PG(n - 1, q^t)$. The structure $\Pi_{n-1}(\mathcal{D})$ is called the $\mathbb{F}_q$-*linear representation* of $PG(n - 1, q^t)$.

A Desarguesian $(t-1)$-spread of $PG(nt - 1, q)$ can also be obtained as follows (see [27, Sect. 27], [20] and [3]). Embed $\Sigma \simeq PG(nt - 1, q)$ in $\Sigma^* = PG(nt - 1, q^t)$ in such a way that Σ is the set of fixed points of a semilinear collineation Ψ of Σ^* of order t. Let $\Theta = PG(n - 1, q^t)$ be a subspace of Σ^* such that $\Theta, \Theta^\Psi, \ldots, \Theta^{\Psi^{t-1}}$ span the whole space Σ^*. If P is a point of Θ, $X^*(P) = \langle P, P^\Psi, \ldots, P^{\Psi^{t-1}} \rangle_{q^t}$ is a $(t-1)$-dimensional subspace of Σ^* defining a $(t-1)$-dimensional subspace $X(P) = X^*(P) \cap \Sigma$ of Σ. As P varies over the subspace Θ we get a set of $q^{t(n-1)} + q^{t(n-2)} +$

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[1]In [27] a Desarguesian spread is called "Sistema Grafico Elementare".

$\cdots + q^t + 1$ mutually disjoint $(t-1)$-dimensional subspaces of Σ. Such a set is denoted by $\mathcal{D} = \mathcal{D}(\Theta)$ and it turns out to be a Desarguesian $(t-1)$-spread of Σ. The $(n-1)$-dimensional subspaces $\Theta, \Theta^{\Psi}, \ldots, \Theta^{\Psi^{t-1}}$ are uniquely defined by the Desarguesian spread $\mathcal{D}$, i.e. $\mathcal{D}(\Theta) = \mathcal{D}(X)$ if and only if $X = \Theta^{\Psi^i}$ for some $i \in \{0, 1, \ldots, t-1\}$ and, following the terminology used by Segre [27, p. 29], we will refer to them as *director spaces* of $\mathcal{D}$ (see also [18, Theorem 3]).

Remark 2.1 Let $\mathcal{S}$ be a $(t-1)$-spread of $\Sigma = PG(nt-1, q)$ embedded in $\Sigma^* = PG(nt-1, q^t)$ in such a way that $\Sigma = Fix(\Psi)$ where Ψ is a semilinear collineation of Σ^* of order t. If H is an $(n-1)$-dimensional subspace of Σ^* such that

(i) $\Sigma^* = \langle H, H^{\Psi}, \ldots, H^{\Psi^{t-1}} \rangle_{q^t}$;
(ii) $X^* \cap H \neq \emptyset$ for each $(t-1)$-dimensional subspace X^* of Σ^* such that $X^* \cap \Sigma \in \mathcal{S}$;

then it is easy to see that $\mathcal{D}(H) = \mathcal{S}$, i.e. $\mathcal{S}$ is a Desarguesian spread and H is one of its director spaces.

2.1 Linear sets

Let $\Lambda = PG(r-1, q^t) = PG(V, \mathbb{F}_{q^t})$, $q = p^h$, p prime, and let L be a set of points of Λ. The set L is said to be an $\mathbb{F}_q$-*linear* set of Λ if it is defined by the non-zero vectors of an $\mathbb{F}_q$-vector subspace U of V, i.e., $L = L_U = \{\langle \mathbf{u} \rangle_{q^t} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}$. If $\dim_{\mathbb{F}_q} U = k$, we say that L has *rank* k. If $\Omega = PG(W, \mathbb{F}_{q^t})$ is a subspace of Λ and L_U is an $\mathbb{F}_q$-linear set of Λ, then $\Omega \cap L_U$ is an $\mathbb{F}_q$-linear set of Ω defined by the $\mathbb{F}_q$-vector subspace $U \cap W$, and we say that Ω has *weight* i in L_U if $\dim_{\mathbb{F}_q}(W \cap U) = i$ and we write $\omega_{L_U}(\Omega) = i$. If $L_U \neq \emptyset$, we have

$$|L_U| \leq q^{k-1} + q^{k-2} + \cdots + q + 1, \tag{1}$$

$$|L_U| \equiv 1 \pmod{q}. \tag{2}$$

For further details on linear sets see [26].

An $\mathbb{F}_q$-linear set L_U of Λ of rank k is *scattered* if all of its points have weight 1, or equivalently, if L_U has maximum size $q^{k-1} + q^{k-2} + \cdots + q + 1$.

In [4], the authors prove the following result on scattered linear sets.

Theorem 2.2 [4, Theorem 4.2] *A scattered $\mathbb{F}_q$-linear set of $PG(r-1, q^t)$ has rank at most $rt/2$.*

A scattered $\mathbb{F}_q$-linear set L of $PG(r-1, q^t)$ of maximum rank $rt/2$ is called a *maximum scattered* linear set.

Remark 2.3 Note that if L_U is a scattered $\mathbb{F}_q$-linear set of $PG(r-1, q^t)$ of rank k containing more than one point, then by $|L_U| = q^{k-1} + q^{k-2} + \cdots + q + 1$ and (2), L_U is not an $\mathbb{F}_{q^s}$-linear set for each subfield $\mathbb{F}_{q^s}$ of $\mathbb{F}_{q^t}$ properly containing $\mathbb{F}_q$. In other words, a scattered $\mathbb{F}_q$-linear set L of rank $k > 1$ of $PG(r-1, q^t)$ is not a linear set of rank $n < k$. Also, by Theorem 2.2, a maximum scattered linear set of $PG(r-1, q^t)$ spans the whole space.

If $\dim_{\mathbb{F}_q} U = \dim_{\mathbb{F}_{q^t}} V = r$ and $\langle U \rangle_{q^t} = V$, then the $\mathbb{F}_q$-linear set L_U is a *subgeometry* of $PG(V, \mathbb{F}_{q^t}) = PG(r-1, q^t)$ isomorphic to $PG(r-1, q)$. If $t = 2$, then L_U is a Baer subgeometry of $PG(r-1, q^2)$.

In [23], the authors give the following characterization of $\mathbb{F}_q$-linear sets. Let $\Sigma = PG(k-1, q)$ be a subgeometry of $\Sigma^* = PG(k-1, q^t)$, let Γ be a $(k-r-1)$-dimensional subspace of Σ^* disjoint from Σ and let $\Lambda = PG(r-1, q^t)$ be an $(r-1)$-dimensional subspace of Σ^* disjoint from Γ. Denote by

$$L = \left\{ \langle \Gamma, P \rangle_{q^t} \cap \Lambda : P \in \Sigma \right\}$$

the projection of Σ from Γ to Λ. We call Γ and Λ, respectively, the *center* and the *axis* of the projection. Denote by $p_{\Gamma, \Lambda}$ the map from Σ to L defined by $P \mapsto \langle \Gamma, P \rangle_{q^t} \cap \Lambda$ for each point P of Σ. By definition $p_{\Gamma, \Lambda}$ is surjective and $L = p_{\Gamma, \Lambda}(\Sigma)$.

Theorem 2.4 [23, Theorems 1 and 2] *If L is a projection of $\Sigma = PG(k-1, q)$ to $\Lambda = PG(r-1, q^t)$, then L is an $\mathbb{F}_q$-linear set of Λ of rank k and $\langle L \rangle_{q^t} = \Lambda$. Conversely, if L is an $\mathbb{F}_q$-linear set of Λ of rank k and $\langle L \rangle_{q^t} = \Lambda$, then either L is a subgeometry of Λ or for each $(k-r-1)$-dimensional subspace Γ of $\Sigma^* = PG(k-1, q^t)$ disjoint from Λ there exists a subgeometry Σ of Σ^* disjoint from Γ such that $L = p_{\Gamma, \Lambda}(\Sigma)$.*

Also, in [17] it has been proven:

Theorem 2.5 [17, Theorem 3] *Let $L_1 = p_{\Gamma_1, \Lambda}(\Sigma_1)$ and $L_2 = p_{\Gamma_2, \Lambda}(\Sigma_2)$ be two $\mathbb{F}_q$-linear sets of rank k of $\Lambda = \langle L_1 \rangle_{q^t} = \langle L_2 \rangle_{q^t}$, and suppose that L_i is not a linear set of rank $n < k$. Then L_1 and L_2 are projectively equivalent if and only if there exists $\beta \in Aut(\Sigma^*)$ such that $\Sigma_1^\beta = \Sigma_2$ and $\Gamma_1^\beta = \Gamma_2$.*

Remark 2.6 Note that, if $S_\Gamma = \Sigma^* / \Gamma \simeq PG(r-1, q^t)$ denotes the $(r-1)$-dimensional space obtained as quotient geometry of Σ^* over Γ, then the set $L_{\Gamma, \Sigma}$ of the $(k-r)$-dimensional subspaces of Σ^* containing Γ and with non-empty intersection with Σ is an $\mathbb{F}_q$-linear set of the space S_Γ isomorphic to $L = p_{\Gamma, \Lambda}(\Sigma)$, for each $(r-1)$-dimensional space Λ disjoint from Γ. This means that $p_{\Gamma, \Lambda}(\Sigma)$ is isomorphic to the $\mathbb{F}_q$-linear set $\{ P + \Gamma : P \in \Sigma \}$ of the quotient space S_Γ, and hence it does not depend on the choice of the axis Λ, and we will simply denote it as $p_\Gamma(\Sigma)$.

3 Maximum scattered $\mathbb{F}_q$-linear sets of pseudoregulus type in $PG(2n-1, q^t)$

In this section we study a family of maximum scattered linear sets to which a geometric structure, called *pseudoregulus*, can be associated. This generalizes results contained in [15, 25] and [18].

Definition 3.1 Let $L = L_U$ be a scattered $\mathbb{F}_q$-linear set of $\Lambda = PG(2n-1, q^t)$ of rank tn, $t, n \geq 2$. We say that L is of *pseudoregulus type* if

(i) there exist $m = \frac{q^{nt}-1}{q^t-1}$ pairwise disjoint lines of Λ, say $s_1, s_2, \ldots, s_m$, such that

$$w_L(s_i) = t, \quad \text{i.e. } |L \cap s_i| = q^{t-1} + q^{t-2} + \cdots + q + 1 \ \forall i = 1, \ldots, m;$$

(ii) there exist exactly two $(n-1)$-dimensional subspaces T_1 and T_2 of Λ disjoint from L such that $T_j \cap s_i \neq \emptyset$ for each $i = 1, \ldots, m$ and $j = 1, 2$.

We call the set of lines $\mathcal{P}_L = \{s_i : i = 1, \ldots, m\}$ the $\mathbb{F}_q$-*pseudoregulus* (or simply *pseudoregulus*) of Λ associated with L and we refer to T_1 and T_2 as *transversal spaces* of $\mathcal{P}_L$ (or transversal spaces of L). Note that, by Remark 2.3, L spans the whole space and hence the transversal spaces T_1 and T_2 are disjoint. When $t = n = 2$, these objects already appeared in [8], where the term "pseudoregulus" was introduced for the first time.

If L is a scattered $\mathbb{F}_q$-linear set of the projective space $PG(r-1, q^t)$, then by Theorem 2.2 every h-dimensional subspace X_h of $PG(r-1, q^t)$ intersects L in a linear set of rank at most $(h+1)t/2$, i.e. the weight of X_h in L is at most $(h+1)t/2$. So we get:

Proposition 3.2 *If $t \geq 3$, then the lines of weight t in a scattered $\mathbb{F}_q$-linear set L of $PG(r-1, q^t)$ of rank h are pairwise disjoint and hence the number of such lines is at most $(q^h - 1)/(q^t - 1)$.*

Proof If ℓ and ℓ' are distinct lines of $PG(r-1, q^t)$ of weight t in L and $\ell \cap \ell' \neq \emptyset$, then the plane π joining ℓ and ℓ' has weight at least $2t - 1$ in L. On the other hand, since $\pi \cap L$ is a scattered $\mathbb{F}_q$-linear set of the plane π, by Theorem 2.2, we also have that the weight of π in L is at most $3t/2$; so we get $t \leq 2$, a contradiction. Hence, the number of lines of $PG(r-1, q^t)$ having weight t in L is at most

$$\frac{q^{h-1} + q^{h-2} + \cdots + q + 1}{q^{t-1} + q^{t-2} + \cdots + q + 1} = \frac{q^h - 1}{q^t - 1}. \qquad \square$$

As a consequence of Proposition 3.2 we get:

Corollary 3.3 *If L is an $\mathbb{F}_q$-linear set of pseudoregulus type of the projective space $\Lambda = PG(2n-1, q^t)$, with $t \geq 3$, then the associated pseudoregulus is the set of all the lines of Λ of weight t in L. Hence, the pseudoregulus associated with L and its transversal spaces are uniquely determined.*

Remark 3.4 If $t = 2$, a scattered $\mathbb{F}_q$-linear set L of rank $2n$ of the projective space Λ is a Baer subgeometry isomorphic to $PG(2n-1, q)$ and each line spread of L produces a set of lines of Λ satisfying (i) of Definition 3.1 and each Desarguesian line spread, say $\mathcal{D}$, of L gives a set of lines of Λ satisfying both (i) and (ii) of Definition 3.1. In this last case the transversal spaces are the two $(n-1)$-dimensional director spaces of the Desarguesian spread $\mathcal{D}$. So each maximum scattered $\mathbb{F}_q$-linear

set L of the projective space $\Lambda = PG(2n - 1, q^2)$ is of pseudoregulus type, but in this case the associated pseudoregulus is not uniquely defined; also, since these linear sets are Baer subgeometries, they are all projectively equivalent. The same happens in the next case, i.e. each maximum scattered $\mathbb{F}_q$-linear set L of the projective space $\Lambda = PG(2n - 1, q^3)$ ($n \geq 2$) is of pseudoregulus type and they are all projectively equivalent (see [25] and [18, Theorem 4, Lemmas 5, 7, Theorem 10]). Whereas, when $t > 3$: (i) there exist maximum scattered $\mathbb{F}_q$-linear sets of the projective space $\Lambda = PG(2n - 1, q^t)$ which are not of pseudoregulus type (see Example 4.6); (ii) $\mathbb{F}_q$-linear sets of pseudoregulus type, in general, are not all projectively equivalent (see Theorem 3.13).

The construction presented in [15, Sect. 2] when $t = 3$ and $n = 2$, can be generalized providing a simple way to construct scattered $\mathbb{F}_q$-linear sets of pseudoregulus type of $PG(2n - 1, q^t)$ for any $t, n \geq 2$.

Let $\Lambda = PG(V, \mathbb{F}_{q^t})$, where $V = V(2n, \mathbb{F}_{q^t}) = U_1 \oplus U_2$, with $\dim U_1 = \dim U_2 = n$ and let $T_1 = PG(U_1, \mathbb{F}_{q^t})$ and $T_2 = PG(U_2, \mathbb{F}_{q^t})$. Now, let Φ_f be a semilinear collineation between T_1 and T_2, induced by the invertible semilinear map $f : U_1 \to U_2$, having as companion automorphism an element $\sigma \in Aut(\mathbb{F}_{q^t})$ such that $Fix(\sigma) = \mathbb{F}_q$. Then, for each $\rho \in \mathbb{F}_{q^t}^*$, the set

$$W_{\rho,f} = \left\{ \underline{u} + \rho f(\underline{u}) : \underline{u} \in U_1 \right\}$$

is an $\mathbb{F}_q$-vector subspace of V of dimension tn and it is not difficult to see that $L = L_{W_{\rho,f}}$ is an $\mathbb{F}_q$-linear set of Λ of rank tn of scattered type.[2] Also, we can see that for each line s_P joining the points $P = \langle \underline{u} \rangle_{q^t}$ and $P^{\Phi_f} = \langle f(\underline{u}) \rangle_{q^t}$ of T_1 and T_2 respectively, we have that

$$s_P \cap L = \left\{ \langle \lambda \underline{u} + \lambda^\sigma f(\underline{u}) \rangle_{q^t} : \lambda \in \mathbb{F}_{q^t}^* \right\}. \tag{3}$$

Hence, the line s_P, for each $P \in T_1$, has weight t in L. Also, if $P \neq Q$, the lines s_P and s_Q are disjoint. This means that L satisfies (i) of Definition 3.1. Moreover, it is clear that $T_1 \cap s_P = \{P\}$ and $T_2 \cap s_P = \{P^{\Phi_f}\}$ for each $P \in T_1$ and that $T_1 \cap L = T_2 \cap L = \emptyset$. In addition, T_1 and T_2 are the only $(n-1)$-dimensional transversal spaces of the lines s_P. Indeed, if $T = PG(U, \mathbb{F}_{q^t}) = PG(n - 1, q^t)$ were another transversal space, then T would be disjoint from T_1 and T_2 and, since $T \cap s_P \neq \emptyset$ for each $P \in T_1$, we have that

$$U = \left\{ \underline{u} + \lambda_{\underline{u}} f(\underline{u}) : \underline{u} \in U_1 \right\},$$

where $\lambda_{\underline{u}} \in \mathbb{F}_{q^t}$ for each $\underline{u} \in U_1$ and $\lambda_{\underline{u}} \neq 0$ for each $\underline{u} \neq \underline{0}$. Now, since U is an $\mathbb{F}_{q^t}$-subspace of V, the map f turns out to be an $\mathbb{F}_{q^t}$-linear map of V, a contradiction. So, also L satisfies (ii) of Definition 3.1 and hence $L_{W_{\rho,f}}$ is a maximum scattered $\mathbb{F}_q$-linear set of Λ of pseudoregulus type and $\mathcal{P}_L = \{s_P : P \in T_1\}$ is its associated pseudoregulus. Hence we have proved the following:

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[2]More generally, if $Fix(\sigma) = \mathbb{F}_{q'}$, then $L_{W_{\rho,f}}$ is an $\mathbb{F}_{q'}$-linear set of Λ of scattered type.

Theorem 3.5 *Let $T_1 = PG(U_1, \mathbb{F}_{q^t})$ and $T_2 = PG(U_2, \mathbb{F}_{q^t})$ be two disjoint $(n-1)$-dimensional subspaces of $\Lambda = PG(V, \mathbb{F}_{q^t}) = PG(2n-1, q^t)$ $(t > 1)$ and let Φ_f be a semilinear collineation between T_1 and T_2 having as companion automorphism an element $\sigma \in Aut(\mathbb{F}_{q^t})$ such that $Fix(\sigma) = \mathbb{F}_q$. Then, for each $\rho \in \mathbb{F}_{q^t}^*$, the set*

$$L_{\rho, f} = \left\{ \langle \underline{u} + \rho f(\underline{u}) \rangle_{q^t} : \underline{u} \in U_1 \setminus \{\underline{0}\} \right\}$$

is an $\mathbb{F}_q$-linear set of Λ of pseudoregulus type whose associated pseudoregulus is $\mathcal{P}_{L_{\rho, f}} = \{ \langle P, P^{\Phi_f} \rangle_{q^t} : P \in T_1 \}$, with transversal spaces T_1 and T_2.

Remark 3.6 Note that, with the notation of the previous theorem, if $L_{\rho, f} \cap L_{\rho', f} \neq \emptyset$, then $L_{\rho, f} = L_{\rho', f}$ and this happens if and only if $N_{q^t/q}(\rho) = N_{q^t/q}(\rho')$.[3] Hence T_1, T_2 and the collineation Φ_f define a set of $q-1$ mutually disjoint linear sets of pseudoregulus type admitting the same associated pseudoregulus $\mathcal{P}$ and covering, together with the transversal spaces T_1 and T_2, the point set of $\mathcal{P}$.

Up to projective equivalence, the scattered $\mathbb{F}_q$-linear sets $L_{\rho, f}$ only depend on the field automorphism associated with f. Indeed, we have:

Theorem 3.7 *The $\mathbb{F}_q$-linear sets of $\Lambda = PG(2n-1, q^t)$ $(n \geq 2, t \geq 2)$ $L_{\rho, f}$ and $L_{\rho', g}$ are $P\Gamma L$-equivalent if and only if $\sigma_f = \sigma_g^{\pm 1}$ where σ_f and σ_g are the automorphisms associated with f and g, respectively.*

Proof If $t = 2$, the assertion follows from Remark 3.4. Let $t > 2$. Then, by Corollary 3.3 the transversal spaces associated with $L_{\rho, f}$ and $L_{\rho', g}$ are uniquely determined. Hence, up to the action of $PGL(2n, q^t)$, we may assume that the transversal spaces of $L_{\rho, f}$ and $L_{\rho', g}$ are the same. Also, since $L_{\rho, f} = L_{1, \rho^{-1} f}$, we may consider scattered $\mathbb{F}_q$-linear sets of the form $L_{1, f}$. Suppose that $L_{1, f}$ and $L_{1, g}$ are projectively equivalent; i.e., there exists a collineation ϕ_F of $\Lambda = PG(2n-1, q^t) = PG(V, \mathbb{F}_{q^t})$ defined by an invertible semilinear map F of the vector space V having companion automorphism τ, such that $\phi_F(L_{1, f}) = L_{1, g}$. By Corollary 3.3 $\phi_F(\{T_1, T_2\}) = \{T_1, T_2\}$. Precisely, either $\phi_F(T_i) = T_i$, $i = 1, 2$ or $\phi_F(T_i) = T_j$, $\{i, j\} = \{1, 2\}$.

In the first case, we have that $F(U_1) = U_1$ and $F(U_2) = U_2$. Since $\phi_F(L_{1, f}) = L_{1, g}$, for each $\underline{u} \in U_1$ we have $\phi_F(\langle \underline{u} + f(\underline{u}) \rangle_{q^t}) = \langle F(\underline{u} + f(\underline{u})) \rangle_{q^t} \in L_{1, g}$; in other words, for each vector $\underline{u} \in U_1$ we have

$$F\left(\underline{u} + f(\underline{u})\right) = \lambda_{\underline{u}}\left(\underline{u}' + g(\underline{u}')\right) = \lambda_{\underline{u}}\underline{u}' + \lambda_{\underline{u}} g(\underline{u}'), \tag{4}$$

where $\underline{u}' \in U_1$ and $\lambda_{\underline{u}} \in \mathbb{F}_{q^t}^*$ if $\underline{u} \neq \underline{0}$. On the other hand, we also have $F(\underline{u} + f(\underline{u})) = F(\underline{u}) + F(f(\underline{u}))$, with $F(\underline{u}) \in U_1$ and $F(f(\underline{u})) \in U_2$. Taking this fact into account, since $V = U_1 \oplus U_2$, Eq. (4) implies that $F(\underline{u}) = \lambda_{\underline{u}}\underline{u}'$ and $F(f(\underline{u})) = \lambda_{\underline{u}} g(\underline{u}')$. Hence

$$F\left(f(\underline{u})\right) = \lambda_{\underline{u}} g\left(\frac{F(\underline{u})}{\lambda_{\underline{u}}}\right) = \frac{\lambda_{\underline{u}}}{\lambda_{\underline{u}}^{\sigma_g}} g\left(F(\underline{u})\right). \tag{5}$$

Let now $\underline{u}$ and $\underline{v}$ be two non-zero vectors of U_1. If $\underline{u}$ and $\underline{v}$ are $\mathbb{F}_{q^t}$-independent, by Eq. (5) we have

$$F\big(f(\underline{u}+\underline{v})\big) = \lambda_{\underline{u}+\underline{v}}^{1-\sigma_g} g\big(F(\underline{u})\big) + \lambda_{\underline{u}+\underline{v}}^{1-\sigma_g} g\big(F(\underline{v})\big).$$

Also, we get

$$F\big(f(\underline{u}+\underline{v})\big) = F\big(f(\underline{u})\big) + F\big(f(\underline{v})\big) = \lambda_{\underline{u}}^{1-\sigma_g} g\big(F(\underline{u})\big) + \lambda_{\underline{v}}^{1-\sigma_g} g\big(F(\underline{v})\big).$$

Hence $\lambda_{\underline{u}}^{1-\sigma_g} = \lambda_{\underline{u}+\underline{v}}^{1-\sigma_g} = \lambda_{\underline{v}}^{1-\sigma_g}$, which implies $\lambda_{\underline{u}}/\lambda_{\underline{v}} \in \mathbb{F}_q$ since $Fix(\sigma_g) = \mathbb{F}_q$.

On the other hand, if $\underline{u}$ and $\underline{v}$ are $\mathbb{F}_{q^t}$-dependent, choosing a vector $\underline{w} \in U_1$, such that $\underline{w} \notin \langle \underline{u} \rangle_{q^t}$, and arguing as above we get $\lambda_{\underline{u}}/\lambda_{\underline{v}} \in \mathbb{F}_q$.

This means that for each $\underline{u}, \underline{v} \in U_1$ there exists an element $\beta_{\underline{u},\underline{v}} \in \mathbb{F}_q$ such that $\lambda_{\underline{u}} = \beta_{\underline{u},\underline{v}}\lambda_{\underline{v}}$.

Let $\underline{u} \in U_1$, $\underline{u} \neq \underline{0}$. Then, by Eq. (5), we get

$$F\big(f(\alpha\underline{u})\big) = \alpha^{\sigma_f \tau} \lambda_{\underline{u}}^{1-\sigma_g} g\big(F(\underline{u})\big) \tag{6}$$

for each $\alpha \in \mathbb{F}_{q^t}$. On the other hand, again by Eq. (5), for each $\alpha \in \mathbb{F}_{q^t}$ we have

$$F\big(f(\alpha\underline{u})\big) = \lambda_{\alpha\underline{u}}^{1-\sigma_g} \alpha^{\tau\sigma_g} g\big(F(\underline{u})\big), \tag{7}$$

where $\lambda_{\alpha\underline{u}} \in \mathbb{F}_{q^t}$. Since $\lambda_{\underline{u}}/\lambda_{\alpha\underline{u}} \in \mathbb{F}_q$, we have $\lambda_{\alpha\underline{u}}^{1-\sigma_g} = \lambda_{\underline{u}}^{1-\sigma_g}$. Taking into account this fact, by Eqs. (6) and (7) we get $\alpha^{\sigma_f \tau} = \alpha^{\tau\sigma_g}$ for each $\alpha \in \mathbb{F}_{q^t}$, which implies $\sigma_f = \sigma_g$.

In the second case we have $F(U_1) = U_2$ and $F(U_2) = U_1$ and arguing as in the previous case we get $\sigma_g = \sigma_f^{-1}$.

Conversely, suppose that $\sigma_f = \sigma_g$ and let ϕ_F be the collineation of Λ defined by the map F of the vector space $V = U_1 \oplus U_2$ defined as follows:

$$F(\underline{u}_1 + \underline{u}_2) = \underline{u}_1 + g\big(f^{-1}(\underline{u}_2)\big),$$

where $\underline{u}_1 \in U_1$ and $\underline{u}_2 \in U_2$. Then $\phi_F(L_{1,f}) = L_{1,g}$. On the other hand, if $\sigma_f = \sigma_g^{-1}$, the collineation ϕ_F of Λ defined by the following map F of $V = U_1 \oplus U_2$

$$F(\underline{u}_1 + \underline{u}_2) = g(\underline{u}_1) + f^{-1}(\underline{u}_2)$$

sends $L_{1,f}$ to $L_{1,g}$. This concludes the proof. $\square$

As a consequence of Theorem 3.7 we have the following:

Corollary 3.8 *In the projective space $\Lambda = PG(2n-1, q^t)$ ($n \geq 2$, $t \geq 3$) there are $\varphi(t)/2$ orbits of scattered $\mathbb{F}_q$-linear sets of Λ of rank tn of type $L_{\rho,f}$ under the action of the collineation group of Λ.*

Proof By the previous theorem, two linear sets $L_{\rho,f}$ and $L_{\rho,g}$ are $P\Gamma L$-equivalent if and only if either $\sigma_f = \sigma_g$ or $\sigma_f = \sigma_g^{-1}$. So the number of orbits of such $\mathbb{F}_q$-linear sets

under the action of $P\Gamma L(2n, q^t)$ is $\chi/2$ where χ is the number of $\mathbb{F}_q$-automorphisms σ of $\mathbb{F}_{q^t}$ with $Fix(\sigma) = \mathbb{F}_q$. This means that χ is the number of generators of the group $Gal(\mathbb{F}_{q^t} : \mathbb{F}_q)$, i.e. $\chi = \varphi(t)$ is the number of positive integers less than t and coprime with t. □

In what follows we will show that each scattered $\mathbb{F}_q$-linear set of pseudoregulus type can be obtained as in Theorem 3.5. Let us start by proving the following:

Theorem 3.9 *Let* $\Sigma \simeq PG(tn − 1, q)$ *be a subgeometry of* $\Sigma^* = PG(V, \mathbb{F}_{q^t}) = PG(tn − 1, q^t)$ *defined by the semilinear collineation* Ψ *of order t of* Σ^*. *Also, let* $\mathcal{D}$ *be a Desarguesian* $(t − 1)$-*spread of* Σ *and denote by* Θ *a director subspace of* $\mathcal{D}$. *Then, for each pair of integers* $i_1, i_2 \in \{0, 1, \ldots, t − 1\}$ *such that* $gcd(i_2 − i_1, t) = 1$, *the linear set obtained by projecting* Σ *from the subspace* $\Gamma = \langle \Theta^{\Psi^i} : i \neq i_1, i_2 \rangle_{q^t}$ *to* $\Lambda = \langle \Theta^{\Psi^{i_1}}, \Theta^{\Psi^{i_2}} \rangle_{q^t}$ *is a scattered* $\mathbb{F}_q$-*linear set of type* $L_{\rho, f}$ *described in Theorem* 3.5.

Proof Since $Fix(\Psi) = \Sigma$, the collineation Ψ is induced by an invertible semilinear map $g : V \longrightarrow V$ of order $t > 1$, with companion automorphism σ such that $Fix(\sigma) = \mathbb{F}_q$. Since $\Theta = PG(U, \mathbb{F}_{q^t})$ is a director subspace of the Desarguesian spread $\mathcal{D}$, we have that $\mathcal{D} = \mathcal{D}(\Theta)$ and $\Sigma^* = \langle \Theta, \Theta^\Psi, \ldots, \Theta^{\Psi^{t-1}} \rangle_{q^t}$. Let $i_1, i_2 \in \{0, 1, \ldots, t − 1\}$ be such that $gcd(i_2 − i_1, t) = 1$ and let $\Gamma = \langle \Theta^{\Psi^i} : i \neq i_1, i_2 \rangle_{q^t}$ and $\Lambda = \langle \Theta^{\Psi^{i_1}}, \Theta^{\Psi^{i_2}} \rangle_{q^t}$. Then dim $\Gamma = n(t − 2) − 1$, dim $\Lambda = 2n − 1$, $\Gamma \cap \Lambda = \Gamma \cap \Sigma = \emptyset$ and hence we can project the subgeometry Σ from the center Γ to the axis Λ. By Theorem 2.4, the projection $L = p_\Gamma(\Sigma)$ is an $\mathbb{F}_q$-linear set of Λ of rank tn and $\langle L \rangle_{q^t} = \Lambda$. Also, it is easy to see that

$$\Sigma = Fix(\Psi)$$
$$= \left\{ \langle \underline{u} + g(\underline{u}) + g^2(\underline{u}) + \cdots + g^{t-1}(\underline{u}) \rangle_{q^t} : \underline{u} \in U \setminus \{\underline{0}\} \right\},$$

and hence the projection of Σ from Γ into Λ is

$$L = p_\Gamma(\Sigma) = \left\{ \langle g^{i_1}(\underline{u}) + g^{i_2}(\underline{u}) \rangle_{q^t} : \underline{u} \in U \setminus \{\underline{0}\} \right\} = \left\{ \langle \underline{v} + f(\underline{v}) \rangle_{q^t} : \underline{v} \in g^{i_1}(U) \setminus \{\underline{0}\} \right\},$$

where $f : \underline{v} \in g^{i_1}(U) \mapsto g^{i_2 − i_1}(\underline{v}) \in g^{i_2}(U)$. Since f is an invertible semilinear map whose companion automorphism is $\sigma^{i_2 − i_1}$ and $gcd(i_2 − i_1, t) = 1$, we have that $Fix(\sigma^{i_2 − i_1}) = \mathbb{F}_q$. So, by Theorem 3.5, L is a scattered $\mathbb{F}_q$-linear set of Λ of pseudoregulus type with $\Theta^{\Psi^{i_1}}$ and $\Theta^{\Psi^{i_2}}$ as transversal spaces. □

Remark 3.10 Note that, if $gcd(i_2 − i_1, t) = s$, in the previous proof we have $Fix(\sigma^{i_2 − i_1}) = \mathbb{F}_{q^s}$, and hence the linear set L obtained projecting Σ from $\Gamma = \langle \Theta^{\Psi^i} : i \neq i_1, i_2 \rangle_{q^t}$, is an $\mathbb{F}_{q^s}$-linear set.

Recall that, by Theorem 2.4, every $\mathbb{F}_q$-linear set L of Λ spanning the whole space can be obtained projecting a suitable subgeometry. If L is of pseudoregulus type we can prove the following:

Theorem 3.11 *Put $\Lambda = PG(2n-1, q^t)$, $\Sigma^* = PG(tn-1, q^t)$ and $\Sigma = Fix(\Psi) \simeq PG(tn-1, q)$ where Ψ is a semilinear collineation of Σ^* of order t. Let L be a scattered $\mathbb{F}_q$-linear set of Λ of pseudoregulus type with associated pseudoregulus $\mathcal{P}$ obtained by projecting Σ into Λ from an $(n(t-2)-1)$-dimensional subspace Γ disjoint from Σ. Then*

(i) *the set*

$$\mathcal{D}_L = \left\{ \langle \Gamma, s \rangle_{q^t} \cap \Sigma : s \in \mathcal{P} \right\}$$

   is a Desarguesian $(t-1)$-spread of Σ;

(ii) *there exists a director space $\bar{\Theta}$ of $\mathcal{D}_L$ such that*

$$\Gamma = \left\langle \bar{\Theta}, \bar{\Theta}^\tau, \ldots, \bar{\Theta}^{\tau^{t-3}} \right\rangle_{q^t}, \tag{8}$$

   where $\tau = \Psi^m$ with $\gcd(m, t) = 1$.

Proof Since each line s of $\mathcal{P}$ has weight t in L, it is clear that $\langle \Gamma, s \rangle_{q^t} \cap \Sigma$ is a $(t-1)$-dimensional subspace of Σ. Also, since the lines of $\mathcal{P}$ are pairwise disjoint and $|\mathcal{P}| = \frac{q^{nt}-1}{q^t-1}$, the set $\mathcal{D}_L$ in (i) is a $(t-1)$-spread of Σ.

Denote by T_1 and T_2 the transversal spaces of $\mathcal{P}$ and let K_1 be the $(n(t-1)-1)$-dimensional subspace of Σ^* joining Γ and T_1. Since L is disjoint from T_1, we have that $K_1 \cap \Sigma = \emptyset$, and hence $K_1 \cap K_1^\Psi \cap \cdots \cap K_1^{\Psi^{t-1}} = \emptyset$. So $\Theta = K_1 \cap K_1^\Psi \cap \cdots \cap K_1^{\Psi^{t-2}}$ is an $(n-1)$-dimensional subspace of Σ^* and $\Sigma^* = \langle \Theta, \Theta^\Psi, \ldots, \Theta^{\Psi^{t-1}} \rangle_{q^t}$.

Now, for each line s of $\mathcal{P}$, let $X_s = \langle \Gamma, s \rangle_{q^t} \cap \Sigma$ be the corresponding element of the spread $\mathcal{D}_L$ and denote by X_s^* the $(t-1)$-dimensional subspace of Σ^* such that $X_s = X_s^* \cap \Sigma$, i.e. X_s^* is the $\mathbb{F}_{q^t}$-extension of X_s in Σ^*. So $X_s^* \subset \langle \Gamma, s \rangle_{q^t}$ and since X_s^* intersects Σ in a subspace of the same dimension, we have that $(X_s^*)^\Psi = X_s^*$ (see [20, Lemma 1]). Also, let P be the point $s \cap T_1$. Then $\langle \Gamma, P \rangle_{q^t}$ is a hyperplane of $\langle \Gamma, s \rangle_{q^t}$ and hence $H_s = \langle \Gamma, P \rangle_{q^t} \cap X_s^*$ is a $(t-2)$-dimensional subspace of X_s^*. Since $H_s \subseteq K_1$, we have that H_s is disjoint from Σ and hence $H_s \cap H_s^\Psi \cap \cdots \cap H_s^{\Psi^{t-1}} = \emptyset$. So $H_s, H_s^\Psi, \ldots, H_s^{\Psi^{t-1}}$ are t independent hyperplanes of X_s^*. This implies that $H_s \cap H_s^\Psi \cap \cdots \cap H_s^{\Psi^{t-2}}$ is a point, say R_s, of X_s^*. So

$$R_s \in X_s^* \cap \left(H_s \cap H_s^\Psi \cap \cdots \cap H_s^{\Psi^{t-2}} \right) \subset X_s^* \cap \left(K_1 \cap K_1^\Psi \cap \cdots \cap K_1^{\Psi^{t-2}} \right) = X_s^* \cap \Theta$$

for each $s \in \mathcal{P}$. By Remark 2.1 we get that $\mathcal{D}_L$ is a Desarguesian spread of Σ with Θ as a director space. Also, $\Theta^{\Psi^i} \subset K_1$ for each $i \neq 1$ and hence $K_1 = \langle \Theta^{\Psi^i} : i \neq 1 \rangle_{q^t}$ and $K_1 \cap \Theta^\Psi = \emptyset$.

Similarly, if $K_2 = \langle \Gamma, T_2 \rangle_{q^t}$, we get that $K_2 \cap K_2^\Psi \cap \cdots \cap K_2^{\Psi^{t-2}}$ is a director space of the Desarguesian spread $\mathcal{D}_L$ and hence there exists $m \in \{1, 2, \ldots, t-1\}$ such that $K_2 \cap K_2^\Psi \cap \cdots \cap K_2^{\Psi^{t-2}} = \Theta^{\Psi^m}$ (see [18, Theorem 3]). So $\Theta^{\Psi^i} \subset K_2$ for each $i \neq m+1$ and hence $K_2 = \langle \Theta^{\Psi^i} : i \neq m+1 \rangle_{q^t}$ and $K_2 \cap \Theta^{\Psi^{m+1}} = \emptyset$. This means that

$$\Gamma = K_1 \cap K_2 = \left\langle \Theta^{\Psi^i} : i \neq 1, m+1 \right\rangle_{q^t}.$$

So, if $\Psi^m = \tau$ and $\bar{\Theta} = \Theta^{\Psi^{2m+1}}$, we get (8) of (ii). Finally, since L is a scattered $\mathbb{F}_q$-linear set, by Theorem 3.9, Remarks 3.10 and 2.3, we have that $gcd(t, m) = 1$. $\square$

By Theorems 3.9 and 3.11 we have the following:

Theorem 3.12 *Each $\mathbb{F}_q$-linear set of $PG(2n - 1, q^t)$ of pseudoregulus type is of the form $L_{\rho, f}$ described in Theorem 3.5.*

Finally, by Theorem 3.12 and by Corollary 3.8 we can state the following classification result which generalizes [18, Theorem 4].

Theorem 3.13 *In the projective space $\Lambda = PG(2n - 1, q^t)$ ($n \geq 2$, $t \geq 3$) there are $\varphi(t)/2$ orbits of maximum scattered $\mathbb{F}_q$-linear sets of pseudoregulus type under the action of the collineation group of Λ.*

4 A class of maximum scattered $\mathbb{F}_q$-linear sets of $PG(1, q^t)$

The arguments proving Theorem 3.5 can be exploited to construct a class of maximum scattered $\mathbb{F}_q$-linear sets of the projective line $\Lambda = PG(V, \mathbb{F}_{q^t}) = PG(1, q^t)$ with a structure resembling that of an $\mathbb{F}_q$-linear set of $PG(2n - 1, q^t)$ ($n, t \geq 2$) of pseudoregulus type. To this aim let $P_1 = \langle \underline{w} \rangle_{q^t}$ and $P_2 = \langle \underline{v} \rangle_{q^t}$ be two distinct points of Λ and let τ be an $\mathbb{F}_q$-automorphism of $\mathbb{F}_{q^t}$ such that $Fix(\tau) = \mathbb{F}_q$; then for each $\rho \in \mathbb{F}_{q^t}^*$ the set

$$W_{\rho, \tau} = \left\{ \lambda \underline{w} + \rho \lambda^\tau \underline{v} : \lambda \in \mathbb{F}_{q^t} \right\},$$

is an $\mathbb{F}_q$-vector subspace of V of dimension t and $L_{\rho, \tau} := L_{W_{\rho, \tau}}$ is a scattered $\mathbb{F}_q$-linear set of Λ.

Definition 4.1 We call the linear sets $L_{\rho, \tau}$ of *pseudoregulus type* and we refer to the points P_1 and P_2 as *transversal points* of $L_{\rho, \tau}$.

If $L_{\rho, \tau} \cap L_{\rho', \tau} \neq \emptyset$, then $L_{\rho, \tau} = L_{\rho', \tau}$. Note that $L_{\rho, \tau} = L_{\rho', \tau}$ if and only if $N_{q^t/q}(\rho) = N_{q^t/q}(\rho')$; so P_1, P_2 and the automorphism τ define a set of $q - 1$ mutually disjoint maximum scattered linear sets of pseudoregulus type admitting the same transversal points. Such maximal scattered linear sets, together with P_1 and P_2, cover the point set of the line $\Lambda = PG(1, q^t)$.

Remark 4.2 Since the group $PGL(2, q^t)$ acts 2-transitively on the points of Λ, we may suppose that all $\mathbb{F}_q$-linear sets of pseudoregulus type of Λ have the same transversal points P_1 and P_2. This means that all such linear sets are only determined by ρ and by the automorphism τ. Moreover, it is easy to see that for each $\rho, \rho' \in \mathbb{F}_{q^t}^*$ the linear sets $L_{\rho, \tau}$ and $L_{\rho', \tau}$ are equivalent. Indeed, it is sufficient to consider the collineation of $\Lambda = PG(V, \mathbb{F}_{q^t})$ induced by the map $a\underline{w} + b\underline{v} \in V \mapsto a\rho^{\tau^{-1}} \underline{w} + b\rho' \underline{v} \in V$. It follows that, up to projectively equivalence, we may only consider $\mathbb{F}_q$-linear sets of type L_{1, σ_i}, where $\sigma_i : x \in \mathbb{F}_{q^t} \mapsto x^{q^i} \in \mathbb{F}_{q^t}$, with $i \in$

$\{1, \ldots, t-1\}$ and $gcd(i, t) = 1$. Now, by observing that, for each $i, j \in \{1, \ldots, t-1\}$ with $gcd(i, t) = gcd(j, t) = 1$,

$$
\begin{aligned}
L_{1,\sigma_i} &= \left\{ \left\langle (x, x^{q^i}) \right\rangle_{q^t} : x \in \mathbb{F}_{q^t}^* \right\} = \left\{ \left\langle (1, x^{q^i-1}) \right\rangle_{q^t} : x \in \mathbb{F}_{q^t}^* \right\} \\
&= \left\{ \left\langle (1, a) \right\rangle_{q^t} : a \in \mathbb{F}_{q^t}^*, N_{q^t/q}(a) = 1 \right\} = \left\{ \left\langle (x, x^{q^j}) \right\rangle_{q^t} : x \in \mathbb{F}_{q^t}^* \right\} = L_{1,\sigma_j},
\end{aligned}
$$

we have that in $\Lambda = PG(1, q^t)$ $(t \geq 2)$ all $\mathbb{F}_q$-linear sets of pseudoregulus type are equivalent to the linear set L_{1,σ_1}, under the action of the collineation group of Λ. This result has also been proven in [7, Remark 2.2].

Proposition 4.3 *If L is an $\mathbb{F}_q$-linear set of pseudoregulus type of $\Lambda = PG(1, q^t)$, $t \geq 3$, then its transversal points are uniquely determined.*

Proof By Remark 4.2, we may consider the $\mathbb{F}_q$-linear set of pseudoregulus type

$$
L := L_{1,\sigma_1} = \left\{ \left\langle (\lambda, \lambda^q) \right\rangle_{q^t} : \lambda \in \mathbb{F}_{q^t}^* \right\}, \tag{9}
$$

having $P_1 = \langle (1, 0) \rangle_{q^t}$ and $P_2 = \langle (0, 1) \rangle_{q^t}$ as transversal points.

Suppose that L has another pair of transversal points $P_1' = \langle \underline{w} \rangle_{q^t}$ and $P_2' = \langle \underline{v} \rangle_{q^t}$, with $\underline{w} = \langle (a, b) \rangle_{q^t}$ and $\underline{v} = \langle (c, d) \rangle_{q^t}$, such that $ad \neq bc$. Then $L = \{ \langle \eta \underline{w} + \rho \eta^\tau \underline{v} \rangle_{q^t} : \eta \in \mathbb{F}_{q^t}^* \}$, with $\tau \in Aut(\mathbb{F}_{q^t})$. Moreover, arguing as in the previous remark, we have that

$$
\begin{aligned}
L &= \left\{ \left\langle \underline{w} + \rho \eta^{\tau-1} \underline{v} \right\rangle_{q^t} : \eta \in \mathbb{F}_{q^t}^* \right\} = \left\{ \left\langle \underline{w} + \rho \mu^{q-1} \underline{v} \right\rangle_{q^t} : \mu \in \mathbb{F}_{q^t}^* \right\} \\
&= \left\{ \left\langle \mu \underline{w} + \rho \mu^q \underline{v} \right\rangle_{q^t} : \mu \in \mathbb{F}_{q^t}^* \right\}. \tag{10}
\end{aligned}
$$

By (9) and (10), we have that for each $\lambda \in \mathbb{F}_{q^t}^*$, there exist $\alpha_\lambda, \mu \in \mathbb{F}_{q^t}^*$ such that

$$
(\lambda, \lambda^q) = \alpha_\lambda (\mu \underline{w} + \rho \mu^q \underline{v}) = \alpha_\lambda (\mu a + \rho \mu^q c, \mu b + \rho \mu^q d).
$$

Then, the above equality implies that

$$
\alpha_\lambda^{q-1} = \frac{\mu b + \rho \mu^q d}{(\mu a + \rho \mu^q c)^q},
$$

which gives $N_{q^t/q}(\mu b + \rho \mu^q d) = N_{q^t/q}(\mu a + \rho \mu^q c)$ for each $\mu \in \mathbb{F}_{q^t}$, i.e.

$$
\prod_{i=0}^{t-1} \left(\mu^{q^i} b^{q^i} + \rho^{q^i} \mu^{q^{i+1}} d^{q^i} \right) = \prod_{i=0}^{t-1} \left(\mu^{q^i} a^{q^i} + \rho^{q^i} \mu^{q^{i+1}} c^{q^i} \right) \tag{11}
$$

for each $\mu \in \mathbb{F}_{q^t}$. From the last equality we get a polynomial identity in the variable μ of degree at most $2q^{t-1} + q^{t-2} + \cdots + q^3 + q^2 + q$. If $q \geq 3$, then $2q^{t-1} + q^{t-2} + \cdots + q^3 + q^2 + q < q^t$, hence the polynomials in (11) are the same. So comparing the coefficients of the terms of maximum degree, we get

$$
d^{1+q+q^2+\cdots+q^{t-2}} b^{q^{t-1}} = c^{1+q+q^2+\cdots+q^{t-2}} a^{q^{t-1}}. \tag{12}
$$

Also, comparing the coefficients of the terms of degree $2q^{t-1} + q^{t-2} + \cdots + q^3 + q^2 + 1$, for $t > 2$, we have

$$d^{q+q^2+\cdots+q^{t-2}} b^{q^{t-1}} b = c^{q+q^2+\cdots+q^{t-2}} a^{q^{t-1}} a. \tag{13}$$

If $bd \neq 0$, then $ac \neq 0$ and dividing both sides of Eqs. (12) and (13), we get $\frac{d}{b} = \frac{c}{a}$, a contradiction since $P_1' \neq P_2'$. If $b = 0$, from (12) we have $c = 0$ and hence $P_1' = P_1$ and $P_2' = P_2$; if $d = 0$, then also $a = 0$ by (12) and hence $P_1' = P_2$ and $P_2' = P_1$.

If $q = 2$, reducing (11) modulo $\mu^{q^t} - \mu$, we get that the two polynomials of (11) have degree at most $q^{t-1} + q^{t-2} + \cdots + q^3 + q^2 + q + 1$. So, comparing the coefficients of the terms of degree $q^{t-1} + q^{t-2} + \cdots + q^3 + q^2 + 2$, and of the terms of degree $q^{t-2} + q^{t-3} + \cdots + q^3 + q^2 + 2$ (for $t > 2$), and arguing as above we get the same result. This completes the proof. $\square$

Remark 4.4 Note that if $t = 2$, then $L_{\rho,\tau}$ is a Baer subline of $\Lambda = PG(1, q^2)$ and P_1 and P_2 are conjugated with respect to the semilinear involution of Λ fixing $L_{\rho,\tau}$ pointwise. Hence, in such a case, the transversal points are not uniquely determined.

Remark 4.5 Let $L_{\rho,f}$ be an $\mathbb{F}_q$-linear set of pseudoregulus type of $PG(2n - 1, q^t)$, $n > 1$, and let $\mathcal{P}_{L_{\rho,f}}$ be the associated $\mathbb{F}_q$-pseudoregulus. By (3) and Definition 4.1, we observe that for each line $s \in \mathcal{P}_{\rho,f}$, the set $L_{\rho,f} \cap s$ is a linear set of pseudoregulus type whose transversal points are the intersections of s with the transversal subspaces of $\mathcal{P}_{L_{\rho,f}}$.

We conclude this section by giving some examples of maximum scattered $\mathbb{F}_q$-linear sets which are not of pseudoregulus type.

Example 4.6

(i) Let

$$L_\rho = \left\{ \langle (x, \rho x^q + x^{q^{t-1}}) \rangle_{q^t} : x \in \mathbb{F}_{q^t}^* \right\},$$

where $\rho \in \mathbb{F}_{q^t}$ such that $N_{q^t/q}(\rho) \neq 1$. By [22, Theorem 2] L_ρ is a scattered $\mathbb{F}_q$-linear set of rank t. Moreover, if $q > 3$, $\rho \neq 0$ and $t \geq 4$, by [22, Theorem 3], there is no collineation of $PG(1, q^t)$ mapping L_ρ to L_{1,σ_1}. Hence, by Remark 4.2, L_ρ is a maximum scattered $\mathbb{F}_q$-linear set which is not of pseudoregulus type when $q > 3$.

(ii) Let

$$L = \left\{ \langle (x_0, x_1, \ldots, x_{n-1}, \rho x_0^q + x_0^{q^{t-1}}, x_1^q, \ldots, x_{n-1}^q) \rangle_{q^t} : x_i \in \mathbb{F}_{q^t} \right\},$$

with $\rho \in \mathbb{F}_{q^t}^*$ and $N_{q^t/q}(\rho) \neq 1$. It is easy to see that L is a scattered $\mathbb{F}_q$-linear set of rank tn. Also, the line r with equations $x_1 = x_2 = \cdots = x_{n-1} = 0$ is a line of weight t in L and, by the previous arguments $r \cap L$ is an $\mathbb{F}_q$-linear set which is not of pseudoregulus type for $q > 3$. So by Remark 4.5 and by point (i), for each $q > 3$, $t \geq 4$ and $n \geq 2$, L is not of pseudoregulus type.

5 Linear sets and the variety $\Omega(\mathcal{S}_{n,n})$

Let $\mathbb{M} = \mathbb{M}(n, q)$ ($n \geq 2$) be the vector space of the matrices of order $n \times n$ with entries in $\mathbb{F}_q$ and let $PG(n^2 - 1, q) = PG(\mathbb{M}, \mathbb{F}_q)$. The Segre variety $\mathcal{S}_{n,n} = \mathcal{S}_{n,n}(q)$ of $PG(n^2 - 1, q)$ is the set of all points $\langle X \rangle_q$ of $PG(n^2 - 1, q)$ such that X is a matrix of rank 1. Here below we list some well known properties of such a variety, that can be found in [9, pp. 98–99], [10] and [11, Sect. 25.5]. Precisely,

- $|\mathcal{S}_{n,n}| = (\frac{q^n - 1}{q - 1})^2$;
- maximal subspaces of $\mathcal{S}_{n,n}$ have dimension $n - 1$;
- there are two families $\mathcal{R}_1$ and $\mathcal{R}_2$ of maximal subspaces of $\mathcal{S}_{n,n}$, which are the *systems* of $\mathcal{S}_{n,n}$. Spaces of the same system are pairwise skew and any two spaces of different systems meet in exactly one point. The elements of each system partition $\mathcal{S}_{n,n}$. Moreover, $|\mathcal{R}_1| = |\mathcal{R}_2| = \frac{q^n - 1}{q - 1}$;
- the automorphism group $Aut(\mathcal{S}_{n,n})$ of $\mathcal{S}_{n,n}$ is isomorphic to $(P\Gamma L(n, q) \times P\Gamma L(n, q)) \rtimes C_2$, and it is the group of all collineations of $PG(n^2 - 1, q)$ fixing or interchanging the two systems of $\mathcal{S}_{n,n}$.

A k-dimensional subspace S of $PG(n^2 - 1, q)$ is a *kth secant subspace* to $\mathcal{S}_{n,n}$ when $S = \langle P_1, P_2, \ldots, P_{k+1} \rangle_q$ and $\{P_1, P_2, \ldots, P_{k+1}\} \subset S \cap \mathcal{S}_{n,n}$. The $(n-2)$th *secant variety* $\Omega(\mathcal{S}_{n,n})$ of $\mathcal{S}_{n,n}$ is the set of all points of $PG(n^2 - 1, q)$ belonging to an $(n-2)$th secant subspace to $\mathcal{S}_{n,n}$. Note that

$$\Omega(\mathcal{S}_{n,n}) = \{\langle X \rangle_q \mid X \in \mathbb{M}(n, q) \setminus \{\mathbf{0}\}, \det X = 0\}, \tag{14}$$

i.e. $\Omega(\mathcal{S}_{n,n})$ is the algebraic variety, also called *determinantal hypersurface*, defined by the non-invertible matrices of $\mathbb{M}(n, q)$.

Regarding $\mathbb{F}_{q^n}$ as an n-dimensional vector space over $\mathbb{F}_q$ and fixing an $\mathbb{F}_q$-basis $\mathcal{B}$ of $\mathbb{F}_{q^n}$, each matrix M of $\mathbb{M} = \mathbb{M}(n, q)$ defines an $\mathbb{F}_q$-endomorphism φ_M of $\mathbb{F}_{q^n}$, and conversely. The map $\phi_M : M \in \mathbb{M} \mapsto \varphi_M \in \mathbb{E}$, where $\mathbb{E} = End(\mathbb{F}_{q^n}, \mathbb{F}_q)$ is the n^2-dimensional vector space of all the $\mathbb{F}_q$-endomorphisms of $\mathbb{F}_{q^n}$, is an isomorphism between the vector spaces $\mathbb{M}$ and $\mathbb{E}$. By using such an isomorphism, we have that the elements of $\mathbb{E}$ with rank 1 define in $PG(\mathbb{E}, \mathbb{F}_q) = PG(n^2 - 1, q)$ the Segre variety $\mathcal{S}_{n,n}$. Recalling that each element $\varphi \in \mathbb{E}$ can be written as $\varphi(x) = \sum_{i=0}^{n-1} \beta_i x^{q^i}$, with $\beta_i \in \mathbb{F}_{q^n}$, we get the following result.

Proposition 5.1 *Let $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_q) = PG(n^2 - 1, q)$ and let $\mathcal{S}_{n,n}$ be the Segre variety of $\mathbb{P}$ defined by the elements of $\mathbb{E}$ with rank 1. Then*[4]

$$\mathcal{S}_{n,n} = \{\langle t_\lambda \circ \mathrm{Tr} \circ t_\mu \rangle_q : \lambda, \mu \in \mathbb{F}_{q^n}^*\},$$

where $t_\alpha : x \in \mathbb{F}_{q^n} \mapsto \alpha x \in \mathbb{F}_{q^n}$, with $\alpha \in \mathbb{F}_{q^n}$ and $\mathrm{Tr} : x \in \mathbb{F}_{q^n} \mapsto x + x^q + \cdots + x^{q^{n-1}} \in \mathbb{F}_q$. Moreover, $\mathcal{R}_1 = \{X(\lambda) : \lambda \in \mathbb{F}_{q^n}^*\}$ and $\mathcal{R}_2 = \{X'(\lambda) : \lambda \in \mathbb{F}_{q^n}^*\}$, where

$$X(\lambda) = \{\langle t_\alpha \circ \mathrm{Tr} \circ t_\lambda \rangle_q : \alpha \in \mathbb{F}_{q^n}^*\} \quad and$$

$$X'(\lambda) = \{\langle t_\lambda \circ \mathrm{Tr} \circ t_\alpha \rangle_q : \alpha \in \mathbb{F}_{q^n}^*\},$$

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[4] $\circ$ stands for composition of maps.

are the two systems of $\mathcal{S}_{n,n}$. Finally, $\Omega(\mathcal{S}_{n,n})$ is defined by the non-invertible elements of $\mathbb{E}$.

Proof Note that, for each $\lambda, \mu \in \mathbb{F}_{q^n}^*$, we have

$$\ker(t_\lambda \circ \mathrm{Tr} \circ t_\mu) = \frac{1}{\mu} \ker \mathrm{Tr},$$

so $\dim(\ker(t_\lambda \circ \mathrm{Tr} \circ t_\mu)) = n - 1$ (i.e., $t_\lambda \circ \mathrm{Tr} \circ t_\mu$ is an element of $\mathbb{E}$ of rank 1) and hence $\langle t_\lambda \circ \mathrm{Tr} \circ t_\mu \rangle_q \in \mathcal{S}_{n,n}$. Also, for each $\lambda', \mu' \in \mathbb{F}_{q^n}^*$, $t_\lambda \circ \mathrm{Tr} \circ t_\mu = t_{\lambda'} \circ \mathrm{Tr} \circ t_{\mu'}$ if and only if $\frac{\lambda}{\lambda'} = \frac{\mu}{\mu'} \in \mathbb{F}_q^*$. Then direct computations show that $|\{\langle t_\lambda \circ \mathrm{Tr} \circ t_\mu \rangle_q : \lambda, \mu \in \mathbb{F}_{q^n}^*\}| = (\frac{q^n - 1}{q - 1})^2$, and hence $\mathcal{S}_{n,n} = \{\langle t_\lambda \circ \mathrm{Tr} \circ t_\mu \rangle_q : \lambda, \mu \in \mathbb{F}_{q^n}^*\}$.[5]

Also, it is easy to prove that for each $\lambda \in \mathbb{F}_{q^n}^*$, the sets $X(\lambda)$ and $X'(\lambda)$ are $(n-1)$-dimensional subspaces of $\mathbb{P}$ contained in $\mathcal{S}_{n,n}$. Moreover, for each $\lambda, \mu \in \mathbb{F}_{q^n}^*$, two subspaces $X(\lambda)$ and $X(\mu)$ are either disjoint or equal, and this latter case holds true if and only if $\frac{\lambda}{\mu} \in \mathbb{F}_q^*$. The same happens for $X'(\lambda)$ and $X'(\mu)$. This implies that $|\{X(\lambda) : \lambda \in \mathbb{F}_{q^n}^*\}| = |\{X'(\lambda) : \lambda \in \mathbb{F}_{q^n}^*\}| = \frac{q^n - 1}{q - 1}$. Also, $X(\lambda) \cap X'(\mu) = \{\langle t_\mu \circ \mathrm{Tr} \circ t_\lambda \rangle_q\}$ is a point. Then $\mathcal{R}_1 = \{X(\lambda) : \lambda \in \mathbb{F}_{q^n}^*\}$ and $\mathcal{R}_2 = \{X'(\lambda) : \lambda \in \mathbb{F}_{q^n}^*\}$ are the systems of $\mathcal{S}_{n,n}$. Finally, by (14) the last part of the assertion follows. □

For each $\varphi \in \mathbb{E}$, where $\varphi(x) = \sum_{i=0}^{n-1} \beta_i x^{q^i}$, the *conjugate* $\overline{\varphi}$ of φ is defined by $\overline{\varphi}(x) = \sum_{i=0}^{n-1} \beta_i^{q^{n-i}} x^{q^{n-i}}$. Precisely, $\bar{\varphi}$ is the *adjoint* map of φ with respect to the non-degenerate bilinear form of $\mathbb{F}_{q^n}$

$$\beta(x, y) = \mathrm{Tr}_{q^n/q}(xy). \tag{15}$$

The map

$$T : \varphi \in \mathbb{E} \mapsto \overline{\varphi} \in \mathbb{E},$$

is an involutory $\mathbb{F}_q$-linear permutation of $\mathbb{E}$ and straightforward computations show that

$$\overline{\varphi \circ \psi} = \overline{\psi} \circ \overline{\varphi}, \qquad \overline{\varphi^{-1}} = (\overline{\varphi})^{-1} \quad \text{for each } \varphi, \psi \in \mathbb{E}; \tag{16}$$

$$\overline{t_\lambda} = t_\lambda \quad \text{for each } \lambda \in \mathbb{F}_{q^n}. \tag{17}$$

Moreover, it can be easily checked that $\ker \varphi = (\mathrm{Im} \, \overline{\varphi})^\perp$, where $\perp$ is the polarity defined by (15), and hence $\dim(\ker \varphi) = \dim(\ker \overline{\varphi})$. Then T induces in $\mathbb{P}$ a linear involutory collineation Φ_T preserving the varieties $\mathcal{S}_{n,n}$ and $\Omega(\mathcal{S}_{n,n})$ and interchanging the systems $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathcal{S}_{n,n}$. Indeed, we have

$$X(\mu)^{\Phi_T} = X'(\mu) \quad \text{for each } \mu \in \mathbb{F}_{q^n}^*. \tag{18}$$

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[5]Alternatively, by [19, Theorem 2.24] it can be easily seen that the maps $t_\lambda \circ \mathrm{Tr} \circ t_\mu$ are all the $\mathbb{F}_q$-endomorphisms of $\mathbb{F}_{q^n}$ with rank 1.

The subgroup $H(\mathcal{S}_{n,n})$ of $P\Gamma L(n^2, q)$ fixing the systems $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathcal{S}_{n,n}$ is isomorphic to $P\Gamma L(n, q) \times P\Gamma L(n, q)$, and such a group has index 2 in the group $Aut(\mathcal{S}_{n,n}) = Aut(\Omega(\mathcal{S}_{n,n})) \simeq (P\Gamma L(n, q) \times P\Gamma L(n, q)) \rtimes C_2$ (see [13, Theorem 3] and [11]). Hence $Aut(\mathcal{S}_{n,n}) = \langle H(\mathcal{S}_{n,n}), \Phi_T \rangle$.

Let $\mathcal{I} := \{\langle t_\lambda \rangle_q : \lambda \in \mathbb{F}_{q^n}^*\}$. Then $\mathcal{I}$ is an $(n-1)$-dimensional subspace of $\mathbb{P}$ disjoint from the variety $\Omega(\mathcal{S}_{n,n})$ and

$$\mathcal{D}_1(\mathcal{I}) = \left\{ \left\{ \langle t_\alpha \circ \varphi \rangle_q : \alpha \in \mathbb{F}_{q^n}^* \right\} : \varphi \in \mathbb{E} \setminus \{\mathbf{0}\} \right\}$$

and

$$\mathcal{D}_2(\mathcal{I}) = \left\{ \left\{ \langle \varphi \circ t_\alpha \rangle_q : \alpha \in \mathbb{F}_{q^n}^* \right\} : \varphi \in \mathbb{E} \setminus \{\mathbf{0}\} \right\}$$

are two Desarguesian spreads of $\mathbb{P}$ (see, e.g., [3, Exercise 3 and Theorem 14]) such that

(I_1) $\mathcal{I} \in \mathcal{D}_i(\mathcal{I})$ and $\mathcal{R}_i \subset \mathcal{D}_i(\mathcal{I})$, for each $i \in \{1, 2\}$.

Also, we explicitly note that

(I_2) Φ_T fixes $\mathcal{I}$ pointwise and, by (18), $\mathcal{D}_1(\mathcal{I})^{\Phi_T} = \mathcal{D}_2(\mathcal{I})$.

Let $\Pi_{n-1}(\mathcal{D}_1(\mathcal{I}))$ be the $\mathbb{F}_q$-linear representation of the projective space $PG(n-1, q^n)$ defined by the Desarguesian spread $\mathcal{D}_1(\mathcal{I})$ of $\mathbb{P}$. Let Υ_1 be the linear collineation of $\mathbb{P}$ defined as

$$\Upsilon_1 : \langle \varphi \rangle_q \in \mathbb{P} \mapsto \langle \varphi' \rangle_q \in \mathbb{P},$$

where $\varphi'(x) = \sum_{i=0}^{n-1} a_{i-1}^q x^{q^i}$ if $\varphi(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$, taking the indices i modulo n.

(I_3) The collineation Υ_1 fixes the Desarguesian spread $\mathcal{D}_1(\mathcal{I})$ and induces a collineation $\bar{\Upsilon}_1$ in $\Pi_{n-1}(\mathcal{D}_1(\mathcal{I}))$ of order n whose fixed point set consists of the elements of $\mathcal{R}_1$. Hence, $\mathcal{R}_1$ turns out to be a subgeometry of $\Pi_{n-1}(\mathcal{D}_1(\mathcal{I}))$ isomorphic to $PG(n-1, q)$.

We explicitly note that

$$\mathcal{I}^{\Upsilon_1^j} = \left\{ \left\langle x \mapsto \lambda x^{q^j} \right\rangle_q : \lambda \in \mathbb{F}_{q^n}^* \right\} \quad \text{and} \quad \left\langle \mathcal{I}, \mathcal{I}^{\Upsilon_1}, \dots, \mathcal{I}^{\Upsilon_1^{n-1}} \right\rangle_q = \mathbb{P}. \tag{19}$$

So $\mathcal{I}$, in $\Pi_{n-1}(\mathcal{D}_1(\mathcal{I}))$, is a point whose orbit under the action of the cyclic group $\langle \bar{\Upsilon}_1 \rangle$ has maximum size n.

In the same way,

(I_4) the collineation $\Upsilon_2 = \Phi_T^{-1} \circ \Upsilon_1 \circ \Phi_T$ fixes the Desarguesian spread $\mathcal{D}_2(\mathcal{I})$ and induces a collineation $\bar{\Upsilon}_2$ in $\Pi_{n-1}(\mathcal{D}_2(\mathcal{I}))$ of order n whose set of fixed points consists of the elements of $\mathcal{R}_2$.

Also,

(I_5) $\mathcal{I}^{\Upsilon_2^i} = \mathcal{I}^{\Upsilon_1^{n-i}}$.

Let $\mathcal{O}_{\mathcal{I}}$ be the orbit, under the action of the group $H(\mathcal{S}_{n,n})$, of the $(n-1)$-dimensional subspace $\mathcal{I}$ of $\mathbb{P}$. A subspace belonging to this orbit will be called a $\mathcal{D}$-*subspace* of $\mathbb{P}$. In the following we will study the geometric properties of the $\mathcal{D}$-subspaces of $\mathbb{P}$ under the action of $H(\mathcal{S}_{n,n})$.

Theorem 5.2 *Let X be a $\mathcal{D}$-subspace of $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_q) = PG(n^2 - 1, q)$, then there exist two Desarguesian spreads $\mathcal{D}_1(X)$ and $\mathcal{D}_2(X)$ of $\mathbb{P}$ such that:*

(D_1) *$X \in \mathcal{D}_i(X)$ and $\mathcal{R}_i \subset \mathcal{D}_i(X)$ for each $i = 1, 2$,*
(D_2) *there is a semilinear collineation $\bar{\Xi}_i$ of $\Pi_{n-1}(\mathcal{D}_i(X))$ of order n induced by a linear collineation Ξ_i of $\mathbb{P}$ fixing the Desarguesian spread $\mathcal{D}_i(X)$. Moreover, $\mathcal{R}_i = Fix\,\bar{\Xi}_i$ is a subgeometry of $\Pi_{n-1}(\mathcal{D}_i(X))$ isomorphic to a $PG(n-1, q)$.*

Also, there exists an involutory collineation Φ of $\mathbb{P}$ such that

(D_3) *Φ fixes X pointwise,*
(D_4) *$\mathcal{D}_1(X)^{\Phi} = \mathcal{D}_2(X)$.*

Proof Let g be an element of $H(\mathcal{S}_{n,n})$ such that $\mathcal{I}^g = X$. By (I_1), $\mathcal{D}_i(X) := \mathcal{D}_i(\mathcal{I})^g$, for each $i \in \{1, 2\}$, is a Desarguesian spread of $\mathbb{P}$ containing X and the system $\mathcal{R}_i$, i.e. (D_1) is satisfied. Putting $\Xi_i := g \circ \Upsilon_i \circ g^{-1}$ and $\Phi := g \circ \Phi_T \circ g^{-1}$ and taking (I_2), (I_3) and (I_4) into account, (D_2), (D_3) and (D_4) follow.   □

This allows us to give the following

Definition 5.3 Let X be a $\mathcal{D}$-subspace of $\mathbb{P}$ and let Ξ_i ($i \in \{1, 2\}$) be one of the two collineations of $\mathbb{P}$ described in (D_2). Each of the $\mathcal{D}$-subspaces $X^{\Xi_i^j}$, with $j \in \{0, 1, \dots, n-1\}$, is said to be a *conjugate* of X. Note that, by (I_5), $X^{\Xi_2^j} = X^{\Xi_1^{n-j}}$.

Remark 5.4 If $n = 2$, then $\mathbb{E} = End(\mathbb{F}_{q^2}, \mathbb{F}_q)$ and $\mathcal{S}_{2,2}$ is the hyperbolic quadric $Q^+(3, q)$ of $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_q) = PG(3, q)$ defined by the quadratic form

$$\varphi \in \mathbb{E} \mapsto a^{q+1} - b^{q+1} \in \mathbb{F}_q,$$

where $\varphi(x) = ax + bx^q$. Hence, the group $H(\mathcal{S}_{2,2})$ is the subgroup of the orthogonal group $P\Gamma O^+(4, q)$ fixing the reguli of $Q^+(3, q)$. Also, the $H(\mathcal{S}_{2,2})$-orbit of the line $\mathcal{I}$, is the set of all external lines to the quadric. Moreover, the involutory linear collineation Υ_1 of $\mathbb{P}$ described above is

$$\langle x \mapsto ax + bx^q \rangle_q \longmapsto \langle x \mapsto b^q x + a^q x^q \rangle_q.$$

This means that the conjugate of $\mathcal{I}$ is the line $\mathcal{I}^{\Upsilon_1} = \{\langle x \mapsto \mu x^q \rangle_q : \mu \in \mathbb{F}_{q^2}^*\}$, which is the polar line of $\mathcal{I}$ with respect to the quadric $Q^+(3, q)$.

5.1 Linear sets and presemifields

A finite semifield is a finite division algebra which is not necessarily associative and throughout this paper the term semifield will always be used to denote a finite semifield (see, e.g., [16, Chap. 6] for definitions and notations on finite semifields). Every

field is a semifield and the term *proper semifield* means a semifield which is not a field. The left nucleus $\mathbb{N}_l$ and the center $\mathbb{K}$ of a semifield $\mathbb{S}$ are fields contained in $\mathbb{S}$ as substructures ($\mathbb{K}$ is a subfield of $\mathbb{N}_l$) and $\mathbb{S}$ is a vector space over $\mathbb{N}_l$ and over $\mathbb{K}$. Semifields are studied up to an equivalence relation called *isotopy* and the dimensions of a semifield over its left nucleus and over its center are invariant up to isotopy.

Let $\mathbb{S}$ be a semifield with center $\mathbb{K}$ and left nucleus $\mathbb{N}_l$ and let $(\mathbb{F}_q, \mathbb{F}_{q^t})$ be a pair of fields such that $\mathbb{F}_q \leq \mathbb{K}$ and $\mathbb{F}_{q^t} \leq \mathbb{N}_l$; then $\mathbb{S}$ is a finite extension of $\mathbb{F}_{q^t}$ and hence it has size q^{nt} for some integer $n \geq 1$. If $\mathbb{S}$ is a proper semifield, then $n \geq 2$. Also, up to isotopy, we may assume that $\mathbb{S} = (\mathbb{F}_{q^{nt}}, +, \star)$, where

$$x \star y = \varphi_y(x)$$

with $\varphi_y \in \mathbb{E} = End(\mathbb{F}_{q^{nt}}, \mathbb{F}_{q^t})$. The set

$$\mathcal{C}_\mathbb{S} = \{\varphi_y : x \in \mathbb{F}_{q^{nt}} \mapsto x \star y \in \mathbb{F}_{q^{nt}} \mid y \in \mathbb{F}_{q^{nt}}\} \subset \mathbb{E}$$

is the *semifield spread set* associated with $\mathbb{S}$ (*spread set* for short): $\mathcal{C}_\mathbb{S}$ is an $\mathbb{F}_q$-subspace of $\mathbb{E}$ of rank nt and each non-zero element of $\mathcal{C}_\mathbb{S}$ is invertible. Hence, for each pair $(\mathbb{F}_q, \mathbb{F}_{q^t}) \subseteq (\mathbb{K}, \mathbb{N}_l)$, we can associate with $\mathbb{S}$ the $\mathbb{F}_q$-linear set of rank nt of the projective space $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_{q^t}) = PG(n^2 - 1, q^t)$ defined by the non-zero elements of $\mathcal{C}_\mathbb{S}$. Such a linear set turns out to be disjoint from the variety $\Omega(\mathcal{S}_{n,n}(q^t))$ of $\mathbb{P}$ defined by the non-invertible elements of $\mathbb{E}$. Isotopic semifields produce in $\mathbb{P} = PG(n^2 - 1, q^t)$ linear sets which are equivalent with respect to the action of the group $H(\mathcal{S}_{n,n}(q^t))$, and conversely (see [21] for $n = 2$ and [14] for $n \geq 2$). Among all the pairs $(\mathbb{F}_q, \mathbb{F}_{q^t})$ such that $\mathbb{F}_q \subseteq \mathbb{K}$ and $\mathbb{F}_{q^t} \subseteq \mathbb{N}_l$, the pair $(\mathbb{K}, \mathbb{N}_l)$ has the following properties: (i) maximizes the field of linearity of the linear set associated with $\mathbb{S}$, (ii) minimizes the dimension of the projective space $\mathbb{P}$ in which the linear set is embedded, and (iii) minimizes the group $H(\mathcal{S}_{n,n})$. For instance, if $\mathbb{S} = \mathbb{F}_q$, then $\mathbb{N}_l = \mathbb{K} = \mathbb{F}_q$ and hence the linear set associated with the field $\mathbb{F}_q$, with respect to the pair $(\mathbb{F}_q, \mathbb{F}_q)$, is the point $PG(\mathbb{F}_q, \mathbb{F}_q)$; whereas, if $\mathbb{F}_{q'}$ is a subfield of $\mathbb{F}_q$, $q = q'^n$, then the linear set associated with $\mathbb{F}_q$, with respect to the pair $(\mathbb{F}_{q'}, \mathbb{F}_{q'})$, is an $(n-1)$-dimensional subspace of $\mathbb{P} = PG(n^2 - 1, q')$ disjoint from the variety $\Omega(\mathcal{S}_{n,n}(q'))$ of $\mathbb{P}$, which is a $\mathcal{D}$-subspace of $\mathbb{P}$, and conversely (see [13, Theorem 20]). In what follows, we will call the linear set associated with $\mathbb{S}$ with respect to the maximum pair $(\mathbb{K}, \mathbb{N}_l)$, the *relevant* linear set associated with $\mathbb{S}$.

In the next sections we will characterize, up to the action of the group $H(\mathcal{S}_{n,n})$, the relevant linear sets associated with some classical semifields: the Generalized Twisted Fields and the Knuth semifields 2-dimensional over their left nucleus.

5.2 Generalized Twisted Fields

If $\mathbb{S}$ satisfies all the axioms for a semifield except, possibly, the existence of the identity element for the multiplication, then it is a *presemifield*. In such a case the nuclei and the center of $\mathbb{S}$ are defined as fields of linear maps contained in $End(\mathbb{S}, \mathbb{F}_p)$ (where p is the characteristic of $\mathbb{S}$) (see, e.g., [24, Theorem 2.2]) and all that we stated and defined above for semifields can be applied to presemifields.

The Generalized Twisted Fields are presemifields constructed by A.A. Albert in [2]. By [1, Lemma 1] a Generalized Twisted Field $\mathbb{G}$ with center of order q, n-dimensional over its left nucleus ($n \geq 2$) and tn-dimensional over its center is of type $\mathbb{G} = (\mathbb{F}_{q^{nt}}, +, \star)$ ($q = p^e$, p prime) with

$$x \star y = yx - cy^{q^m} x^{q^{tl}}, \tag{20}$$

where $c \in \mathbb{F}_{q^{nt}}^*$, $c \neq x^{q^{tl}-1} y^{q^m-1}$ for every $x, y \in \mathbb{F}_{q^{nt}}$, and $1 \leq l \leq n-1$, $1 \leq m \leq nt-1$, $m \neq tl$. Since we required $\dim_{\mathbb{K}} \mathbb{G} = nt$ and $\dim_{\mathbb{N}_l} \mathbb{G} = n$, we also have $gcd(l, n) = gcd(t, m) = 1$. From the previous conditions we get $q > 2$ and, if $t = 1$, then $n \geq 3$. In terms of linear maps, by [24, Theorem 2.2] and by [1, Lemma 1] we can describe the left nucleus and the center of $\mathbb{G}$ as follows:

$$\mathbb{N}_l = \{t_\lambda : x \in \mathbb{F}_{q^{nt}} \mapsto \lambda x \in \mathbb{F}_{q^{nt}} \,|\, \lambda \in \mathbb{F}_{q^t}\} \subset \mathbb{E},$$

$$\mathbb{K} = \{t_\lambda : x \in \mathbb{F}_{q^{nt}} \mapsto \lambda x \in \mathbb{F}_{q^{nt}} \,|\, \lambda \in \mathbb{F}_q\} \subset \mathbb{E},$$

where $\mathbb{E} = End(\mathbb{F}_{q^{nt}}, \mathbb{F}_{q^t}) = V(n^2, \mathbb{F}_{q^t})$. The *spread set* associated with $\mathbb{G}$ is

$$\mathbb{C} = \{\varphi_y : x \in \mathbb{F}_{q^{nt}} \mapsto x \star y \in \mathbb{F}_{q^{nt}} \,|\, y \in \mathbb{F}_{q^{nt}}\} \subset \mathbb{E}$$

and it is an $\mathbb{F}_q$-subspace of $\mathbb{E}$ of dimension nt. Hence $\mathbb{C}$ defines an $\mathbb{F}_q$-linear set of rank nt in the projective space $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_{q^t}) = PG(n^2-1, q^t)$; precisely

$$L(\mathbb{G}) = L_{\mathbb{C}} = \{\langle \varphi_y \rangle_{q^t} : y \in \mathbb{F}_{q^{nt}}^*\},$$

which is the relevant linear set associated with $\mathbb{G}$. Since the non-zero elements of $\mathbb{C}$ are invertible, $L(\mathbb{G})$ is disjoint from the variety $\Omega(\mathcal{S}_{n,n}(q^t))$ of $\mathbb{P}$ defined by the non-invertible elements of $\mathbb{E}$. By (20) it is clear that $L(\mathbb{G})$ is contained in the subspace $\Lambda = PG(2n-1, q^t)$ of $\mathbb{P}$ joining the $\mathcal{D}$-space $\mathcal{I} = \{\langle t_\lambda \rangle_{q^t} : \lambda \in \mathbb{F}_{q^{nt}}^*\}$ and its conjugate $\mathcal{I}^{\gamma_1^l} = \{\langle x \mapsto \lambda x^{q^{tl}} \rangle_{q^t} : \lambda \in \mathbb{F}_{q^{nt}}^*\}$, precisely

$$\Lambda = \{\langle x \mapsto Ax + Bx^{q^{lt}} \rangle_{q^t} : A, B \in \mathbb{F}_{q^{nt}}\}.$$

Note that Λ defines a line $PG(1, q^{nt})$ in the $\mathbb{F}_{q^t}$-linear representation $\Pi_{n-1}(\mathcal{D}_1(\mathcal{I}))$. Also, since $gcd(t, m) = 1$, it is easy to verify that, if $t \geq 2$, then $L(\mathbb{G})$ is a maximum scattered $\mathbb{F}_q$-linear set of Λ and, hence, $\Lambda = \langle L(\mathbb{G}) \rangle_{q^t}$.

Proposition 5.5 *Let* $\mathbb{G} = (\mathbb{F}_{q^{nt}}, +, \star)$ *be a Generalized Twisted Field n-dimensional over its left nucleus and tn-dimensional over its center. Let* $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_{q^t}) = PG(n^2-1, q^t)$ *(where* $\mathbb{E} = End(\mathbb{F}_{q^{nt}}, \mathbb{F}_{q^t})$*),* $\Lambda = \{\langle x \mapsto Ax + Bx^{q^{lt}} \rangle_{q^t} : A, B \in \mathbb{F}_{q^{nt}}\}$ *and* $\Pi_{n-1}(\mathcal{D}_1(\mathcal{I}))$ *be the $\mathbb{F}_q$-linear representation of* $PG(n-1, q^n)$.

(a) *If $t = 1$, then (a.i) $L(\mathbb{G})$ is an $(n-1)$-dimensional subspace of $\mathbb{P} = PG(n^2-1, q)$ contained in Λ and in the linear representation $\Pi_{n-1}(\mathcal{D}_1(\mathcal{I})) \simeq PG(n-1, q^n)$; (a.ii) $L(\mathbb{G})$ induces an $\mathbb{F}_{q^t}$-linear set of pseudoregulus type with transversal points $\mathcal{I}$ and $\mathcal{I}^{\gamma_1^l}$.*

(b) *If $t \geq 2$, then $L(\mathbb{G})$ is a scattered $\mathbb{F}_q$-linear set of rank tn of pseudoregulus type of Λ with $\mathcal{I}$ and $\mathcal{I}^{\Upsilon_1^l}$ as transversal spaces.*

Proof (a) If $t = 1$, then $\mathbb{C}$ is an $\mathbb{F}_q$-subspace of $\mathbb{E} = End(\mathbb{F}_{q^n}, \mathbb{F}_q)$; i.e. $L(\mathbb{G})$ is just an $(n-1)$-dimensional subspace of $\mathbb{P}$ contained in Λ. Note that the map

$$\Phi : \left\{ \langle t_\lambda \circ \varphi \rangle_q : \lambda \in \mathbb{F}_{q^n}^* \right\} \in \mathcal{D}_1(\mathcal{I}) \mapsto \langle (a_0, a_1, \dots, a_{n-1}) \rangle_{q^n} \in PG(n-1, q^n),$$

where $\varphi(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ is a linear collineation between $\Pi_{n-1}(\mathcal{D}_1(\mathcal{I}))$ and $PG(n-1, q^n)$ such that $\mathcal{I}^\Phi = \langle (1, 0, \dots, 0) \rangle_{q^n}$, $\mathcal{I}^{\Upsilon_1^l \Phi} = \langle (0, \dots, 1, \dots, 0) \rangle_{q^n}$ and Λ^Φ is the line of $PG(n-1, q^n)$ with equations $x_i = 0$ for $i \neq 0, l$. Also $L(\mathbb{G})^\Phi = \{ \langle (y, 0, 0, \dots, -cy^{q^m}, 0, \dots, 0) \rangle_{q^n} : y \in \mathbb{F}_{q^n}^* \}$ is an $\mathbb{F}_q$-linear set contained in the line Λ^Φ of $PG(n-1, q^n)$. By Definition 4.1, $L(\mathbb{G})^\Phi$ is a maximum scattered $\mathbb{F}_q$-linear set of pseudoregulus type of Λ^Φ, with transversal points $\mathcal{I}^\Phi$ and $\mathcal{I}^{\Upsilon_1^l \Phi}$. This proves (a).

(b) If $t \geq 2$, then the collineation

$$\Phi_f : \langle t_y : x \mapsto yx \rangle_{q^t} \in \mathcal{I} \mapsto \left\langle f(t_y) : x \mapsto -cy^{q^m} x^{q^{lt}} \right\rangle_{q^t} \in \mathcal{I}^{\Upsilon_1^l},$$

is a semilinear collineation between $\mathcal{I}$ and $\mathcal{I}^{\Upsilon_1^l}$ with companion automorphism $\sigma : \alpha \in \mathbb{F}_{q^t} \mapsto \alpha^{q^m} \in \mathbb{F}_{q^t}$ and, since $gcd(t, m) = 1$, $Fix(\sigma) = \mathbb{F}_q$. Hence, by Theorem 3.5, $L_{W_{1,f}}$ is an $\mathbb{F}_q$-linear set of Λ of pseudoregulus type with transversal spaces $\mathcal{I}$ and $\mathcal{I}^{\Upsilon_1^l}$, and since

$$W_{1,f} = \left\{ t_y + f(t_y) : y \in \mathbb{F}_{q^{nt}} \right\} = \left\{ x \mapsto yx - cy^{q^m} x^{q^{tl}} \mid y \in \mathbb{F}_{q^{nt}} \right\} = \mathbb{C}.$$

Case (b) follows.   □

Now, we will prove that the properties of $L(\mathbb{G})$ described in Proposition 5.5 completely characterize, up to isotopy, the Generalized Twisted Fields.

Theorem 5.6 *Let $\mathbb{S}$ be a presemifield of order q^{nt} with $\mathbb{F}_q$ contained in its center and $\mathbb{F}_{q^t}$ contained in its left nucleus and let $L(\mathbb{S})$ be the associated linear set with respect to the pair $(\mathbb{F}_q, \mathbb{F}_{q^t})$. Also, assume that $L(\mathbb{S})$ is contained in a $(2n-1)$-dimensional subspace of $\mathbb{P} = PG(n^2 - 1, q^t)$ joining two conjugated $\mathcal{D}$-spaces X and X' of $\mathbb{P}$. Suppose that either Case (a) or Case (b) below holds:*

(a) *$t = 1$ and $L(\mathbb{S})$ induces, in the linear representation $\Pi_{n-1}(\mathcal{D}_1(X)) \simeq PG(n-1, q^n)$, an $\mathbb{F}_q$-linear set of pseudoregulus type of the line $PG(1, q^n)$ of $\Pi_{n-1}(\mathcal{D}_1(X))$ joining the points X and X', with transversal points X and X';*

(b) *$t \geq 2$ and $L(\mathbb{S})$ is a maximum scattered $\mathbb{F}_q$-linear set of pseudoregulus type of $\langle X, X' \rangle_{q^t}$ with X and X' as transversal spaces;*

then $\mathbb{S}$ is isotopic to a Generalized Twisted Field.

Proof Without loss of generality we may assume that $\mathbb{S} = (\mathbb{F}_{q^{nt}}, +, *)$ with $\mathbb{F}_q$ contained in $\mathbb{K}$ and $\mathbb{F}_{q^t}$ contained in $\mathbb{N}_l$. Let $\mathbb{E} = End(\mathbb{F}_{q^{nt}}, \mathbb{F}_q)$ and let $\mathbb{C} = \{ \varphi_y : x \in$

$\mathbb{F}_{q^{nt}} \mapsto x * y \in \mathbb{F}_{q^{nt}} \mid y \in \mathbb{F}_{q^{nt}}\}$ be the spread set defined by $\mathbb{S}$. Note that describing $\mathbb{S}$ corresponds, up to isotopy, to describing the associated linear set $L(\mathbb{S}) = L_{\mathbb{C}}$ in the projective space $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_{q^t}) = PG(n^2 - 1, q^t)$, up to the action of the group $H(\mathcal{S}_{n,n}(q^t))$. Since all the $\mathcal{D}$-spaces of $\mathbb{P}$ belong to the same $H(\mathcal{S}_{n,n}(q^t))$-orbit, we may assume, up to isotopy, that $X = \mathcal{I}$, so $X' = \mathcal{I}^{\gamma_1^l}$ for some $l \in \{1, \ldots, n - 1\}$ (see (19)).

(a) By Definition 4.1, $L(\mathbb{S}) = L_{\rho, \tau}$ where $\rho \in \mathbb{F}_{q^n}^*$ and $\tau : x \mapsto x^{q^m}$ is an automorphism of $\mathbb{F}_{q^n}$ such that $gcd(m, n) = 1$. This implies that

$$\mathbb{C} = \left\{ \varphi_y : x \in \mathbb{F}_{q^n} \mapsto xy + \rho y^{q^m} x^{q^l} \in \mathbb{F}_{q^n} \mid y \in \mathbb{F}_{q^n} \right\}.$$

Hence $x * y = xy - cy^{q^m} x^{q^l}$ where $c = -\rho$, i.e. $\mathbb{S}$, up to isotopy, is a Generalized Twisted Field.

(b) By Theorem 3.12, $L(\mathbb{S})$ is of type $L_{\rho, f}$ with transversal spaces $\mathcal{I}$ and $\mathcal{I}^{\gamma_1^l}$. Hence, there exist a semilinear collineation

$$\Phi_f : \langle t_y \rangle_{q^t} \in \mathcal{I} \mapsto \left\langle f(t_y) \right\rangle_{q^t} \in \mathcal{I}^{\gamma_1^l}$$

with companion automorphism $\sigma \in Aut(\mathbb{F}_{q^t})$ such that $Fix(\sigma) = \mathbb{F}_q$ and an element $\rho \in \mathbb{F}_{q^{nt}}^*$ such that

$$\mathbb{C} = \left\{ t_y + \rho f(t_y) : y \in \mathbb{F}_{q^{nt}} \right\}.$$

This implies that

$$f(t_y) : x \mapsto \eta y^{q^m} x^{q^{tl}},$$

where $\eta \in \mathbb{F}_{q^{nt}}^*$, $1 \le m \le nt - 1$ and $gcd(t, m) = 1$. Hence, putting $c = -\eta\rho$, we have

$$\mathbb{C} = \left\{ \varphi_y : x \in \mathbb{F}_{q^{nt}} \mapsto xy - cy^{q^m} x^{q^{tl}} \in \mathbb{F}_{q^{nt}} \mid y \in \mathbb{F}_{q^{nt}} \right\},$$

this means that $x * y = xy - cy^{q^m} x^{q^{tl}}$, and hence $\mathbb{S}$, up to isotopy, is a Generalized Twisted Field. $\qquad\square$

Note that if $n = 2$ then $t \ge 2$ and by Remark 5.4, we can restate Theorem 5.6 as follows, which is a generalization of [5, Theorems 4.3, 3.7] and [25, Theorems 4.12, 4.13].

Corollary 5.7 *Let $\mathbb{S}$ be a presemifield of order q^{2t} with center $\mathbb{F}_q$ and left nucleus $\mathbb{F}_{q^t}$. If $L(\mathbb{S})$ is an $\mathbb{F}_q$-linear set of $PG(3, q^t)$ of pseudoregulus type with transversal lines external to the quadric $\mathcal{S}_{2,2} = Q^+(3, q^t)$ pairwise polar with respect to the polarity defined by $Q^+(3, q^t)$, then $\mathbb{S}$ is isotopic to a Generalized Twisted Field.*

5.3 2-Dimensional Knuth semifields

The Knuth semifields 2-dimensional over the left nucleus and $2t$-dimensional ($t \ge 2$) over the center $\mathbb{F}_q$ are the following:

$\mathbb{K}_{17} = (\mathbb{F}_{q^t} \times \mathbb{F}_{q^t}, +, *)$ and $\mathbb{K}_{19} = (\mathbb{F}_{q^t} \times \mathbb{F}_{q^t}, +, \star)$ (see [6, p. 241 (multiplications (17) and (19))]), with

$$(u, v) * (x, y) = (u, v) \begin{pmatrix} x & y \\ fy^\sigma & x^\sigma + y^\sigma g \end{pmatrix}$$

and

$$(u, v) \star (x, y) = (u, v) \begin{pmatrix} x & y \\ fy^{\sigma^{-1}} & x^\sigma + yg \end{pmatrix},$$

where $\sigma \in Aut(\mathbb{F}_{q^t})$, $Fix\,\sigma = \mathbb{F}_q$, and f and g are non-zero elements in $\mathbb{F}_{q^t}$ such that the polynomial $x^{q+1} + gx - f$ has no root in $\mathbb{F}_{q^t}$.

The spread sets (of matrices) associated with $\mathbb{K}_{17}$ and $\mathbb{K}_{19}$ are

$$\mathbb{C}_{17} = \left\{ \begin{pmatrix} x & y \\ fy^\sigma & x^\sigma + y^\sigma g \end{pmatrix} : x, y \in \mathbb{F}_{q^t} \right\} \subset \mathbb{M}$$

and

$$\mathbb{C}_{19} = \left\{ \begin{pmatrix} x & y \\ fy^{\sigma^{-1}} & x^\sigma + yg \end{pmatrix} : x, y \in \mathbb{F}_{q^t} \right\} \subset \mathbb{M},$$

respectively, where $\mathbb{M} = \mathbb{M}(2, q^t)$ is the vector space of the 2×2-matrices over $\mathbb{F}_{q^t}$.

The sets $\mathbb{C}_{17}$ and $\mathbb{C}_{19}$ are $\mathbb{F}_q$-subspaces of $\mathbb{M}$ of dimension $2t$ and hence they define $\mathbb{F}_q$-linear sets of rank $2t$ in the projective space $\mathbb{P} = PG(\mathbb{M}, \mathbb{F}_{q^t}) = PG(3, q^t)$. Precisely, using the coordinatization $\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \mapsto (x_0, x_1, x_2, x_3)$,

$$L(\mathbb{K}_{17}) = \left\{ \langle (x, y, fy^\sigma, x^\sigma + gy^\sigma) \rangle_{q^t} : x, y \in \mathbb{F}_{q^t}, (x, y) \neq (0, 0) \right\} \qquad (21)$$

and

$$L(\mathbb{K}_{19}) = \left\{ \langle (x, y, fy^{\sigma^{-1}}, x^\sigma + gy) \rangle_{q^t} : x, y \in \mathbb{F}_{q^t}, (x, y) \neq (0, 0) \right\} \qquad (22)$$

are the relevant linear sets associated with the semifields $\mathbb{K}_{17}$ and $\mathbb{K}_{19}$, respectively.

Recall that $L(\mathbb{K}_{17})$ and $L(\mathbb{K}_{19})$ are disjoint from the hyperbolic quadric $Q^+(3, q^t)$ of $\mathbb{P}$ defined by the non-invertible matrices of $\mathbb{M}$. Let $\mathcal{R}_1$ be the regulus of $Q^+(3, q^t)$ containing the line $x_2 = x_3 = 0$ and let $\mathcal{R}_2$ be the opposite one.

Remark 5.8 Note that the collineation Φ_T of $\mathbb{P}$ defined by the transpose operation on matrices fixes the quadric $Q^+(3, q^t)$ and interchanges the reguli $\mathcal{R}_1$ and $\mathcal{R}_2$ and $\Phi_T(L(\mathbb{K}_{17}(\sigma, f, g))) = L(\mathbb{K}_{19}(\sigma, \frac{1}{f^{\sigma^{-1}}}, \frac{g}{f}))$. In other words, the family $\mathbb{K}_{19}$ is the transpose family of $\mathbb{K}_{17}$ (see [12, Sect. 5]).

Proposition 5.9 (1) $L(\mathbb{K}_{17})$ *is an* $\mathbb{F}_q$-*linear set of* $\mathbb{P} = PG(3, q^t)$ *of pseudoregulus type, whose transversal lines belong to* $\mathcal{R}_1$.

(2) $L(\mathbb{K}_{19})$ *is an* $\mathbb{F}_q$-*linear set of* $\mathbb{P} = PG(3, q^t)$ *of pseudoregulus type, whose transversal lines belong to* $\mathcal{R}_2$.

Proof Let r and r' be the lines of $\mathcal{R}_1$ with equations $r : x_2 = x_3 = 0$ and $r' : x_0 = x_1 = 0$. Then the map

$$f : (x, y, 0, 0) \mapsto \left(0, 0, fy^\sigma, x^\sigma + gy^\sigma\right)$$

induces a semilinear collineation Φ_f between r and r' having σ as a companion automorphism. Then, by Theorem 3.5, $L_{1,f}$ is an $\mathbb{F}_q$-linear set of pseudoregulus type. Since $L_{1,f} = L(\mathbb{K}_{17})$ we get (1).

Case (2) follows from Remark 5.8 and Case (1). □

In the next theorem we prove that the descriptions of $L(\mathbb{K}_{17})$ and $L(\mathbb{K}_{19})$ given in Proposition 5.9 characterize the semifields $\mathbb{K}_{17}$ and $\mathbb{K}_{19}$ up to isotopism, generalizing some results contained in [5] and [25] for $t = 2$.

Theorem 5.10 *Let $\mathbb{S}$ be a presemifield of order q^{2t} with $\mathbb{F}_q$ contained in its center and $\mathbb{F}_{q^t}$ contained in its left nucleus and let $L(\mathbb{S})$ be the associated linear set with respect to the pair $(\mathbb{F}_q, \mathbb{F}_{q^t})$. If $L(\mathbb{S})$ is an $\mathbb{F}_q$-linear set of pseudoregulus type of $\mathbb{P} = PG(3, q^t)$ with associated transversal lines r and r' contained in $Q^+(3, q^t)$, then $\mathbb{S}$ is isotopic to a Knuth semifield $\mathbb{K}_{17}$ or $\mathbb{K}_{19}$. Precisely, if $r, r' \in \mathcal{R}_1$, then $\mathbb{S}$ is isotopic to a semifield $\mathbb{K}_{17}$, whereas, if $r, r' \in \mathcal{R}_2$, then $\mathbb{S}$ is isotopic to a semifield $\mathbb{K}_{19}$.*

Proof Without loss of generality, we may assume that $\mathbb{S} = (\mathbb{F}_{q^t} \times \mathbb{F}_{q^t}, +, \circ)$, with $\mathbb{F}_{q^t} \times \{0\}$ contained in its left nucleus. This implies that

$$(u, v) \circ (x, y) = (u, v)M,$$

where $M = M_{x,y} \in \mathbb{M}$. So, the spread set of matrices associated with $\mathbb{S}$ is

$$\mathbb{C} = \left\{ M_{x,y} = \begin{pmatrix} m_0(x, y) & m_1(x, y) \\ m_2(x, y) & m_3(x, y) \end{pmatrix} : x, y \in \mathbb{F}_{q^t} \right\}$$

and

$$L_{\mathbb{C}} = L(\mathbb{S})$$
$$= \left\{ \langle (m_0(x, y), m_1(x, y), m_2(x, y), m_3(x, y)) \rangle_{q^t} : x, y \in \mathbb{F}_{q^t}, (x, y) \neq (0, 0) \right\},$$

where $m_i(x, y)$ are $\mathbb{F}_q$-linear maps. Assume that the transversal lines r and r' of $L(\mathbb{S})$ are contained in $\mathcal{R}_1$. Since the group $H(\mathcal{S}_{2,2}) = G$ (see Remark 5.4) acts 2-transitively on the lines of $\mathcal{R}_1$, we can suppose that $r = \{ \langle (x_0, x_1, 0, 0) \rangle_{q^t} : x_0, x_1 \in \mathbb{F}_{q^t}, (x, y) \neq (0, 0) \}$ and $r' = \{ \langle (0, 0, x_2, x_3) \rangle_{q^t} : x_2, x_3 \in \mathbb{F}_{q^t}, (x, y) \neq (0, 0) \}$. Note that the stabilizer $G_{\{r,r'\}}$ in the group G of the lines r and r' acts transitively on the points of r. If P is any point of r, then the stabilizer $G_{\{r,r',P\}}$ of P in $G_{\{r,r'\}}$ fixes the point $P^\perp \cap r'$ and acts transitively on the remaining points of r'. This means that we can suppose, without loss of generality, that the line s with equations $x_1 = x_2 = 0$ belongs to the pseudoregulus associated with $L(\mathbb{S})$. Let $R = r \cap s = \langle (1, 0, 0, 0) \rangle_{q^t}$ and $R' = r' \cap s = \langle (0, 0, 0, 1) \rangle_{q^t}$. By Theorems 3.13 and 3.5 there exist a semilinear collineation $\Phi : \langle (x, y, 0, 0) \rangle_{q^t} \in r \mapsto \langle (0, 0, h(x, y), g(x, y)) \rangle_{q^t} \in r'$ having

$\sigma \in Aut(\mathbb{F}_{q^t})$, with $Fix\,\sigma = \mathbb{F}_q$, as companion automorphism, and an element $\rho \in \mathbb{F}_{q^t}^*$ such that

$$L(\mathbb{S}) = \left\{ \langle (x, y, \rho h(x, y), \rho g(x, y)) \rangle_{q^t} : x, y \in \mathbb{F}_{q^t}, (x, y) \neq (0, 0) \right\}.$$

Since Φ is semilinear with companion automorphism σ, we have that

$$h(x, y) = a_1 x^\sigma + a_2 y^\sigma \quad \text{and} \quad g(x, y) = b_1 x^\sigma + b_2 y^\sigma,$$

where $a_1, a_2, b_1, b_2 \in \mathbb{F}_{q^t}$. Also, since the line s belongs to the pseudoregulus associated with $L(\mathbb{S})$, we have $\Phi(R) = R'$, i.e. $h(1, 0) = a_1 = 0$. So $L(\mathbb{S}) = \{\langle (x, y, \alpha y^\sigma, \beta x^\sigma + \gamma y^\sigma) \rangle_{q^t} : x, y \in \mathbb{F}_{q^t}, (x, y) \neq (0, 0)\}$, where $\alpha = \rho a_2$, $\beta = \rho b_1$, $\gamma = \rho b_2$. Note that $L(\mathbb{S}) \cap Q^+(3, q^t) = \emptyset$ implies $\beta \neq 0$. The collineation ω of $\mathbb{P}$ defined as $\omega : \langle (x_0, x_1, x_2, x_3) \rangle_{q^t} \mapsto \langle (x_0, x_1, x_2/\beta, x_3/\beta) \rangle_{q^t}$ fixes the reguli of $Q^+(3, q^t)$ and

$$L(\mathbb{S})^\omega = \left\{ \langle (x, y, f y^\sigma, x^\sigma + g y^\sigma) \rangle_{q^t} : x, y \in \mathbb{F}_{q^t}, (x, y) \neq (0, 0) \right\},$$

where $f = \frac{\alpha}{\beta}$ and $g = \frac{\gamma}{\beta}$, i.e., up to isotopy, $\mathbb{S}$ is a $\mathbb{K}_{17}$ Knuth semifield. Finally, if the transversal lines of $L(\mathbb{S})$ belong to $\mathcal{R}_2$, arguing as in the previous case, we get that $\mathbb{S}$ is isotopic to a $\mathbb{K}_{19}$ Knuth semifield. $\qquad \square$

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