J Algebr Comb (2013) 38:225–242
DOI 10.1007/s10801-012-0401-1

Bipartite Q-polynomial distance-regular graphs and uniform posets

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Received: 16 August 2011 / Accepted: 24 September 2012 / Published online: 9 October 2012

Abstract Let Γ denote a bipartite distance-regular graph with vertex set X and diameter $D \geq 3$. Fix $x \in X$ and let L (resp., R) denote the corresponding lowering (resp., raising) matrix. We show that each Q-polynomial structure for Γ yields a certain linear dependency among RL^2, LRL, L^2R, L. Define a partial order $\leq$ on X as follows. For $y, z \in X$ let $y \leq z$ whenever $\partial(x, y) + \partial(y, z) = \partial(x, z)$, where ∂ denotes path-length distance. We determine whether the above linear dependency gives this poset a uniform or strongly uniform structure. We show that except for one special case a uniform structure is attained, and except for three special cases a strongly uniform structure is attained.

Keywords Distance-regular graphs · Q-Polynomial structure · Uniform posets

1 Introduction

In his thesis [12], Delsarte introduced the Q-polynomial property for a distance-regular graph Γ (see Sect. 2 for formal definitions). Since then the Q-polynomial property has been investigated by many authors, such as Bannai and Ito [1], Brouwer, Cohen and Neumaier [3], Caughman [4–9], Curtin [10, 11], Jurišić, Terwilliger, and Žitnik [14], Lang [15, 16], Lang and Terwilliger [17], Miklavič [18–21], Pascasio [22, 23], Tanaka [24, 25], Terwilliger [26, 27, 30, 32], and Weng [33, 34].

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To simplify this investigation, it is sometimes assumed that Γ is bipartite [4–9, 15, 16, 19, 20] and this is the point of view taken in the present paper. For the rest of this Introduction, assume Γ is bipartite and Q-polynomial. To avoid trivialities, assume Γ has diameter $D \geq 3$ and valency $k \geq 3$.

In [28], Terwilliger introduced the subconstituent algebra of Γ. For each vertex x of Γ, the corresponding subconstituent algebra $T = T(x)$ is generated by the adjacency matrix A and a certain diagonal matrix $A^* = A^*(x)$. The eigenspaces of A^* are the subconstituents of Γ with respect to x. The matrices A and A^* satisfy two relations called the tridiagonal relations [29, Lemma 5.4], [31]. The first (resp., second) tridiagonal relation is of degree 3 in A (resp., A^*) and of degree 1 in A^* (resp., A). In [29], the tridiagonal relations are used to describe the combinatorics of Γ. In this description, it is natural to view Γ as the Hasse diagram for a ranked poset. The partial order $\leq$ is defined as follows. For vertices y, z of Γ, let $y \leq z$ whenever $\partial(x, y) + \partial(y, z) = \partial(x, z)$, where ∂ denotes path-length distance. The poset structure induces a decomposition $A = L + R$, where $L = L(x)$ (resp., $R = R(x)$) is the lowering matrix (resp., raising matrix) of Γ with respect to x. For vertices y, z of Γ, the (y, z)-entry of L is 1 if z covers y, and 0 otherwise. The matrix R is the transpose of L. In the first tridiagonal relation, if one eliminates A using $A = L + R$, one finds that on each x-subconstituent of Γ the elements

$$RL^2, \quad LRL, \quad L^2R, \quad L$$

are linearly dependent. The coefficients in this linear dependence depend on the subconstituent. We call this collection of dependencies an R/L dependency structure.

Motivated by these R/L dependency structures, in [27] Terwilliger introduced the uniform property for a partially ordered set. In that work, he described the algebraic structure of the uniform posets and displayed eleven infinite families of examples.

In spite of the known connection between the Q-polynomial property and uniform posets, a careful study of this connection was not completed until now. The goal of the present paper is to provide this study. As part of this study we introduce a variation on the uniform property called strongly uniform. Strongly uniform implies uniform. For each Q-polynomial structure on Γ we determine precisely when the corresponding R/L dependency structure is uniform or strongly uniform. To describe our results, let $\{\theta_i\}_{i=0}^D$ denote the ordering of the eigenvalues of Γ for the given Q-polynomial structure. Consider the following cases:

(i) Γ is the hypercube $H(D, 2)$ with D even and $\theta_i = (-1)^i(D-2i)$ for $0 \leq i \leq D$;
(ii) Γ is the antipodal quotient $\overline{H}(2D, 2)$ and $\theta_i = 2D - 4i$ for $0 \leq i \leq D$;
(iii) $D = 3$ and Γ is of McFarland type with parameters $(1, t)$ for some integer $t \geq 2$, and $\theta_0, \theta_1, \theta_2, \theta_3$ are $t(t+1), t, -t, -t(t+1)$ respectively.

(See Sect. 4 for the meaning of McFarland type.) In Case (i), the corresponding R/L dependency structure is not uniform. In Cases (ii) and (iii), this structure is uniform but not strongly uniform. In all other cases, this structure is strongly uniform.

The paper is organized as follows. In Sects. 2 and 3, we discuss the Bose–Mesner algebra and the dual Bose–Mesner algebra of a distance-regular graph. In Sects. 4 and 5, we consider the bipartite case and discuss the associated poset structure. In Sect. 6, we consider R/L dependency structures. In Sect. 7, we review the uniform property and define the strongly uniform property. In Sects. 8–11, we consider a given Q-polynomial structure for our graph. We determine precisely when the corresponding R/L dependency structure is uniform or strongly uniform. Our main result is Theorem 11.9.

2 Preliminaries

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with entries in $\mathbb{R}$, and rows and columns indexed by X. Let $V = \mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of the column vectors with entries in $\mathbb{R}$ and rows indexed by X. Observe that $\mathrm{Mat}_X(\mathbb{R})$ acts on V by left multiplication. We refer to V as the *standard module* of $\mathrm{Mat}_X(\mathbb{R})$. We endow V with the bilinear form $\langle\, ,\rangle : V \times V \to \mathbb{R}$ that satisfies $\langle u, v\rangle = u^t v$ for $u, v \in V$, where t denotes transpose. For $y \in X$ let $\hat{y}$ denote the vector in V that has y-coordinate 1 and all other coordinates 0. Observe that $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V.

Throughout the paper, let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set $\mathcal{R}$, path-length distance function ∂, and diameter $D := \max\{\partial(x, y) \mid x, y \in X\}$. For $x \in X$ and an integer i, let $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \ge 0$, we say Γ is *regular with valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$. We say Γ is *distance-regular* whenever for all integers $0 \le h, i, j \le D$ and all $x, y \in X$ with $\partial(x, y) = h$ the number $p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1,i-1}$ $(1 \le i \le D)$, $a_i := p^i_{1i}$ $(0 \le i \le D)$, $b_i := p^i_{1,i+1}$ $(0 \le i \le D-1)$, $k_i := p^0_{ii}$ $(0 \le i \le D)$, and $c_0 := 0$, $b_D := 0$. For the rest of this paper, assume Γ is distance-regular with diameter $D \ge 3$. By the triangle inequality, for $0 \le h, i, j \le D$ we have $p^h_{ij} = 0$ (resp., $p^h_{ij} \ne 0$) whenever one of h, i, j is greater than (resp., equal to) the sum of the other two. In particular, $c_i \ne 0$ for $1 \le i \le D$ and $b_i \ne 0$ for $0 \le i \le D-1$. Observe that Γ is regular with valency $k = b_0 = k_1$ and that $c_i + a_i + b_i = k$ for $0 \le i \le D$.

We recall the Bose–Mesner algebra of Γ. For $0 \le i \le D$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (y, z)-entry

$$(A_i)_{yz} = \begin{cases} 1 & \text{if } \partial(y, z) = i, \\ 0 & \text{if } \partial(y, z) \ne i \end{cases} \qquad (y, z \in X). \tag{1}$$

We call A_i the ith *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. We observe (ai) $A_0 = I$; (aii) $J = \sum_{i=0}^D A_i$; (aiii) $A_i^t = A_i$ $(0 \le i \le D)$; (aiv) $A_i A_j = \sum_{h=0}^D p^h_{ij} A_h$ $(0 \le i, j \le D)$, where I (resp., J) denotes the identity matrix (resp., all 1s matrix) in $\mathrm{Mat}_X(\mathbb{R})$. Using these facts we find $\{A_i\}_{i=0}^D$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{R})$. We call M the *Bose–Mesner algebra* of Γ. By [1, p. 190], A generates M. By [3, p. 45], M has a basis $\{E_i\}_{i=0}^D$ such that (ei) $E_0 = |X|^{-1} J$; (eii) $I = \sum_{i=0}^D E_i$; (eiii) $E_i^t = E_i$ $(0 \le i \le D)$;

(eiv) $E_i E_j = \delta_{ij} E_i$ $(0 \le i, j \le D)$. We call $\{E_i\}_{i=0}^D$ the *primitive idempotents* of Γ. The primitive idempotent E_0 is said to be *trivial*.

We recall the eigenvalues of Γ. Since $\{E_i\}_{i=0}^D$ form a basis for M, there exist scalars $\{\theta_i\}_{i=0}^D$ in $\mathbb{R}$ such that $A = \sum_{i=0}^D \theta_i E_i$. Combining this with (eiv), we find

$$A E_i = E_i A = \theta_i E_i \quad (0 \le i \le D).$$

We call θ_i the *eigenvalue* of Γ associated with E_i. The $\{\theta_i\}_{i=0}^D$ are mutually distinct since A generates M. By (ei) we have $\theta_0 = k$. By (eii)–(eiv),

$$V = E_0 V + E_1 V + \cdots + E_D V \quad \text{(orthogonal direct sum).} \tag{2}$$

For $0 \le i \le D$ the space $E_i V$ is the eigenspace of A associated with θ_i. Let m_i denote the rank of E_i and note that m_i is the dimension of $E_i V$. We call m_i the *multiplicity* of θ_i.

We recall the Krein parameters of Γ. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{R})$. Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \le i, j \le D$, so M is closed under $\circ$. Thus there exist scalars $q_{ij}^h \in \mathbb{R}$ $(0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \le i, j \le D).$$

The parameters q_{ij}^h are called the *Krein parameters of* Γ. By [3, Proposition 4.1.5], these parameters are nonnegative. The given ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents is said to be *Q-polynomial* whenever for $0 \le h, i, j \le D$ the Krein parameter $q_{ij}^h = 0$ (resp., $q_{ij}^h \ne 0$) whenever one of h, i, j is greater than (resp., equal to) the sum of the other two. Let E denote a nontrivial primitive idempotent of Γ and let θ denote the corresponding eigenvalue. We say that Γ is *Q-polynomial with respect to E* (or θ) whenever there exists a Q-polynomial ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of Γ such that $E_1 = E$.

3 The dual Bose–Mesner algebra

We continue to discuss the distance-regular graph Γ from Sect. 2. In this section, we recall the dual Bose–Mesner algebra of Γ. For the rest of the paper, fix $x \in X$. For $0 \le i \le D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \ne i \end{cases} \quad (y \in X). \tag{3}$$

We call E_i^* the ith *dual idempotent* of Γ with respect to x [28, p. 378]. For convenience, set $E_i^* = 0$ for $i < 0$ or $i > D$. We observe (esi) $I = \sum_{i=0}^D E_i^*$; (esii) $E_i^{*t} = E_i^*$ $(0 \le i \le D)$; (esiii) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i, j \le D)$. By these facts, $\{E_i^*\}_{i=0}^D$

forms a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{R})$. We call M^* the *dual Bose–Mesner algebra* of Γ with respect to x [28, p. 378]. By (esi)–(esiii),

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \quad \text{(orthogonal direct sum)}. \tag{4}$$

For $0 \le i \le D$ the subspace $E_i^* V$ has basis $\{\hat{y} \mid y \in \Gamma_i(x)\}$. Moreover, the dimension of $E_i^* V$ is k_i.

The algebras M and M^* are related as follows. By [28, Lemma 3.2],

$$E_i^* A_j E_h^* = 0 \quad \text{if and only if} \quad p_{ij}^h = 0 \quad (0 \le h, i, j \le D). \tag{5}$$

Let E denote a nontrivial primitive idempotent of Γ and assume Γ is Q-polynomial with respect to E. Let $A^* = A^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (y, y)-entry

$$A^*_{yy} = |X| E_{xy} \quad (y \in X).$$

We call A^* the *dual adjacency matrix of* Γ that corresponds to E and x. By [28, Lemma 3.11(ii)], A^* generates M^*. We recall the dual eigenvalues for our Q-polynomial structure. Since $\{E_i^*\}_{i=0}^D$ forms a basis for M^* there exist scalars $\{\theta_i^*\}_{i=0}^D$ in $\mathbb{R}$ such that $A^* = \sum_{i=0}^D \theta_i^* E_i^*$. Combining this with (esiii), we find

$$A^* E_i^* = E_i^* A^* = \theta_i^* E_i^* \quad (0 \le i \le D). \tag{6}$$

We call $\{\theta_i^*\}_{i=0}^D$ the *dual eigenvalue sequence* for the given Q-polynomial structure. The $\{\theta_i^*\}_{i=0}^D$ are mutually distinct since A^* generates M^*. For $0 \le i \le D$ the space $E_i^* V$ is the eigenspace of A^* associated with θ_i^*. By [1, Proposition 3.4.(iv)], we have that $\theta_0^* = \mathrm{rank}(E)$. Let θ denote the eigenvalue of Γ associated with E. By [3, p. 128],

$$c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta \theta_i^* \quad (0 \le i \le D), \tag{7}$$

where θ_{-1}^* and θ_{D+1}^* are indeterminants.

Lemma 3.1 ([29, Lemma 5.4]) *Let $\{E_i\}_{i=0}^D$ denote a Q-polynomial ordering of the primitive idempotents of Γ and for $0 \le i \le D$ let θ_i denote the eigenvalue of Γ for E_i. Let $\{\theta_i^*\}_{i=0}^D$ denote the dual eigenvalue sequence for the given Q-polynomial structure. Then the following* (i)–(iii) *hold.*

(i) *There exists $\beta \in \mathbb{R}$ such that*

$$\beta + 1 = \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \tag{8}$$

for $2 \le i \le D - 1$.

(ii) *There exist $\gamma, \gamma^* \in \mathbb{R}$ such that both*

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}, \qquad \gamma^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* \tag{9}$$

for $1 \le i \le D - 1$.

(iii) *There exist* $\varrho, \varrho^* \in \mathbb{R}$ *such that both*

$$\begin{aligned} \varrho &= \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i), \\ \varrho^* &= \theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) \end{aligned} \tag{10}$$

for $1 \le i \le D$.

Lemma 3.2 ([29, Lemma 5.4]) *Let* E *denote a* Q*-polynomial primitive idempotent of* Γ *and let* $A^* = A^*(x)$ *denote the corresponding dual adjacency matrix. Then both*

$$\left[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma\left(AA^* + A^*A\right) - \varrho A^*\right] = 0, \tag{11}$$

$$\left[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*\left(A^*A + AA^*\right) - \varrho^* A\right] = 0, \tag{12}$$

where $[r, s] = rs - sr$ *and* $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ *are from Lemma* 3.1.

4 Bipartite distance-regular graphs

We continue to discuss the distance-regular graph Γ from Sect. 2. Recall that Γ is *bipartite* whenever $a_i = 0$ for $0 \le i \le D$. For Γ bipartite, $p_{ij}^h = 0$ if $h + i + j$ is odd ($0 \le h, i, j \le D$). In this case,

$$E_i^* A E_h^* = 0 \quad \text{if } |h - i| \ne 1 \quad (0 \le h, i \le D). \tag{13}$$

The case in which Γ is bipartite with $D = 3$ will play an important role.

By [3, Theorem 1.6.1.], Γ is bipartite with $D = 3$ if and only if Γ is the incidence graph of a square 2-(v, k, λ) design. In this case, $c_2 = \lambda$ and $v = 1 + k(k - 1)/\lambda$. See [2] for more information and background on square 2-designs.

Pick integers $d \ge 1$ and $t \ge 2$. A square 2-(v, k, λ) design is said to have *McFarland type with parameters* (d, t) whenever

$$v = t^{d+1}\left(1 + \frac{t^{d+1} - 1}{t - 1}\right), \qquad k = t^d \frac{t^{d+1} - 1}{t - 1}, \qquad \lambda = t^d \frac{t^d - 1}{t - 1}.$$

For the moment assume that t is a prime power. By [2, Corollary II.8.17], a square 2-design of McFarland type with parameters (d, t) exists for every integer $d \ge 1$. By [2, p. 982], this design can be realized as a McFarland difference set.

Our graph Γ is said to have *McFarland type with parameters* (d, t) whenever Γ is the incidence graph of a square 2-design of McFarland type with parameters (d, t).

5 The bipartite case; lowering and raising matrices

We continue to discuss the distance-regular graph Γ from Sect. 2. For the rest of this paper, assume that Γ is bipartite.

Define a partial order $\leq$ on X such that for all $y, z \in X$,

$$y \leq z \quad \text{if and only if} \quad \partial(x, y) + \partial(y, z) = \partial(x, z).$$

For $y, z \in X$ define $y < z$ whenever $y \leq z$ and $y \neq z$. We say that z *covers* y whenever $y < z$ and there does not exist $w \in X$ such that $y < w < z$. Note that z covers y if and only if y, z are adjacent and $\partial(x, y) + 1 = \partial(x, z)$. For $0 \leq i \leq D$ each vertex in $\Gamma_i(x)$ covers exactly c_i vertices from $\Gamma_{i-1}(x)$, and is covered by exactly b_i vertices in $\Gamma_{i+1}(x)$. Therefore, the partition $\{\Gamma_i(x)\}_{i=0}^D$ of X is a *grading* of the poset $(X, \leq)$ in the sense of [27, Sect. 1].

Definition 5.1 Define matrices $L = L(x)$ and $R = R(x)$ by

$$L = \sum_{i=1}^{D} E^*_{i-1} A E^*_i, \qquad R = \sum_{i=0}^{D-1} E^*_{i+1} A E^*_i.$$

Note that $R = L^t$ and $L + R = A$.

We have three observations.

Lemma 5.2 *Let L, R be as in Definition 5.1. Then the following* (i), (ii) *hold for $y \in X$.*

(i) $L\hat{y} = \sum \hat{z}$, *where the sum is over all $z \in X$ that are covered by y*;
(ii) $R\hat{y} = \sum \hat{z}$, *where the sum is over all $z \in X$ that cover y.*

Motivated by Lemma 5.2, we call L (resp., R) the *lowering matrix* (resp., *raising matrix*) *of Γ with respect to x.*

Lemma 5.3 *Let L, R be as in Definition 5.1. Then the following* (i), (ii) *hold.*

(i) $RE^*_i V \subseteq E^*_{i+1} V$ *for* $0 \leq i \leq D - 1$, *and* $RE^*_D V = 0$;
(ii) $LE^*_i V \subseteq E^*_{i-1} V$ *for* $1 \leq i \leq D$, *and* $LE^*_0 V = 0$.

Lemma 5.4 *Let L, R be as in Definition 5.1. Then for $1 \leq i \leq D$ the following* (i)–(iv) *hold.*

(i) $E^*_{i-1} A E^*_i = L E^*_i$;
(ii) $E^*_{i-1} A E^*_i = E^*_{i-1} L$;
(iii) $E^*_i A E^*_{i-1} = R E^*_{i-1}$;
(iv) $E^*_i A E^*_{i-1} = E^*_i R$.

Moreover

$$L E^*_0 = 0, \qquad E^*_D L = 0, \qquad R E^*_D = 0, \qquad E^*_0 R = 0. \tag{14}$$

Lemma 5.5 *Let L, R be as in Definition 5.1. Then*

$$E^*_{i-1} A^3 E^*_i = RL^2 E^*_i + LRLE^*_i + L^2 R E^*_i$$

for $1 \leq i \leq D$.

Proof Straightforward using $A = L + R$ and Lemma 5.3. □

From now on we use the following notational convention.

Notation 5.6 For the rest of this paper, we assume our distance-regular graph Γ is bipartite with valency $k \geq 3$. Let $\{E_i\}_{i=0}^D$ denote a Q-polynomial ordering of the primitive idempotents of Γ and let $\{\theta_i\}_{i=0}^D$ denote the corresponding eigenvalues. Abbreviate $E = E_1$. Recall our fixed vertex $x \in X$ from Sect. 3. For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the ith dual idempotent of Γ with respect to x. Let $A^* = A^*(x)$ denote the dual adjacency matrix of Γ that corresponds to E and x. Let $\{\theta_i^*\}_{i=0}^D$ denote the dual eigenvalue sequence for the given Q-polynomial structure. Let the scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ be from Lemma 3.1. Let the matrices $L = L(x)$ and $R = R(x)$ be as in Definition 5.1.

With reference to Notation 5.6, we have $\gamma = 0$ by [3, Theorem 8.2.1] and since Γ is bipartite. Thus by (11),

$$\big[A, A^2A^* - \beta AA^*A + A^*A^2 - \varrho A^*\big] = 0. \tag{15}$$

6 The R/L dependency structure

In this section, we display certain linear dependencies among RL^2, RLR, L^2R, L.

Lemma 6.1 *With reference to Notation 5.6 the following* (i), (ii) *hold for* $1 \leq i \leq D$.

(i) $E_{i-1}^* A^2A^*AE_i^* = \theta_{i-1}^* RL^2E_i^* + \theta_{i-1}^* LRLE_i^* + \theta_{i+1}^* L^2RE_i^*$;
(ii) $E_{i-1}^* AA^*A^2E_i^* = \theta_{i-2}^* RL^2E_i^* + \theta_i^* LRLE_i^* + \theta_i^* L^2RE_i^*$.

Proof Straightforward using $A = L + R$ along with (6) and Lemma 5.3. □

Proposition 6.2 *With reference to Notation* 5.6, *for* $1 \leq i \leq D$ *the equation*

$$\begin{aligned} &\frac{\theta_i^* - \theta_{i-1}^* + (\beta+1)(\theta_{i-2}^* - \theta_{i-1}^*)}{\theta_i^* - \theta_{i-1}^*} RL^2 + (\beta+2)LRL \\ &\quad + \frac{\theta_i^* - \theta_{i-1}^* + (\beta+1)(\theta_i^* - \theta_{i+1}^*)}{\theta_i^* - \theta_{i-1}^*} L^2R = \varrho L \end{aligned} \tag{16}$$

holds on E_i^*V.

Proof Multiply (15) by E_{i-1}^* on the left and by E_i^* on the right. Divide the result by $\theta_{i-1}^* - \theta_i^*$ and simplify using (6) along with Lemmas 5.4(i), 5.5, 6.1. □

We call the equations (16) the *R/L dependency structure* that corresponds to the given Q-polynomial structure. We have a comment about the coefficients in line (16).

Lemma 6.3 *With reference to Notation 5.6 the following* (i), (ii) *hold.*

(i) *For* $3 \le i \le D$,

$$\frac{\theta^*_i - \theta^*_{i-1} + (\beta+1)(\theta^*_{i-2} - \theta^*_{i-1})}{\theta^*_i - \theta^*_{i-1}} = \frac{\theta^*_{i-3} - \theta^*_{i-1}}{\theta^*_i - \theta^*_{i-1}}.$$

(ii) *For* $1 \le i \le D-2$,

$$\frac{\theta^*_i - \theta^*_{i-1} + (\beta+1)(\theta^*_i - \theta^*_{i+1})}{\theta^*_i - \theta^*_{i-1}} = \frac{\theta^*_i - \theta^*_{i+2}}{\theta^*_i - \theta^*_{i-1}}.$$

Proof (i) Evaluate the left-hand side using $\beta+1 = (\theta^*_{i-3} - \theta^*_i)/(\theta^*_{i-2} - \theta^*_{i-1})$.
(ii) Evaluate the left-hand side using $\beta+1 = (\theta^*_{i-1} - \theta^*_{i+2})/(\theta^*_i - \theta^*_{i+1})$. □

7 Uniform structures on a poset

In this section, we discuss the uniform property for a partially ordered set [27]. This property involves the notion of a parameter matrix. With reference to Notation 5.6, by a *parameter matrix* we mean a tridiagonal matrix $U = (e_{ij})_{1 \le i,j \le D}$ with entries in $\mathbb{R}$ such that

1. $e_{ii} = 1$ for $1 \le i \le D$;
2. $e_{i,i-1} \ne 0$ for $2 \le i \le D$ or $e_{i-1,i} \ne 0$ for $2 \le i \le D$;
3. the principal submatrix $(e_{ij})_{r \le i,j \le p}$ is nonsingular for $1 \le r \le p \le D$.

We abbreviate $e^-_i := e_{i,i-1}$ for $2 \le i \le D$ and $e^+_i := e_{i,i+1}$ for $1 \le i \le D-1$. For notational convenience, define $e^-_1 := 0$ and $e^+_D := 0$.

By a *uniform structure* for Γ we mean a pair (U, f) where $U = (e_{ij})_{1 \le i,j \le D}$ is a parameter matrix and $f = \{f_i\}_{i=1}^D$ is a vector in $\mathbb{R}^D$ such that the equation

$$e^-_i RL^2 + LRL + e^+_i L^2 R = f_i L \tag{17}$$

holds on $E^*_i V$ for $1 \le i \le D$. By a *strongly uniform structure* for Γ we mean a uniform structure (U, f) for Γ such that $e_{i,i-1} \ne 0$ and $e_{i-1,i} \ne 0$ for $2 \le i \le D$. Note that a strongly uniform structure is uniform.

Lemma 7.1 *With reference to Notation 5.6, let* (U, f) *denote a uniform structure for* Γ. *Then the equation*

$$e^-_i R^2 L + RLR + e^+_i LR^2 = f_i R$$

holds on $E^*_{i-1} V$ *for* $1 \le i \le D$.

Proof The equation (17) holds on $E^*_i V$ so

$$\big(e^-_i RL^2 + LRL + e^+_i L^2 R - f_i L\big) E^*_i = 0. \tag{18}$$

By Lemma 5.4, we have $LE^*_j = E^*_{j-1}L$ and $E^*_j R = RE^*_{j-1}$ for $1 \le j \le D$. Evaluating (18) using this and (14), we find

$$E^*_{i-1}\big(e^-_i RL^2 + LRL + e^+_i L^2R - f_i L\big) = 0. \tag{19}$$

In line (19), apply the transpose map to each term and recall $R = L^t$. This yields

$$\big(e^-_i R^2L + RLR + e^+_i LR^2 - f_i R\big)E^*_{i-1} = 0,$$

and the result follows. □

See [27] for more information on uniform posets.

Recall our Q-polynomial structure from Notation 5.6. Our next goal is to determine in which cases the corresponding R/L dependency structure is uniform or strongly uniform. We first consider the case in which $\beta = -2$, where β is from line (8).

8 The case $\beta = -2$

Recall our Q-polynomial structure from Notation 5.6. In this section, we determine whether the corresponding R/L dependency structure is uniform or strongly uniform, for the case $\beta = -2$. We will be discussing the D-dimensional hypercube $H(D,2)$. By [3, Theorem 9.2.1], $H(D,2)$ is distance-regular with diameter D and intersection numbers

$$b_i = D - i, \quad c_i = i \quad (0 \le i \le D). \tag{20}$$

By [3, Theorem 9.2.1], the eigenvalues of $H(D,2)$ are $\{D-2i\}_{i=0}^D$. By [1, p. 304], the ordering $\{D-2i\}_{i=0}^D$ is Q-polynomial. For this Q-polynomial structure $\beta = 2$. If D is odd then this Q-polynomial structure is unique. If D is even then $H(D,2)$ has exactly one more Q-polynomial structure, with eigenvalue ordering $\{(-1)^i(D-2i)\}_{i=0}^D$ [1, p. 305]. For this Q-polynomial structure $\beta = -2$.

Proposition 8.1 ([26, Theorem 2]) *With reference to Notation 5.6, assume $\beta = -2$. Then D is even and Γ is $H(D,2)$ with the following Q-polynomial ordering of the eigenvalues*:

$$\theta_i = (-1)^i(D-2i) \quad (0 \le i \le D). \tag{21}$$

Lemma 8.2 *With reference to Notation 5.6, assume Γ is $H(D,2)$ and let $\{\theta_i\}_{i=0}^D$ denote the Q-polynomial ordering of the eigenvalues (21). Let $\{\theta^*_i\}_{i=0}^D$ denote the corresponding dual eigenvalue sequence. Then $\theta^*_i = \theta_i$ for $0 \le i \le D$. Also*

$$\beta = -2, \qquad \gamma^* = 0, \qquad \varrho = 4, \qquad \varrho^* = 4.$$

Proof We have $\theta^*_i = \theta_i$ by [4, Theorem 1.1]. The remaining assertions follow from Lemma 3.1. □

Proposition 8.3 *With reference to Notation 5.6, assume Γ is $H(D,2)$ and consider the Q-polynomial ordering of the eigenvalues (21). Then the corresponding R/L dependency structure is that the equation*

$$\frac{2}{2i-D-1}RL^2 - \frac{2}{2i-D-1}L^2R = 4L \tag{22}$$

holds on E_i^*V for $1 \le i \le D$.

Proof Evaluate (16) using Lemma 8.2. □

Proposition 8.4 *With reference to Notation 5.6, assume Γ is $H(D,2)$ and consider the Q-polynomial ordering of the eigenvalues (21). Then the corresponding R/L dependency structure is not uniform.*

Proof The equation (22) does not match the form (17). □

9 The case $\beta \neq -2$

Recall our Q-polynomial structure from Notation 5.6. Until further notice assume $\beta \neq -2$. Under this assumption, we show that the corresponding R/L dependency structure is uniform. Moreover, we show that this structure is strongly uniform except in two special cases. The following definition is for notational convenience.

Definition 9.1 With reference to Notation 5.6, assume $\beta \neq -2$. Let $U = (e_{ij})_{1\le i,j\le D}$ denote the tridiagonal matrix with entries

$$\begin{aligned} e_{ii} &= 1 && (1 \le i \le D),\\ e_{i,i-1} &= \frac{\theta_i^* - \theta_{i-1}^* + (\beta+1)(\theta_{i-2}^* - \theta_{i-1}^*)}{(\beta+2)(\theta_i^* - \theta_{i-1}^*)} && (2 \le i \le D),\\ e_{i-1,i} &= \frac{\theta_{i-1}^* - \theta_{i-2}^* + (\beta+1)(\theta_{i-1}^* - \theta_i^*)}{(\beta+2)(\theta_{i-1}^* - \theta_{i-2}^*)} && (2 \le i \le D). \end{aligned}$$

For notational convenience, write $e_i^- = e_{i,i-1}$ for $2 \le i \le D$ and $e_1^- = 0$, and also $e_i^+ = e_{i,i+1}$ for $1 \le i \le D-1$ and $e_D^+ = 0$. Define a vector $\{f_i\}_{i=1}^D$ in $\mathbb{R}^D$ such that $f_i = \varrho/(\beta+2)$ for $1 \le i \le D$.

Proposition 9.2 *With reference to Notation 5.6 and Definition 9.1, the equation*

$$e_i^- RL^2 + LRL + e_i^+ L^2R = f_i L \tag{23}$$

holds on E_i^*V for $1 \le i \le D$.

Proof Divide (16) by $\beta+2$ and use Definition 9.1. □

Table 1 Special cases

Case	Γ	Q-polynomial structure
I	$H(D,2)$	$\theta_i = D-2i\ (0 \le i \le D)$
II	$\overline{H}(2D,2)$	$\theta_i = 2D-4i\ (0 \le i \le D)$
III	$D=3, b_2=1$	$\theta_0=k, \theta_1=1, \theta_2=-1, \theta_3=-k$
IV	$D=3, b_2>1$	$\theta_0=k, \theta_1=\sqrt{b_2}, \theta_2=-\sqrt{b_2}, \theta_3=-k$
V	$D=3, b_2>1$	$\theta_0=k, \theta_1=-\sqrt{b_2}, \theta_2=\sqrt{b_2}, \theta_3=-k$

Our next general goal is to determine whether the equations (23) give a uniform or strongly uniform structure. In order to do this, we introduce some parameters q and s^*.

10 The parameters q and s^*

Recall our Q-polynomial structure from Notation 5.6. We would like to write the corresponding data in terms of two parameters q and s^*. However, it will be convenient to exclude several special cases. The first special case is $H(D,2)$ with eigenvalue ordering $\{D-2i\}_{i=0}^D$. The next special case concerns the antipodal quotient of $H(2D,2)$. We denote this quotient graph by $\overline{H}(2D,2)$. By [3, p. 264], $\overline{H}(2D,2)$ is distance-regular with diameter D and intersection numbers

$$b_i = 2D-i, \quad c_i = i \quad (0 \le i \le D-1),$$

and $c_D = 2D$. By [3, p. 264], the eigenvalues of $\overline{H}(2D,2)$ are

$$\theta_i = 2D-4i \quad (0 \le i \le D). \tag{24}$$

By [1, p. 306], the ordering (24) is the unique Q-polynomial structure for $\overline{H}(2D,2)$. In order to describe some more special cases, we turn our attention to Notation 5.6 with $D=3$. By [3, Proposition 4.2.2.(ii)], $b_2=1$ if and only if Γ is antipodal. In this case, $b_1=k-1$, $c_2=k-1$, $c_3=k$. Moreover, Γ has a unique Q-polynomial structure with eigenvalues $\theta_0=k$, $\theta_1=1$, $\theta_2=-1$, $\theta_3=-k$ [3, p. 432]. For $b_2>1$, Γ has exactly two Q-polynomial structures: $\theta_0=k$, $\theta_1=\sqrt{b_2}$, $\theta_2=-\sqrt{b_2}$, $\theta_3=-k$ and $\theta_0=k$, $\theta_1=-\sqrt{b_2}$, $\theta_2=\sqrt{b_2}$, $\theta_3=-k$ [3, p. 432]. In the following table, we summarize the cases discussed so far.

Lemma 10.1 *With reference to Notation 5.6, assume the Q-polynomial structure is listed in Table 1. Then the corresponding dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^D$ is given in the table below.*

Case	Dual eigenvalue sequence
I	$\theta_i^* = D-2i\ (0 \le i \le D)$
II	$\theta_i^* = 2(D-i)^2-D\ (0 \le i \le D)$
III	$\theta_0^*=k, \theta_1^*=1, \theta_2^*=-1, \theta_3^*=-k$
IV, V	$\theta_0^*=\frac{k(k-1)}{c_2}, \theta_1^*=\frac{\theta_1(k-1)}{c_2}, \theta_2^*=-1, \theta_3^*=-\frac{k}{\theta_1}$

Proof The $\{\theta^*_i\}_{i=0}^D$ are computed using (7) with $\theta = \theta_1$, once θ^*_0 is known. Recall that θ^*_0 is the rank of E_1. In Case I, the rank of E_1 is D by [3, Theorem 9.2.1]. In Case II, the rank of E_1 is $2D^2 - D$ by [3, p. 264]. In Case III, the rank of E_1 is k by [3, p. 432]. In Cases IV and V, the rank of E_1 is $k(k-1)/c_2$ by [3, p. 432]. The result follows. □

Lemma 10.2 *With reference to Notation 5.6, assume the Q-polynomial structure is listed in Table 1. Then β, γ^*, ϱ, ϱ^* are given in the table below.*

Case	β	γ^*	ϱ	ϱ^*
I	2	0	4	4
II	2	4	16	$4(2D-1)$
III	$k-1$	0	$k+1$	$k+1$
IV, V	$\frac{k-\theta_1}{\theta_1}$	$\frac{(k-1)\theta_1-c_2}{c_2}$	$\theta_1(\theta_1+k)$	$\frac{(k-1)(k+\theta_1)}{c_2}$

Proof Use Lemma 3.1 and Lemma 10.1. □

We have now completed our description of the special cases.

Lemma 10.3 ([9, Lemma 3.2, Lemma 3.3]) *With reference to Notation 5.6, assume the Q-polynomial structure is not listed in Table 1 and $\beta \neq -2$. Then there exist $q, s^* \in \mathbb{R}$ such that the following* (i)–(iii) *hold.*

(i) $|q| > 1$, $s^* q^i \neq 1$ $(2 \le i \le 2D+1)$;

(ii) $b_0 = h(q^D - 1) = c_D$,

$$b_i = \frac{h(q^D - q^i)(1 - s^* q^{i+1})}{1 - s^* q^{2i+1}}, \quad c_i = \frac{h(q^i - 1)(1 - s^* q^{D+i+1})}{1 - s^* q^{2i+1}}$$
$$(1 \le i \le D-1);$$

(iii) $\theta_i = h(q^{D-i} - q^i)$, $\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^* q^{i+1})q^{-i}$ $(0 \le i \le D)$, *where*

$$h = \frac{1 - s^* q^3}{(q-1)(1 - s^* q^{D+2})}, \qquad h^* = \frac{(q^D + q^2)(q^D + q)}{q(q^2-1)(1 - s^* q^{2D})},$$
$$\theta^*_0 = \frac{h^*(q^D - 1)(1 - s^* q^2)}{q(q^{D-1}+1)}.$$

Note 10.4 With reference to Notation 5.6, assume the Q-polynomial structure is not listed in Table 1 and $\beta \neq -2$. Then by [9, Corollary 6.7] the scalar s^* from Lemma 10.3 is zero provided $D \ge 12$.

Lemma 10.5 *With reference to Notation 5.6, assume the Q-polynomial structure is not listed in Table 1 and $\beta \neq -2$. Then*

$$\beta = q + q^{-1},$$

$$\gamma^* = \frac{(q-1)(q^{D-2}+1)(1+s^*q^{D+1})}{1-s^*q^{2D}},$$

$$\varrho = \frac{q^{D-2}(q+1)^2(1-s^*q^3)^2}{(1-s^*q^{D+2})^2},$$

$$\varrho^* = \frac{q(q^{D-2}+1)^2(1-s^*q^2)}{1-s^*q^{2D}},$$

where q, s^* are from Lemma 10.3.

Proof Use Lemma 3.1 and Lemma 10.3. □

In Lemma 10.3(i), we cited some inequalities involving q and s^*. We now prove one more inequality involving q and s^*.

Lemma 10.6 *With reference to Notation 5.6, assume the Q-polynomial structure is not listed in Table 1 and $\beta \neq -2$. Then the scalars q and s^* from Lemma 10.3 satisfy $s^*q \neq 1$.*

Proof We assume $s^*q = 1$ and get a contradiction. Recall $q \in \mathbb{R}$ and $|q| > 1$. By [5, Theorem 15.6(ii)], the scalar

$$\frac{q(q^{D-1}-1)}{q^{D+1}-1}$$

is nonnegative. The factors $q^{D-1}-1$ and $q^{D+1}-1$ have the same sign, since $D-1$ and $D+1$ have the same parity. Therefore, $q > 0$. By these comments, $q > 1$. By [5, Theorem 15.6(iii)], the scalar

$$\frac{(q^D-1)(q^D-q)(1-q^3)(1+q^D)}{q(q^2-1)(1-q^{D+1})(1-q^{2D-1})} \tag{25}$$

is nonnegative. Since $q > 1$ the expression (25) is negative, for a contradiction. □

11 The main result

Recall our Q-polynomial structure from Notation 5.6. We are now ready to determine whether the corresponding R/L dependency structure is uniform or strongly uniform. We begin with some computations involving the matrix U from Definition 9.1.

Proposition 11.1 *With reference to Notation 5.6 and Definition 9.1, the scalars $\{e_i^-\}_{i=2}^D$, $\{e_i^+\}_{i=1}^{D-1}$, $\{f_i\}_{i=1}^D$ are given in the following table:*

Case	e_i^-	e_i^+	f_i
I	$-\frac{1}{2}$	$-\frac{1}{2}$	1
II	$\frac{i-D-2}{2D-2i+1}$	$\frac{i-D+1}{2D-2i+1}$	4
III	$e_2^- = \frac{2-k}{2}, e_3^- = \frac{1}{1-k}$	$e_1^+ = \frac{1}{1-k}, e_2^+ = \frac{2-k}{2}$	1
IV, V	$e_2^- = \frac{\theta_1-k+1}{\theta_1+1}, e_3^- = -\frac{\theta_1^2}{c_2}$	$e_1^+ = \frac{1}{1-k}, e_2^+ = \frac{\theta_1-c_2}{\theta_1(\theta_1+1)}$	θ_1^2
other	$-\frac{q^2(1-s^*q^{2i-3})}{(q+1)(1-s^*q^{2i})}$	$-\frac{1-s^*q^{2i+3}}{q(q+1)(1-s^*q^{2i})}$	$\frac{q^{D-1}(1-s^*q^3)^2}{(1-s^*q^{D+2})^2}$

The scalars q, s^* are from Lemma 10.3.

Proof For Cases I–V use Definition 9.1, Lemma 10.1, and Lemma 10.2. For the remaining case use Definition 9.1, Lemma 10.3, and Lemma 10.5. □

With reference to Notation 5.6, assume for the moment that $D = 3$. For an integer $t \geq 2$, the following are equivalent: (i) Γ is of McFarland type with parameters $(1, t)$; (ii) the intersection numbers of Γ satisfy $k = t(t+1)$ and $c_2 = t$. Assume that (i), (ii) hold. Then $b_1 = t^2 + t - 1$, $b_2 = t^2$, $c_3 = t(t+1)$. Moreover, the eigenvalue θ_1 is either t or $-t$. The case $\theta_1 = t$ is contained in Case IV. We call this situation Case IV′. Let us examine Case IV′ in more detail.

Lemma 11.2 *With reference to Notation 5.6, assume the Q-polynomial structure is in Case* IV′. *Then the following* (i)–(iii) *hold.*

(i) *The eigenvalues $\{\theta_i\}_{i=0}^3$ are*

$$\theta_0 = t(t+1), \qquad \theta_1 = t, \qquad \theta_2 = -t, \qquad \theta_3 = -t(t+1).$$

(ii) *The dual eigenvalues $\{\theta_i^*\}_{i=0}^3$ are*

$$\theta_0^* = (t+1)\bigl(t^2 + t - 1\bigr), \qquad \theta_1^* = t^2 + t - 1, \qquad \theta_2^* = -1, \qquad \theta_3^* = -t - 1.$$

(iii) *The parameters $\beta, \gamma^*, \varrho, \varrho^*$ from Lemma 3.1 are*

$$\beta = t, \qquad \gamma^* = t^2 + t - 2, \qquad \varrho = t^2(t+2), \qquad \varrho^* = t^3 + 3t^2 + t - 2.$$

Proof (i) Immediate from Table 1.
(ii) Immediate from Lemma 10.1.
(iii) Immediate from Lemma 10.2. □

Lemma 11.3 *With reference to Notation 5.6, assume the Q-polynomial structure is in Case IV'. Then the following* (i)–(iii) *hold.*

(i) *$e_2^- = 1 - t$ and $e_3^- = -t$;*
(ii) *$e_1^+ = -(t^2 + t - 1)^{-1}$ and $e_2^+ = 0$;*
(iii) *$f_i = t^2$ for $1 \leq i \leq 3$.*

Proof From Case IV of the table of Proposition 11.1, using $k = t(t+1)$, $c_2 = t$, and $\theta_1 = t$. □

Corollary 11.4 *With reference to Notation 5.6 and Definition 9.1, the following* (i)–(iii) *hold.*

(i) *Assume the Q-polynomial structure is in Case II. Then* $e^+_{D-1} = 0$.
(ii) *Assume the Q-polynomial structure is in Case IV′. Then* $e^+_2 = 0$.
(iii) *For all other cases* $e^-_i \neq 0$ *for* $2 \le i \le D$ *and* $e^+_i \neq 0$ *for* $1 \le i \le D-1$.

Proof (i) Immediate from Case II of the table in Proposition 11.1.
(ii) Immediate from Lemma 11.3(ii).
(iii) Immediate from Proposition 11.1, using Lemma 10.3(i) and Lemma 10.6. □

We recall a result from linear algebra.

Lemma 11.5 ([13, p. 29]) *Pick an integer $d \ge 3$ and let $B = (B_{ij})_{1\le i,j\le d}$ denote a tridiagonal matrix. Then*

$$\det(B) = B_{dd}\det\bigl((B_{ij})_{1\le i,j\le d-1}\bigr) - B_{d-1,d}B_{d,d-1}\det\bigl((B_{ij})_{1\le i,j\le d-2}\bigr).$$

Recall the principal submatrices $(e_{ij})_{r\le i,j\le p}$ from the beginning of Sect. 7.

Proposition 11.6 *With reference to Notation 5.6 and Definition 9.1, for $1 \le r \le p \le D$ the determinant of $(e_{ij})_{r\le i,j\le p}$ is given in the following table*:

Case	Determinant of $(e_{ij})_{r\le i,j\le p}$
I	$\frac{p-r+2}{2^{p-r+1}}$
II	$\frac{(p-r+2)(2D-r-p+1)(D-p+1)_{p-r}}{2^{p-r+2}(D-p+1/2)_{p-r+1}}$
III	1 if $p=r$; $\frac{k}{2(k-1)}$ if $p=r+1$; $\frac{1}{k-1}$ if $p=r+2$
IV, V	1 if $p=r$; $\frac{k\theta_1}{(k-1)(\theta_1+1)}$ if $(r,p)=(1,2)$ $\frac{k}{c_2(\theta_1+1)}$ if $(r,p)=(2,3)$; $\frac{\theta_1(k-\theta_1)}{(k-1)c_2}$ if $(r,p)=(1,3)$
other	$\frac{(q^{p-r+2}-1)(1-s^*q^{p+r})(s^*q^{2r+1};q^2)_{p-r}}{(q+1)^{p-r+1}(q-1)(s^*q^{2r};q^2)_{p-r+1}}$

We are using the notation

$$(a)_n = a(a+1)\cdots(a+n-1),$$
$$(a;q)_n = (1-a)(1-qa)\cdots\bigl(1-q^{n-1}a\bigr).$$

Proof For Cases III–V the result follows from a straightforward computation using Proposition 11.1. For the other cases use Proposition 11.1, Lemma 11.5, and induction on $p-r$. □

Corollary 11.7 *With reference to Notation 5.6 and Definition 9.1, for* $1 \le r \le p \le D$ *the principle submatrix* $(e_{ij})_{r \le i, j \le p}$ *is nonsingular.*

Proof Immediate from Lemma 10.3(i) and Proposition 11.6. □

Proposition 11.8 *With reference to Notation 5.6 assume* $\beta \neq -2$. *For Case* II *and Case* IV′ *the corresponding* R/L*-dependency structure is uniform but not strongly uniform. In all other cases, the corresponding* R/L*-dependency structure is strongly uniform.*

Proof Immediate from Proposition 9.2, Corollary 11.4 and Corollary 11.7. □

Theorem 11.9 *Let* Γ *denote a bipartite distance-regular graph with diameter* $D \ge 3$ *and valency* $k \ge 3$. *Fix a vertex* x *and let* L (*resp.*, R) *denote the corresponding lowering* (*resp., raising*) *matrix from Definition 5.1. Let* $\{\theta_i\}_{i=0}^{D}$ *denote a* Q*-polynomial ordering of the eigenvalues of* Γ. *Consider the following cases*:

(i) Γ *is the hypercube* $H(D,2)$ *with* D *even and* $\theta_i = (-1)^i(D-2i)$ *for* $0 \le i \le D$;
(ii) Γ *is the antipodal quotient* $\overline{H}(2D,2)$ *and* $\theta_i = 2D - 4i$ *for* $0 \le i \le D$;
(iii) $D = 3$ *and* Γ *is of McFarland type with parameters* $(1,t)$ *for some integer* $t \ge 2$, *and* $\theta_0, \theta_1, \theta_2, \theta_3$ *are* $t(t+1), t, -t, -t(t+1)$ *respectively.*

In Case (i), *the corresponding* R/L*-dependency structure is not uniform. In Cases* (ii) *and* (iii), *this structure is uniform but not strongly uniform. In all other cases this structure is strongly uniform.*

Proof Immediate from Propositions 8.4 and 11.8. □

Acknowledgements Š. Miklavič was supported in part by ARRS—Javna agencija za raziskovalno dejavnost Republike Slovenije, program no. P1-0285, project no. J1-4010-1669 and project no. BI-USA/09-12-009.

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