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A Prioritized Information Fusion Method for Handling Fuzzy Decision-Making Problems

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Abstract. Although Yager has presented a prioritized operator for fuzzy subsets, called the non-monotonic operator, it can not be used to deal with multi-criteria fuzzy decision-making problems when generalized fuzzy numbers are used to represent the evaluating values of criteria. In this paper, we present a prioritized information fusion algorithm based on the similarity measure of generalized fuzzy numbers. The proposed prioritized information fusion algorithm has the following advantages: (1) It can handle prioritized multi-criteria fuzzy decision-making problems in a more flexible manner due to the fact that it allows the evaluating values of criteria to be represented by generalized fuzzy numbers or crisp values between zero and one, and (2) it can deal with prioritized information filtering problems based on generalized fuzzy numbers.

Keywords: prioritized information fusion algorithm, generalized fuzzy numbers, similarity measures, information filtering, prioritized operator

1. Introduction

Yager [5] presented an important prioritized operator for fuzzy subsets, called the non-monotonic operator. Bordogna et al. [1] regarded the process of information retrieval as a multi-criteria fuzzy decision-making activity, and they used the prioritized operator to deal with fuzzy information retrieval. Hirota et al. [2] used the prioritized operator to deal with two applications, i.e., an estimation of default fuzzy sets and a default-driven extension of fuzzy reasoning. Yager [3] used the prioritized operator to deal with multi-criteria fuzzy decision-making problems and presented a type of criterion called *a second order criterion*. Furthermore, he pointed out that the second order criterion acts as an additional selector or a filter. Yager [4] used the prioritized operator in fuzzy information fusion structures. From [1–5], we can see that the prioritized operator is useful to deal with multi-criteria fuzzy decision-making problems.

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The multi-criteria fuzzy decision-making models provide a useful way to deal with the subjectiveness and vagueness in multi-criteria decision-making problems [1, 6]. In many situations, fuzzy numbers are very useful to represent evaluating values of criteria to deal with multi-criteria fuzzy decision-making problems [7–10]. Some researchers used generalized fuzzy numbers [11] to deal with multi-criteria fuzzy decision-making problems [12–15], where the generalized fuzzy numbers are non-normal fuzzy numbers. Chen [11] presented the operations of generalized fuzzy numbers with the function principle. Dubois [14] pointed out that when a fuzzy number is not normalized, this situation could be interpreted as a lack of confidence in the information provided by such numbers. Some researchers also used generalized fuzzy numbers to represent fuzzy information and their associated degrees of uncertainty [12], degrees of confidence [13], and the certainty degrees of the represented values [15]. However, the prioritized operator presented in [5] can not deal with multi-criteria fuzzy decision-making problems when generalized fuzzy numbers [11] are used to represent evaluating values of criteria due to the fact that the operator can not measure the degree of intersection between two sets of generalized fuzzy numbers. Thus, it is obvious that to extend the prioritized operator presented in [5] and to develop a prioritized information fusion algorithm for aggregating the evaluating values of the criteria represented by generalized fuzzy numbers are important research topics in multi-criteria fuzzy decision-making problems.

In this paper, we extend the prioritized operator presented in [5] to propose a prioritized information fusion algorithm based on similarity measures of generalized fuzzy numbers for dealing with multi-criteria fuzzy decision-making problems. The proposed prioritized information fusion algorithm plays the role of a filter in the multi-criteria fuzzy decision-making problems. It allows the evaluating values of the criteria of decision-maker's subjective assessments to be represented by generalized fuzzy numbers or crisp values between zero and one. The proposed information fusion algorithm can overcome the drawback of the prioritized operator presented in [5]. The proposed algorithm is useful to deal with query and filtering problems for fuzzy information retrieval and is useful to deal with decision-making problems based on second order structures [3] when users need to use fuzzy numbers to represent linguistic evaluating values.

The rest of this paper is organized as follows. In Section 2, we briefly review the concepts of generalized fuzzy numbers from [11, 12], the simple center of gravity method (SCGM) [16], the similarity measure based on SCGM we presented in [17], and the ranking method of generalized fuzzy numbers based on the SCGM we presented in [18]. In Section 3, we briefly review Yager's prioritized operator [5] and some properties of the prioritized operator. In Section 4, we extend Yager's prioritized operator to propose a prioritized information fusion algorithm. In Section 5, we apply the proposed information fusion algorithm for handling prioritized multi-criteria fuzzy decision-making problems. The conclusions are discussed in Section 6.

2. Preliminaries

In this section, we briefly review the concepts of generalized fuzzy numbers from [11, 12], the simple center of gravity method (SCGM) [16], the similarity measure of generalized fuzzy numbers based on the SCGM we presented in [13], and the ranking method of generalized fuzzy numbers based on the SCGM we presented in [18].

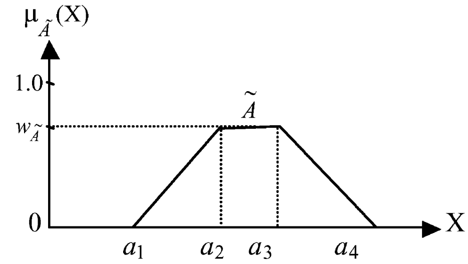


Figure 1. A generalized trapezoidal fuzzy number.

2.1. Generalized Fuzzy Numbers

Chen [11] represented generalized fuzzy numbers and their operations. Figure 1 shows a generalized trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}})$, where $0 < w_{\tilde{A}} \leq 1$, and a_1, a_2, a_3 and a_4 are real values. The value of $w_{\tilde{A}}$ represents the certainty degree of the evaluating value of a decision-maker's opinion. The generalized fuzzy number $\tilde{A}$ of the universe of discourse X is characterized by a membership function $\mu_{\tilde{A}}$, where X takes its number on the real line R and $\mu_{\tilde{A}}: X \to [0, 1]$. If $w_{\tilde{A}} = 1$, then the generalized fuzzy number $\tilde{A}$ is called a normal trapezoidal fuzzy number, denoted as $\tilde{A} = (a_1, a_2, a_3, a_4)$. If $a_1 = a_2, a_3 = a_4$ and $w_{\tilde{A}} = 1$, then $\tilde{A}$ is called a crisp interval. If $a_2 = a_3$, then $\tilde{A}$ is called a generalized triangular fuzzy number. If $a_1 = a_2 = a_3 = a_4$ and $w_{\tilde{A}} = 1$, then $\tilde{A}$ is called a crisp value. From [12–15], we can see that some researchers are using generalized fuzzy numbers to represent fuzzy information and their associated degrees of uncertainty [12], degrees of confidence [13], the certainty degrees of the represented values [15], and lack of confidence in the information provided by such numbers [14].

2.2. A Simple Center of Gravity Method (SCGM) [16]

The center-of-gravity (COG) of an object is a geometric property of the object, and it is the average location of the weight of an object [19]. The center-of-gravity (COG) method can be used to deal with defuzzification problems [20–22] and fuzzy ranking problems [8, 23]. In [16], we have pointed out that there are some drawbacks in the traditional COG method, i.e., it cannot

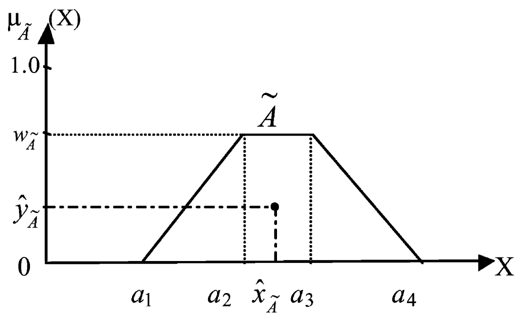


Figure 2. A COG point $(\hat{x}_{\tilde{A}},\ \hat{y}_{\tilde{A}})$ of the generalized trapezoidal fuzzy number $\tilde{A}$.

directly calculate the COG point of a crisp interval or a real number, and it is very time-consuming to calculate the COG point. Thus, in [16], we presented the simple center of gravity method (SCGM) to calculate the COG point of a generalized fuzzy number based on the concept of the medium curve [24]. The method we presented in [16] can overcome the drawbacks of the traditional COG method. Let $\tilde{A}$ be a generalized trapezoidal fuzzy number, where $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}})$. The SCGM method for calculating the COG point $(\hat{x}_{\tilde{A}},\ \hat{y}_{\tilde{A}})$ of the generalized trapezoidal fuzzy number $\tilde{A}$ is shown as follows:

$$\hat{y}_{\tilde{A}} = \begin{cases} \dfrac{w_{\tilde{A}} \times \left(\frac{a_3-a_2}{a_4-a_1} + 2\right)}{6}, & \text{if } a_1 \neq a_4 \text{ and } 0 < w_{\tilde{A}} \leq 1, \\ \dfrac{w_{\tilde{A}}}{2}, & \text{if } a_1 = a_4 \text{ and } 0 < w_{\tilde{A}} \leq 1, \end{cases} \tag{1}$$

$$\hat{x}_{\tilde{A}} = \frac{\hat{y}_{\tilde{A}}(a_3 + a_2) + (a_4 + a_1)(w_{\tilde{A}} - \hat{y}_{\tilde{A}})}{2w_{\tilde{A}}}. \tag{2}$$

Based on formulas (1) and (2), we can obtain the COG point $(\hat{x}_{\tilde{A}},\ \hat{y}_{\tilde{A}})$ of a generalized trapezoidal fuzzy number $\tilde{A}$. The SCGM method is used for calculating the COG point $(\hat{x}_{\tilde{A}},\ \hat{y}_{\tilde{A}})$ of the generalized trapezoidal fuzzy number $\tilde{A}$ as shown in Fig. 2. Furthermore, in [13, 18], we have used the proposed SCGM method to calculate the ranking order of generalized fuzzy numbers and to measure the degree of similarity between generalized fuzzy numbers, respectively.

2.3. *A Similarity Measure of Generalized Fuzzy Numbers Based on the SCGM [13]*

Some similarity measures of trapezoidal fuzzy numbers have been presented to calculate the degree of similarity between trapezoidal fuzzy numbers [7, 10, 25]. In [13], we have pointed out that there are some drawbacks in the existing similarity measures, i.e., they can not correctly calculate the degree of similarity between trapezoidal fuzzy numbers in some situations. Therefore, in [13], we presented a method to evaluate the degree of similarity between generalized trapezoidal fuzzy numbers and we used the method to deal with fuzzy risk analysis problems. The proposed similarity measure can overcome the drawbacks of the existing similarity measures presented in [7, 10, 25].

In [13], we let the universe of discourse U of generalized fuzzy numbers be between zero and one (i.e., standardized generalized fuzzy numbers). Assume that there are two generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, where $\tilde{A} = (a_1, a_2, a_3, a_4;\ w_{\tilde{A}})$, $\tilde{B} = (b_1, b_2, b_3, b_4;\ w_{\tilde{B}})$, $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$ and $0 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 1$. First, we use formulas (1) and (2) to obtain the COG points COG$(\tilde{A})$ and COG$(\tilde{B})$ of the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, respectively, where COG$(\tilde{A}) = (\hat{x}_{\tilde{A}}, \hat{y}_{\tilde{A}})$ and COG$(\tilde{B}) = (\hat{x}_{\tilde{B}}, \hat{y}_{\tilde{B}})$. Then, the degree of similarity $S(\tilde{A}, \tilde{B})$ between the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ can be calculated as follows:

$$S(\tilde{A}, \tilde{B}) = \left(1 - \frac{\sum_{i=1}^{4} |a_i - b_i|}{4}\right) \times (1 - |\hat{x}_{\tilde{A}} - \hat{x}_{\tilde{B}}|)^{B(S_{\tilde{A}},\ S_{\tilde{B}})} \times \frac{\min(\hat{y}_{\tilde{A}}, \hat{y}_{\tilde{B}})}{\max(\hat{y}_{\tilde{A}}, \hat{y}_{\tilde{B}})}, \tag{3}$$

where $S(\tilde{A}, \tilde{B}) \in [0, 1]$, the values $\hat{y}_{\tilde{A}}$ and $\hat{y}_{\tilde{B}}$ are obtained by formula (1), the values $\hat{x}_{\tilde{A}}$ and $\hat{x}_{\tilde{B}}$ are obtained by formula (2), and $B(S_{\tilde{A}}, S_{\tilde{B}})$ is defined as follows:

$$B(S_{\tilde{A}}, S_{\tilde{B}}) = \begin{cases} 1, & \text{if } S_{\tilde{A}} + S_{\tilde{B}} > 0, \\ 0, & \text{if } S_{\tilde{A}} + S_{\tilde{B}} = 0, \end{cases} \tag{4}$$

where $S_{\tilde{A}}$ and $S_{\tilde{B}}$ are the bases of the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, respectively, defined as follows:

$$S_{\tilde{A}} = a_4 - a_1, \tag{5}$$

$$S_{\tilde{B}} = b_4 - b_1. \tag{6}$$

The larger the value of $S(\tilde{A}, \tilde{B})$, the more the similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$. In formula (3), the values of $|a_i - b_i|$ and $|\hat{x}_{\tilde{A}} - \hat{x}_{\tilde{B}}|$ are positive values, respectively, due to the fact that the symbol "| |" denotes the "absolute" operator. Furthermore, because $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$ and $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$, from formulas (5) and (6), we can see that the values of $S_{\tilde{A}}$ and $S_{\tilde{B}}$ are positive values between zero and one, respectively. Therefore, from formula (4), we can see that the value of $B(S_{\tilde{A}}, S_{\tilde{B}})$ is either zero or one. Because the values of $\hat{x}_{\tilde{A}}$ and $\hat{x}_{\tilde{B}}$ obtained by formula (2), respectively, are between zero and one, and because the values of $|a_i - b_i|$ and $|\hat{x}_{\tilde{A}} - \hat{x}_{\tilde{B}}|$ are between zero and one, we can see that formula (3) is not a negative value. In [13], we have shown that the value of $S(\tilde{A}, \tilde{B})$ in formula (3) is between zero and one.

The proposed similarity measure of generalized fuzzy numbers has the following properties [13]:

Property 2.1. *Two generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are identical if and only if $S(\tilde{A}, \tilde{B}) = 1$.*

Property 2.2. $S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A})$.

Property 2.3. *If $\tilde{A} = (a, a, a, a; 1.0)$ and $\tilde{B} = (b, b, b, b; 1.0)$ are two real numbers, then $S(\tilde{A}, \tilde{B}) = 1 - |a - b|$.*

In [17], we have pointed out that formula (3) can be further simplified. Thus, in [17], we modified formula (3) into formula (7) to simplify the calculation process shown as follows:

$$S(\tilde{A}, \tilde{B}) = \left[\left(1 - \frac{\sum_{i=1}^{4} |a_i - b_i|}{4}\right) \times (1 - |\hat{x}_{\tilde{A}} - \hat{x}_{\tilde{B}}|)\right]^{\frac{1}{2}} \times \frac{\min(\hat{y}_{\tilde{A}}, \hat{y}_{\tilde{B}})}{\max(\hat{y}_{\tilde{A}}, \hat{y}_{\tilde{B}})}, \quad (7)$$

where $S(\tilde{A}, \tilde{B}) \in [0, 1]$. The larger the value of $S(\tilde{A}, \tilde{B})$, the more the similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$. From the previous discussions, we can see that the values of $|a_i - b_i|$ and $|\hat{x}_{\tilde{A}} - \hat{x}_{\tilde{B}}|$ are positive values, respectively, and the values $\hat{x}_{\tilde{A}}$ and $\hat{x}_{\tilde{B}}$ obtained by formula (2) are between zero and one. Therefore, in formula (7), we can see that:

$$\left(1 - \frac{\sum_{i=1}^{4} |a_i - b_i|}{4}\right) \times (1 - |\hat{x}_{\tilde{A}} - \hat{x}_{\tilde{B}}|) \ge 0.$$

In [17], we have shown that the value of $S(\tilde{A}, \tilde{B})$ in formula (7) is between zero and one.

2.4. A Ranking Method of Generalized Fuzzy Numbers Based on the SCGM [18]

From [23, 26, 27], we can see that some centroid-index ranking methods of fuzzy numbers have been presented to calculate the ranking order between fuzzy numbers. However, according to [18], we can see that there are some drawbacks in the existing methods, i.e., they cannot correctly calculate the ranking order of the generalized fuzzy numbers in some situations. Thus, in [18], we presented a method to evaluate the ranking order between generalized fuzzy numbers, based on center-of-gravity (COG) points and standard deviations of generalized fuzzy numbers. The proposed ranking method can overcome the drawbacks of the existing ranking methods presented in [23, 26, 27]. Assume that there is a generalized fuzzy number $\tilde{A}_i$, where $\tilde{A}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}; w_{\tilde{A}_i})$, represented as an evaluating value of an alternative x_i, where $1 \le i \le n$. The ranking value $R(\tilde{A}_i)$ of the generalized fuzzy number $\tilde{A}_i$ is calculated as follows:

$$R(\tilde{A}_i) = \hat{x}_{\tilde{A}_i} + \left(w_{\tilde{A}_i} - \hat{y}_{\tilde{A}_i}\right)^{\hat{s}_{\tilde{A}_i}} \times \left(\hat{y}_{\tilde{A}_i} + 0.5\right)^{1 - w_{\tilde{A}_i}}, \quad (8)$$

where the values of $\hat{x}_{\tilde{A}_i}$ and $\hat{y}_{\tilde{A}_i}$ are calculated by formulas (1) and (2), respectively, and the value of $\hat{s}_{\tilde{A}_i}$ is the standard deviation of the generalized fuzzy number $\tilde{A}_i$, defined as follows [18]:

$$\begin{aligned} \hat{s}_{\tilde{A}_i} &= \sqrt{\frac{1}{3}\sum_{j=1}^{4}(a_{ij} - \bar{a}_i)^2} \\ &= \sqrt{\frac{1}{3}\sum_{j=1}^{4} a_{ij}^2 - 2\bar{a}_i^2 + \bar{a}_i^2} \\ &= \sqrt{\frac{1}{3}\sum_{j=1}^{4} a_{ij}^2 - \bar{a}_i^2} \\ &= \sqrt{\frac{1}{3}\sum_{j=1}^{4} a_{ij}^2 - \left(\sum_{j=1}^{4} a_{ij} \Big/ 3\right)^2}. \end{aligned} \quad (9)$$

The larger the value of $R(\tilde{A}_i)$, the better the ranking of $\tilde{A}_i$ (i.e., x_i is the better alternative), where $1 \le i \le n$. In [18], we have used the proposed ranking method for handling the outsourcing targets selection problems. In [28], we have used the proposed ranking method for handling multi-criteria fuzzy

Table 1. Three t-norms and t-conorms [4].

T-Norms		T-Conorms	
$\min(a, b)$	(Logical Product)	$\max(a, b)$	(Logical Sum)
$a \times b$	(Algebraic Product)	$a + b - a \times b$	(Algebraic Sum)
$\max(a + b - 1, 0)$	(Bounded Product)	$\min(a + b, 1)$	(Bounded Sum)

decision-making problems based on the FN-IOWA operator.

3. A Prioritized Intersection Operator

Zadeh [29] developed the theory of approximate reasoning. Yager [30] introduced non-monotonic logics into the theory of approximate reasoning for representing default knowledge or commonsense reasoning. Furthermore, in [5], Yager presented a prioritized intersection operator, called the non-monotonic intersection operator η, to emulate common sense reasoning. Let X be the universe of discourse, and let A and B be two fuzzy subsets in X. The non-monotonic intersection operator η is defined as follows:

$$\eta(A, B) = D, \tag{10}$$

where D is also a fuzzy subset of X, such that

$$\mu_D(x) = \mu_A(x) \wedge (\mu_B(x) \vee (1 - \text{Poss}[B \mid A])), \tag{11}$$

where $\text{Poss}[B \mid A]$ is defined as follows:

$$\text{Poss}[B \mid A] = \text{Max}_x[\mu_A(x) \wedge \mu_B(x)]. \tag{12}$$

Dubois and Prade [14] pointed out that $\text{Poss}[B \mid A]$ denotes the possibility of B given A. Yager [4] summarized three t-norms (i.e., $\wedge$) and t-conorms (i.e., $\vee$) as shown in Table 1, where $a \in [0, 1]$ and $b \in [0, 1]$.

Yager [4] pointed out that $\text{Poss}[B \mid A]$ essentially measures the degree of intersection between the fuzzy subsets A and B, and $1-$ $\text{Poss}[B \mid A]$ can be seen as a measure of conflict between the fuzzy subsets A and B. In [3], the two operators $\wedge$ and $\vee$ can be replaced by any t-normss and t-conormss as shown in Table 1, respectively. As explained in [3, 21], we can see that there are two properties of Yager's non-monotonic intersection operator shown as follows:

Property 3.1. *If Poss*$[B \mid A] = 1$, *then* $\mu_D(x) = \mu_A(x) \wedge \mu_B(x)$. *It means that the fuzzy subset A can be revised by the fuzzy subset B entirely due to the fact that the fuzzy subset B is completely compatible with the fuzzy subset A.*

Property 3.2. *If Poss*$[B \mid A] = 0$, *then* $\mu_D(x) = \mu_A(x)$. *It means that the fuzzy subset A can not be revised by the fuzzy subset B due to the fact that the fuzzy subset B is completely incompatible with the fuzzy subset A. Thus, we disregard the fuzzy subset B.*

Yager [33], used the non-monotonic intersection operator to deal with multi-criteria fuzzy decision-making problems and presented a type of criterion called *a second order criterion*. From [3], we can see that the second order criterion is a criterion that does not interfere with our getting satisfactory solution using the first order criterion. Let us consider the following statement [3]:

> "I want a car that is luxurious and inexpensive and *if possible I would like to buy it from my brother*."

The statement reflects a natural language formulation of a combination of the first order criteria (i.e., a car that is luxurious and inexpensive) and the second order criterion (i.e., if possible I would like to buy it from my brother). Yager [3] pointed out that the criterion A and the criterion B of formula (11) are called the first order criterion and the second order criterion, respectively, and he modified formula (11) into

$$\mu_D(x) = \mu_A(x) \wedge (\mu_B(x) \vee (1 - \omega(\text{Poss}[B \mid A]))), \tag{13}$$

where ω is a qualifier defined as follows:

$$\omega(r) = \begin{cases} 0, & \text{if } r < \alpha, \\ 1, & \text{if } r \geq \alpha, \end{cases} \tag{14}$$

where $\alpha \in [0, 1]$. Yager [3] pointed out that the second order criterion B acts as an additional selector (or a filter) for those elements which satisfy the first order criterion A. Yager [4] used the operator in fuzzy information fusion structures and he called the operator "non-monotonic/prioritized intersection operator" because the operator supports a concept of priority, i.e., in $\eta(A, B)$, A has the priority over B [5].

Example 3.1. Assume that we use the algebraic product (i.e., $a \times b$) and the algebraic sum (i.e., $a+b-a \times b$) to represent t-norms and t-conorms (i.e., $\wedge$ and $\vee$), respectively. Assume that there are two different criteria A and B, three alternatives x_1, x_2, x_3, and their evaluating values are shown as follows:

$$A = \left\{ \frac{1.0}{x_1}, \frac{0.4}{x_2}, \frac{0.5}{x_3} \right\}, \quad B = \left\{ \frac{0.2}{x_1}, \frac{1.0}{x_2}, \frac{0.3}{x_3} \right\}.$$

We can see that $\mathrm{Poss}[B \mid A] = \underset{x}{\mathrm{Max}}\, [A(x) \wedge B(x)] = \mathrm{Max}\, [(1.0 \wedge 0.2), (0.4 \wedge 1.0), (0.5 \wedge 0.3)] = 0.4$. By applying formula (11), we can see that

$$\mu_D(x_1) = 1.0 \wedge (0.2 \vee (1.0 - 0.4)) = 0.68,$$
$$\mu_D(x_2) = 0.4 \wedge (1.0 \vee (1.0 - 0.4)) = 0.4,$$
$$\mu_D(x_3) = 0.5 \wedge (0.3 \vee (1.0 - 0.4)) = 0.36.$$

Because $\mu_D(x_1) > \mu_D(x_2) > \mu_D(x_3)$, we can see that the preferring order of the alternatives is $x_1 > x_2 > x_3$. Thus, the best alternative among the alternatives x_1, x_2 and x_3 is x_1.

4. A Prioritized Information Fusion Algorithm

From [1–5], we can see that the prioritized operator shown in formula (11) is very useful to deal with prioritized multi-criteria fuzzy decision-making problems. However, the prioritized operator shown in formula (11) can not deal with multi-criteria fuzzy decision-making problems if we use generalized fuzzy numbers to represent evaluating values of criteria due to the fact that the operator can not measure the degree of intersection (i.e., formula (12)) between two sets of generalized fuzzy numbers. Thus, to develop a prioritized information fusion algorithm for aggregating the evaluating values represented by generalized fuzzy numbers is one of the important research topics of multi-criteria fuzzy decision-making problems.

In [31, 32], we can see that t-norms and t-conorm operators can apply to the fuzzy number environment. The results of t-norms and t-conorms operations of fuzzy numbers are still fuzzy numbers [33]. Assume that there are two trapezoidal fuzzy numbers $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$, where $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$ and $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$. Let T be a t-norm and let S be a t-conorm, where T: $[0, 1] \times [0, 1] \to [0, 1]$ and $S : [0, 1] \times [0, 1] \to [0, 1]$. The t-norm and t-conorm operations between the trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are shown as follows:

$$T(\tilde{A}, \tilde{B}) = (a_1 \wedge b_1, a_2 \wedge b_2, a_3 \wedge b_3, a_4 \wedge b_4), \tag{15}$$

$$S(\tilde{A}, \tilde{B}) = (a_1 \vee b_1, a_2 \vee b_2, a_3 \vee b_3, a_4 \vee b_4), \tag{16}$$

where $\wedge$ and $\vee$ are any t-norm and t-conorm as shown in Table 1, respectively. According to formulas (15) and (16), we can see that the results of t-norm and t-conorm operations of fuzzy numbers are linear. However, the traditional t-norm and t-conorm operators with fuzzy numbers do not preserve the linearity [33]. Thus, in [34], Junghanns et al. pointed out that the results of t-norm and t-conorm operations of fuzzy numbers using formulas (15) and (16) are approximate results.

In the following, we extend Yager's prioritized operator to propose a prioritized information fusion algorithm for aggregating evaluating values represented by generalized fuzzy numbers. Let X be the universe of discourse, $X = [0, k]$. Assume that there are n alternatives $x_1, x_2, \ldots,$ and x_n, and assume that there are two different prioritized criteria $\boldsymbol{A}$ and $\boldsymbol{B}$ as shown in Table 2, where $\boldsymbol{A} = (\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_i, \ldots, \tilde{A}_n)$ is called the first order criterion, $\boldsymbol{B} = (\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_i, \ldots, \tilde{B}_n)$ is called the second order criterion, $\tilde{A}_i$ and $\tilde{B}_i$ are generalized fuzzy numbers, $\tilde{A}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}; w_{\tilde{A}_i})$, $\tilde{B}_i = (b_{i1}, b_{i2}, b_{i3}, b_{i4}; w_{\tilde{B}_i})$, $0 \le a_{ij} \le k$, $0 \le b_{ij} \le k$, $0 < w_{\tilde{A}_i} \le 1$, $0 < w_{\tilde{B}_i} \le 1$, $1 \le i \le n$, and $1 \le j \le 4$.

The proposed information fusion algorithm is now presented as follows:

Table 2. Evaluating values of the alternatives using two different criteria $\boldsymbol{A}$ and $\boldsymbol{B}$.

	Alternatives					
Different criteria	x_1	x_2	$\cdots$	x_i	$\cdots$	x_n
First order criterion $\boldsymbol{A}$	$\tilde{A}_1$	$\tilde{A}_2$	$\cdots$	$\tilde{A}_i$	$\cdots$	$\tilde{A}_n$
Second order criterion $\boldsymbol{B}$	$\tilde{B}_1$	$\tilde{B}_2$	$\cdots$	$\tilde{B}_i$	$\cdots$	$\tilde{B}_n$

Step 1: If the universe of discourse $X = [0, k]$ and $k \neq 1$, then translate the generalized trapezoidal fuzzy numbers $\tilde{A}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}; w_{\tilde{A}_i})$ and $\tilde{B}_i = (b_{i1}, b_{i2}, b_{i3}, b_{i4}; w_{\tilde{B}_i})$ into standardized generalized trapezoidal fuzzy numbers $\tilde{A}_i^*$ and $\tilde{B}_i^*$, respectively, where $1 \leq i \leq n$, shown as follows:

$$\begin{aligned}\tilde{A}_i^* &= \left(\frac{a_{i1}}{k},\ \frac{a_{i2}}{k},\ \frac{a_{i3}}{k},\ \frac{a_{i4}}{k};\ w_{\tilde{A}_i^*}\right)\\ &= \left(a_{i1}^*,\ a_{i2}^*,\ a_{i3}^*,\ a_{i4}^*;\ w_{\tilde{A}_i^*}\right), \qquad (17)\\ \tilde{B}_i^* &= \left(\frac{b_{i1}}{k},\ \frac{b_{i2}}{k},\ \frac{b_{i3}}{k},\ \frac{b_{i4}}{k};\ w_{\tilde{B}_i^*}\right)\\ &= \left(b_{i1}^*,\ b_{i2}^*,\ b_{i3}^*,\ b_{i4}^*;\ w_{\tilde{B}_i^*}\right). \qquad (18)\end{aligned}$$

where $0 \leq a_{ij}^* \leq 1$, $0 \leq b_{ij}^* \leq 1$, $w_{\tilde{A}_i^*} = w_{\tilde{A}_i}$, $w_{\tilde{B}_i^*} = w_{\tilde{B}_i}$, $1 \leq i \leq n$, and $1 \leq j \leq 4$. The standardized generalized fuzzy numbers mean that the universes of discourse U of the generalized fuzzy numbers are between zero and one.

Step 2: Based on formulas (1), (2) and (7), evaluate the degree of compatibility $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ between the first order criterion $\boldsymbol{A}$ and the second order criterion $\boldsymbol{B}$ shown as follows:

$$\begin{aligned}\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}) = \mathrm{Max}\{&\mathrm{Min}[(\hat{x}_{\tilde{A}_1^*} \wedge \hat{x}_{\tilde{B}_1^*}), S(\tilde{A}_1^*, \tilde{B}_1^*)],\\ &\mathrm{Min}[(\hat{x}_{\tilde{A}_2^*} \wedge \hat{x}_{\tilde{B}_2^*}), S(\tilde{A}_2^*, \tilde{B}_2^*)],\\ &\vdots\\ &\mathrm{Min}[(\hat{x}_{\tilde{A}_i^*} \wedge \hat{x}_{\tilde{B}_i^*}), S(\tilde{A}_i^*, \tilde{B}_i^*)],\\ &\vdots\\ &\mathrm{Min}[(\hat{x}_{\tilde{A}_n^*} \wedge \hat{x}_{\tilde{B}_n^*}), S(\tilde{A}_n^*, \tilde{B}_n^*)]\}, \qquad (19)\end{aligned}$$

where

1. the operators $\wedge$ can be replaced by any t-norms as shown in Table 1;
2. the value $\hat{x}_{\tilde{A}_i^*}$ of the standardized generalized fuzzy number $\tilde{A}_i^*$ and the value $\hat{x}_{\tilde{B}_i^*}$ of the standardized generalized fuzzy number $\tilde{B}_i^*$ are calculated by formula (2), respectively;
3. $S(\tilde{A}_i^*, \tilde{B}_i^*)$ denotes the degree of similarity between the standardized generalized fuzzy numbers $\tilde{A}_i^*$ and $\tilde{B}_i^*$ calculated by formula (7).

The value of $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ of formula (19) is a real value between zero and one due to the fact that values $\hat{x}_{\tilde{A}_i^*} \in [0, 1]$, $\hat{x}_{\tilde{B}_i^*} \in [0, 1]$, $S(\tilde{A}_i^*, \tilde{B}_i^*) \in [0, 1]$ and $1 \leq i \leq n$. The value $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ essentially measures the degree of compatibility between two sets of generalized fuzzy numbers $\boldsymbol{A}$ and $\boldsymbol{B}$, where $\boldsymbol{A} = \{\tilde{A}_1^*, \tilde{A}_2^*, \ldots, \tilde{A}_n^*\}$ and $\boldsymbol{B} = \{\tilde{B}_1^*, \tilde{B}_2^*, \ldots, \tilde{B}_n^*\}$.

Step 3: Based on formulas (15), (16) and (19), extend formula (11) into

$$\tilde{D}_i = \tilde{A}_i^* \wedge (\tilde{B}_i^* \vee (1 - \boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}))), \qquad (20)$$

where $\tilde{D}_i$ is a generalized fuzzy number denoting the fusion result of the generalized fuzzy numbers $\tilde{A}_i^*$ and $\tilde{B}_i^*$, $1 \leq i \leq n$, and the operators $\wedge$ and $\vee$ can be replaced by any t-norm and t-conorms as shown in Table 1, respectively. 1-$\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ can be seen as a measure of conflict between two sets of generalized fuzzy numbers $\boldsymbol{A}$ and $\boldsymbol{B}$. Assume that $1 - \boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}) = p$, where $p \in [0, 1]$, using formulas (15) and (16) to calculate $\tilde{D}_i$ as follows:

$$\begin{aligned}\tilde{D}_i &= \tilde{A}_i^* \wedge (\tilde{B}_i^* \vee (1 - \boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}))) = \tilde{A}_i^* \wedge (\tilde{B}_i^* \vee p)\\ &= \Big(\left(a_{i1}^*,\ a_{i2}^*,\ a_{i3}^*,\ a_{i4}^*;\ w_{\tilde{A}_i^*}\right)\\ &\quad \wedge \left(\left(b_{i1}^*,\ b_{i2}^*,\ b_{i3}^*,\ b_{i4}^*;\ w_{\tilde{B}_i^*}\right) \vee p\right)\\ &= \Big(a_{i1}^* \wedge (b_{i1}^* \vee p),\ a_{i2}^* \wedge (b_{i2}^* \vee p),\ a_{i3}^*\\ &\quad \wedge (b_{i3}^* \vee p),\ a_{i4}^* \wedge (b_{i4}^* \vee p);\\ &\quad \frac{w_{\tilde{A}_i^*} + w_{\tilde{B}_i^*} \times C(A,\ B)}{1 + C(A,\ B)}\Big)\\ &= (d_{i1},\ d_{i2},\ d_{i3},\ d_{i4};\ w_{\tilde{D}_i}), \qquad (21)\end{aligned}$$

where $d_{ik} = (a_{ik}^* \wedge (b_{ik}^* \vee p)$, $w_{\tilde{D}_i} = \frac{w_{\tilde{A}_i^*} + w_{\tilde{B}_i^*} \times C(A,\ B)}{1 + C(A,\ B)}$, $1 \leq k \leq 4$, and $1 \leq i \leq n$. From formula (21), we can see that the calculation result $\tilde{D}_i$ is still a generalized fuzzy number. In formula (21), we use the mean value $\frac{w_{\tilde{A}_i^*} + w_{\tilde{B}_i^*} \times C(A,\ B)}{1 + C(A,\ B)}$ to represent the fusion result of the two certainty degrees of the evaluating values $w_{\tilde{A}_i^*}$ and $w_{\tilde{B}_i^*}$ in different degrees of compatibility $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$, where $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}) \in [0, 1]$.

Step 4: Apply formula (8) to obtain the ranking order $R(\tilde{D}_i)$ of the fusion results $\tilde{D}_i$, where $1 \leq i \leq n$. According to formula (8), the ranking value $R(\tilde{D}_i)$ of a generalized fuzzy number $\tilde{D}_i$, where $\tilde{D}_i = (d_{i1}, d_{i2}, d_{i3}, d_{i4}; w_{\tilde{D}_i})$, can be calculated as follows:

$$R(\tilde{D}_i) = \hat{x}_{\tilde{D}_i} + \left(w_{\tilde{D}_i} - \hat{y}_{\tilde{D}_i}\right)^{\hat{s}_{\tilde{D}_i}} \times \left(\hat{y}_{\tilde{D}_i} + 0.5\right)^{1 - w_{\tilde{D}_i}}. \qquad (22)$$

The larger the value of $R(\tilde{D}_i)$, the better the ranking of $\tilde{D}_i$, where $1 \leq i \leq n$. Assume that $R(\tilde{D}_i)$ is the largest value among the values of $R(\tilde{D}_1)$, $R(\tilde{D}_2), \ldots$ and $R(\tilde{D}_n)$, then the alternative x_i is the best choice, where $1 \le i \le n$.

In Step 2 of the proposed algorithm, we use the degree of compatibility $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ (i.e., formula (19)) to replace Poss[$B \mid A$] (i.e., formula (12)). The degree of compatibility $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ has considered two different situations shown as follows:

Situation 1: If we use the crisp values between zero and one to represent evaluating values of criteria, then the degrees of similarity $S(\tilde{A}_1^*, \tilde{B}_1^*)$, $S(\tilde{A}_2^*, \tilde{B}_2^*), \ldots, S(\tilde{A}_n^*, \tilde{B}_n^*)$ play a minor role and the result of the degree of compatibility $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ is equal to the result of Poss[$B \mid A$] in formula (12).

Situation 2: If we use the generalized fuzzy numbers to represent evaluating values of criteria, then the degrees of similarity $S(\tilde{A}_1^*, \tilde{B}_1^*)$, $S(\tilde{A}_2^*, \tilde{B}_2^*), \ldots, S(\tilde{A}_n^*, \tilde{B}_n^*)$ play an important role to obtain the degree of compatibility $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$, due to the fact that the generalized fuzzy numbers have different types of shape, as discussed in Section 2.

In the following, we prove that formula (20) still satisfies the two properties of Yager's non-monotonic intersection operator (i.e., Property 3.1 and Property 3.2) as follows:

Property 4.1. *If* $\boldsymbol{C}(\mathbf{A}, \mathbf{B}) = 0$, *then* $\tilde{D}_i = \tilde{A}_i^*$, *where* $1 \le i \le n$.

Proof: From formula (21), if $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}) = 0$, then we can see that

$$\begin{aligned}
\tilde{D}_i &= \tilde{A}_i^* \wedge (\tilde{B}_i^* \vee (1 - \mathbf{C}(\mathbf{A}, \mathbf{B}))) \\
&= \Bigg(a_{i1}^* \wedge (b_{i1}^* \vee 1),\ a_{i2}^* \wedge (b_{i2}^* \vee 1), \\
&\qquad a_{i3}^* \wedge (b_{i3}^* \vee 1),\ a_{i4}^* \wedge (b_{i4}^* \vee 1);\ \frac{w_{\tilde{A}_i^*} + w_{\tilde{B}_i^*} \times 0}{1+0}\Bigg) \\
&= \left(a_{i1}^* \wedge 1,\ a_{i2}^* \wedge 1,\ a_{i3}^* \wedge 1,\ a_{i4}^* \wedge 1; \frac{w_{\tilde{A}_i^*} + 0}{1}\right) \\
&= (a_{i1}^*,\ a_{i2}^*,\ a_{i3}^*,\ a_{i4}^*;\ w_{\tilde{A}_i^*}) \\
&= \tilde{A}_i^*,
\end{aligned}$$

□

where the operators $\wedge$ and $\vee$ can be replaced by any t-norms and t-conorms, respectively, and $1 \le i \le n$.

Property 4.2. *If* $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}) = 1$, *then* $\tilde{D}_i = \tilde{A}_i^* \wedge \tilde{B}_i^*$, *where* $1 \le i \le n$.

Proof: From formula (21), if $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}) = 1$, then we can see that

$$\begin{aligned}
\tilde{D}_i &= \tilde{A}_i^* \wedge (\tilde{B}_i^* \vee (1 - \boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}))) \\
&= \Bigg(a_{i1}^* \wedge (b_{i1}^* \vee 0),\ a_{i2}^* \wedge (b_{i2}^* \vee 0),\ a_{i3}^* \\
&\qquad \wedge (b_{i3}^* \vee 0),\ a_{i4}^* \wedge (b_{i4}^* \vee 0);\ \frac{w_{\tilde{A}_i^*} + w_{\tilde{B}_i^*} \times 1}{1+1}\Bigg) \\
&= \Bigg(a_{i1}^* \wedge b_{i1}^*,\ a_{i2}^* \wedge b_{i2}^*,\ a_{i3}^* \wedge b_{i3}^*,\ a_{i4}^* \\
&\qquad \wedge b_{i4}^*;\ \frac{w_{\tilde{A}_i^*} + w_{\tilde{B}_i^*}}{2}\Bigg) \\
&= \tilde{A}_i^* \wedge\ \tilde{B}_i^*,
\end{aligned}$$

where the two operators $\wedge$ and $\vee$ can be replaced by any t-norms and t-conorms, respectively, and $1 \le i \le n$. □

The proposed information fusion algorithm can handle multi-criteria fuzzy decision-making problems in a more flexible and more intelligent manner due to the fact that it allows the evaluating values of the criteria to be represented by generalized fuzzy numbers or crisp values between zero and one. In the following, we use an example to compare the proposed information fusion method with Yager's operator η [5], where the evaluating values are represented by crisp values between zero and one.

Example 4.1. We use the proposed information fusion algorithm to deal with Example 3.1. Assume that the algebraic product (i.e., $a \times b$) and the algebraic sum (i.e., $a + b - a \times b$) are used to represent the t-norm and the t-conorm (i.e., $\wedge$ and $\vee$), respectively. Assume that there are two different criteria $\boldsymbol{A}$ and $\boldsymbol{B}$ and assume that there are three alternatives, x_1, x_2 and x_3, where their evaluating values are as follows:

$$\boldsymbol{A} = \left\{\frac{1.0}{x_1},\ \frac{0.4}{x_2},\ \frac{0.5}{x_3}\right\},$$
$$\boldsymbol{B} = \left\{\frac{0.2}{x_1},\ \frac{1.0}{x_2},\ \frac{0.3}{x_3}\right\}.$$

Table 3. A comparison of the prioritized information fusion method with the prioritized intersection operator proposed by Yager [5].

	Properties		
Methods	Satisfy the properties of non-monotonic intersection operator	Deal with crisp values	Deal with fuzzy numbers
Prioritized intersection operator [5]	Yes	Yes	No
Prioritized information fusion method	Yes	Yes	Yes

The evaluating values of the two criteria $\boldsymbol{A}$ and $\boldsymbol{B}$ can be seen as follows:

$$\boldsymbol{A} = \left\{\frac{(1.0,\ 1.0,\ 1.0,\ 1.0;\ 1.0)}{x_1}, \frac{(0.4,\ 0.4,\ 0.4,\ 0.4;\ 1.0)}{x_2}, \frac{(0.5,\ 0.5,\ 0.5,\ 0.5;\ 1.0)}{x_3}\right\} = \left\{\frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \frac{\mu_A(x_3)}{x_3}\right\},$$

$$\boldsymbol{B} = \left\{\frac{(0.2, 0.2,\ 0.2,\ 0.2;\ 1.0)}{x_1}, \frac{(1.0,\ 1.0,\ 1.0,\ 1.0;\ 1.0)}{x_2}, \frac{(0.3,\ 0.3,\ 0.3,\ 0.3;\ 1.0)}{x_3}\right\} = \left\{\frac{\mu_B(x_1)}{x_1}, \frac{\mu_B(x_2)}{x_2}, \frac{\mu_B(x_3)}{x_3}\right\}.$$

By applying formulas (2), (7) and (19), we can obtain the degree of compatibility $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ between the criteria $\boldsymbol{A}$ and $\boldsymbol{B}$ as follows:

$$\begin{aligned}\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}) &= \text{Max}\{\text{Min}[(1.0 \wedge 0.2), S(\mu_A(x_1), \mu_B(x_1))], \\ &\quad \text{Min}[(0.4 \wedge 1.0), S(\mu_A(x_2), \mu_B(x_2))], \\ &\quad \text{Min}[(0.5 \wedge 0.3), S(\mu_A(x_3), \mu_B(x_3))]\} \\ &= \text{Max}\{\text{Min}[0.2, 0.2], \text{Min}[0.4, 0.4], \\ &\quad \text{Min}[0.15, 0.8]\} \\ &= 0.4.\end{aligned}$$

It is obvious that the value $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ is equal to the value of Poss[$\boldsymbol{B}|\boldsymbol{A}$] shown in Example 3.1, and we can see that the degree of compatibility between the criteria $\boldsymbol{A}$ and $\boldsymbol{B}$ is 0.4. Based on the value of $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ and formula (21), we can calculate the fusion result $\tilde{D}_1$ shown as follows:

$$\begin{aligned}\tilde{D}_1 &= \tilde{A}_1^* \wedge (\tilde{B}_1^* \vee (1 - \boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}))) \\ &= (1.0,\ 1.0,\ 1.0,\ 1.0;\ 1.0) \\ &\quad \wedge ((0.2,\ 0.2,\ 0.2,\ 0.2;\ 1.0) \vee 0.6) \\ &= (0.68,\ 0.68,\ 0.68,\ 0.68;\ 1) \\ &= 0.68.\end{aligned}$$

In the same way, we can calculate the fusion result $\tilde{D}_2$ and $\tilde{D}_3$ shown as follows:

$$\begin{aligned}\tilde{D}_2 &= \tilde{A}_2^* \wedge (\tilde{B}_2^* \vee (1 - \boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}))) \\ &= 0.4, \\ \tilde{D}_3 &= \tilde{A}_3^* \wedge (\tilde{B}_3^* \vee (1 - \mathbf{C}(\boldsymbol{A}, \boldsymbol{B}))) \\ &= 0.36.\end{aligned}$$

Because of $\tilde{D}_1 > \tilde{D}_2 > \tilde{D}_3$, we can see that the preferring order of the alternatives x_1, x_2 and x_3 is $x_1 > x_2 > x_3$. Thus, x_1 is the best alternative among the alternatives x_1, x_2 and x_3. This result coincides with Yager's method [5] shown in Example 3.1.

In the following, we compare the proposed prioritized information fusion method with the prioritized intersection operator proposed by Yager [5], as shown in Table 3.

In the next section, we will illustrate that the proposed information fusion algorithm can be used in the multi-criteria decision-making environment, where generalized fuzzy numbers are used to represent evaluating values.

5. A Numerical Example

In this section, we apply the proposed information fusion algorithm to deal with a multi-criteria decision-making (MCDM) problem, where we use generalized

Table 4. Evaluating values represented by generalized fuzzy numbers of the priority criteria ***A*** and ***B***.

Different criteria	Alternatives	
	First order criterion ***A***	Second order criterion ***B***
	Evaluating values	
x_1	$\tilde{A}_1 = (10, 10, 10, 10; 1.0)$	$\tilde{B}_1 = (3, 4, 4, 5; 1.0)$
x_2	$\tilde{A}_2 = (0, 0, 0, 0; 1.0)$	$\tilde{B}_2 = (10, 10, 10, 10; 1.0)$
x_3	$\tilde{A}_3 = (3, 4, 4, 5; 1.0)$	$\tilde{B}_3 = (3, 3.5, 4.5, 5; 1.0)$
x_4	$\tilde{A}_4 = (0, 1, 1, 2; 1.0)$	$\tilde{B}_4 = (1, 2, 2, 3; 1.0)$
x_5	$\tilde{A}_5 = (10, 10, 10, 10; 1.0)$	$\tilde{B}_5 = (0, 0, 0, 0; 1.0)$
x_6	$\tilde{A}_6 = (3, 4, 4, 5; 1.0)$	$\tilde{B}_6 = (5, 6, 6, 7; 0.9)$
x_7	$\tilde{A}_7 = (4, 5, 5, 6; 1.0)$	$\tilde{B}_7 = (9, 9.5, 10, 10; 1.0)$
x_8	$\tilde{A}_8 = (0, 1, 1, 2; 0.9)$	$\tilde{B}_8 = (6, 7, 7, 8; 1.0)$

fuzzy numbers to represent the evaluating values of two different criteria ***A*** and ***B***. Assume that there are eight alternatives $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ and two different criteria ***A*** and ***B***, where ***A*** is the first order criterion and ***B*** is the second order criterion. We use the generalized fuzzy numbers to represent the evaluating values $\tilde{A}_i$ and $\tilde{B}_i$ of the alternatives x_i, where $1 \le i \le 8$, and the generalized fuzzy numbers are defined in the universe of discourse $X = [0, 10]$ as shown in Table 4.

From [3], we can see that there are two multi-criteria decision-making (MCDM) frameworks, i.e., (i) without considering the second order criterion ***B*** and (ii) considering the second order criterion ***B*** as shown in Fig. 3, respectively.

Yager [3] pointed out that the second order criterion could be seen as an additional selector or a filter as illustrated in Fig. 3(ii). In the following, we use two different cases to discuss two different conditions, i.e., (1) The MCDM framework only considers the first order criterion ***A***; (2) The MCDM framework both considers the first order criterion ***A*** and the second order criterion ***B***, and we see the second order criterion ***B*** as a filter. Firstly, we use formulas (17) and (18) to translate the generalized trapezoidal fuzzy numbers shown in Table 4 into standardized generalized fuzzy numbers shown in Table 5.

Then, we use the algebraic product (i.e., $a \times b$) and the algebraic sum (i.e., $a + b - a \times b$) as shown in Table 1 to represent the t-norm and the t-conorm (i.e., $\wedge$ and $\vee$), respectively. In the two cases, the alternatives $x_1, x_2, \ldots, x_8$ can be seen as eight different books and a decision maker wants to choose a most suitable book from these books.

Case 1: The MCDM framework only considers the first order criterion ***A***. Assume that the decision maker's statement is as follows:

> "I want a book which is talking about **fuzzy theory**."

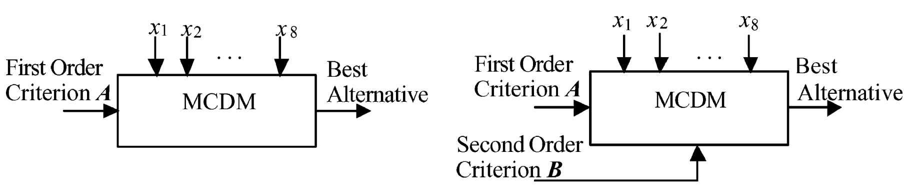
In this statement, the first order criterion ***A*** is "fuzzy theory". According to Table 5, we can see that the books x_1 and x_5 completely satisfy the first order criterion ***A***, the books x_3 and x_6 have the same satisfying degrees with respect to the first order criterion ***A***, and the book x_2 does not satisfy the first order criterion ***A*** at all. In this case, because the MCDM framework only considers the first order criterion ***A***, we use formula (22) to calculate the ranking values $R(\tilde{A}_i^*)$ of other evaluating values $\tilde{A}_i^*$, where the ranking values $R(\tilde{A}_i^*)$ of the evaluating values $\tilde{A}_i^*$, where $1 \le i \le 8$, of the first order criterion ***A*** are shown in Table 6.

From Table 6, we can see that the ranking order of these ranking values $R(\tilde{A}_i^*)$ of evaluating values $\tilde{A}_i^*$ is $R(\tilde{A}_1^*) = R(\tilde{A}_5^*) > R(\tilde{A}_7^*) > R(\tilde{A}_6^*) = R(\tilde{A}_3^*) > R(\tilde{A}_4^*) > R(\tilde{A}_8^*) > R(\tilde{A}_2^*)$. Thus, the preferring order of these books is as follows:

$$x_1 = x_5 > x_7 > x_6 = x_3 > x_4 > x_8 > x_2,$$

and we can see that the most suitable books for the decision maker are x_1 and x_5.

Case 2: The MCDM framework both considers the first order criterion ***A*** and the second order criterion ***B***,



(i) (ii)

Figure 3. The MCDM frameworks: (i) without considering the second order criterion ***B***, and (ii) considering the second order criterion ***B***.

Table 5. Standardized generalized fuzzy numbers of the priority criteria $\boldsymbol{A}$ and $\boldsymbol{B}$.

Different criteria	Alternatives	
	First order criterion $\boldsymbol{A}$	Second order criterion $\boldsymbol{B}$
	Evaluating values	
x_1	$\tilde{A}_1^* = (1.0, 1.0, 1.0, 1.0; 1.0)$	$\tilde{B}_1^* = (0.3, 0.4, 0.4, 0.5; 1.0)$
x_2	$\tilde{A}_2^* = (0.0, 0.0, 0.0, 0.0; 1.0)$	$\tilde{B}_2^* = (1.0, 1.0, 1.0, 1.0; 1.0)$
x_3	$\tilde{A}_3^* = (0.3, 0.4, 0.4, 0.5; 1.0)$	$\tilde{B}_3^* = (0.3, 0.35, 0.45, 0.5; 1.0)$
x_4	$\tilde{A}_4^* = (0.0, 0.1, 0.1, 0.2; 1.0)$	$\tilde{B}_4^* = (0.1, 0.2, 0.2, 0.3; 1.0)$
x_5	$\tilde{A}_5^* = (1.0, 1.0, 1.0, 1.0; 1.0)$	$\tilde{B}_5^* = (0.0, 0.0, 0.0, 0.0; 1.0)$
x_6	$\tilde{A}_6^* = (0.3, 0.4, 0.4, 0.5; 1.0)$	$\tilde{B}_6^* = (0.5, 0.6, 0.6, 0.7; 0.9)$
x_7	$\tilde{A}_7^* = (0.4, 0.5, 0.5, 0.6; 1.0)$	$\tilde{B}_7^* = (0.9, 0.95, 1.0, 1.0; 1.0)$
x_8	$\tilde{A}_8^* = (0.0, 0.1, 0.1, 0.2; 0.9)$	$\tilde{B}_8^* = (0.6, 0.7, 0.7, 0.8; 1.0)$

Table 6. Ranking values $R(\tilde{A}_i^*)$ of each evaluating value $\tilde{A}_i^*$ of the first order criterion $\boldsymbol{A}$.

Evaluating values $\tilde{A}_i^*$	Ranking values $R(\tilde{A}_i^*)$
$\tilde{A}_1^*$	2.0
$\tilde{A}_2^*$	1.0
$\tilde{A}_3^*$	1.3674
$\tilde{A}_4^*$	1.0674
$\tilde{A}_5^*$	2.0
$\tilde{A}_6^*$	1.3674
$\tilde{A}_7^*$	1.4674
$\tilde{A}_8^*$	1.038

and we see the second order criterion $\boldsymbol{B}$ as a filter. Assume that the decision maker's statement is as follows:

> "I want the most suitable book which is talking about **fuzzy theory** and *if possible I want this book to be concerning management*."

In this statement, the first order criterion $\boldsymbol{A}$ is "fuzzy theory", the second order criterion $\boldsymbol{B}$ is "management". In this case, we apply the proposed information fusion algorithm to deal with the MCDM framework as follows:

Step 1. Based on formulas (17), (18) and (22), we can translate the generalized trapezoidal fuzzy numbers shown in Table 4 into the standardized generalized fuzzy numbers as shown in Table 5.

Step 2. Based on formulas (2), (7) and (19), we can evaluate the degree of compatibility $\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B})$ between the first order criterion $\boldsymbol{A}$ and the positive criterion $\boldsymbol{B}$ as follows:

$$\begin{aligned}
\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}) = \mathrm{Max}\big\{&\mathrm{Min}\big[\big(\hat{x}_{\tilde{A}_1^*} \wedge \hat{x}_{\tilde{B}_1^*}\big), S(\tilde{A}_1^*, \tilde{B}_1^*)\big],\\
&\mathrm{Min}\big[\big(\hat{x}_{\tilde{A}_2^*} \wedge \hat{x}_{\tilde{B}_2^*}\big), S(\tilde{A}_2^*, \tilde{B}_2^*)\big],\\
&\mathrm{Min}\big[\big(\hat{x}_{\tilde{A}_3^*} \wedge \hat{x}_{\tilde{B}_3^*}\big), S(\tilde{A}_3^*, \tilde{B}_3^*)\big],\\
&\mathrm{Min}\big[\big(\hat{x}_{\tilde{A}_4^*} \wedge \hat{x}_{\tilde{B}_4^*}\big), S(\tilde{A}_4^*, \tilde{B}_4^*)\big],\\
&\mathrm{Min}\big[\big(\hat{x}_{\tilde{A}_5^*} \wedge \hat{x}_{\tilde{B}_5^*}\big), S(\tilde{A}_5^*, \tilde{B}_5^*)\big],\\
&\mathrm{Min}\big[\big(\hat{x}_{\tilde{A}_6^*} \wedge \hat{x}_{\tilde{B}_6^*}\big), S(\tilde{A}_6^*, \tilde{B}_6^*)\big],\\
&\mathrm{Min}\big[\big(\hat{x}_{\tilde{A}_7^*} \wedge \hat{x}_{\tilde{B}_7^*}\big), S(\tilde{A}_7^*, \tilde{B}_7^*)\big],\\
&\mathrm{Min}\big[\big(\hat{x}_{\tilde{A}_8^*} \wedge \hat{x}_{\tilde{B}_8^*}\big), S(\tilde{A}_8^*, \tilde{B}_8^*)\big]\big\}\\
= \mathrm{Max}\{&0.2667, 0.0, 0.16, 0.02, 0.0, 0.24,\\
&0.4308, 0.07\}\\
= 0.4308&.
\end{aligned}$$

Step 3. Based on formulas (15), (16) and (21), we can calculate the fusion results $\tilde{D}_1$, $\tilde{D}_2$, $\tilde{D}_3$, $\tilde{D}_4$, $\tilde{D}_5$, $\tilde{D}_6$, $\tilde{D}_7$ and $\tilde{D}_8$, respectively, shown as follows:

$$\begin{aligned}
\tilde{D}_1 &= \tilde{A}_1^* \wedge (\tilde{B}_1^* \vee (1 - 0.4308))\\
&= (0.6984, 0.7415, 0.7415, 0.7846; 1.0),\\
\tilde{D}_2 &= \tilde{A}_2^* \wedge (\tilde{B}_2^* \vee (1 - 0.4308))\\
&= (0.0, 0.0, 0.0, 0.0; 1.0),\\
\tilde{D}_3 &= \tilde{A}_3^* \wedge (\tilde{B}_3^* \vee (1 - 0.4308))\\
&= (0.2095, 0.288, 0.3052, 0.3923; 1.0),\\
\tilde{D}_4 &= \tilde{A}_4^* \wedge (\tilde{B}_4^* \vee (1 - 0.4308))\\
&= (0.0, 0.0655, 0.0655, 0.1397; 1.0),\\
\tilde{D}_5 &= \tilde{A}_5^* \wedge (\tilde{B}_5^* \vee (1 - 0.4308))\\
&= (0.5692, 0.5692, 0.5692, 0.5692; 1.0),
\end{aligned}$$

Table 7. Ranking values $R(\tilde{D}_i)$ of each fusion result $\tilde{D}_i$.

Fusion results $\tilde{D}_i$	Ranking values $R(\tilde{D}_i)$
$\tilde{D}_1$	1.7273
$\tilde{D}_2$	1.0
$\tilde{D}_3$	1.2677
$\tilde{D}_4$	1.0455
$\tilde{D}_5$	1.5692
$\tilde{D}_6$	1.2933
$\tilde{D}_7$	1.4561
$\tilde{D}_8$	1.0523

$$\begin{aligned}\tilde{D}_6 &= \tilde{A}_6^* \wedge (\tilde{B}_6^* \vee (1 - 0.4308)) \\ &= (0.2354, 0.3311, 0.3311, 0.4354; 0.9699), \\ \tilde{D}_7 &= \tilde{A}_7^* \wedge (\tilde{B}_7^* \vee (1 - 0.4308)) \\ &= (0.3828, 0.4893, 0.5, 0.6; 1.0), \\ \tilde{D}_8 &= \tilde{A}_8^* \wedge (\tilde{B}_8^* \vee (1 - 0.4308)) \\ &= (0.0, 0.0871, 0.0871, 0.1828; 0.9699).\end{aligned}$$

where we use the t-norm "$a \times b$" and the t-conorm "$a + b - a \times b$" shown in Table 1 to denote the operations of the operators $\wedge$ and $\vee$, respectively.

Step 4. By applying formula (22), we can get the ranking values $R(\tilde{D}_1)$, $R(\tilde{D}_2)$, $R(\tilde{D}_3)$, $R(\tilde{D}_4)$, $R(\tilde{D}_5)$, $R(\tilde{D}_6)$, $R(\tilde{D}_7)$ and $R(\tilde{D}_8)$ of the fusion results $\tilde{D}_1$, $\tilde{D}_2$, $\tilde{D}_3$, $\tilde{D}_4$, $\tilde{D}_5$, $\tilde{D}_6$, $\tilde{D}_7$ and $\tilde{D}_8$ respectively, shown in Table 7.

From Table 7, we can see that the ranking order of these ranking values is $R(\tilde{D}_1) > R(\tilde{D}_5) > R(\tilde{D}_7) > R(\tilde{D}_6) > R(\tilde{D}_3) > R(\tilde{D}_8) > R(\tilde{D}_4) > R(\tilde{D}_2)$. Thus, the preferring order of these books is as follows:

$$x_1 > x_5 > x_7 > x_6 > x_3 > x_8 > x_4 > x_2,$$

and we can see that the most suitable book for the decision maker is x_1.

From the results of Case 1 and Case 2, we can see that if we do not consider the second order criterion ***B*** as a filter, we can not distinguish the preferring order between the books x_1 and x_5; x_3 and x_6. However, if we use the proposed information fusion algorithm to consider the second order criterion ***B*** in the MCDM framework, we not only can distinguish the preferring order between the books x_1 and x_5; x_3 and x_6, but also can hold the preferring order of the other books using the first order criterion ***A***. Furthermore, in Case 1, we can see that the book x_4 is more suitable than the book x_8 for the decision maker's need if we only consider the first order criterion ***A***. However, in Case 2, we can see that the book x_8 is more suitable than the book x_4 for the decision maker's need if we consider the two different criteria ***A*** and ***B*** simultaneously. The main reason is that the evaluating values $\tilde{A}_4^*$ and $\tilde{A}_8^*$ of the books x_4 and x_8, respectively, of the first order criterion ***A*** are very similar, where $\tilde{A}_4^* > \tilde{A}_8^*$; the evaluating values $\tilde{B}_4^*$ and $\tilde{B}_8^*$ of the books x_4 and x_8, respectively, of the positive criterion ***B*** are very dissimilar, where $\tilde{B}_4^* < \tilde{B}_8^*$. Thus, the preferring order between the books x_4 and x_8 will be changed when we consider the two different criteria ***A*** and ***B*** simultaneously.

6. Conclusions

In this paper, we have extended the prioritized operator presented in [5] to propose a prioritized information fusion algorithm based on similarity measures of generalized fuzzy numbers for handling prioritized multi-criteria fuzzy decision-making problems. The prioritized multi-criteria fuzzy decision-making problems have two different criteria ***A*** and ***B***, where ***A*** is the first order criterion and ***B*** is the second order criterion. From Section 5, we can see that the second order criterion ***B*** can be seen as a filter to choose the most suitable book from those books which satisfy the first order criterion ***A***. Furthermore, the proposed information fusion method can handle the decision-making problems in a more flexible manner due to the fact that it allows the evaluating values of the criteria to be represented by generalized fuzzy numbers or crisp values between zero and one, where the generalized fuzzy numbers can indicate certainty degrees of the evaluating values of the decision maker's opinions.

The prioritized information fusion method presented in this paper can deal with single-level information filtering problems. In the future, we will extend the proposed prioritized information fusion algorithm to develop a recursive algorithm for handling multi-level information filtering problems due to the fact that the decision makers may deal with multi-level information filtering in the real world.

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References

1. R.E. Bellman and L.A. Zadeh, "Decision making in a fuzzy environment," *Management Science*, vol. 17, no. 4, pp. 141–164, 1970.
2. K. Hirota and W. Pedrycz, "Non-monotonic fuzzy set operations: A generalization and some applications," *International Journal of Intelligent Systems*, vol. 12, no. 15, pp. 483–493, 1997.
3. R.R. Yager, "Second order structures in multi-criteria decision making," *International Journal of Man-Machine Studies*, vol. 36, no. 16, pp. 553–570, 1992.
4. R.R. Yager, "Structures for prioritized fusion of fuzzy information," *Information Sciences*, vol. 108, no. 1, pp. 71–90, 1998.
5. R.R. Yager, "Non-monotonic set theoretic operations," *Fuzzy Sets and Systems*, vol. 42, no. 2, pp. 173–190, 1991.
6. H.J. Zimmermann, *Fuzzy Sets, Decision Making, and Expert Systems*, Kluwer Academic Publishers: Boston, USA, 1987.
7. S.M. Chen, "Aggregating fuzzy opinions in the group decision-making environment," *Cybernetics and Systems: An International Journal*, vol. 29, no. 4, pp. 363–376, 1998.
8. S.J. Chen and C.L. Hwang, *Fuzzy Multiple Attribute Decision Making*, Springer-Verlag: Berlin, Heidelberg, 1992.
9. F. Herrera, E. Herrera-Viedma, and J.L. Verdegay, "A rational consensus model in group decision making using linguistic assessments," *Fuzzy Sets and Systems*, vol. 88, no. 1, pp. 31–49, 1997.
10. H.S. Lee, "An optimal aggregation method for fuzzy opinions of group decision," in *Proceedings of the 1999 IEEE International Conference on Systems, Man, and Cybernetics*, Tokyo, Japan, 1999, pp. 314–319.
11. S.H. Chen, "Operations on fuzzy numbers with function principle," *Tamkang Journal of Management Sciences*, vol. 6, no. 1, pp. 13–25, 1985.
12. S.H. Chen, "Ranking generalized fuzzy number with graded mean integration," in *Proceedings of the Eighth International Fuzzy Systems Association World Congress*, Taipei, Taiwan, Republic of China, 1999, pp. 899–902.
13. S.J. Chen and S.M. Chen, "Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers," *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 1, pp. 45–56, 2003.
14. D. Dubois, "Modeles mathematiques l'imprecis et de l'incertain en vue d'applications aux techniques d'aide a la decision," Ph.D. Dissertation, Université Scientifique et Médicale de Grenoble, 1983.
15. A. Gonzalez, O. Pons, and M.A, "Vila. Dealing with uncertainty and imprecision by means of fuzzy numbers," *International Journal of Approximate Reasoning*, vol. 21, no. 2, pp. 233–256, 1999.
16. S.J. Chen and S.M. Chen, "A new method for handling the fuzzy ranking and the defuzzification problems," in *Proceedings of the Eighth National Conference on Fuzzy Theory and Its Applications*, Taipei, Taiwan, Republic of China, 2000.
17. S.J. Chen and S.M. Chen, "Aggregating fuzzy opinions in the heterogeneous group decision-making environment," *Cybernetics and Systems: An International Journal*, vol. 36, no. 3, 2005.
18. S.J. Chen and S.M. Chen, "A new ranking method for handling outsourcing targets selection problems," in *Proceedings of the 13th International Conference on Information Management*, Taipei, Taiwan, Republic of China, 2002, pp. 103–110.
19. B. Rich, *Theory and Problems of Geometry*, McGraw-Hill: New York, U.S.A., 1989.
20. M.J. Wierman, "Central values of fuzzy numbers—Defuzzification," *Information Sciences*, vol. 100, nos. 1–4, pp. 207–215, 1997.
21. R.R. Yager, "Fuzzy sets and approximate reasoning in decision and control," in *Proceedings of the 1992 IEEE International Conference on Fuzzy Systems*, San Diego, California, 1992, pp. 415–428.
22. R.R. Yager and D.P. Filev, "On the issue of defuzzification and selection based on a fuzzy set," *Fuzzy Sets and Systems*, vol. 55, no. 3, pp. 255–272, 1993.
23. C.H. Cheng, "A new approach for ranking fuzzy numbers by distance method," *Fuzzy Sets and Systems*, vol. 95, no. 2, pp. 307–317, 1998.
24. P. Subasic and K. Hirota, "Similarity rules and gradual rules for analogical and interpolative reasoning with imprecise data,"*Fuzzy Sets and Systems*, vol. 96, no. 1, pp. 53–75, 1998.
25. C.H. Hsieh and S.H. Chen, "Similarity of generalized fuzzy numbers with graded mean integration representation," in *Proceedings of the Eighth International Fuzzy Systems Association World Congress*, Taipei, Taiwan, Republic of China, 1999, pp. 551–555.
26. S. Murakami, S. Maeda, and S. Imamura, "Fuzzy decision analysis on the development of centralized regional energy control systems," in *Proceedings of the IFAC Symposium on Fuzzy Information, Knowledge Representation and Decision Analysis*, Pergamon Press: New York, 1983, pp. 363–368.
27. R.R. Yager, "On a general class of fuzzy connectives," *Fuzzy Sets and Systems*, vol. 4, no. 2, pp. 235–242, 1980.
28. S.J. Chen and S.M. Chen, "A new method for handling multi-criteria fuzzy decision making problems using FN-IOWA operators," *Cybernetics and Systems: An International Journal*, vol. 34, no. 2, pp. 109–137, 2003.
29. L.A. Zadeh, "A theory of approximate reasoning," *Machine Intelligence*, vol. 9, no. 1, pp. 149–194, 1979.
30. R.R. Yager, "Using approximate reasoning to represent default knowledge," *Artificial Intelligence*, vol. 31, no. 1, pp. 99–112, 1987.
31. G.Q. Zhang, "On fuzzy number-valued fuzzy measures defined by fuzzy number-valued fuzzy integrals I,"*Fuzzy Sets and Systems*, vol. 45, no. 2, pp. 227–237, 1992.
32. G.Q. Zhang, "On fuzzy number-valued fuzzy measures defined by fuzzy number-valued fuzzy integrals II," *Fuzzy Sets and Systems*, vol. 48, no. 2, pp. 257–265, 1992.
33. A. Kaufmann and M.M. Gupta, *Fuzzy Mathematical Models in Engineering and Management Science*, North-Holland: Amsterdam, 1988.
34. A. Junghanns, C. Posthoff, and M. Schlosser, "Search with fuzzy numbers," in *Proceedings of the Fourth IEEE International Conference on Fuzzy Systems*, Yokohama, Japan, 1995, pp. 979–986.
35. G. Bordogna and G. Pasi, "Linguistic aggregation operators of selection criteria in fuzzy information retrieval," *International Journal of Intelligent Systems*, vol. 10, no. 2, pp. 233–248, 1995.
36. S.J. Chen and S.M. Chen, "A prioritized information fusion algorithm for handling multi-criteria fuzzy decision-making problems," in *Proceedings of the 2002 International Conference on Fuzzy Systems and Knowledge Discovery*, Singapore, 2002.

37. D. Dubois and H. Prade, *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Plenum Press: New York, 1988.
38. M.L. Ginsberg, *Nonmonotonic Reasoning*, Morgan Kaufman: Los Altos, California, 1987.
39. M.M. Gupta, A. Kandel, W. Bandler, and J.B. Kiszka, *Approximate Reasoning in Expert Systems*, North-Holland, Amsterdam, 1985.
40. S. Miyamoto, *Fuzzy Sets in Information Retrieval and Cluster Analysis*, Kluwer Academic Publishers: Massachusetts, 1990.
41. D. Ramot, R. Milo, M. Friedman, and A. Kandel, "On fuzzy correlations," *IEEE Transactions on Systems, Man, and Cybernetics- Part B: Cybernetics*, vol. 31, no. 3, pp. 381–390, 2001.
42. L.A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.