Would the Real Lakatos Please Stand Up

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ABSTRACT: Although Imre Lakatos described the work published in his book *Proofs and Refutations* as a study of mathematical methodology, work which has been responded to and criticized by philosophers and historians of mathematics more on its own terms, a significant body of writing in the 30 years since its appearance has used it as a pertinent cognate text appropriable for school mathematics education. In this paper, we contrast the responses these two fields have generated, with an emphasis on that of mathematics education. Doing so offers a potentially salutary case study of how challenging and fraught it can be at times to undertake work at the nexus of history and philosophy on the one hand, while at the same time seeking to explore its possible relevance and significance for education. As our title suggests, we are concerned about the proliferating Lakatos personas that seem to exist, including a growing range of self-styled reform or progressive educational practices which get attributed to him.

KEYWORDS: Mathematical method, proof, criticism, Lakatos, Lakatosian, philosophy of mathematics, philosophy of mathematics education.

Prelude

In 2006, Bharath Sriraman published an article in this journal entitled "An Ode to Imre Lakatos: Quasi-thought Experiments to Bridge the Ideal and Actual Classrooms." In it, he claims that a "universal pedagogical goal of mathematics teachers" could be realized if classroom discourse could be structured around the "Lakatosian (thought-experimental) view of mathematics" (p. 151). Similar to Lampert (1990), Sriraman attempted to simulate the Lakatosian classroom, implying that not only did a "Lakatosian method" and "Lakatosian vision" of classroom discourse exist, but that they were somehow *ideal* for the teaching and learning of mathematics. He also asserted that the dialogue in *Proofs and Refutations* contains "an imaginary account of

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classroom discourse between the students and the teacher in the ideal classroom" (pp. 151-152).

In a certain sense, such nods (and more) of nominalised attribution, as well as the adjectival reifying of someone's work, are academic commonplaces. However, we have been seeing more and more of them in relation to Lakatos and on occasions in a manner increasingly distant from and implausible in relation to our understanding of both the actual content and declared purpose and target of Lakatos's work seen as a study of mathematical method. In consequence, we decided to start to document this particular traversing of fields (and the traducing that perhaps necessarily happens along the way) as a potential case study – one that we can only scratch the surface of here – of attempts to take up work from one field and transport it to achieve a different task in another.

Introduction

In 1961, some 45 years ago, Imre Lakatos completed his Ph.D. *Essays in the Logic of Mathematical Discovery* at the University of Cambridge[1], and 30 years ago, in 1976, shortly after Lakatos's death two years previously, *Proofs and Refutations: the Logic of Mathematical Discovery* appeared. The bulk of this book documents the history of the Descartes-Euler formula for polyhedra offered as a generic example of the development of mathematics, but does so in a strikingly novel form, namely as a fictionalized dialogue (or playscript) set in an imagined classroom (as it was in his doctoral dissertation).[2] Although Lakatos described his work as a study of mathematical methodology, and his argument with philosophers of mathematics[3], a non-trivial amount of writing since then has used it as a font of suggestions concerning mathematics education, especially school mathematics education.

A few mathematics education authors have previously signaled their awareness that in a translation to a mathematics education setting Lakatos's work might not be straightforward: for instance Nunokawa (1996) acknowledges,

> Although Lakatos (1976) employs the dialogue form of description, his method is the reflection of 'rationally constructed' or 'distilled' history (p. 5) ... it is not immediately possible to deduce from his theory the importance of social interactions or discussions in the construction of mathematical knowledge. (pp. 269-270)

This is after he has done a thorough job of pointing to the considerable variety of ways in which Lakatos's *Proofs and Refutations* has been claimed to be influential in relation to school mathematics classrooms (including the introduction of exploration and observation, emphasizing students creation of mathematical knowledge, paying attention to informal mathematics, to social aspects of mathematical proof, and emphasizing classroom discussion). In the decade since this paper, this list continues to grow. Nunokawa summarizes his opening observations as follows: “there is something vague concerning the relationship between Lakatos’s theory and mathematics education” (p. 270), before going on to draw on Lakatos’s theory of scientific research programmes instead.

Thus, near the outset, we wonder had Lakatos not chosen this dialogue-apparently-set-in-a-classroom form for the presentation of his Descartes-Euler essay whether his work would have been seen as so relevant for mathematics education. It is this sense of appropriation, akin perhaps to that of Piaget’s work taken far beyond its intended setting (Piaget, a self-styled “genetic epistemologist,” explicitly refrained from pronouncements about the institutional education of children based on his work), that set us off on our own case study. We attempt to identify some of the various Lakatosia who have been created and argued about, cited and drawn upon, lauded and vilified within mathematics education as a field. But our title itself contains a fallacy, namely the suggestion that there is a unitary *real* Lakatos to be identified.

In a superficially similar way to Lacan’s “return to Freud,” we look once again at what Lakatos said about what he was about. However, we can only do so in light of what has been written subsequently (recall the David Lodge character in his novel *Small World* whose Ph.D. thesis was on the influence of T.S. Eliot on Shakespeare). Samuel Weber (1991), writing about Lacan’s influence and work, has observed: “What often happens … is that what we call ‘proper’ names begin to circulate widely, suggesting a sort of permanence or at least durability. But these names, far from rendering what they name accessible, function as *screens”* (p. xi). He is also at pains to document how much of a “dislocation” Lacan’s reading of Freud was.

In brief outline, we first take a look at mathematics education writers who have drawn on Lakatos, both as proper name – author and authority – and as metonymic label for a set of ideas about mathematics and, it is frequently assumed, its teaching at the school level. Then, we revisit *Proofs and Refutations* (1976) in order to see what evidence there is for some of these personas. Lastly, we attempt to see where such dislocations may lead us, not least in terms of this being a potentially generic study of the fate of attempts to draw lessons from the history and philosophy of mathematics for mathematics education theory.

Mathematics Educators Look to Lakatos

For the past 30 years, philosophers and mathematics educators alike have appropriated the dialogue in *Proofs and Refutations* and inferred great meaning for the classroom practices of both teachers and students (Sierpinska & Lerman, 1996). In general, many of the approaches and recommendations by the following educators for school teaching and learning mathematics are not inherently disagreeable to us. But we are struck by the seeming rhetorical need to attribute them to Lakatos himself. However, we certainly agree with Sierpinska and Lerman who claim:

> Although the "classroom" in Lakatos' (1976) *Proofs and Refutations* was perhaps not intended to suggest that mathematics proceeds by negotiation, or that the heuristics are the essence of mathematics, not the outcomes, it has been taken in that sense by mathematics educators. (p. 838)

But we go further, in feeling some mathematics educators' claims represents a signal misreading of Lakatos's actual work and intentions and so the "perhaps" in this quotation seems to us rather disingenuous.

Not long after the publication of *Proofs and Refutations*, the philosopher Joseph Agassi (1980), in his address to the Canadian Mathematics Education Study Group, spoke of a "Lakatosian Revolution" in mathematics education. Going so far as criticizing the state of mathematics teaching and recommending ways in which mathematics teachers could work and communicate with their students, Agassi saw the world of mathematics education in desperate need of Lakatos-inspired teaching. Agassi spoke of a Lakatos method for the classroom, which had "the merit of taking the student from where he stands and using his interruptions of the lecture as a chief vehicle of his progress, rather than worrying about the teacher's progress" (p. 30).

Ernest (1991) claimed that the fallibilist philosophy and social construction of mathematics presented by Lakatos not only had educational implications, but that Lakatos was even aware of these implications (p. 208). Ernest argued that school mathematics should take on the socially constructed nature presented by Lakatos, and also that teacher and students should engage in ways identical to those in his dialogue, specifically posing and solving problems, articulating and confronting assumptions, and participating in genuine discussion.

Lampert (1990) relied on Lakatos as a guide in a project "to develop and implement new forms of teacher-student interaction" (p. 33). Thus, Lampert applied Lakatos's dialogue to her own classroom in an experiment to test whether the qualities of Lakatos's mathematics could be observed in a classroom setting. She concluded that she had observed the students as having "learned to do mathematics together in a way that is consonant with Lakatos's and Polya's assertions about what doing and knowing mathematics entails" (p. 33). For the teacher's behavior, Lampert surmised that a teaching practice similar to that demonstrated by the fictionalized teacher in *Proofs and Refutations* was necessary for students to "see what sort of knowing mathematics involves" (p. 41). Later, Lampert's work was used to help justify the NCTM standards for classrooms modeled using Lakatos's dialogue (Yackel & Hanna, 2003), illustrating the influence that Lakatos has had on teaching and learning in mathematics (or at least that he is claimed to have had by others).

Brodie (2000), in her attempt to interpret a teacher's actions through Lakatosian eyes, relied heavily on his historical account of mathematicians' work to explore questions such as:

> How do teachers intervene and mediate appropriately in pupils' interactions? …. How long should they wait before challenging and trying to avoid misconceptions? How long should they avoid attempts to reorient discussions in more mathematically fruitful directions? How do teachers manage to hear the contributions of all pupils in the group and enable pupils to communicate effectively with each other? (p. 9)

Further, Brodie went so far as to claim that "Lakatos' teacher might represent the kind of teacher envisaged by curriculum reforms" (p. 11). With regard to student behavior, she interpreted the dialogue of students as taking on the form of a Lakatosian dialogue, having interpreted Lakatos's dialogue as an "imaginary dialogue between a mathematical teacher and a group of (university) students" (p. 10).

These are but a few authors among many who cite Lakatos's name and work.

Lakatos on Lakatos

As with any real classroom dialogue, Lakatos's dialogue from *Proofs and Refutations* (1976) exists as an entity unto itself now, and is therefore subject to analysis and interpretation with or without regard to the intention of the writer.[4] Those who interpret his classroom construction as an example of a generic ideal are free to make their case; we certainly do not argue against the benefits of mathematical discussion in the classroom. It is the attribution of this interpretation to Lakatos himself which we find suspect. We believe that a return to Lakatos can illuminate the notions he was actually attending to in his book.

In the Preface and, in particular, in the two seldom-cited Appendices, Lakatos makes a few references to (real) students, teachers, or textbooks. But his discussion is almost entirely around the development of new mathematical ideas and the ways in which mathematicians choose – or could choose – to present the results of their labours to fellow mathematicians and students alike. According to Lakatos, the method of "proofs and refutations" is a general heuristic pattern of mathematical discovery which consists of several stages from primitive conjecture to "proof" to the consideration of counterexamples resulting in an "improved proof." He gives a number of examples of mathematical concepts whose development can be described by such a pattern, including that of *polyhedron* considered in the main body of this work.

In Appendix 2, Lakatos contrasts this "heuristic" approach with the traditional "deductivist" approach to the presentation of mathematical results. The deductivist style is familiar to those of us who experienced a university mathematics education.

> The style starts with a painstakingly stated list of *axioms*, *lemmas* and/or *definitions*. The axioms and definitions frequently look artificial and mystifyingly complicated. One is never told how these complications arose. The list of axioms and definitions is followed by the carefully worded *theorems*. These are loaded with heavy-going conditions; it seems impossible that anyone should ever have guessed them. The theorem is followed by the *proof*. (p. 142)

In the deductivist style, "all propositions are true and all inferences valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. Counterexamples, refutations, criticism cannot possibly enter" (p. 142). This "authoritarian air" is itself a fiction, achieved artificially:

> by beginning with disguised monster-barring and proof-generated definitions and with the fully-fledged theorem, and by suppressing the primitive conjecture, the refutations, and the criticism of the proof. Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility. (1976, p. 142)

In fact, this is the crux of Lakatos's argument in *Proofs and Refutations*: that progress at the frontiers of mathematics *does not occur* by a deductivist process, but rather, by the very heuristic process he has scripted his students to illustrate. He is skeptical of the claim that:

> mathematicians start with an empty mind, set up their axioms and definitions at their pleasure, in the course of a playful free creative activity, and it is only at a later stage that they deduce the theorems from these axioms and definitions. (p. 143)

In terms of his declared purpose, Lakatos states clearly that, "these essays intend to show that those who claim that deduction is the logic of mathematical discovery are wrong" (1976, p. 143).

In all of his examples, Lakatos is referring to the work of mathematicians, not students, in the discovery and development of these ideas. Lakatos was well aware – having done the "distillation," as he called it – that the dialogue among these students actually took place among experienced mathematicians over a period of years, not hours. Nowhere does he suggest that it is necessary – or appropriate, or even possible – that students should re-invent every mathematical concept in the classroom. Nor is he arguing against logic or proof in favour of "shabby inductive reasoning." Rather, he calls for making the process of proving transparent:

> One can easily give more examples, where stating the primitive conjecture, showing the proof, the counterexamples, and following the heuristic order up to the theorem and to the proof-generated definition would dispel the authoritarian mysticism of abstract mathematics. (1976, p. 154)

The teacher in Lakatos's dialogue is a literary device which, by necessity, had to mediate a fictionalized discussion which had already taken place over more than a century in reality. Short of altering history, Lakatos was constrained to some extent by the way in which the development of the Descartes-Euler formula actually occurred (though his rendering of the history has been challenged). Clearly it did not result from a traditional, deductivist, classroom experience; the character of the teacher reflects this reality.

A heuristic approach to teaching, in contrast, emphasizes the "problem-situation: it emphasizes the 'logic' which gave birth to the new concept." Lakatos argues that there is no theory in mathematics which has not passed through a period of growth in which "growing concepts are the vehicles of progress, where the most exciting developments come from exploring the boundary regions of concepts, from stretching them, and from differentiating formerly undifferentiated concepts" (p 140).

Such periods are “the most exciting from the historical point of view and should be the most important from the teaching point of view” (1976, p. 140).

He refers in detail to an example from a textbook in which the Riemann-Stieltjes theorem is delivered in a deductive style, and considers how differently the same theorem would be introduced heuristically. By making explicit that two “mystical concepts” which appear in it are actually proof-generated, originating in a proof by Dirichlet, a heuristic approach would deprive these two concepts of “their authoritarian magic” (1976, p. 140). Their development could be traced back to pre-existing problems and to criticism of previous attempts at solving them.

Lakatos does refer occasionally to the “student of mathematics,” who, if unfortunate enough to wonder how the “definitions, lemmas and the theorem can possibly precede the proof,” will be labeled mathematically “immature.” In typical tongue-in-cheek style, he interprets “mathematical maturity” in the context of some textbooks as being “endowed by nature with the ‘ability’ to take a Euclidean argument without any unnatural interest in the problem-background, in the heuristic behind the argument” (1976, p. 142).

In Appendix 1, Lakatos gives a second detailed example of the development of a mathematical concept by the method of “proofs and refutations.” However, this example is presented without scenario, teacher, or students, suggesting that a commentary on classroom practice was not remotely his objective in this work. The *teacher* in this fiction plays a central role in educating the *reader* about the process and language of proof-analysis.

Philosophers of Mathematics on Lakatos

The work of Lakatos certainly did not go unnoticed within philosophy of mathematics either, though despite the assessment of Ian Hacking (given in footnote 3), its reception there was far less one of general accolade (see Hanna & Jahnke, 1996 for some references). Nevertheless, work on Lakatos’s ideas continue both in terms of critique and extension, most notably in that of United Kingdom philosopher of mathematics David Corfield (2002). His chapter “Argumentation and the Mathematical Process” in *Appraising Lakatos: Mathematics, Methodology and the Man* gives an outline of ways of engaging with Lakatos’s philosophy of mathematics, while both critiquing it and extending it (something he does more significantly in Chapter 7 and

Chapter 8 of his 2003 book *Towards a Philosophy of Real Mathematics*). Interestingly, from our perspective, in another chapter entitled "Lakatos and Aspects of Mathematics Education" (Reichel, 2002), the only overlap with mathematics education authors cited here is that of Gila Hanna. This short chapter, presents a more European rather than North American view of things, identifying proof as the most significant context of influence for Lakatos's work, as compared with that of classroom discussion and non-teacher centredness in North America. One of the biggest challenges we see in this area is knowing sufficient both about the philosophy of mathematics and mathematics education.

An Ending of Sorts

To some considerable extent, we believe it was a conceit, in an earlier sense of that word as "an extravagant, fanciful, and elaborate construction or structure," that Lakatos chose to embed what he had discovered about the Descartes-Euler conjecture (a dissertation topic suggested to him by George Polyà) in the form of a classroom dialogue for his doctoral dissertation. This was a fine conceit, both witty and telling, one that allowed the character of the teacher, among other things, to make generic comments about the "method of proofs and refutations" (a method discovered in the 1840s, Lakatos claims) that rose up above the particular detail of the historical and mathematical content under discussion. Had Lakatos written his dissertation in the more conventional form that the outline of his second case study takes (concerned with the proof-generated concept of uniform continuity of functions), we continue to wonder whether his work would have been seen as anywhere near so relevant to issues of the classroom teaching of mathematics and, in particular, how classrooms *should* proceed.

The issue of generality always besets case studies, even – perhaps especially – mathematical ones. Using the various moves and processes involving counter-examples that Lakatos drew attention to (and so compellingly named), some even seek to monster-bar or monster-adjust his own study. Hanna and Jahnke (1996), for instance, reiterate a claim that has been intermittently made for the past three decades about the particularity of the mathematical setting of this conjecture. In their 32-page chapter on "proof and proving" in the *International Handbook of Mathematics Education*, they wrote:

> Mathematics educators may have assumed that Lakatos' approach is more widely applicable than in fact it is. The case for heuristic proof analysis as a general method rests only upon its successful use in the study of polyhedra, an area in which it is relatively easy

> to suggest the counterexamples which this method requires. … Should one really generalise from a sample of one? (p. 888)

In both a move and a tone that are themselves deserving of a Greek-letter character, they go on to monster-adjust the original case study itself: "Even the proof of Euler's theorem cited by Lakatos, for example, is a case where refutation is redundant; as soon as adequate definitions are formulated the theorem can be proved for all possible cases without further discussion" (p. 888).

But that of course is the whole point: where are these "adequate definitions" to come from, in conjunction with Lakatos's strong mathematical focus on ascertaining what "all possible cases" are? And also in showing so clearly how definitions determine that and how definitions can be criticized and *improved*. Psychoanalyst Adam Phillips (1993) asks the question where all this certainty comes from:

> The psychoanalytic question becomes not, Is that true? but what in your personal history disposes you to believe that? ... always an interesting question to ask someone in a state of conviction, What kind of person would you be if you no longer believed that? (p. 112)

Lakatos's book *Proofs and Refutations* (1976) presented an innovative, captivating, and powerful context for a reconstructed historical debate and proof of the Descartes-Euler theorem. We have found ourselves entranced by it, reading aloud to each other and reading to ourselves, fascinated by what might come next and intrigued by the discourse of the "class" presented by Lakatos. We have also found ourselves equally engaged by Lakatos's purpose in writing *Proofs and Refutations*, described in the preface and appendices of the book. And, as a result, we have been concerned by the various interpretations of this wonderful text as having great meaning for real mathematics classrooms, teachers, and students. Thus, through this brief article, we hope we have provided a thoughtful reminder, a recollection of Lakatos and his stated purposes in writing *Proofs and Refutations*, that might encourage a new look at and recognition of Lakatos and his intentions as opposed to what they have become.

Coda

Given Lakatos's notoriously mordant sense of humour (seen most recently at play with the publication of the Lakatos-Feyerabend correspondence – see Motterlini, 1999, and also Ian Hacking's beautifully crafted and informative review of it in the *London Review of Books* in 2000), we can only imagine the Swiftian pleasure he would have taken at seeing – and skewering – some of what has been made of his writing. His own background in the Hungarian Ministry of Higher Education as well as his response to educational political struggles at the London School of Economics where he was a professor in the late 1960s certainly indicated his interest in higher education (and politics). But his connection to the early Black papers on education make it seem unlikely that he would have agreed to lend his name to some (even many) mathematics education proposals (modest and otherwise) that now have been made to bear his name.

NOTES

1. Much of this work was published in 1963-1964 in a series of articles in the *British Journal for the Philosophy of Science*.
2. The question of the "truth" of this "rationally reconstructed" dialogue and where it lies has also proven of interest. In his author's introduction, Lakatos claimed, "the dialogue form should reflect the dialectic of the story. … *The real history will chime in in the footnotes, most of which are to be taken, therefore as an organic part of the essay"* (1976, p. 5). In a more recent enterprise, the question of veracity in relation to an artifact also surfaces, namely the classroom animations used in Herbst and Chazan's (2003) research project exploring what they term a mathematics teacher's "practical rationality." While they have many actual transcribed lessons to draw on, their scripts are nonetheless fictional. Just as with a playscript or a poem, one may ask where the sense of "truth" resides, in what is it grounded.
3. Hacking (2000, p. 28) observes, "Lakatos's contribution to the philosophy of mathematics was, to put it simply, definitive: the subject will never be the same again."
4. One of us (DP) regularly assigns his graduate students in mathematics education the task of analyzing the role of Lakatos's "teacher" character, presuming the essay to be a classroom transcript.

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