DOI 10.1007/s10778-023-01215-z
International Applied Mechanics, Vol. 59, No. 2, March, 2023

FREE VIBRATIONS OF COMPOSITE PLATES MADE OF FUNCTIONALLY GRADED MATERIAL ON AN ELASTIC OR PERFECTLY RIGID FOUNDATION

O. V. Marchuk, O. O. Plisov, and T. G. Tamoyan

Two approaches to studying the free vibrations of functionally graded plates using a three-dimensional problem statement are developed. The first approach does not have the error of approximation of the unknown functions across the plate thickness. The distribution of the elastic modulus in the first approach is modeled by the exponential law. In the second approach, the polynomial approximation of the unknown functions across the plate thickness is used. The elastic modulus changes as a fourth-degree polynomial. The second approach reduces the unknown functions to the external surfaces of the layers, allowing the partitioning of the layers into sublayers to improve the results. These approaches are used to analyze the free vibrations of a plate made of functionally graded material on a rigid foundation and on a foundation in the form of a finite-thickness layer.

Keywords: functionally graded materials, 3D problem statement, analytical solution, free vibrations, inertial properties, rigid foundation

Introduction. The use of composite structures, particularly those made of functionally graded materials with a high elastic modulus gradient across the thickness of the structure, is increasing in modern engineering. These composite structures are subjected to complex deformation conditions. They have different edge conditions and contacts on their outer surfaces, and they are often exposed to nearly resonant dynamic loads. These characteristics can lead to significant three-dimensional dynamic deformation.

The use of accurate methods of spatial elasticity theory to study the free vibrations of these structures is mainly limited to the consideration of hinged plates. In particular, problems related to materials with constant stiffness characteristics were considered in [2, 3, 15–17]. The problem of finding the frequencies of free vibrations using the discrete orthogonalization method was considered in [1]. The design of functionally graded structures based on this approach is addressed in [1, 9]. The classical Kirchhoff–Love plate theory, which is based on simplified hypothesis, can lead to significant errors of the characteristics of free vibrations of plates made of functionally graded materials in certain cases.

The free vibrations of plates made of functionally graded materials were studied using various refined models in [5, 7, 8, 11]. Various numerical-analytical methods for the three-dimensional analysis of plates made of functionally graded materials continue to be developed [6, 10, 12, 13, 18]. In these studies, the elastic modulus is assumed to change either exponentially or polynomially. Despite the large number of works on the dynamic deformation of multilayered plates, we failed to find exact solutions based on a three-dimensional statement for finding the natural frequencies of plates made of functionally graded materials on a rigid foundation or on a foundation in the form of a finite thickness layer considering the inertial properties of the foundation. Static loading of structures with elastic modulus dependent on the thickness was considered in [14]. Note that previous studies using applied approaches mainly focused on structures on a foundation modeled by one modulus of subgrade reaction (Fuss–Winkler model) or two moduli of subgrade reaction (Pasternak model).

National Transport University, 1 Omelianovych-Pavlenko St., Kyiv, Ukraine, 01010; e-mail: mksmntu386@gmail.com. Translated from Prykladna Mekhanika, Vol. 59, No. 2, pp. 110–122, March–April 2023. Original article submitted May 15, 2019.

In what follows, we will study, for the first time, the free vibrations of plates made of functionally graded materials on a rigid foundation and on a foundation in the form of a layer of finite thickness, taking into account the inertial properties of the elastic foundation and using a three-dimensional problem statement. We will show that shear models for plates and models of elastic foundations that disregard their inertial properties are inapplicable to such problems.

1. Problem Statement. Consider a layered structure described in a Cartesian coordinate system. Direction 1 is identical to the direction *X*, direction 2 is identical to the direction *Y*, and direction 3 is identical to the direction *Z*. The axis *Z* is directed downwards. The superscript "(*k*)" is the layer number. The physical and mechanical characteristics of the functionally graded layers are related as follows:

$$e_{11}^{(k)}=\frac{1}{E^{(k)}(z)}(\sigma_{11}^{(k)}+\nu^{(k)}(\sigma_{22}^{(k)}+\sigma_{33}^{(k)})),\qquad e_{22}^{(k)}=\frac{1}{E^{(k)}(z)}(\sigma_{22}^{(k)}+\nu^{(k)}(\sigma_{11}^{(k)}+\sigma_{33}^{(k)})),$$

$$e_{33}^{(k)}=\frac{1}{E^{(k)}(z)}(\sigma_{33}^{(k)}+\nu^{(k)}(\sigma_{22}^{(k)}+\sigma_{11}^{(k)})),\qquad 2e_{23}^{(k)}=\frac{\sigma_{23}^{(k)}}{G^{(k)}(z)},$$

$$2e_{13}^{(k)}=\frac{\sigma_{13}^{(k)}}{G^{(k)}(z)},\qquad 2e_{12}^{(k)}=\frac{\sigma_{12}^{(k)}}{G^{(k)}(z)}. \tag{1.1}$$

We study the free vibrations of functionally graded plates on an elastic and perfectly rigid foundation using two approaches. The first approach is based on variable separation, according to which the displacement vector and the transverse components of the stress tensor are represented as follows:

$$U_i^{(k)}(x,y,z,t)=V_i^{(k)}(x,y,t)\,f_i^{(k)}(z),$$

$$\sigma_{i3}^{(k)}(x,y,z,t)=\tau_{i3}^{(k)}(x,y,t)f_{i+3}^{(k)}(z)\qquad (i=1,2,3). \tag{1.2}$$

This approach is a development of the work [14], which addressed the statics of plates on an elastic foundation with a thickness-dependent elastic modulus, and the works [15, 16], which studied the free vibrations of plates and shallow shells using an exact problem statement.

The system of integral-differential equations for this approach was derived variationally and can be solved analytically in the case of hinged support. The distribution of the elastic modulus over the thickness is modeled by an exponential law. Since this approach does not have any approximation error, the calculated results can be considered as a benchmark for testing various approximate methods for analyzing the free vibrations of functionally graded plates.

In the second approach, the unknown functions are approximated by polynomials across the thickness of the structure. The change in the elastic modulus is modeled using a fourth-degree polynomial. The displacement vector is represented as follows:

$$U_1^{(k)}(x,y,z,t)=U_{1l}^{(k)}(x,y,t)f_{1l}^{(k)}(z),$$

$$U_2^{(k)}(x,y,z,t)=U_{2l}^{(k)}(x,y,t)f_{2l}^{(k)}(z),$$

$$U_3^{(k)}(x,y,z,t)=W_p^{(k)}(x,y,t)\beta_p^{(k)}(z). \tag{1.3}$$

Here, $U_{11}^{(k)}(x,y,t)$ are the tangential displacements on the upper surface of the *k*th layer along the axis X; $U_{12}^{(k)}(x,y,t)$ are the tangential displacements on the lower surface of the *k*th layer along the axis X, $U_{13}^{(k)}(x,y,t)$ are the shear functions along the axis X; $U_{21}^{(k)}(x,y,t)$, $U_{22}^{(k)}(x,y,t)$, $U_{23}^{(k)}(x,y,t)$ are the tangential displacements on the face surfaces of the *k*th layer and shear functions along the axis *Y*; $W_1^{(k)}(x,y,t)$, $W_2^{(k)}(x,y,t)$ are the normal displacements on the face surfaces of the *k*th layer; $W_3^{(k)}(x,y,t)$ is the compression function; $f_{11}^{(k)}(z)$, $f_{12}^{(k)}(z)$, $f_{21}^{(k)}(z)$, $f_{22}^{(k)}(z)$, $\beta_1^{(k)}(z)$, $\beta_2^{(k)}(z)$ are given polynomials of the first degree; $\beta_3^{(k)}(z)$ are polynomials of the second degree; $f_{13}^{(k)}(z)$, $f_{23}^{(k)}(z)$ are polynomials of the third degree [16] ($l = 1, 2, 3$, $p = 1, 2, 3$).

The second approach reduces the unknown functions to the face surfaces of the layer. This allows us to partition them into sublayers across the thickness, if necessary, to minimize the approximation error.

Since free vibrations are being studied, the unknown displacement functions in both approaches are assumed to change as $e^{i\omega t}$. We will apply the approaches to the case of hinged support.

2. Approach 1 (A1) to Studying Free Vibrations Involving Analytical Search for the Distribution of the Unknown Functions over the Thickness of the Structure. Using the kinematic equations and the expressions for displacements and transverse stresses (1.2), we can write the expressions of strains:

$$e_{ii}^{(k)} = V_{i,i}^{(k)} f_i^{(k)}, \quad e_{33}^{(k)} = V_3^{(k)} f_{3,3}^{(k)},$$

$$2e_{12}^{(k)} = V_{1,2}^{(k)} f_1^{(k)} + V_{2,1}^{(k)} f_2^{(k)},$$

$$2e_{i3}^{(k)} = V_i^{(k)} f_{i,3}^{(k)} + V_{3,i}^{(k)} f_3^{(k)} \quad (i = 1, 2). \tag{2.1}$$

Hooke's law [4] and relations (2.1) allow us to determine the longitudinal components of the stress tensor $\sigma_{11}^{(k)}, \sigma_{22}^{(k)}, \sigma_{12}^{(k)}$:

$$\sigma_{11}^{(k)} = B_{11}^{(k)} V_{1,1}^{(k)} f_1^{(k)} + B_{12}^{(k)} V_{2,2}^{(k)} f_2^{(k)} + B_{13}^{(k)} \tau_{33}^{(k)} f_6^{(k)},$$

$$\sigma_{22}^{(k)} = B_{21}^{(k)} V_{1,1}^{(k)} f_1^{(k)} + B_{22}^{(k)} V_{2,2}^{(k)} f_2^{(k)} + B_{23}^{(k)} \tau_{33}^{(k)} f_6^{(k)},$$

$$\sigma_{12}^{(k)} = G_{12}^{(k)} (V_{1,2}^{(k)} f_1^{(k)} + V_{2,1}^{(k)} f_2^{(k)}). \tag{2.2}$$

We obtain the equations of vibrations using the variational approach. In the layer plane, they take the form

$$-V_{1,11}^{(k)} \int_{a_{k-1}}^{a_k} B_{11}^{(k)} f_1^{(k)} f_1^{(k)} dz - V_{1,22}^{(k)} \int_{a_{k-1}}^{a_k} G_{12}^{(k)} f_1^{(k)} f_1^{(k)} dz$$

$$-V_{2,12}^{(k)} \left(\int_{a_{k-1}}^{a_k} B_{12}^{(k)} f_1^{(k)} f_2^{(k)} dz + \int_{a_{k-1}}^{a_k} G_{12}^{(k)} f_1^{(k)} f_2^{(k)} dz \right)$$

$$+\tau_{13}^{(k)} \int_{a_{k-1}}^{a_k} f_{1,3}^{(k)} f_4^{(k)} dz - \tau_{33,1}^{(k)} \int_{a_{k-1}}^{a_k} B_{13}^{(k)} f_1^{(k)} f_6^{(k)} dz - \ddot{V}_1^{(k)} \int_{a_{k-1}}^{a_k} \rho^{(k)} f_1^{(k)} f_1^{(k)} dz = 0,$$

$$-V_{2,22}^{(k)} \int_{a_{k-1}}^{a_k} B_{22}^{(k)} f_2^{(k)} f_2^{(k)} dz - V_{2,11}^{(k)} \int_{a_{k-1}}^{a_k} G_{12}^{(k)} f_2^{(k)} f_2^{(k)} dz$$

$$-V_{1,12}^{(k)} \left(\int_{a_{k-1}}^{a_k} B_{12}^{(k)} f_1^{(k)} f_2^{(k)} dz + \int_{a_{k-1}}^{a_k} G_{12}^{(k)} f_1^{(k)} f_2^{(k)} dz \right)$$

$$+\tau_{23}^{(k)} \int_{a_{k-1}}^{a_k} f_{2,3}^{(k)} f_5^{(k)} dz - \tau_{33,2}^{(k)} \int_{a_{k-1}}^{a_k} B_{23}^{(k)} f_1^{(k)} f_6^{(k)} dz - \ddot{V}_2^{(k)} \int_{a_{k-1}}^{a_k} \rho^{(k)} f_2^{(k)} f_2^{(k)} dz = 0,$$

$$-\tau_{13,1}^{(k)}\int_{a_{k-1}}^{a_k} f_3^{(k)} f_4^{(k)} dz-\tau_{23,2}^{(k)}\int_{a_{k-1}}^{a_k} f_3^{(k)} f_5^{(k)} dz+\tau_{33}^{(k)}\int_{a_{k-1}}^{a_k} f_{3,3}^{(k)} f_6^{(k)} dz-\ddot{V}_3^{(k)}\int_{a_{k-1}}^{a_k}\rho^{(k)} f_3^{(k)} f_3^{(k)} dz=0,$$

$$V_1^{(k)}\int_{a_{k-1}}^{a_k} f_4^{(k)} f_{1,3}^{(k)} dz+V_{3,1}^{(k)}\int_{a_{k-1}}^{a_k} f_4^{(k)} f_3^{(k)} dz-\tau_{13}^{(k)}\int_{a_{k-1}}^{a_k}\frac{1}{G_{13}^{(k)}} f_4^{(k)} f_4^{(k)} dz=0,$$

$$V_2^{(k)}\int_{a_{k-1}}^{a_k} f_5^{(k)} f_{2,3}^{(k)} dz+V_{3,2}^{(k)}\int_{a_{k-1}}^{a_k} f_5^{(k)} f_3^{(k)} dz-\tau_{23}^{(k)}\int_{a_{k-1}}^{a_k}\frac{1}{G_{23}^{(k)}} f_5^{(k)} f_5^{(k)} dz=0,$$

$$V_{1,1}^{(k)}\int_{a_{k-1}}^{a_k} B_{13}^{(k)} f_6^{(k)} f_1^{(k)} dz+V_{2,2}^{(k)}\int_{a_{k-1}}^{a_k} B_{23}^{(k)} f_6^{(k)} f_2^{(k)} dz$$

$$+V_3^{(k)}\left(\int_{a_{k-1}}^{a_k} f_6^{(k)} f_{3,3}^{(k)}\ dz\right)-\tau_{33}^{(k)}\int_{a_{k-1}}^{a_k} B_{33}^{(k)} f_6^{(k)} f_6^{(k)} dz=0, \tag{2.3}$$

with respect to the layer thickness

$$-f_{1,3}^{(k)}\iint_S\int_{t_1}^{t_2} V_1^{(k)}\tau_{13}^{(k)} dtdS-f_3^{(k)}\iint_S\int_{t_1}^{t_2} V_{3,1}^{(k)}\tau_{13}^{(k)} dtdS+f_4^{(k)}\iint_S\int_{t_1}^{t_2}\frac{1}{G_{13}^{(k)}}\tau_{13}^{(k)}\tau_{13}^{(k)} dtdS=0,$$

$$-f_{2,3}^{(k)}\iint_S\int_{t_1}^{t_2} V_2^{(k)}\tau_{23}^{(k)} dtdS-f_3^{(k)}\iint_S\int_{t_1}^{t_2} V_{3,2}^{(k)}\tau_{23}^{(k)} dtdS+f_5^{(k)}\iint_S\int_{t_1}^{t_2}\frac{1}{G_{23}^{(k)}}\tau_{23}^{(k)}\tau_{23}^{(k)} dtdS=0,$$

$$-f_{3,3}^{(k)}\iint_S\int_{t_1}^{t_2} V_3^{(k)}\tau_{33}^{(k)} dtdS-f_1^{(k)}\iint_S\int_{t_1}^{t_2} B_{13}^{(k)} V_{1,1}^{(k)}\tau_{33}^{(k)} dtdS-f_2^{(k)}\iint_S\int_{t_1}^{t_2} B_{23}^{(k)} V_{2,2}^{(k)}\tau_{33}^{(k)} dtdS$$

$$-f_6^{(k)}\iint_S\int_{t_1}^{t_2} B_{33}^{(k)}\tau_{33}^{(k)}\tau_{33}^{(k)} dtdS=0,$$

$$-f_{4,3}^{(k)}\iint_S\int_{t_1}^{t_2} V_1^{(k)}\tau_{13}^{(k)} dtdS+f_1^{(k)}\left(\iint_S\int_{t_1}^{t_2} B_{11}^{(k)} V_{1,1}^{(k)} V_{1,1}^{(k)} dtdS\right.$$

$$\left.+\iint_S\int_{t_1}^{t_2} G_{12}^{(k)} V_{1,2}^{(k)} V_{1,2}^{(k)} dtdS+\iint_S\int_{t_1}^{t_2}\rho^{(k)}(z)\ddot{V}_1^{(k)} V_1^{(k)} dtdS\right)$$

$$+f_2^{(k)}\left(\iint_S\int_{t_1}^{t_2} B_{12}^{(k)} V_{1,1}^{(k)} V_{2,2}^{(k)}\ dtdS+\iint_S\int_{t_1}^{t_2} G_{12}^{(k)} V_{1,2}^{(k)} V_{2,1}^{(k)} dtdS\right)+f_6^{(k)}\iint_S\int_{t_1}^{t_2} B_{13}^{(k)} V_{1,1}^{(k)}\tau_{33}^{(k)} dtdS=0,$$

$$-f_{5,3}^{(k)}\iint_S\int_{t_1}^{t_2} V_2^{(k)}\tau_{23}^{(k)} dtdS+f_1^{(k)}\left(\iint_S\int_{t_1}^{t_2} B_{21}^{(k)} V_{2,2}^{(k)}\, V_{1,1}^{(k)} dtdS+\iint_S\int_{t_1}^{t_2} G_{12}^{(k)} V_{1,2}^{(k)} V_{2,1}^{(k)} dtdS\right)$$

$$+f_2^{(k)}\left(\iint_S \int_{t_1}^{t_2} B_{22}^{(k)} V_{2,2}^{(k)} V_{2,2}^{(k)} dtdS + \iint_S \int_{t_1}^{t_2} G_{12}^{(k)} V_{2,1}^{(k)} V_{2,1}^{(k)} dtdS + \iint_S \int_{t_1}^{t_2} \rho^{(k)}(z)\ddot{V}_2^{(k)} V_2^{(k)} dtdS\right)$$

$$+f_6^{(k)} \iint_S \int_{t_1}^{t_2} B_{23}^{(k)} V_{2,2}^{(k)} \tau_{33}^{(k)} dtdS = 0,$$

$$-f_{6,3}^{(k)} \iint_S \int_{t_1}^{t_2} V_3^{(k)} \tau_{33}^{(k)} dtdS + f_3^{(k)} \iint_S \int_{t_1}^{t_2} \rho^{(k)}(z)\ddot{V}_3^{(k)} V_3^{(k)} dtdS$$

$$+f_4^{(k)} \iint_S \int_{t_1}^{t_2} V_{3,1}^{(k)} \tau_{13}^{(k)} dtdS + f_5^{(k)} \iint_S \int_{t_1}^{t_2} V_{3,2}^{(k)} \tau_{23}^{(k)} dtdS = 0, \tag{2.4}$$

and the boundary conditions:

on the layer edge:

$$(\sigma_{11}^{(k)} - q_{11}^{(k)})\delta V_1^{(k)}\Big|_0^a = 0, \quad (\sigma_{12}^{(k)} - q_{12}^{(k)})\delta V_2^{(k)}\Big|_0^a = 0, \quad (\sigma_{13}^{(k)} - q_{13}^{(k)})\delta V_3^{(k)}\Big|_0^a = 0,$$

$$(\sigma_{21}^{(k)} - q_{21}^{(k)})\delta V_1^{(k)}\Big|_0^b = 0, \quad (\sigma_{22}^{(k)} - q_{22}^{(k)})\delta V_2^{(k)}\Big|_0^b = 0, \quad (\sigma_{23}^{(k)} - q_{23}^{(k)})\delta V_3^{(k)}\Big|_0^b = 0,$$

on the layer surfaces:

$$(\sigma_{13}^{(k)} - q_{13}^{(k)})\delta f_1^{(k)}\Big|_{a_{k-1}}^{a_k} = 0, \quad (\sigma_{23}^{(k)} - q_{23}^{(k)})\delta f_2^{(k)}\Big|_{a_{k-1}}^{a_k} = 0, \quad (\sigma_{33}^{(k)} - q_{33}^{(k)})\delta f_3^{(k)}\Big|_{a_{k-1}}^{a_k} = 0,$$

at the ends of the time interval:

$$\dot{V}_1^{(k)}\delta(V_1^{(k)})\Big|_{t_1}^{t_2} = 0, \quad \dot{V}_2^{(k)}\delta(V_2^{(k)})\Big|_{t_1}^{t_2} = 0, \quad \dot{V}_3^{(k)}\delta(V_3^{(k)})\Big|_{t_1}^{t_2} = 0.$$

In the cases of hinged edge and free vibrations, we can write

$$V_1^{(k)} = \tau_{13}^{(k)} = \cos\frac{\pi m x}{a}\sin\frac{\pi n y}{b}e^{i\omega t},$$

$$V_2^{(k)} = \tau_{23}^{(k)} = q_{2l}\sin\frac{\pi m x}{a}\cos\frac{\pi n y}{b}e^{i\omega t},$$

$$V_3^{(k)} = \tau_{33}^{(k)} = q_{3l}\sin\frac{\pi m x}{a}\sin\frac{\pi n y}{b}e^{i\omega t}. \tag{2.5}$$

Substituting expressions (2.5) into Eqs. (2.4), we obtain

$$f_{1,3}^{(k)} = -f_3^{(k)}\left(\frac{\pi m}{a}\right) + f_4^{(k)}\left(\frac{1}{\overline{G}_{13}^{(k)} e^{z\gamma^{(k)}}}\right),$$

$$f_{2,3}^{(k)} = -f_3^{(k)}\left(\frac{\pi n}{b}\right) + f_5^{(k)}\left(\frac{1}{\overline{G}_{23}^{(k)} e^{z\gamma^{(k)}}}\right),$$

$$f_{3,3}^{(k)} = f_1^{(k)} B_{13}^{(k)} \left(\frac{\pi m}{a}\right) + f_2^{(k)} B_{23}^{(k)} \left(\frac{\pi n}{b}\right) + f_6^{(k)} \left(\frac{\overline{B}_{33}^{(k)}}{e^{z\gamma^{(k)}}}\right),$$

$$f_{4,3}^{(k)} = f_1^{(k)} \left[\overline{B}_{11}^{(k)} \left(\frac{\pi m}{a}\right)^2 + \overline{G}_{12}^{(k)} \left(\frac{\pi n}{b}\right)^2 - \overline{\rho}^{(k)} \omega^2\right] e^{z\gamma^{(k)}}$$

$$+ f_2^{(k)} (\overline{B}_{12}^{(k)} + \overline{G}_{12}^{(k)}) e^{z\gamma^{(k)}} \left(\frac{\pi m}{a}\right)\left(\frac{\pi n}{b}\right) - f_6^{(k)} B_{13}^{(k)} \left(\frac{\pi m}{a}\right),$$

$$f_{5,3}^{(k)} = f_1^{(k)} (\overline{B}_{12}^{(k)} + \overline{G}_{12}^{(k)}) e^{z\gamma^{(k)}} \left(\frac{\pi m}{a}\right)\left(\frac{\pi n}{b}\right)$$

$$+ f_2^{(k)} \left[\overline{B}_{22}^{(k)} \left(\frac{\pi n}{b}\right)^2 + \overline{G}_{12}^{(k)} \left(\frac{\pi m}{a}\right)^2 - \overline{\rho}^{(k)} \omega^2\right] e^{z\gamma^{(k)}} - f_6^{(k)} B_{23}^{(k)} \left(\frac{\pi n}{b}\right),$$

$$f_{6,3}^{(k)} = -f_3^{(k)} (\overline{\rho}^{(k)} \omega^2) e^{z\gamma^{(k)}} + f_4^{(k)} \left(\frac{\pi m}{a}\right) + f_5^{(k)} \left(\frac{\pi n}{b}\right). \tag{2.6}$$

The solution of system (2.6) is represented in the following form [14]:

$$f_1^{(k)} = \mu_1^{(k)} e^{(\beta^{(k)} - \gamma^{(k)})z}, \quad f_2^{(k)} = \mu_2^{(k)} e^{(\beta^{(k)} - \gamma^{(k)})z}, \quad f_3^{(k)} = \mu_3^{(k)} e^{(\beta^{(k)} - \gamma^{(k)})z},$$

$$f_4^{(k)} = \mu_4^{(k)} e^{\beta^{(k)} z}, \quad f_5^{(k)} = \mu_5^{(k)} e^{\beta^{(k)} z}, \quad f_6^{(k)} = \mu_6^{(k)} e^{\beta^{(k)} z}.$$

We obtain the system of homogeneous algebraic equations

$$-\mu_1^{(k)} (\beta^{(k)} - \gamma^{(k)}) - \mu_3^{(k)} \left(\frac{\pi m}{a}\right) + \mu_4^{(k)} \left(\frac{1}{\overline{G}_{13}^{(k)}}\right) = 0,$$

$$-\mu_2^{(k)} (\beta^{(k)} - \gamma^{(k)}) - \mu_3^{(k)} \left(\frac{\pi n}{b}\right) + \mu_5^{(k)} \left(\frac{1}{\overline{G}_{23}^{(k)}}\right) = 0,$$

$$\mu_1^{(k)} B_{13}^{(k)} \left(\frac{\pi m}{a}\right) + \mu_2^{(k)} B_{23}^{(k)} \left(\frac{\pi n}{b}\right) - \mu_3^{(k)} (\beta^{(k)} - \gamma^{(k)}) + \mu_6^{(k)} \overline{B}_{33}^{(k)} = 0,$$

$$\mu_1^{(k)} \left(\overline{B}_{11}^{(k)} \left(\frac{\pi m}{a}\right)^2 + \overline{G}_{12}^{(k)} \left(\frac{\pi n}{b}\right)^2 - \overline{\rho}^{(k)} \omega^2\right) + \mu_2^{(k)} (\overline{B}_{12}^{(k)} + \overline{G}_{12}^{(k)}) \left(\frac{\pi m}{a}\right)\left(\frac{\pi n}{b}\right) - \mu_4^{(k)} \beta^{(k)} - \mu_6^{(k)} B_{13}^{(k)} \left(\frac{\pi m}{a}\right) = 0,$$

$$\mu_1^{(k)} (\overline{B}_{12}^{(k)} + \overline{G}_{12}^{(k)}) \left(\frac{\pi m}{a}\right)\left(\frac{\pi n}{b}\right) + \mu_2^{(k)} \left(\overline{B}_{22}^{(k)} \left(\frac{\pi n}{b}\right)^2 + \overline{G}_{12}^{(k)} \left(\frac{\pi m}{a}\right)^2 - \overline{\rho}^{(k)} \omega^2\right) - \mu_5^{(k)} \beta^{(k)} - \mu_6^{(k)} B_{23}^{(k)} \frac{\pi n}{b} = 0,$$

$$-\mu_3^{(k)} (\overline{\rho}^{(k)} \omega^2) + \mu_4^{(k)} \left(\frac{\pi m}{a}\right) + \mu_5^{(k)} \left(\frac{\pi m}{b}\right) - \mu_6^{(k)} \beta^{(k)} = 0. \tag{2.7}$$

Expanding the determinant of system (2.7), we obtain the relations between the parameters $\beta^{(k)}$ and ω^2. Then from system (2.7) we obtain the coefficients $\mu^{(k)}$. The unknown functions $f_i^{(k)}$ are determined as follows [14]:

$$f_1^{(k)} = \mu_{11}^{(k)} C_1^{(k)} e^{(\beta_1^{(k)} - \gamma)z} + \mu_{12}^{(k)} C_2^{(k)} e^{(\beta_2^{(k)} - \gamma)z} + \mu_{13}^{(k)} C_3^{(k)} e^{(\beta_3^{(k)} - \gamma)z}$$

$$+ \mu_{14}^{(k)} C_4^{(k)} e^{(\beta_4^{(k)} - \gamma)z} + \mu_{15}^{(k)} C_5^{(k)} e^{(\beta_5^{(k)} - \gamma)z} + \mu_{16}^{(k)} C_6^{(k)} e^{(\beta_6^{(k)} - \gamma)z},$$

$$f_2^{(k)} = \mu_{21}^{(k)} C_1^{(k)} e^{(\beta_1^{(k)} - \gamma)z} + \mu_{22}^{(k)} C_2^{(k)} e^{(\beta_2^{(k)} - \gamma)z} + \mu_{23}^{(k)} C_3^{(k)} e^{(\beta_3^{(k)} - \gamma)z}$$

$$+ \mu_{24}^{(k)} C_4^{(k)} e^{(\beta_4^{(k)} - \gamma)z} + \mu_{25}^{(k)} C_5^{(k)} e^{(\beta_5^{(k)} - \gamma)z} + \mu_{26}^{(k)} C_6^{(k)} e^{(\beta_6^{(k)} - \gamma)z},$$

$$f_3^{(k)} = \mu_{31}^{(k)} C_1^{(k)} e^{(\beta_1^{(k)} - \gamma)z} + \mu_{32}^{(k)} C_2^{(k)} e^{(\beta_2^{(k)} - \gamma)z} + \mu_{33}^{(k)} C_3^{(k)} e^{(\beta_3^{(k)} - \gamma)z}$$

$$+ \mu_{34}^{(k)} C_4^{(k)} e^{(\beta_4^{(k)} - \gamma)z} + \mu_{35}^{(k)} C_5^{(k)} e^{(\beta_5^{(k)} - \gamma)z} + \mu_{36}^{(k)} C_6^{(k)} e^{(\beta_6^{(k)} - \gamma)z},$$

$$f_4^{(k)} = \mu_{41}^{(k)} C_1^{(k)} e^{\beta_1^{(k)} z} + \mu_{42}^{(k)} C_2^{(k)} e^{\beta_2^{(k)} z} + \mu_{43}^{(k)} C_3^{(k)} e^{\beta_3^{(k)} z}$$

$$+ \mu_{44}^{(k)} C_4^{(k)} e^{\beta_4^{(k)} z} + \mu_{45}^{(k)} C_5^{(k)} e^{\beta_5^{(k)} z} + \mu_{46}^{(k)} C_6^{(k)} e^{\beta_6^{(k)} z},$$

$$f_5^{(k)} = \mu_{51}^{(k)} C_1^{(k)} e^{\beta_1^{(k)} z} + \mu_{52}^{(k)} C_2^{(k)} e^{\beta_2^{(k)} z} + \mu_{53}^{(k)} C_3^{(k)} e^{\beta_3^{(k)} z}$$

$$+ \mu_{54}^{(k)} C_4^{(k)} e^{\beta_4^{(k)} z} + \mu_{55}^{(k)} C_5^{(k)} e^{\beta_5^{(k)} z} + \mu_{56}^{(k)} C_6^{(k)} e^{\beta_6^{(k)} z},$$

$$f_6^{(k)} = \mu_{61}^{(k)} C_1^{(k)} e^{\beta_1^{(k)} z} + \mu_{62}^{(k)} C_2^{(k)} e^{\beta_2^{(k)} z} + \mu_{63}^{(k)} C_3^{(k)} e^{\beta_3^{(k)} z}$$

$$+ \mu_{64}^{(k)} C_4^{(k)} e^{\beta_4^{(k)} z} + \mu_{65}^{(k)} C_5^{(k)} e^{\beta_5^{(k)} z} + \mu_{66}^{(k)} C_6^{(k)} e^{\beta_6^{(k)} z}.$$

We derive the governing system of equations for integration constants $C_i^{(k)}$ by satisfying the interface conditions between layers and the conditions on the surfaces of the sandwich (when determining the frequencies of free vibrations, the load on the surface is absent). The parameter ω^2 is determined by equating the determinant of this system to zero.

3. Approach 2 (A2) to Studying Free Vibrations with Polynomial Approximation of the Unknown Functions across the Layer Thickness. The components of the strain tensor of the layer, using the introduced approximation (1.3), are determined from the following relations:

$$e_{11}^{(k)} = U_{1l,1}^{(k)} f_{1l}^{(k)}, \quad e_{22}^{(k)} = U_{2l,2}^{(k)} f_{2l}^{(k)}, \quad e_{33}^{(k)} = W_p^{(k)} \beta_{p,3}^{(k)},$$

$$2e_{12}^{(k)} = U_{1l,2}^{(k)} f_{1l}^{(k)} + U_{2l,1}^{(k)} f_{2l}^{(k)}, \quad 2e_{13}^{(k)} = U_{1l}^{(k)} f_{1l,3}^{(k)} + W_{p,1}^{(k)} \beta_p^{(k)},$$

$$2e_{23}^{(k)} = U_{2l}^{(k)} f_{2l,3}^{(k)} + W_{p,2}^{(k)} \beta_p^{(k)}. \tag{3.1}$$

With (3.1), the stresses can be expressed using Hooke's law:

$$\sigma_{11}^{(k)} = C_{11}^{(k)}(z) U_{1l,1}^{(k)} f_{1l}^{(k)} + C_{12}^{(k)}(z) U_{2l,2}^{(k)} f_{2l}^{(k)} + C_{13}^{(k)}(z) W_p^{(k)} \beta_{p,3}^{(k)},$$

$$\sigma_{22}^{(k)} = C_{21}^{(k)}(z) U_{1l,1}^{(k)} f_{1l}^{(k)} + C_{22}^{(k)}(z) U_{2l,2}^{(k)} f_{2l}^{(k)} + C_{23}^{(k)}(z) W_p^{(k)} \beta_{p,3}^{(k)},$$

$$\sigma_{33}^{(k)} = C_{31}^{(k)}(z) U_{1l,1}^{(k)} f_{1l}^{(k)} + C_{32}^{(k)}(z) U_{2l,2}^{(k)} f_{2l}^{(k)} + C_{33}^{(k)}(z) W_p^{(k)} \beta_{p,3}^{(k)},$$

$$\sigma_{12}^{(k)}=G^{(k)}(z)(U_{1l,2}^{(k)}f_{1l}^{(k)}+U_{2l,1}^{(k)}f_{2l}^{(k)}),\qquad \sigma_{13}^{(k)}=G^{(k)}(z)(U_{1l}^{(k)}f_{1l,3}^{(k)}+W_{p,1}^{(k)}\beta_{p}^{(k)}),$$

$$\sigma_{23}^{(k)}=G^{(k)}(z)(U_{2l}^{(k)}f_{2l,3}^{(k)}+W_{p,2}^{(k)}\beta_{p}^{(k)}). \tag{3.2}$$

Here $C_{11}^{(k)}(z)=C_{22}^{(k)}(z)=C_{33}^{(k)}(z), C_{12}^{(k)}(z)=C_{13}^{(k)}(z)=C_{23}^{(k)}(z).$

In view of (3.1) and (3.2), the variation of the potential strain energy takes the following form:

$$\delta\Pi=\iint_{s}\int_{a_{k-1}}^{a_k}\int_{t_1}^{t_2}\Big\{\Big[C_{11}^{(k)}(z)U_{1l,1}^{(k)}f_{1l}^{(k)}+C_{12}^{(k)}(z)U_{2l,2}^{(k)}f_{2l}^{(k)}+C_{13}^{(k)}(z)W_{p}^{(k)}\beta_{p,3}^{(k)}\Big]\delta U_{1\bar{l},1}^{(k)}f_{1\bar{l}}^{(k)}$$

$$+[C_{21}^{(k)}(z)U_{1l,1}^{(k)}f_{1l}^{(k)}+C_{22}^{(k)}(z)U_{2l,2}^{(k)}f_{2l}^{(k)}+C_{23}^{(k)}(z)W_{p}^{(k)}\beta_{p,3}^{(k)}]\delta U_{2\bar{l},2}^{(k)}f_{2\bar{l}}^{(k)}$$

$$+[C_{31}^{(k)}(z)U_{1l,1}^{(k)}f_{1l}^{(k)}+C_{32}^{(k)}(z)U_{2l,2}^{(k)}f_{2l}^{(k)}+C_{33}^{(k)}(z)W_{p}^{(k)}\beta_{p,3}^{(k)}]\delta W_{\bar{p}}^{(k)}\beta_{p,3}^{(k)}$$

$$+\Big[G_{12}^{(k)}(z)(U_{1l,2}^{(k)}f_{1l}^{(k)}+U_{2l,1}^{(k)}f_{2l}^{(k)})\Big]\delta(U_{1l,2}^{(k)}f_{1l}^{(k)}+U_{2l,1}^{(k)}f_{2l}^{(k)})$$

$$+\Big[G_{13}^{(k)}(z)(U_{1l}^{(k)}f_{1l,3}^{(k)}+W_{p,1}^{(k)}\beta_{p}^{(k)})\Big]\delta(U_{1l}^{(k)}f_{1l,3}^{(k)}+W_{p,1}^{(k)}\beta_{p}^{(k)})$$

$$+\Big[G_{23}^{(k)}(z)(U_{2l}^{(k)}f_{2l,3}^{(k)}+W_{p,2}^{(k)}\beta_{p}^{(k)})\Big]\delta(U_{2l}^{(k)}f_{2l,3}^{(k)}+W_{p,2}^{(k)}\beta_{p}^{(k)})\Big\}dtdzdS. \tag{3.3}$$

The variation of kinetic energy can be written as follows:

$$\delta T^{(k)}=-\iint_{S}\int_{a_{k-1}}^{a_k}\rho^{(k)}(z)\Bigg\{\int_{t_1}^{t_2}\Big[(\ddot{U}_{1l}^{(k)}f_{1l}^{(k)})\delta\Big(U_{1\bar{l}}^{(k)}f_{1\bar{l}}^{(k)}\Big)+(\ddot{U}_{2l}^{(k)}f_{2l}^{(k)})\delta(U_{2\bar{l}}^{(k)}f_{2\bar{l}}^{(k)})$$

$$+(\ddot{W}_{p}^{(k)}\beta_{p}^{(k)})\delta(W_{\bar{p}}^{(k)}\beta_{\bar{p}}^{(k)})\Big]dt-\Big[(\dot{U}_{1l}^{(k)}f_{l}^{(k)})\delta\Big(U_{1\bar{l}}^{(k)}f_{\bar{l}}^{(k)}\Big)$$

$$+(\dot{U}_{2l}^{(k)}f_{l}^{(k)})\delta(U_{2\bar{l}}^{(k)}f_{\bar{l}}^{(k)})+(\dot{W}_{p}^{(k)}\beta_{p}^{(k)})\delta(W_{\bar{p}}^{(k)}\beta_{\bar{p}}^{(k)})\Big]_{t_1}^{t_2}\Bigg\}dzdS. \tag{3.4}$$

The equation of free vibrations can be derived from the following variational relation:

$$\delta\Pi-\delta T=0. \tag{3.5}$$

Taking into account

$$U_{1l}^{(k)}(x,y,t)=U_{1l}^{(k)}(x,y)e^{-i\omega t},$$

$$U_{2l}^{(k)}(x,y,t)=U_{2l}^{(k)}(x,y)e^{-i\omega t},$$

$$W_{p}^{(k)}(x,y,t)=W_{p}^{(k)}(x,y)e^{-i\omega t},$$

expressions (3.3) and (3.4) and performing transformations of Eq. (3.5), we obtain the differential equations of free vibrations

$$-(B11_{\bar{l}l}^{(k)}U_{1l,11}^{(k)}+B611_{\bar{l}l}^{(k)}U_{1l,22}^{(k)}-TU1_{\bar{l}l}^{(k)}U_{1l}^{(k)})-(B12_{\bar{l}l}^{(k)}+B612_{\bar{l}l}^{(k)})U_{2l,12}^{(k)}$$

$$-(SD1_{\bar{l}p}^{(k)}-CUW1_{\bar{l}p}^{(k)})W_{p,1}^{(k)}-\omega^2 BT1_{\bar{l}l}^{(k)}U_{1l}^{(k)}=0,$$

TABLE 1

$\overline{\omega}^2 = \omega^2(\overline{\rho}h^2 / \overline{E})$					
Frequency number	Approach (A2). Each layer has one sublayer; no compression	Approach (A2). Each layer has one sublayer	Approach (A2). Each layer has 10 sublayers	Approach (A1)	Approach (A1), no inertia of the foundation $\rho^{(3)} = 0.00001\overline{\rho}$
1	1.8819e–001	1.6393e–001	1.5198e–001	1.5167e–001	1.8547e–001
2	3.0878e–001	3.0878e–001	3.0877e–001	3.0877e–001	3.7367e–001
3	3.6094e–001	3.6094e–001	3.6064e–001	3.6015e–001	9.3689e–001

$$-(B12_{\bar{l}l}^{(k)} + B612_{\bar{l}l}^{(k)})U_{1l,12}^{(k)} - (B22_{\bar{l}l}^{(k)} U_{2l,22}^{(k)} + B622_{\bar{l}l}^{(k)} U_{2l,11}^{(k)} - TU2_{\bar{l}l}^{(k)} U_{2l}^{(k)})$$

$$-(SD2_{\bar{l}p}^{(k)} - CUW2_{\bar{l}p}^{(k)})W_{p,2}^{(k)} - \omega^2 BT2_{\bar{l}l}^{(k)} U_{2l}^{(k)} = 0,$$

$$(SD1_{\bar{p}l}^{(k)} - CUW1_{\bar{p}l}^{(k)})U_{1l,1}^{(k)} + (SD2_{\bar{p}l}^{(k)} - CUW2_{\bar{p}l}^{(k)})U_{2l,2}^{(k)}$$

$$-(CC1_{\bar{p}p}^{(k)} W_{p,11}^{(k)} + CC2_{\bar{p}p}^{(k)} W_{p,22}^{(k)} - ZZ_{\bar{p}p}^{(k)} W_{p}^{(k)}) - \omega^2 ZT_{\bar{p}p}^{(k)} W_{p}^{(k)} = 0. \quad (3.6)$$

Numerical analysis based on this system will be conducted approximating the elastic modulus by a fourth-degree polynomial. If necessary, the layers of the structure can be partitioned into sublayers.

4. Numerical Results. Let us test the applicability of the developed approach with polynomial approximation of the unknown functions with respect to the thickness (A2) to studying the free vibrations of inhomogeneous plates made of functionally graded material on an elastic foundation in the form of a finite thickness layer. We will consider a two-layer plate as an example. The physical and mechanical characteristics of the material layers of the plate and the elastic foundation are as follows:

$$E^{(1)}(z) = \overline{E}e^{\gamma^{(1)}z}, \quad E^{(2)}(z) = \overline{E}e^{\gamma^{(2)}z}, \quad E^{(3)} = \overline{E} \text{ (third layer is foundation);}$$

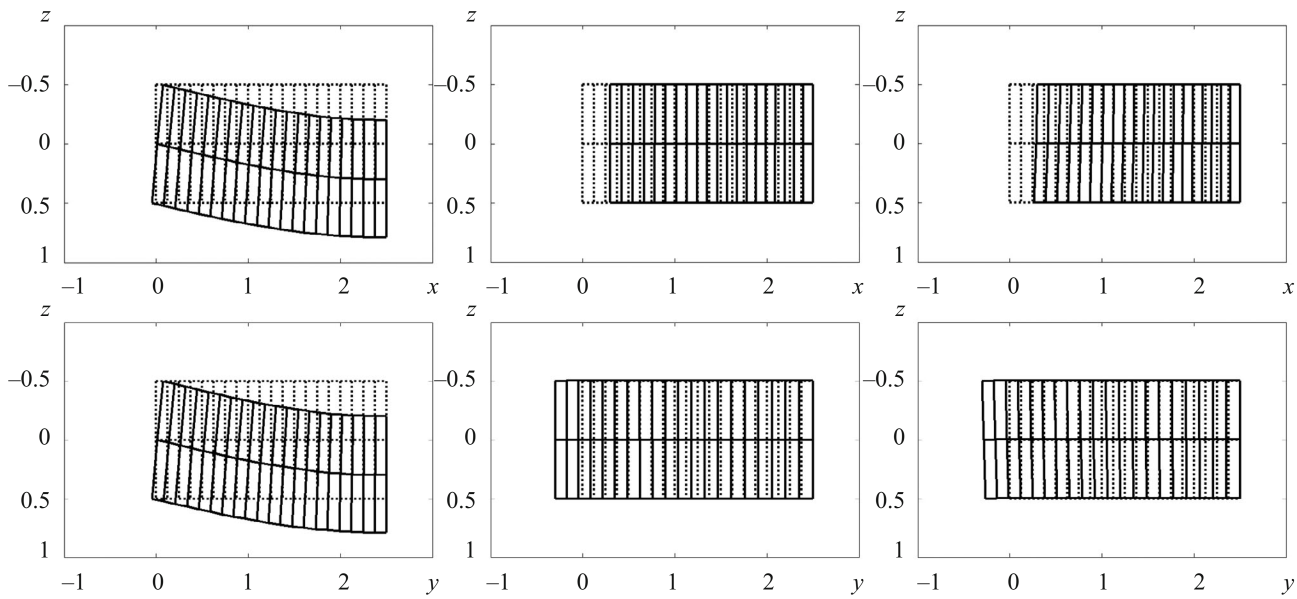
$$\rho^{(1)}(z) = \overline{\rho}e^{\gamma^{(1)}z}, \quad \rho^{(2)}(z) = \overline{\rho}e^{\gamma^{(2)}z}, \quad \rho^{(3)} = \overline{\rho}, \quad \overline{E} = 1, \quad \overline{\rho} = 1,$$

$$\gamma^{(1)} = -5, \quad \gamma^{(2)} = 5, \quad h^{(1)} = h^{(2)} = h/2, \quad h^{(3)} = 19h/2, \quad L/h = 5, \quad L = a = b.$$

Let us consider hinged support. For this case, the solution obtained with the approach (A1) can be considered as a reference since it does not have any approximation error. In the (A2), the exponential law of change in the elastic modulus is modeled by a fourth-degree polynomial.

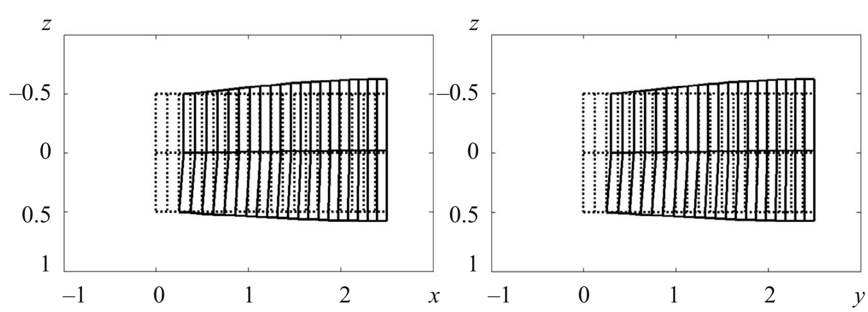
Table 1 compares the first natural frequencies of the structure, obtained using the two approaches. Using the approach (A2), we considered each layer to have one sublayer, either without or with compression, and each layer divided into 10 sublayers. For comparison, the table includes the natural frequencies determined with the approach (A1) with and without regard to the inertial properties of the foundation.

Figure 1 shows the distribution of displacements of the two-layer plate with the inertial characteristics of the elastic foundation taken into account. The foundation is not shown in the figure. The top part of the figure shows the displacements for $Y = b/2$, and the bottom part shows displacements for $X = a/2$. The figures have been plotted using the approach (A1). They do not differ from those plotted using the approach (A2) (omitted to save space).



a *b*

Fig. 1



a *b*

Fig. 2

Transverse bending vibrations occur at the first frequency (Fig. 1*a*). Planar vibrations occur at the second frequency. The plate is compressed along one coordinate axis and expands along the other axis, and vice versa, while the displacements of the upper and lower surfaces of the plate are equal (Fig. 1*b*). Planar vibrations occur at the third frequency, similarly to the vibrations at the second resonant frequency, but the outer surfaces vibrate in opposite directions simultaneously (Fig. 1*c*). At the second and third frequencies, there is no compression. The frequencies determined with the approaches (A1) and (A2) are in good agreement only in the fourth significant digit, even when compression is neglected.

The error of the square of the first frequency is the highest. In the case of one sublayer, the approach (A2) leads to an error of 8.1% of the square of the natural frequency. With compression is neglected, the error is 24.1%. An important feature of the approach (A2) is the possibility to consider a layer to have many sublayers. When the layer is divided into 10 sublayers, the results obtained with the approach (A2) are practically indistinguishable from those obtained with the approach (A1), which confirms their reliability.

If the inertial properties of the elastic foundation are neglected, transverse bending vibrations occur at the first resonant frequency (Fig. 1*a*), as in the case of taking the inertia of the elastic foundation into account. The error of the first bending frequency is 22.3% in this case. The vibrations occurring at the second resonant frequency when the inertial properties of the elastic foundation are neglected are similar to the vibrations at the third frequency when the inertial properties of the elastic foundation are taken into account (Fig. 1*c*). At the third resonant frequency (Fig. 2), planar vibrations occur with simultaneous

TABLE 2

$\overline{\omega}^2 = \omega^2(\overline{\rho}h^2 / \overline{E})$				
Frequency number	Approach (A2). Each layer has one sublayer; no compression	Approach (A2). Each layer has one sublayer	Approach (A2). Each layer has 10 sublayers	Approach (A1)
1	1.2110e+000	1.21098e+000	8.26195e–001	8.23023e–001
2	1.7745e+000	1.81464e+000	1.38638e+000	1.38283e+000
3	1.4358e+001	1.88245e+000	1.79349e+000	1.79319e+000

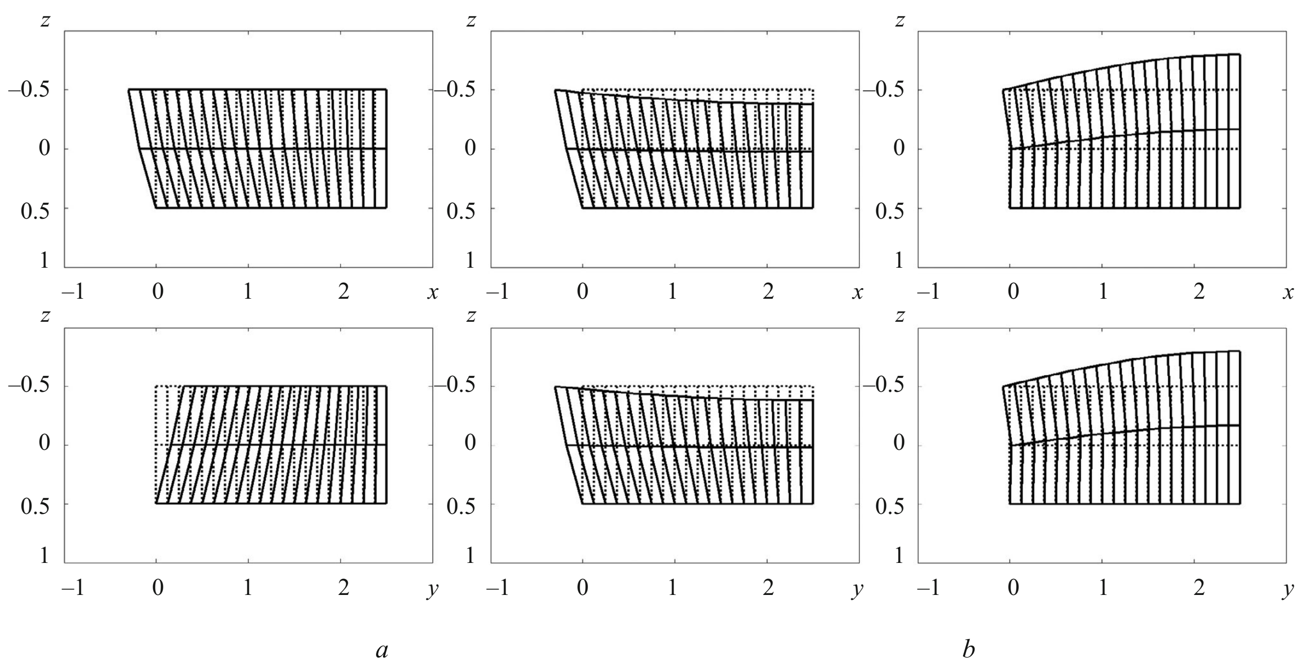


Fig. 3

compression (tension) along the axes X and Y, with noticeable tensile (compressive) displacements in the transverse direction. If the inertial characteristics of the foundation are taken into account, such vibrations occur at the ninth frequency.

Let us consider a two-layer plate with the same physical and mechanical properties as in the previous example on a perfectly rigid foundation. Table 2 summarizes the calculated results.

Figure 3 shows the displacement distribution.

At the first frequency, planar vibrations occur with compression along one coordinate axis and expansion along the other axis. The vibrations at the second frequency involve simultaneous compression (tension) along the two coordinate axes. At the third frequency, bending vibrations occur. In this case, the approach (A2) applied to the layer with one sublayer produces a sufficient accuracy only in determining the third frequency. The error of the squared frequency is 5%. Partitioning the layers into 10 sublayers allows the approach (A2) to accurately describe free vibrations. In this case, it is necessary to consider transverse compression.

The first three natural frequencies differ significantly from the real values. The displacement distribution at the third frequency obtained neglecting compression (Fig. 4) does not coincide with that obtained using the three-dimensional approach (Fig. 3*c*). With compression taken into account, such vibrations occur at the fourth frequency.

Conclusion. The developed approach with polynomial approximation of the unknown functions across the thickness of the structure ensures sufficient accuracy when considering each layer to have one sublayer in studying the free vibrations of a

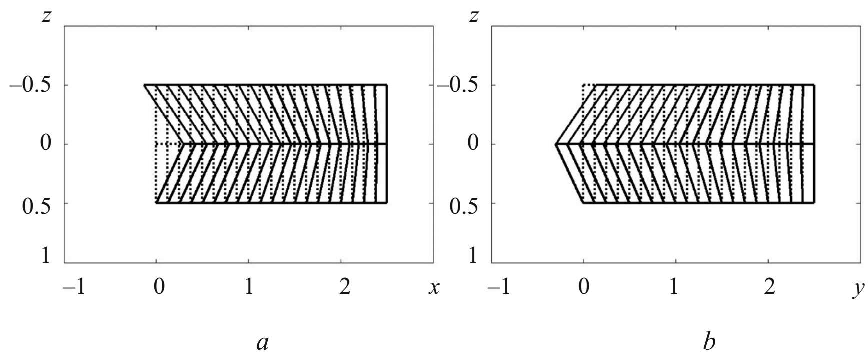


Fig. 4

plate made of functionally graded material on an elastic foundation in the form of a finite thickness layer, except for the squared frequency of bending vibrations. Applying the approach without regard to compression leads to a significant error. Neglecting the inertial properties of the foundation can result in both quantitative and qualitative errors of the parameters of free vibrations. Applying the approach with polynomial approximation and without partitioning the layers into sublayers to a plate on a perfectly rigid foundation may lead to significant errors. These approaches should be applied taking into account compression. Partitioning the layers into ten sublayers increases the accuracy to the level ensured by the analytical determination of the distribution of functions over the thickness of the plate.

REFERENCES

1. Ya. M. Grigorenko, E. I. Bespalova, A. B. Kitaigorodskii, and A. I. Shinkar’, *Free Vibrations of Elements of Shell Structures* [in Russian], Naukova Dumka, Kyiv (1986).
2. P. A. Zhilin and T. P. Il’icheva, “Spectrum and vibration modes of a rectangular parallelepiped obtained using three-dimensional elasticity theory and plate theory,” *Izv. AN SSSR, Mekh. Komp. Tel*, No. 2, 94–103 (1980).
3. V. N. Moskalenko, “Natural vibrations of sandwich plates,” *Izv. AN SSSR, Mekh. Mashinostr.*, No. 4, 125–130 (1962).
4. A. O. Rasskazov, I. I. Sokolovskaya, and N. A. Shul’ga, *Theory and Design of Layered Orthotropic Plates and Shells* [in Russian], Vyshcha Shkola, Kyiv (1986).
5. Z. Q. Cheng and R. C. Batra, “Exact correspondence between eigenvalues of membranes and functionally graded simply supported polygonal plates,” *J. Sound Vibr.*, **229**, No. 4, 879–895 (2000).
6. S. Dastjerdi and B. Akgöz, “New static and dynamic analyses of macro and nano FGM plates using exact three-dimensional elasticity in thermal environment,” *Comp. Struct.*, **192**, No. 5, pp. 626–641 (2018).
7. E. Efraim and M. Eisenberger, “Exact vibration analysis of variable thickness thick annular isotropic and FGM plates,” *J. Sound Vibr.*, **299**, No. 4–5, 720–738 (2007).
8. T. Fu, Z. Chen, H. Yu, Z. Wang, and X. Liu, “Mechanical behavior of laminated functionally graded carbon nanotube reinforced composite plates resting on elastic foundations in thermal environments,” *J. Compos. Mater.*, **53**, No. 9, 1159–1179 (2019).
9. A. Ya. Grigorenko, T. L. Efimova, and Yu. A. Korotkikh, “Free axisymmetric vibrations of cylindrical shells made of functionally graded materials,” *Int. Appl. Mech.*, **51**, No. 6, 654–663 (2015).
10. S. Kitipornchai, J. Yang, and K. M. Liew, “Semi-analytical solution for nonlinear vibration of laminated functionally graded material plates with geometric imperfections,” *Int. J. Solids Struct.*, **41**, No. 9–10, 2235–2257 (2004).
11. J. S. Kumar, B. S. Reddy, S. E. Reddy, and K. V. Kumar, “Higher order theory for free vibration analysis of functionally graded material plates,” *ARPN J. Eng. Appl. Sci.*, **6**, No. 10, 105–111 (2011).
12. Q. Li, V. P. Iu, and K. P. Kou, “Three-dimensional vibration analysis of functionally graded material sandwich plates,” *J. Sound Vibr.*, **311**, No. 1–2, 498–515 (2008).
13. X. Liang, Z. Wang, L. Wang, and G. Liu. “Semi-analytical solution for three-dimensional transient response of functionally graded annular plate on a two parameter viscoelastic foundation,” *J. Sound Vibr.*, **333**, No. 12, 2649–2663 (2014).

14. A. V. Marchuk, “Three-dimensional analytic solution for a hinged slab on an elastic half-space,” *Int. Appl. Mech.*, **33**, No. 10, 794–798 (1997).
15. A. V. Marchuk. “Determination of the natural frequencies of vibration of nonuniform slabs,” *Int. Appl. Mech.*, **35**, No. 2, 152–158 (1999).
16. A. V. Marchuk and V. G. Piskunov, “Statics, vibrations and stability of composite panels with gently curved orthotropic layers. 1. Statics and vibrations,” *Mech. Comp. Mater.*, **35**, No. 4, 285–292 (1999).
17. S. Srinivas, A. K. Rao, and C. V. Joga Rao, “Flexure of simple supported thick homogeneous and laminated rectangular plates,” *ZAMM*, No. 49, 449–458 (1969).
18. P. Tan, N. Nguyen-Thanh, T. Rabczuk, and K. Zhou, “Static, dynamic and buckling analyses of 3D FGM plates and shells via an isogeometric-meshfree coupling approach,” *Comp. Struct.*, No. 198, 35–50 (2018).