DOI 10.1007/s10778-023-01201-5
International Applied Mechanics, Vol. 59, No. 1, January, 2023

STRESS–STRAIN STATE OF A THICK-WALLED ANISOTROPIC CYLINDRICAL SHELL

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The system of partial inhomogeneous differential equations of elasticity of a three-dimensional anisotropic body in a cylindrical coordinate system is obtained using the modified Hu–Washizu variational principle. To reduce it to a one-dimensional one, the Bubnov–Galerkin method is employed. The discrete orthogonalization method is applied to solve the one-dimensional problem along the normal to the shell mid-surface. The stress state of an anisotropic thick-walled composite layered cylindrical shell acted upon by lateral pressure is analyzed. The dependence of the stress state on the angle of rotation of the principal axes of elasticity of a unidirectional fibrous material and the number of cross-stacked layers is analyzed.

Keywords: anisotropic layered shell, stress–strain state, Bubnov–Galerkin method, variational principle, discrete orthogonalization method

Introduction. Variational principles are effectively used in solving problems of solid mechanics [1–3, 5, 8, 12, 14, 15, 22, 23]. They are used to model complex structures acted upon by various physical factors. The method of discrete orthogonalization, whose main stage is solving a canonical system of ordinary differential equations [17–21, 24], used for solving elasticity problems with separable variables was only combined with the variational principle in [3]. The Reissner principle represents the Hamiltonian form of Lagrange's principle [14]. In this case, the set of necessary stationarity conditions is formulated as a system of equations for first-order partial derivatives of the components of the displacement vector and stress tensor. This system can be reduced to a usual normal form only by excluding the dependence of the functions on two coordinates. The difficulties arising in deriving the canonical system based on the Reissner principle are the same as without it, but its usage is better justified in this case [3, 12, 22]. The functional transformation methods [1, 4, 8, 14] make it possible to establish the stationarity conditions for a functional in the form of a system of differential equations for the chosen variables. After reducing the dimension of the problem, the system becomes of normal form. Its dimension can be reduced either by representing the solution of the tree-dimensional problem as a series expansion in two variables or by using simplifying hypotheses of various applied theories. In what follows, we will present a technique of modifying the functional of the generalized Hu–Washizu principle to the required form using the linear anisotropic elasticity theory. This technique is applied to a hollow composite thick-walled layered cylinder (Fig. 1). It is assumed that its material has one plane of elastic symmetry.

1. Basic Equations.

1.1. Variational Hu–Washizu Principle. In line with the Hu–Washizu variational principle [6], we can derive the constitutive equations, kinematic equations, and appropriate boundary conditions using the stationarity condition of the functional

$$\Pi_1 = \int_V \left\{ W(e_{ij}) + \Phi(u_i) - \sigma_{ij}\left[e_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}) \right] \right\} dV + \int_{S_1} \Psi(u_i)\, dS_1 - \int_{S_2} p_i (u_i - \bar{u}_i)\, dS_2, \qquad (1.1)$$

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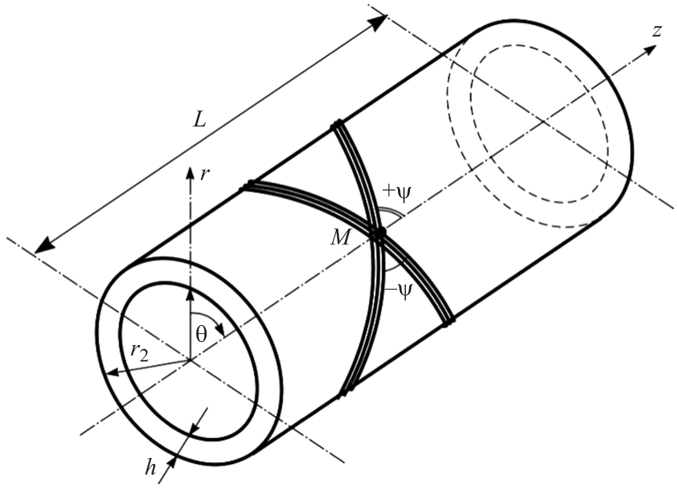

Fig. 1

where the following quantities are varied without addition conditions: u_i are displacements, e_{ij} are strains, σ_{ij} are stresses, p_i are the stresses on the surface S_2 caused by the displacements $\overline{u}_i$. Also, $W(e_{ij})$ is the potential strain energy, $\Phi(u_i)$ and $\Psi(u_i)$ are the potentials of volume and surface loads; u_i are the components of the displacement vector. Here the semicolon before the parameters i and j denotes covariant differentiation with respect to the coordinate with index $i, j = 1, 2, 3$. The potential strain energy has the following vector-matrix representation:

$$W(e_{ij}) = \frac{1}{2}\varepsilon^T B\varepsilon, \tag{1.2}$$

where $\varepsilon^T = (\varepsilon_{zz}, \varepsilon_{\theta\theta}, \varepsilon_{rr}, 2\varepsilon_{r\theta}, 2\varepsilon_{rz}, 2\varepsilon_{z\theta})$; B is the stiffness matrix.

Introducing a vector $\sigma^T = (\sigma_{zz}, \sigma_{\theta\theta}, \sigma_{rr}, \tau_{r\theta}, \tau_{rz}, \tau_{z\theta})$, we obtain the following equations from the stationarity condition for $\delta\Pi_1$:

$$\sigma = B\varepsilon, \tag{1.3}$$

$$\varepsilon = \varepsilon(u), \tag{1.4}$$

$$\sigma_{ij,j} + f_i = 0, \tag{1.5}$$

as well as the boundary conditions $\sigma_{ij} n_j = \overline{F}_i$ on the surface S_1 and the displacements $u_i = \overline{u}_i$ and the stresses $p_i = \sigma_{ij} n_j$ on S_2.

Equations (1.4) relate strains and displacements. From (1.3) it follows that

$$\varepsilon = A\sigma, \tag{1.6}$$

where $A = B^{-1}$.

The elements of the matrices A and B are denoted by a_{ij} and b_{ij} $(i, j = \overline{1,6})$, respectively. Since $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, the matrices A and B are symmetric. Let us establish the relationship between these matrices.

1.2. Modified Mixed Variational Principle. According to [3, 13, 16], the Hu–Washizu mixed variational principle can be modified by splitting the vectors σ and ε into two parts:

$$\sigma_1^T = (\sigma_{rr}, \tau_{r\theta}, \tau_{rz}), \quad \sigma_2^T = (\sigma_{zz}, \sigma_{\theta\theta}, \tau_{z\theta}),$$

$$\varepsilon_1^T = (\varepsilon_{rr}, \varepsilon_{r\theta}, \varepsilon_{rz}), \quad \varepsilon_2^T = (\varepsilon_{zz}, \varepsilon_{\theta\theta}, \varepsilon_{z\theta}). \tag{1.7}$$

Equations (1.3) have the following matrix form:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}, \tag{1.8}$$

where, according to (1.7), the blocks A_{ij} of the matrix A in (1.3) have the following expressions for an anisotropic material with one plane of elastic symmetry:

$$A_{11} = \begin{bmatrix} a_{33} & 0 & 0 \\ 0 & a_{44} & a_{45} \\ 0 & a_{45} & a_{55} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} a_{31} & a_{32} & a_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} a_{13} & 0 & 0 \\ a_{23} & 0 & 0 \\ a_{36} & 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix}. \tag{1.9}$$

Using $\varepsilon = A\sigma$ and (1.8), we obtain

$$\varepsilon_1 = A_{11}\sigma_1 + A_{12}\sigma_2, \tag{1.10}$$

$$\varepsilon_2 = A_{21}\sigma_1 + A_{22}\sigma_2, \tag{1.11}$$

Then, from (1.11) we get

$$\sigma_2 = A_{22}^{-1}\varepsilon_2 - A_{22}^{-1}A_{12}\sigma_1. \tag{1.12}$$

Substituting this expression into (1.10), we arrive at the expressions

$$\varepsilon_1 = A_{11}\sigma_1 + A_{12}A_{22}^{-1}\varepsilon_2 - A_{12}A_{22}^{-1}A_{21}\sigma_1 = A_{12}A_{22}^{-1}\varepsilon_2 + (A_{11} - A_{12}A_{22}^{-1}A_{21})\sigma_1, \tag{1.13}$$

$$\sigma_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}\varepsilon_1 - (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1}\varepsilon_2. \tag{1.14}$$

Considering

$$\sigma = B\varepsilon \tag{1.15}$$

we obtain

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}. \tag{1.16}$$

Then, for the anisotropic material we get

$$\sigma_1 = B_{11}\varepsilon_1 + B_{12}\varepsilon_2, \tag{1.17}$$

$$\sigma_2 = B_{21}\varepsilon_1 + B_{22}\varepsilon_2. \tag{1.18}$$

Comparison of (1.15) and (1.14) gives

$$B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}, \tag{1.19}$$

$$B_{12} = -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1}. \tag{1.20}$$

Substituting (1.14) into (1.12), we obtain

$$\sigma_2 = A_{22}^{-1}\varepsilon_2 - A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}\varepsilon_1 + A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})A_{12}A_{22}^{-1}\varepsilon_2$$

$$=-A_{22}^{-1}A_{21}(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1}\varepsilon_1+\left[A_{22}^{-1}+A_{22}^{-1}A_{21}(A_{11}-A_{12}A_{22}^{-1}A_{21})A_{12}A_{22}^{-1}\right]\varepsilon_2. \quad (1.21)$$

According to (1.21) and (1.18), we have

$$B_{21}=-A_{22}^{-1}A_{21}(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1}, \quad (1.22)$$

$$B_{22}=A_{22}^{-1}+A_{22}^{-1}A_{21}(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1}. \quad (1.23)$$

Thus, expressions (1.19), (1.20), (1.22), and (1.23) relate the matrices in (1.3) and (1.6) of the generalized Hooke's law for the material under consideration.

Unlike the Hu–Washizu principle, we assume that the displacements u_r, u_θ, u_z, strains $\varepsilon_{zz}, \varepsilon_{z\theta}, \varepsilon_{\theta\theta}$, and stresses σ_{rr}, $\tau_{r\theta}, \tau_{rz}$ appearing in (1.1) are independent. From the equations

$$\sigma_1=B_{11}\varepsilon_1+B_{12}\varepsilon_2, \quad (1.24)$$

$$\varepsilon_2=A_{21}\sigma_1+A_{22}\sigma_2 \quad (1.25)$$

we find

$$\varepsilon_1=B_{11}^{-1}\sigma_1-B_{11}^{-1}B_{12}\varepsilon_2,$$

$$\sigma_2=A_{22}^{-1}\varepsilon_2-A_{22}^{-1}A_{21}\sigma_1. \quad (1.26)$$

With the new notation, the potential $W(e_{ij})$ takes the form

$$W(\varepsilon)=\frac{1}{2}(\varepsilon_1^T B_{11}\varepsilon_1+\varepsilon_1^T B_{12}\varepsilon_2+\varepsilon_2^T B_{12}^T\varepsilon_1+\varepsilon_2^T B_{22}\varepsilon_2). \quad (1.27)$$

Considering (1.26) and eliminating ε_1 from (1.27), we get

$$W(\sigma_1,\varepsilon_2)=\frac{1}{2}\Big[\left(B_{11}^{-1}\sigma_1-B_{11}^{-1}B_{12}\varepsilon_2\right)^T B_{11}\left(B_{11}^{-1}\sigma_1-B_{11}^{-1}B_{12}\varepsilon_2\right)$$

$$+(B_{11}^{-1}\sigma_1-B_{11}^{-1}B_{12}\varepsilon_2)^T B_{12}\varepsilon_2+\varepsilon_2^T B_{12}^T(B_{11}^{-1}\sigma_1-B_{11}^{-1}B_{12}\varepsilon_2)+\varepsilon_2^T B_{22}\varepsilon_2\Big].$$

After simple transformations, we finely obtain

$$W(\sigma_1,\varepsilon_2)=\frac{1}{2}\sigma_1^T B_{11}^{-1}\sigma_1+\frac{1}{2}\varepsilon_2^T(B_{22}-B_{12}^T B_{11}^{-1}B_{12})\varepsilon_2. \quad (1.28)$$

We transform the expression $\sigma_{ij}\varepsilon_{ij}$ in a similar way. Comparing the matrix expressions $\varepsilon_1=A_{11}\sigma_1+A_{12}\sigma_2$ and $\sigma_1=B_{11}\varepsilon_1+B_{12}\varepsilon_2$, we see that $B_{12}B_{22}^{-1}=-A_{11}^{-1}A_{12}$.

Then

$$\sigma_{ij}\varepsilon_{ij}=\sigma_1^T B_{11}^{-1}\sigma_1+\varepsilon_2^T(B_{22}-B_{12}^T B_{11}^{-1}B_{12})\varepsilon_2. \quad (1.29)$$

Eliminating σ_2 from the expression $\sigma_{ij}\varepsilon_{ij}(u)$, we get

$$\sigma_{ij}\varepsilon_{ij}(u)=(\varepsilon_1^T(u)+\varepsilon_2^T(u)B_{12}^T B_{11}^{-1})\sigma_1+\varepsilon_2^T(u)(B_{22}-B_{12}^T B_{11}^{-1}B_{12})\varepsilon_2. \quad (1.30)$$

With (1.28)–(1.30), the potential becomes

$$W_1=W(\sigma_1,\varepsilon_2)-\sigma_{ij}(\varepsilon_{ij}-\varepsilon_{ij}(u))=-\frac{1}{2}\sigma_1^T B_{11}^{-1}\sigma_1-\frac{1}{2}\varepsilon_2^T(B_{22}-B_{12}^T B_{11}^{-1}B_{12})\varepsilon_2$$

$$+(\varepsilon_1^T(u)+\varepsilon_2^T(u)B_{12}^T B_{11}^{-1})\sigma_1+\varepsilon_2^T(u)(B_{22}-B_{12}^T B_{11}^{-1}B_{12})\varepsilon_2. \tag{1.31}$$

Also, according to (1.1), we have

$$\int_{S_1}\psi(u_i)dS_1-\int_{S_2}p_i(u_i-\bar{u}_i)dS_2$$

$$=\int_{S_1}\Big[(q_r^- u_r+q_\theta^- u_\theta+q_z^- u_z,h_1)+(q_r^+ u_r+q_\theta^+ u_\theta+q_z^+ u_z,h_{n+1})\Big]dS_1-\int_{S_2}p_i(u_i-\bar{u}_i)dS_2, \tag{1.32}$$

where u_r, u_θ, u_z are the displacements that coincide with the axes of the cylindrical coordinate system shown in Fig. 1; h_1 and h_{n+1} are the thicknesses of the first and $(n+1)$th layers of the shell, respectively.

Varying the potential of the surface loads (1.32), we arrive at the variation of the work done by external forces:

$$\delta\int_{S_1}\psi(u_i)dS_1=\int_{S_1}(q_r\delta u_r+q_\theta\delta u_\theta+q_z\delta u_z)dS_1+\int_{S_2}\sum_{i=1}^{3}p_i(\delta u_i-\delta\bar{u}_i)dS_2, \tag{1.33}$$

where $q_r=q_r^-+q_r^+$, $q_\theta=q_\theta^-+q_\theta^+$, $q_z=q_z^-+q_z^+$, while $p_i=0,\ i=\overline{1,3}$.

Then the potential Π_1 appearing in (1.1) can be represented in final form:

$$\Pi_1=\int_V\big[W(\sigma_1,\varepsilon)-\Phi(u_i)\big]dV-\int_{S_1}\psi(u_i)dS_1-\int_{S_2}p_i(u_i-\bar{u}_i)dS_2. \tag{1.34}$$

The expression for Π_1 is a part of functional (1.1). Then, the variation of functional (1.34) caused by the change in the components of the displacement vector u and stresses σ_1 becomes

$$\delta\Pi_1=\int_V\Bigg\{\Bigg[-\frac{1}{2}\sigma_1^T B_{11}^{-1}\sigma_1+\Big(\varepsilon_1^T(u)+\varepsilon_2(u)B_{11}^T B_{12}^{-1}\Big)\sigma_1\Bigg]\delta\sigma_1$$

$$-\Bigg[\frac{1}{2}\varepsilon_2^T(B_{22}-B_{12}^T B_{11}^{-1}B_{12})\,\varepsilon_2\Bigg]\delta\varepsilon_2+\Big[\varepsilon_2^T(u)(B_{22}-B_{12}^T B_{11}^{-1}B_{12})\varepsilon_2\Big]\delta u-T(u)\delta u\Bigg\}dV$$

$$+\int_{S_1}(\psi(u)\delta u)dS_1-\int_{S_2}p_i(u-\bar{u})\delta p\,dS_2. \tag{1.35}$$

In what follows, we will use the following linear kinematic equations from [11]:

$$e_{rr}^i=\frac{\partial u_r^i}{\partial r},\qquad e_{rz}^i=\frac{\partial u_r^i}{\partial z}+\frac{\partial u_z^i}{\partial r},\qquad e_{r\theta}^i=\frac{\partial u_\theta^i}{\partial r}-\frac{1}{r}u_\theta^i+\frac{1}{r}\frac{\partial u_r^i}{\partial\theta}, \tag{1.36}$$

where e_{rr}^i are the linear strains along the coordinate axis r; e_{rz}^i and $e_{r\theta}^i$ are the shear strains tangent to the corresponding coordinate surfaces.

Using the stationarity condition of functional (1.35), the expressions for the stresses $\sigma_1^T=(\sigma_{rr},\tau_{r\theta},\tau_{rz})$, displacements $u^T=(u_r,u_\theta,u_z)$, the kinematic equations (1.36), and the variations of the work done by the external forces (1.33) and equating the coefficients of the independent variations $\delta\sigma_{rr}$, $\delta\tau_{r\theta}$, $\delta\tau_{rz}$ and δu_r, δu_θ, δu_z in the integral over the volume V, we get

$$\frac{\partial\sigma_{rr}^i}{\partial r}=-\frac{c_{23}^i+1}{r}\sigma_{rr}^i-\frac{\partial\tau_{rz}^i}{\partial z}-\frac{1}{r}\frac{\partial\tau_{r\theta}^i}{\partial\theta}+\frac{c_{22}^i}{r^2}u_r^i+\frac{c_{12}^i}{r}\frac{\partial u_z^i}{\partial z}+\frac{c_{26}^i}{r^2}\frac{\partial u_z^i}{\partial\theta}+\frac{c_{26}^i}{r}\frac{\partial u_\theta^i}{\partial z}+\frac{c_{22}^i}{r^2}\frac{\partial u_\theta^i}{\partial\theta}+q_r,$$

$$\frac{\partial \tau_{rz}^i}{\partial r} = c_{13}^i \frac{\partial \sigma_{rr}^i}{\partial z} - \frac{1}{r}\tau_{rz}^i - \frac{c_{12}^i}{r}\frac{\partial u_r^i}{\partial z} - c_{11}^i \frac{\partial^2 u_z^i}{\partial z^2} - \frac{c_{66}^i}{r^2}\frac{\partial^2 u_z^i}{\partial \theta^2} - \frac{c_{12}^i + c_{66}^i}{r}\frac{\partial^2 u_\theta^i}{\partial z \partial \theta}$$

$$+ \frac{c_{36}^i}{r}\frac{\partial \sigma_{rr}^i}{\partial \theta} - \frac{c_{26}^i}{r^2}\frac{\partial u_r^i}{\partial \theta} - \frac{2c_{16}^i}{r}\frac{\partial^2 u_z^i}{\partial z \partial \theta} - c_{16}^i \frac{\partial^2 u_\theta^i}{\partial z^2} - \frac{c_{26}^i}{r^2}\frac{\partial^2 u_\theta^i}{\partial \theta^2} + q_z,$$

$$\frac{\partial \tau_{r\theta}^i}{\partial r} = \frac{c_{23}^i}{r}\frac{\partial \sigma_{rr}^i}{\partial \theta} - \frac{2}{r}\tau_{r\theta}^i - \frac{c_{22}^i}{r^2}\frac{\partial u_r^i}{\partial \theta} - \frac{c_{12}^i + c_{66}^i}{r}\frac{\partial^2 u_z^i}{\partial z \partial \theta} - c_{66}^i \frac{\partial^2 u_\theta^i}{\partial z^2} - \frac{c_{22}^i}{r^2}\frac{\partial^2 u_\theta^i}{\partial \theta^2}$$

$$+ c_{36}^i \frac{\partial \sigma_{rr}^i}{\partial z} - \frac{c_{26}^i}{r}\frac{\partial u_r^i}{\partial z} - c_{16}^i \frac{\partial^2 u_z^i}{\partial z^2} - \frac{c_{26}^i}{r^2}\frac{\partial^2 u_z^i}{\partial \theta^2} - \frac{2c_{26}^i}{r}\frac{\partial^2 u_\theta^i}{\partial z \partial \theta} + q_\theta,$$

$$\frac{\partial u_r^i}{\partial r} = c_{33}^i \sigma_{rr}^i + \frac{c_{23}^i}{r} u_r^i + c_{13}^i \frac{\partial u_z^i}{\partial z} + \frac{c_{36}^i}{r}\frac{\partial u_z^i}{\partial \theta} + c_{36}^i \frac{\partial u_\theta^i}{\partial z} + \frac{c_{23}^i}{r}\frac{\partial u_\theta^i}{\partial \theta},$$

$$\frac{\partial u_z^i}{\partial r} = a_{55}^i \tau_{rz}^i + a_{45}^i \tau_{r\theta}^i - \frac{\partial u_r^i}{\partial z}, \qquad \frac{\partial u_\theta^i}{\partial r} = a_{45}^i \tau_{rz}^i + a_{44}^i \tau_{r\theta}^i - \frac{1}{r}\frac{\partial u_r^i}{\partial \theta} + \frac{1}{r} u_\theta^i, \tag{1.37}$$

where r is the cylinder radius independent of the coordinates z and θ; σ_{rr}^i, τ_{rz}^i, $\tau_{r\theta}^i$ are the components of the stress tensor (1.7); u_z^i, u_θ^i, u_r^i are the displacements of points of the ith layer of the shell along the axes of the cylindrical coordinate system z, θ, r; q_r, q_z, q_θ are the projections of the vector of specific volume forces onto the tangents to the coordinate lines r, z, θ; c_{kl}^i ($k, l = 1, 2, 3, 6$) are constants characterizing the ith layer determined from the mechanical constants a_{kl}^i [9, 13] of the shell material as follows:

$$c_{11}^i = \frac{1}{|A_{22}^i|}(a_{22}^i a_{66}^i - a_{26}^{i\,2}), \qquad c_{12}^i = \frac{1}{|A_{22}^i|}(a_{16}^i a_{26}^i - a_{12}^i a_{66}^i),$$

$$c_{22}^i = \frac{1}{|A_{22}^i|}(a_{11}^i a_{66}^i - a_{16}^{i\,2}), \qquad c_{16}^i = \frac{1}{|A_{22}^i|}(a_{12}^i a_{26}^i - a_{22}^i a_{16}^i),$$

$$c_{26}^i = \frac{1}{|A_{22}^i|}(a_{12}^i a_{16}^i - a_{11}^i a_{26}^i), \qquad c_{66}^i = \frac{1}{|A_{22}^i|}(a_{11}^i a_{22}^i - a_{12}^{i\,2}),$$

$$|A_{22}^i| = a_{66}^i (a_{11}^i a_{22}^i - a_{12}^{i2}) + a_{26}^i (a_{12}^i a_{16}^i - a_{11}^i a_{26}^i) + a_{16}^i (a_{12}^i a_{26}^i - a_{22}^i a_{16}^i),$$

$$c_{13}^i = a_{13}^i c_{11}^i + a_{23}^i c_{12}^i + a_{36}^i c_{16}^i, \qquad c_{23}^i = a_{13}^i c_{12}^i + a_{23}^i c_{22}^i + a_{36}^i c_{26}^i,$$

$$c_{36}^i = a_{13}^i c_{16}^i + a_{23}^i c_{26}^i + a_{36}^i c_{66}^i, \qquad c_{33}^i = a_{33}^i - (a_{13}^i c_{13}^i + a_{23}^i c_{23}^i + a_{36}^i c_{36}^i).$$

Thus, using the variational equation (1.35), we have derived the system of six inhomogeneous differential equations of linear elasticity for the three-dimensional shell model (1.37). It includes partial derivatives with respect to six components of the vectors $\sigma_1^T = (\sigma_{rr}, \tau_{r\theta}, \tau_{rz})$ and $u^T = (u_r, u_\theta, u_z)$ and can be used to analyze the stress–strain state of an anisotropic layered thick-walled composite cylindrical shell.

The solution of system (1.37) must satisfy the boundary conditions on the lateral surfaces:

$$\sigma_{rr}^0(r_1, z, \theta) = \pm q_r^0(z); \quad \tau_{rz}^0(r_1, z, \theta) = 0; \quad \tau_{r\theta}^0(r_1, z, \theta) = 0 \quad \text{at} \quad r = r_1;$$

$$\sigma_{rr}^n(r_2, z, \theta) = \pm q_r^n(z); \quad \tau_{rz}^n(r_2, z, \theta) = 0; \quad \tau_{r\theta}^n(r_2, z, \theta) = 0 \quad \text{at} \quad r = r_2; \tag{1.38}$$

the conditions at the ends $z = 0$, $z = L$ (Fig. 1):

$$\sigma_{zz}^i = u_r^i = u_\theta^i = 0 \tag{1.39}$$

and the perfect bonding conditions for layers:

$$\sigma_{rr}^i(r_i) = \sigma_{rr}^{i+1}(r_i), \quad \tau_{rz}^i(r_i) = \tau_{rz}^{i+1}(r_i),$$

$$\tau_{r\theta}^i(r_i) = \tau_{r\theta}^{i+1}(r_i), \quad u_r^i(r_i) = u_r^{i+1}(r_i),$$

$$u_z^i(r_i) = u_z^{i+1}(r_i), \quad u_\theta^i(r_i) = u_\theta^{i+1}(r_i), \tag{1.40}$$

where i is the layer number. Conditions (1.39) mean that there are diaphragm perfectly rigid in its plane at the cylinder ends [7]. In (1.38), $q_r^0(z)$ and $q_r^n(z)$ are the internal and external pressures, respectively, on the lateral surfaces of the shell.

2. Problem-Solving Technique. Reduction of the elasticity equations of the three-dimensional shell model to a unidimensional one. To solve the system of equations (1.37) subject to the boundary conditions (1.38) and (1.39), we will use the Bubnov–Galerkin procedure. Following it, we expand all the functions into trigonometric series [7] in coordinates z and θ so that they satisfy the boundary conditions (1.39):

$$\sigma_{rr}(r,z,\theta) = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}[y_{1,pk}(r)\cos k\theta + y'_{1,mk}(r)\sin k\theta]\sin l_m z,$$

$$\tau_{rz}(r,z,\theta) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}[y_{2,pk}(r)\cos k\theta + y'_{2,mk}(r)\sin k\theta]\cos l_m z,$$

$$\tau_{r\theta}(r,z,\theta) = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}[y_{3,pk}(r)\sin k\theta + y'_{3,mk}(r)\cos k\theta]\sin l_m z,$$

$$u_r(r,z,\theta) = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}[y_{4,pk}(r)\cos k\theta + y'_{4,mk}(r)\sin k\theta]\sin l_m z,$$

$$u_z(r,z,\theta) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}[y_{5,pk}(r)\cos k\theta + y'_{5,mk}(r)\sin k\theta]\cos l_m z,$$

$$u_\theta(r,z,\theta) = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}[y_{6,pk}(r)\sin k\theta + y'_{6,mk}(r)\cos k\theta]\sin l_m z, \tag{2.1}$$

where $y_{i,pk}$ and $y'_{i,mk}$ $(i = \overline{1,6})$ are the components of the stress–strain state in the Fourier series; p, m, and k are the wave numbers in the series; $l_m = m\pi / L$ is the parameter where L is the length of the cylinder generatrix (Fig. 1).

After mathematical rearrangements and separation of variables in (1.37) and use of formulas (2.1), we obtain, for the ith layer, the following system of twelfth-order differential equations in Cauchy form:

$$\frac{d\bar{y}^i}{dr} = T^i(r)\bar{y}^i + f^i, \quad T^i(r) = t_{n,l}^i(r) \quad (n,\ l = 1,\ldots,12), \tag{2.2}$$

where $\bar{y}^i = \left\{ y_{1,p}^i, y_{2,p}^i, y_{3,p}^i, y_{4,p}^i, y_{5,p}^i, y_{6,p}^i, y_{1,m}'^i, y_{2,m}'^i, y_{3,m}'^i, y_{4,m}'^i, y_{5,m}'^i, y_{6,m}'^i \right\}$ is the unknown vector function; f^i is the load. The nonzero elements of the matrix $T^i(r)$ and the coefficients of the unknowns in (2.2) are given in [21].

The one-dimensional stress–strain state problem for a thick-walled cylindrical shell can be solved using the numerical discrete-orthogonalization method [17–20].

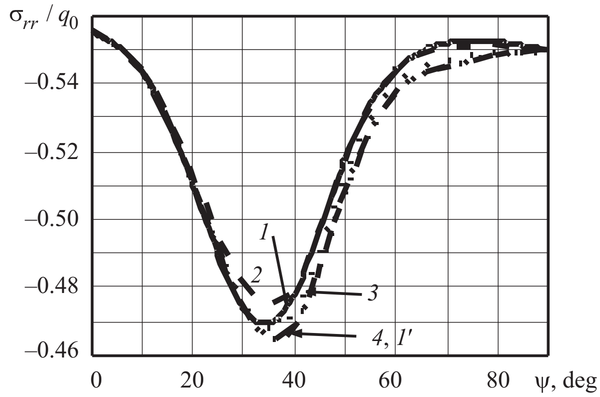

Fig. 2

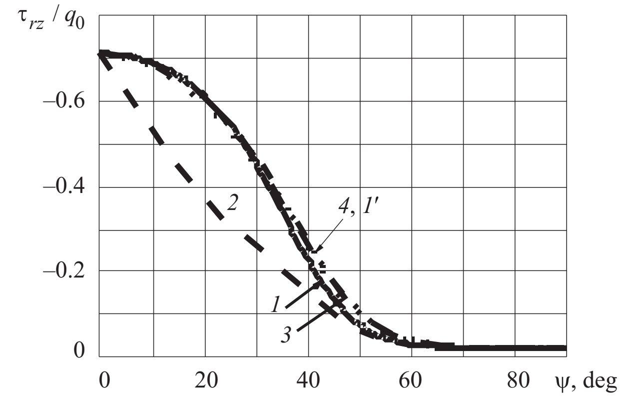

Fig. 3

System (2.2) with the boundary conditions (1.38) can be solved to find the stresses σ^i_{rr}, τ^i_{rz}, $\tau^i_{r\theta}$ and displacements u^i_r, u^i_z, u^i_θ for the variable z by substituting the corresponding coordinate z along the cylinder generatrix $0 \le z \le L$ into the trigonometric series (2.1).

The stresses $\sigma_2^T = (\sigma_{zz}, \sigma_{\theta\theta}, \tau_{z\theta})$ for the chosen material can be expressed in terms of the unknown functions of the generalized Hooke's law (1.15):

$$\sigma^i_{zz} = c^i_{11} e^i_{zz} + c^i_{12} e^i_{\theta\theta} + c^i_{16} e^i_{z\theta} - c^i_{13} \sigma^i_{rr},$$

$$\sigma^i_{\theta\theta} = c^i_{12} e^i_{zz} + c^i_{22} e^i_{\theta\theta} + c^i_{26} e^i_{z\theta} - c^i_{23} \sigma^i_{rr},$$

$$\tau^i_{z\theta} = c^i_{16} e^i_{zz} + c^i_{26} e^i_{\theta\theta} + c^i_{66} e^i_{z\theta} - c^i_{36} \sigma^i_{rr}. \tag{2.3}$$

To determine these components, we will employ kinematic equations from [11] for e^i_{zz}, $e^i_{\theta\theta}$, $e^i_{z\theta}$:

$$e^i_{zz} = \frac{\partial u^i_z}{\partial z}, \qquad e^i_{\theta\theta} = \frac{1}{r}\frac{\partial u^i_\theta}{\partial \theta} + \frac{1}{r} u^i_r, \qquad e^i_{z\theta} = \frac{\partial u^i_\theta}{\partial z} + \frac{1}{r}\frac{\partial u^i_z}{\partial \theta}, \tag{2.4}$$

where e^i_{zz} and $e^i_{\theta\theta}$ are the linear strains along the coordinate axes z, θ; $e^i_{z\theta}$ are the shear strains tangent to the corresponding coordinate surface.

3. Analysis of Numerical Results. Let us consider, as an example, a layered thick-walled anisotropic cylindrical shell made of a fibrous composite (Fig. 1). The shell is acted upon by distributed pressure $q = -q_0 \sin(\pi z / L)$, $q_0 = 1.0$ MPa. The principal axes of elasticity of the orthotropic material can rotate through an angle ±ψ about the shell generatrix (Fig. 1).

The shell material is boroplastic with the following mechanical characteristics: $E_{11} = 280E_0$, $E_{22} = E_{33} = 31E_0$, $G_{12} = G_{23} = 10.5E_0$, $G_{13} = 21.2E_0$, $\nu_{21} = 0.25$, $\nu_{12} = 0.0277$, $E_0 = 10000$ MPa. The shell dimensions are: the radius of the inner and outer lateral surfaces $r_1 = 0.54$ m and $r_2 = 0.66$ m, the length $L = 1.2$ m.

The stress state of the shell was analyzed for varying number of layers (from one to eight) and for the angle ψ changing from 0 to 90°. The orthotropic case was also considered by equating the mechanical characteristics c_{16}, c_{26}, c_{36}, and a_{45} to zero.

The results are presented in the figures below. Figures 2–7 represent the following stresses: σ_{rr} (Fig. 2), τ_{rz} (Fig. 3), $\tau_{r\theta}$ (Fig. 4), σ_{zz} (Fig. 5), $\sigma_{\theta\theta}$ (Fig. 6), and $\tau_{z\theta}$ (Fig. 7). The stresses σ_{rr}, σ_{zz}, and $\sigma_{\theta\theta}$ were calculated at the point $z = 0.5L$ on the outer surface, while the stresses τ_{rz}, $\tau_{z\theta}$, and $\tau_{r\theta}$ were determined at the points $z = 0$ (τ_{rz}, $\tau_{z\theta}$) and $z = 0.25L$ ($\tau_{r\theta}$) of the mid-surface. The curves are denoted as follows: *1* for single layer, *2* for two layers, *3* for three layers, *4* for four layers, and *1′* for the orthotropic case.

The results for an anisotropic (orthotropic) thick-walled cylindrical shell with five and more layers are omitted here. This is because a thick-walled anisotropic cylinder with four symmetrically cross-stacked layers with the boundary conditions

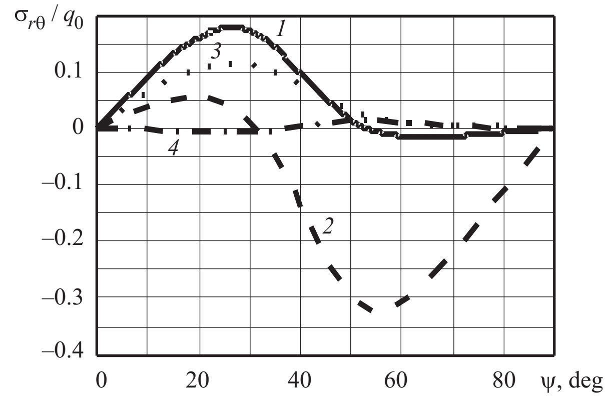

Fig. 4

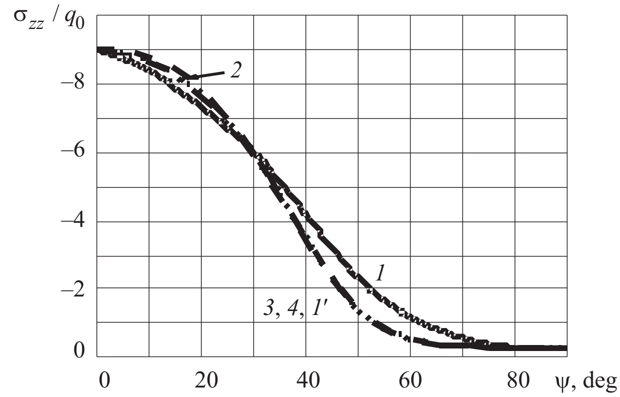

Fig. 5

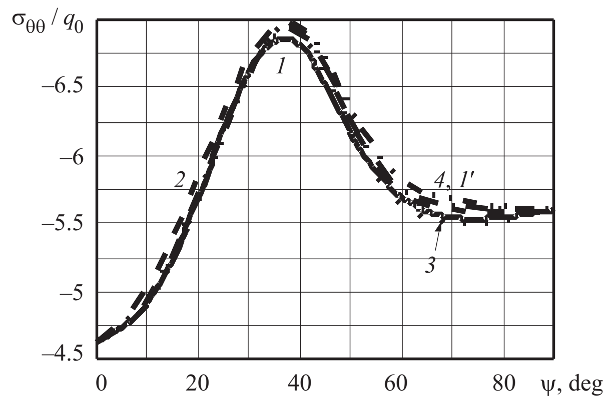

Fig. 6

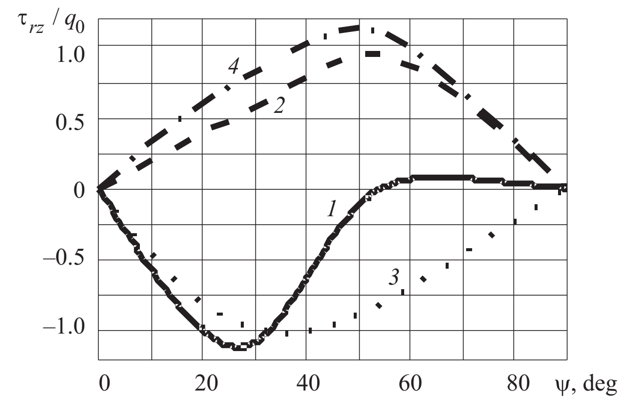

Fig. 7

(1.39) under a lateral symmetric load is considered virtually orthotropic. Such orthotropic solutions were obtained in [3, 10] for a cylindrical shell with $L/r = 2$ (r is the mid-surface radius) which was found to become orthotropic when the number of layers is equal to 14 and 7, respectively.

As can be seen from Figs. 2–7, the stresses obtained for one- and two-layer anisotropic cylinders using the above approach differ from those determined without usage of the anisotropic mechanical characteristics of the material. Figure 2 shows that this difference for the normal stress σ_{rr} in the single-layer shell is equal to 2.5% and decreases with increasing number of layers, becoming zero for four layers. The stress σ_{rr} depends on the angle ψ. For example, the stress σ_{rr} reaches its minimum for $30° \le \psi \le 40°$, the difference between it and the values at $\psi = 0$ and 90° being 16%.

The difference between the values of the shear stress τ_{rz} (Fig. 3) found with and without allowance for the anisotropic mechanical characteristics increases from 16% to 43% as the number of layers changes from one to two. As the number of layers increases to four, this difference tends to zero. The shear stress τ_{rz} decreases in absolute magnitude with increase in the angle of rotation of the principal axes of elasticity from $\psi = 0$ to 90°.

The shear stress $\tau_{r\theta}$ (Fig. 4) that is absent in the orthotropic case arises in solving the problem with the proposed approach. In this case, increasing the number of layers to two increases the maximum values of the stress. With further increase in the number of layers, this stress decreases to zero.

Figure 5 shows that the normal stress σ_{zz} varies similarly to τ_{rz}. The value of σ_{zz} at $\psi = 90°$ is 3% of the same value at $\psi = 0$. Also, the values of σ_{zz} for single-layer shells obtained with the proposed approach differ from those in the orthotropic case. This difference is maximum (40%) is the range $50° \le \psi \le 60°$.

Unlike σ_{rr}, the normal stress $\sigma_{\theta\theta}$ (Fig. 6) become maximum in the range of angles from 30° to 40°. The maximum difference between these stresses is at $\psi = 0$ and equal to 32%. The effect of the number of layers ψ on $\sigma_{\theta\theta}$ is insufficient. The maximum difference between single-layer and four-layer shells is 2%.

Figure 7 shows how the shear stress $\tau_{z\theta}$ varies with the angle of rotation of the principal axes of elasticity, when load acts on the outer lateral surface of the shell. In this case, the stress is maximum for $20° \le \psi \le 40°$ if the number of layer is odd and for $50° \le \psi \le 60°$ if the number of layers is even.

Conclusions. Based on the modified Hy–Washizu variational principle, we have obtained a three-dimensional system of equations describing the stress–strain state of a thick-walled anisotropic cylindrical shell. To solve the system, the Bubnov–Galerkin and the discrete-orthogonalization methods were used. The approach we have developed makes it possible to solve spatial problems of the stress–strain state of thick-walled layered cylindrical shells made of anisotropic material with one plane of elastic symmetry. As an example, the stress state of an anisotropic thick-walled composite cylindrical shell under a lateral external load was analyzed for different angles of rotation of the principal exes of elasticity and different the number of layers.

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