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QUATERNION ATTITUDE DETERMINATION BY VECTOR MEASUREMENT

L. M. Ryzhkov

Quaternion attitude determination algorithms based on vector measurements are proposed. The projections of the normalized vectors in the reference and the body-fixed coordinate systems are assumed to be known. The task is to determine the quaternion of rotation of the body-fixed coordinate system relative to the reference coordinate system. The accuracies of the proposed algorithms and the QUEST algorithm are compared. It is shown that the proposed algorithms are practically equivalent in terms of attitude determination accuracy.

Keywords: attitude; quaternion; algorithm

Introduction. Let us determine the quaternion of transformation from the reference coordinate system to the body-fixed coordinate system. We will use information on the projections of vectors in these coordinate systems [1, 2]. At the initial stage of analysis, the q-method [4] and the QUEST algorithm [5] employ the direction cosine matrix, which is then expressed in terms of the quaternion. We will develop and generalize the approach of [6, 7] without the need to introduce the direction cosine matrix.

Problem Solution. If a rotation quaternion $\boldsymbol{q}$ corresponds to the transformation from a fixed coordinate system to a moving one, then the expressions of the same vector in the moving ($\boldsymbol{r}$) and fixed ($\boldsymbol{r}_0$) coordinated systems are related by

$$\boldsymbol{r} = \tilde{\boldsymbol{q}} \circ \boldsymbol{r}_0 \circ \boldsymbol{q}, \tag{1}$$

where $\tilde{\boldsymbol{q}}$ is the conjugate quaternion.

Represent formula (1) in the form

$$\boldsymbol{q} \circ \boldsymbol{r} = \boldsymbol{r}_0 \circ \boldsymbol{q}. \tag{2}$$

Formula (2) has the following matrix form:

$$\boldsymbol{V}\boldsymbol{q} = \boldsymbol{V}_0 \boldsymbol{q}_0, \tag{3}$$

where

$$\boldsymbol{V} = \begin{bmatrix} 0 & -\boldsymbol{r}^T \\ \boldsymbol{r} & \boldsymbol{D}^T \end{bmatrix}, \quad \boldsymbol{V}_0 = \begin{bmatrix} 0 & -\boldsymbol{r}_0^T \\ \boldsymbol{r}_0 & \boldsymbol{D}_0 \end{bmatrix},$$

$$\boldsymbol{D} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}, \quad \boldsymbol{D}_0 = \begin{bmatrix} 0 & -r_{oz} & r_{oy} \\ r_{oz} & 0 & -r_{ox} \\ -r_{oy} & r_{ox} & 0 \end{bmatrix}.$$

National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute," 37 Peremogy Av., Kyiv, Ukraine, 03056; e-mail: lev_ryzhkov@rambler.ru. Translated from Prikladnaya Mekhanika, Vol. 57, No. 5, pp. 140–144, September–October 2021. Original article submitted September 8, 2020.

This expression can be represented as

$$\boldsymbol{W}\boldsymbol{q} = \boldsymbol{0}_{4\times 1}, \tag{4}$$

where

$$\boldsymbol{W} = \boldsymbol{V}_0 - \boldsymbol{V} = \begin{bmatrix} 0 & -\boldsymbol{a}^T \\ \boldsymbol{a} & \boldsymbol{U} \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}, \quad \boldsymbol{u} = \boldsymbol{r}_0 + \boldsymbol{r}, \quad \boldsymbol{a} = \boldsymbol{r}_0 - \boldsymbol{r}.$$

For n vectors, we have

$$\boldsymbol{W}_i \boldsymbol{q} = \boldsymbol{0}_{4\times 1} \quad (i = 1 \ldots n).$$

Considering the measurement errors, we introduce the loss function

$$l(\boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{n} \mu_i (\boldsymbol{W}_i \boldsymbol{q})^2 = \frac{1}{2}\sum_{i=1}^{n} (\widetilde{\boldsymbol{W}}_i \boldsymbol{q})^2 = \frac{1}{2}\sum_{i=1}^{n} (\widetilde{\boldsymbol{W}}_i \boldsymbol{q})^T (\widetilde{\boldsymbol{W}}_i \boldsymbol{q}) = \frac{1}{2}\boldsymbol{q}^T \boldsymbol{G}\boldsymbol{q}, \tag{5}$$

where μ_i are weight coefficients;

$$\boldsymbol{G} = \sum_{i=1}^{n} \widetilde{\boldsymbol{W}}_i^T \widetilde{\boldsymbol{W}}_i = -\sum_{i=1}^{n} \widetilde{\boldsymbol{W}}_i^2 = \sum_{i=1}^{n} \begin{bmatrix} \|\tilde{\boldsymbol{a}}_i\|^2 & \tilde{\boldsymbol{a}}_i^T \widetilde{\boldsymbol{U}}_i \\ \widetilde{\boldsymbol{U}}_i^T \boldsymbol{a}_i & -\widetilde{\boldsymbol{U}}_i^2 \end{bmatrix},$$

$$\widetilde{\boldsymbol{W}}_i = \frac{\boldsymbol{W}_i}{\sqrt{\mu_i}}, \quad \tilde{\boldsymbol{a}}_i = \frac{\boldsymbol{a}_i}{\sqrt{\mu_i}}, \quad \widetilde{\boldsymbol{U}}_i = \frac{\boldsymbol{U}_i}{\sqrt{\mu_i}}.$$

Since a quaternion must be normalized, we use the following form of losses:

$$l_1(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{q}^T \boldsymbol{G}\boldsymbol{q} - \lambda(\boldsymbol{q}^T \boldsymbol{q} - 1), \tag{6}$$

where λ is the Lagrange multiplier.

We have

$$\frac{\partial l_1(\boldsymbol{q})}{\partial \boldsymbol{q}} = \boldsymbol{G}\boldsymbol{q} - \lambda \boldsymbol{q} = \boldsymbol{0}_{4\times 1}.$$

The minimum condition is

$$\boldsymbol{G}\boldsymbol{q} = \lambda \boldsymbol{q}. \tag{7}$$

Then

$$l(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{q}^T \boldsymbol{G}\boldsymbol{q} = \frac{1}{2}\boldsymbol{q}^T \lambda \boldsymbol{q} = \frac{1}{2}\lambda. \tag{8}$$

This means that we are interested in the minimum value of the parameter λ. Thus, the problem is reduced to finding the eigenvector (quaternion) of the matrix $\boldsymbol{G}$ that corresponds to the minimum of the eigenvalue λ. To this end, it is convenient to use the eig Matlab routine.

Let us solve the problem without this function.

Expressing the quaternion as a vector $\boldsymbol{q} = [q_0 \ \ \boldsymbol{q}_v^T]$, we reduce (8) to the form

$$\begin{bmatrix} b & \boldsymbol{Z}^T \\ \boldsymbol{Z} & \boldsymbol{H} \end{bmatrix} \begin{bmatrix} q_0 \\ \boldsymbol{q}_v \end{bmatrix} = \lambda \begin{bmatrix} q_0 \\ \boldsymbol{q}_v \end{bmatrix}, \tag{9}$$

where

$$\boldsymbol{H} = -\sum_{i=1}^{n} \tilde{\boldsymbol{U}}_i^2, \qquad \boldsymbol{Z} = \sum_{i=1}^{n} \tilde{\boldsymbol{U}}_i^T \tilde{\boldsymbol{a}}_i, \qquad b = \sum_{i=1}^{n} ||\tilde{\boldsymbol{a}}_i||^2.$$

Let us represent system (9) in the form

$$q_0 b + \boldsymbol{Z}^T \boldsymbol{q}_v = \lambda q_0, \qquad q_0 \boldsymbol{Z} + \boldsymbol{H}\boldsymbol{q}_v = \lambda \boldsymbol{q}_v. \tag{10}$$

As in [5], we will consider the second equation of this system:

$$(\lambda \boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y} = \boldsymbol{Z},$$

where $\boldsymbol{Y} = \boldsymbol{q}_v / q_0$ is the Gibbs vector. Then

$$\boldsymbol{Y} = (\lambda \boldsymbol{I} - \boldsymbol{H})^{-1} \boldsymbol{Z}. \tag{11}$$

Assuming that $\lambda = \lambda_{\min}$, we find $\boldsymbol{Y}$ and $\boldsymbol{q}$:

$$\boldsymbol{q} = \frac{1}{\sqrt{1 + |\boldsymbol{Y}|^2}} \begin{bmatrix} 1 \\ \boldsymbol{Y} \end{bmatrix}. \tag{12}$$

A shortcoming of this expression is the presence of the Gibbs vector, which does not allow us to use this expression in the case of a 180° rotation. To do away with it, we will do the following.

The eigenvalues of the matrix $\boldsymbol{G}$ are the roots of the following characteristic equation [3]:

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0, \tag{13}$$

where

$$c_1 = -T_1, \quad c_2 = -(c_1 T_1 + T_2), \quad c_3 = -(c_2 T_1 + c_1 T_2 + T_3),$$
$$c_4 = \det \boldsymbol{G}, \quad T_1 = \operatorname{tr} \boldsymbol{G}, \quad T_2 = \operatorname{tr} \boldsymbol{G}^2, \quad T_3 = \operatorname{tr} \boldsymbol{G}^3.$$

Let us derive a similar equation for the matrix $\boldsymbol{H}_{3\times 3}$:

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{14}$$

where $\boldsymbol{G}$ in (13) should be replaced by $\boldsymbol{H}$.

According to the Cayley–Hamilton theorem, each matrix corresponds to its characteristic equation:

$$\boldsymbol{H}^3 + c_1 \boldsymbol{H}^2 + c_2 \boldsymbol{H} + c_3 = 0. \tag{15}$$

Let us represent the expression $(\boldsymbol{H} - \lambda \boldsymbol{I})^{-1}$ as

$$(\boldsymbol{H} - \lambda \boldsymbol{I})^{-1} = \gamma^{-1}(\alpha + \beta \boldsymbol{H} + \boldsymbol{H}^2). \tag{16}$$

We have

$$\boldsymbol{H}^3 + (\beta - \lambda)\boldsymbol{H}^2 + (\alpha - \beta\lambda)\boldsymbol{H} - (\gamma + \alpha\lambda) = 0. \tag{17}$$

Equating the coefficients of like powers of $\boldsymbol{H}$ in (15) and (17), we get

$$\beta = c_1 + \lambda, \quad \alpha = c_2 + \beta\lambda, \quad \gamma = -(c_3 + \alpha\lambda). \tag{18}$$

We can write

$$\boldsymbol{Y} = \boldsymbol{L} / \gamma, \tag{19}$$

where $\boldsymbol{L} = -(\alpha\boldsymbol{I} + \beta\boldsymbol{H} + \boldsymbol{H}^2)\boldsymbol{Z}$.

Thus,

$$\boldsymbol{q} = \frac{1}{\sqrt{\gamma^2 + |\boldsymbol{L}|^2}} \begin{bmatrix} \gamma \\ \boldsymbol{L} \end{bmatrix}. \tag{20}$$

It is important that the Gibbs vector is absent in (20).

Assuming that $\lambda_{\min} \approx 0$, we can find the minimum eigenvalue from the simplified characteristic equation

$$\Delta \approx c_3 \lambda_{\min} + c_4 = 0.$$

Then

$$\lambda_{\min} \approx -\frac{c_4}{c_3}. \tag{21}$$

To analyze the accuracy of the algorithms, we assume that $\psi = 30°$, $\theta = 20°$, $\varphi = 10°$, $\boldsymbol{r}_{01} = [1\ 20\ 30]^T$, $\boldsymbol{r}_{02} = [4\ 5\ 0]^T$, $\tilde{\boldsymbol{r}}_1 = \boldsymbol{E}_1 \boldsymbol{r}_1$, $\tilde{\boldsymbol{r}}_2 = \boldsymbol{E}_2 \boldsymbol{r}_2$, where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the values of the vectors without measurement errors.

The matrix

$$\boldsymbol{E}_1 = [0.95\ 0\ 0;\ 0\ 1\ 0;\ 0\ 0\ 1.01], \qquad \boldsymbol{E}_2 = [1.01\ 0\ 0;\ 0\ 1\ 0;\ 0\ 0\ 0.95]$$

characterize the measurement errors. The weight coefficients are assumed equal to unity.

Using the eig function, we find $\lambda_{\min 1} = 9.7717 \cdot 10^{-5}$. If we use formula (20), then $\lambda_{\min 2} = 1.0760 \cdot 10^{-4}$.

The following angles have been obtained:

$\psi = 29.7226°$, $\theta = 19.4205°$, $\varphi = 9.7095°$, with the eig function;
$\psi = 29.7226°$, $\theta = 19.4205°$, $\varphi = 9.7095°$, with formula (12);
$\psi = 29.7226°$, $\theta = 19.4205°$, $\varphi = 9.7095°$, with formula (20) and $\lambda_{\min 1}$;
$\psi = 29.7227°$, $\theta = 19.4206°$, $\varphi = 9.7096°$, with formula (20) and $\lambda_{\min 2}$.

For comparison: the QUEST algorithm yields

$$\psi = 29.7229°, \quad \theta = 19.4085°, \quad \varphi = 9.7140°.$$

It can be seen that the results are very similar.

Since $\lambda_{\min} \approx 0$, we can reduce the amount of calculation by setting $\lambda_{\min} = 0$. Then, using formulas (12) and (20), we get

$$\psi = 29.7214°, \quad \theta = 19.4198°, \quad \varphi = 9.7086°.$$

That is, we can set $\lambda_{\min} = 0$ for practical purposed. Then

$$\alpha\boldsymbol{I} + \beta\boldsymbol{H} + \boldsymbol{H}^2 = \gamma\boldsymbol{H}^{-1},$$

i.e., $\boldsymbol{L} = -\gamma\boldsymbol{H}^{-1}\boldsymbol{Z}$.

Formula (19) becomes simpler:

$$\boldsymbol{q} = \frac{1}{\sqrt{1+\left|\boldsymbol{X}^2\right|}}\begin{bmatrix} 1 \\ \boldsymbol{X} \end{bmatrix}, \tag{22}$$

where

$$\boldsymbol{X} = -\boldsymbol{H}^{-1}\boldsymbol{Z} = \left(\sum_{i=1}^{n} \widetilde{\boldsymbol{U}}_i^T \widetilde{\boldsymbol{U}}_i\right)^{-1} \sum_{i=1}^{n} \widetilde{\boldsymbol{U}}_i^T \widetilde{\boldsymbol{a}}_i .$$

Using formula (22), we get

$$\psi = 29.7214°, \quad \theta = 19.4206°, \quad \varphi = 9.7086°.$$

Conclusions. Effective quaternion attitude determination algorithms based on vector measurements have been proposed. The algorithm that involves the determination of the eigenvalues and eigenvectors of the matrix $\boldsymbol{G}$ is the most general. The algorithms are practically equivalent in terms of the attitude determination accuracy.

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