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APPLICATION OF THE INHOMOGENEOUS ELASTICITY THEORY TO THE DESCRIPTION OF THE MECHANICAL STATE OF A SINGLE-ROOTED TOOTH*

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Some preliminary experience in application of the inhomogeneous elasticity theory to the description of the mechanical state of a single-rooted tooth is outlined. A few simple models of the axisymmetric state of cylindrical isotropic and transversely isotropic bodies are considered and discussed.

Keywords: inhomogeneous elasticity theory, radius-dependent properties, simplest models of cylindrical bodies

Introduction. Our goal here is to study features of the stress distribution in a single-rooted tooth, which is initially modeled by an elastic cylindrical rod with mechanical characteristics varying continuously along the radius. Actually, a tooth is a more complex structure.

A tooth has three layers of calcified tissues: enamel, dentin, and cementum. The dental cavity is filled with pulp. The pulp is covered by a dentin, which represents the main calcified tissue. The dentin on the visual portion of the tooth is covered by enamel, while the tooth roots in the jaw are covered by cementum. A more complete picture of the real structure is shown in Fig. 1.

Biomechanics actively studies a tooth as a composite structure. For example, the recent publications [1, 2] are devoted to the experimental investigation of the strength properties of a tooth and draw the following conclusion: "… hard tooth tissues are made of the same protein-mineral material but have unlike structure that is responsible for dissimilar mechanical characteristics. For example, the enamel strength is 7 to 10 times higher than that of dentine; however, the elastic moduli of materials that compose the hard tooth tissues smoothly vary under certain operation conditions in such a way that stresses and strains do not grow to parasite levels under loading…". In this connection, the publications [8, 9] are noteworthy.

The representation of a tooth as a composition of many layers with dissimilar properties resembles the geometric structure of either bamboo or onion and can be considered adequate at the initial stages of the tooth analysis within the framework of modern biomechanics. For this reason, modeling a tooth within the framework of the inhomogeneous elasticity theory as a multilayer cylindrical structure can be considered permissible if we take into account the fact that the unremovable radius dependence of the mechanical properties of the structure has already been established in composite mechanics.

Cylindrical shapes are found frequently in nature, engineering, and even in household use. A classic example from nature is either tree trunk or bone. A like example from engineering is a circular bolt. In private life a man constantly uses

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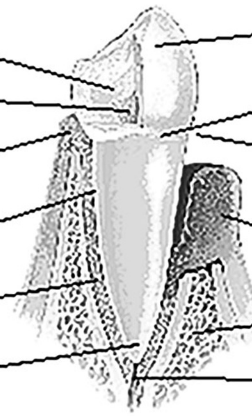

Fig. 1

something round and long, from a stick, pencil, and water pipe to a rolling pin to shape and flatten dough. For the most part, all of these objects are subject to various mechanical loads such as tension, compression, bending, twisting, shearing, etc. Because of this, the mechanics of materials and structures has always much attention to cylindrical bodies. Their mechanical behavior has been described in a great many scientific publications. As a rule, bodies in the form of solid or hollow cylinders are considered as homogeneous in their mechanical properties. However, in many cases cylindrical objects are essentially inhomogeneous [3, 4, 28]. The most frequently such inhomogeneity is manifested as radius dependence of density and other mechanical properties. In this connection, we recall bamboo or bone, which have higher density and are stronger on the outer surface, the density and the tensile and shear moduli of elasticity decreasing with depth. Because the inhomogeneity is observed not only in natural materials, it would be technologically appropriate to introduce it in other materials. For example, the theory of functionally graded materials (FGM), which has been formulated not so long and is actively developed, focuses on artificially made inhomogeneous materials and is the main user of results on inhomogeneous materials.

Remark 1. The success and relevancy of the FGM theory are supported by the first publications [19–21, 27, 31], the review [5], informative monographs [26, 30], and a number of useful papers published recently [10, 22, 23, 29].

While the mechanics of homogeneous bodies can be considered as a fully developed division of science, the mechanics of inhomogeneous bodies is yet to be adequately investigated. This especially applies to the analytical mechanics of inhomogeneous bodies which develops strict mathematical models described either by differential or integral equations (to which analytical methods are applied). In what follows, we address one of the above-mentioned fragments as a continuation of the publications [4, 12 – 18, 28]. From the standpoint of the general theory of materials, we will analyze cylindrical bodies using the axisymmetric theory of inhomogeneous isotropic and transversely isotropic elasticity for the type of inhomogeneity where the mechanical properties of an elastic medium vary along the radius [12–18].

1. Basic Static Equations of an Elastic Cylindrical Body. Let us consider the case where the elastic parameters are functionally dependent of the coordinate *r*, while the Lame elastic constants λ and μ are functions

$$\lambda(r)=\lambda_o l(r), \quad \mu(r)=\mu_o m(r) \quad \left(\lambda_o, \mu_o = \text{const}, \nu_o = \frac{\lambda_o}{2(\lambda_o+\mu_o)}\right), \tag{1}$$

that are differentiable at least two times.

The axisymmetric state is described using circular cylindrical coordinates (r, ϑ, z) in the case where the *z*-axis is the symmetry axis, while the ϑ-coordinate is absent. In this case [3, 6, 7, 24, 25], the displacement vector includes only two nonzero components $u=(u_r, u_\vartheta = 0, u_z)$; the strain tensor has four nonzero components:

$$\varepsilon_{rr}=u_{r,r}, \quad \varepsilon_{\vartheta\vartheta}=(u_r/r), \quad \varepsilon_{zz}=u_{z,z},$$

$$\varepsilon_{rz}=(1/2)(u_{r,z}+u_{z,r}), \quad \varepsilon_{r\vartheta}=0, \quad \varepsilon_{\vartheta z}=0,$$

The expression of dilatation is simpler as well:

$$e = \varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{zz} = u_{r,r} + u_{z,z} + (u_r / r) = (1/r)(ru_r)_{,r} + u_{z,z}.$$

The linear stress tensor includes four nonzero components:

$$\sigma = (\sigma_{rr}, \sigma_{\vartheta\vartheta}, \sigma_{zz}, \sigma_{rz}, \sigma_{r\vartheta} = 0, \sigma_{\vartheta z} = 0).$$

Then constitutive equations become

$$\sigma_{rr} = \lambda e + 2\mu\varepsilon_{rr}, \qquad \sigma_{\vartheta\vartheta} = \lambda e + 2\mu\varepsilon_{\vartheta\vartheta},$$

$$\sigma_{zz} = \lambda e + 2\mu\varepsilon_{zz}, \qquad \sigma_{rz} = 2\mu\varepsilon_{rz}. \tag{2}$$

The system of equilibrium equations (without body forces) includes only two equations:

$$\sigma_{rr,r} + \sigma_{rz,z} + (1/r)(\sigma_{rr} - \sigma_{\vartheta\vartheta}) = 0,$$

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0. \tag{3}$$

The system of Lame's equations also consists of two equations:

$$\mu\left[\Delta - (1/r^2)\right]u_r + (\lambda + \mu)e_{,r} = 0,$$

$$\mu\Delta u_z + (\lambda + \mu)e_{,z} = 0,$$

$$\Delta e = e_{,rr} + (1/r)e_{,r} + e_{,zz}, \tag{4}$$

which take the following form without dilatation:

$$(\lambda + 2\mu)\left[\Delta - (1/r^2)\right]u_r - (\lambda + \mu)u_{r,zz} + (\lambda + \mu)u_{z,rz} = 0, \tag{5}$$

$$\mu\Delta u_z + (\lambda + \mu)u_{z,zz} + (\lambda + \mu)\left[u_{r,r} + (u_r / r)\right]_{,z} = 0. \tag{6}$$

The first fundamental difference from the problem of inhomogeneous theory is in the constitutive equations:

$$\sigma_{rr}(r,z) = \lambda(r)e(r,z) + 2\mu(r)\varepsilon_{rr}(r,z),$$

$$\sigma_{\vartheta\vartheta}(r,z) = \lambda(r)e(r,z) + 2\mu(r)\varepsilon_{\vartheta\vartheta}(r,z),$$

$$\sigma_{zz}(r,z) = \lambda(r)e(r,z) + 2\mu(r)\varepsilon_{zz}(r,z),$$

$$\sigma_{rz}(r,z) = 2\mu(r)\varepsilon_{rz}(r,z). \tag{7}$$

These formulas indicate that both homogeneous and inhomogeneous elasticity theories are formally identical in the kinematic part of the description of the mechanical state (displacements and strains). In other words, the inhomogeneity of physical properties as the functional dependence of density and elastic characteristics causes stress redistribution, according to the laws of inhomogeneity. Note that in any boundary-value problem, all the mechanical fields vary from point to point, becoming inhomogeneous.

2. The Simplest Problem Modeling the Longitudinal Compression of a Tooth. Consider the classical homogeneous elasticity problem of universal uniaxial tension–compression as applied to the inhomogeneous elasticity theory with radius-dependent elastic parameters. Let such strain occur in a long straight rod with circular cross-section whose axis coincides with the $0z$-axis in the cylindrical coordinate system $0r\vartheta z$. The lateral surface of the cylinder is free of stresses. Under such conditions, the stress–strain state is homogeneous and axisymmetric in all cross-sections except for the areas near the rod ends. The state is characterized by only one nonzero component σ_{zz} of the stress tensor and two nonzero components ε_{rr} and ε_{zz} of the strain tensor (or two principal elongations λ_r and λ_z). As in the case of homogeneous elasticity theory, the displacements, strains, and stresses are independent of the axial coordinate. However, they vary along the radius within all cross-sections identically. Analyzing the problem, we can introduce new elastic parameters corresponding to Young's modulus and Poisson's ratio in the homogeneous elasticity theory. Here the constitutive relations (7) are primary. Under the above conditions, they are simplified:

$$0 = \lambda(r) e^{o}(r) + 2\mu(r)\varepsilon^{o}_{rr}(r),$$

$$\sigma^{o}_{zz} = \lambda(r) e^{o}(r) + 2\mu(r)\varepsilon^{o}_{zz}(r),$$

$$e^{o} = \varepsilon^{o}_{rr} + \varepsilon^{o}_{zz}. \tag{8}$$

It is assumed that the rod undergoes longitudinal deformation of fixed magnitude and constant in the axial coordinate, when the longitudinal displacement varies slightly in comparison with the length and diameter of the rod. This produces the stress # identical along the length and similarly varying along the radius.

Adding the first two equations from (8), we arrive at the expression for dilatation:

$$\sigma^{o}_{zz}(r) = [3\lambda(r) + 2\mu(r)] e^{o}(r) \to e^{o}(r) = \sigma^{o}_{zz}(r) / [3\lambda(r) + 2\mu(r)]. \tag{9}$$

Substituting (9) into the second equation in (8) yields the following relation between the axial stress and strain:

$$\sigma^{o}_{zz}(r) = \left[\lambda(r) / (3\lambda(r) + 2\mu(r))\right]\sigma^{o}_{zz}(r) + 2\mu(r)\varepsilon^{o}_{zz} \to$$

$$\to \sigma^{o}_{zz} = \left[\mu(r)(3\lambda(r) + 2\mu(r)) / (\lambda(r) + \mu(r))\right]\varepsilon^{o}_{zz}. \tag{10}$$

Formula (10) represents the elementary law $\sigma^{o}_{zz}(r) = E(r)\varepsilon^{o}_{zz}$ of the relation between the stress and longitudinal strain of the rod which employs the variable (radius-dependent) Young modulus expressed in the terms of variable Lame moduli:

$$E(r) = \frac{\mu(r)[3\lambda(r) + 2\mu(r)]}{\lambda(r) + \mu(r)}. \tag{11}$$

Substituting the expression for dilatation into the first equation in (8), we obtain

$$-\varepsilon_{zz} = \frac{\lambda(r)}{2(\lambda(r) + \mu(r))}\varepsilon_{rr}(r). \tag{12}$$

This formula demonstrates the classic Poisson law for inhomogeneous materials concerning the relation between the transverse compression of the rod cross-section under longitudinal uniaxial compression and the radial strain. Formula (12) allows introducing a variable Poisson's ratio:

$$\upsilon(r) = -\frac{\varepsilon_{rr}(r)}{\varepsilon_{zz}(r)} = \frac{\lambda(r)}{2(\lambda(r) + \mu(r))} \tag{13}$$

and inhomogeneity laws into the kinematic parameters.

3. Functionally Gradient Model of Inhomogeneous Elasticity Theory. Consider the case where the elastic parameters depend functionally on the coordinate r, while the Lame elastic constants λ and μ are functions

$$\lambda(r) = \lambda_{o} l(r), \quad \mu(r) = \mu_{o} m(r) \quad (\lambda_{o}, \mu_{o} = \text{const}), \tag{14}$$

that are differentiable at least two times. Then formulas (11) and (13) becomes

$$E(r) = \frac{\mu_{o} m(r)[3\lambda_{o} l(r) + 2\mu_{o} m(r)]}{\lambda_{o} l(r) + \mu_{o} m(r)}, \tag{15}$$

$$\upsilon(r) = \frac{\lambda_{o} l(r)}{2[\lambda_{o} l(r) + \mu_{o} m(r)]}. \tag{16}$$

Thus, all the four elastic constants (variable elastic parameters in the inhomogeneous isotropic elasticity theory) vary differently with radius. If the Lame moduli vary similarly, the Young modulus varies in the same way, while Poisson's ratio remains constant. It should be noted that the assumption of the constancy of Poisson's ratio is used rather frequently in inhomogeneous elasticity.

Let us assume that the exponential dependence of the Lame moduli, which is frequently used in the inhomogeneous elasticity theory, is described as

$$\lambda(r)=\lambda_o e^{-lr}, \qquad \mu(r)=\mu_o e^{-mr} \qquad (l,\ m=\text{const}). \tag{17}$$

Then formulas (15) and (16) become simpler:

$$E(r)=\frac{\mu_o(3\lambda_o+2\mu_o e^{-(m-l)r})}{\lambda_o+\mu_o e^{-(m-l)r}}e^{-mr}, \tag{18}$$

$$\upsilon(r)=\frac{\lambda_o}{2(\lambda_o+\mu_o e^{-(m-l)r})}. \tag{19}$$

An analysis of the last two formulas shows that the distinction in the radial dependence of all the elastic parameters remains in the case of exponential dependence. If the difference between the parameters is small (when the difference between the ways the Lame parameters vary with the radius is also small), the same small difference *e* will be for the Young modulus, while Poisson's ratio remains almost constant.

Let us consider some features of the deformation of an inhomogeneous (over the thickness) rod under uniaxial compression. It is assumed that its elastic parameters depend exponentially on the radius and decrease substantially with distance from the outer surface $r=R^o$ as follows according to (17):

$$\lambda(r)=\lambda_o e^{-l(R^o-r)}, \quad \mu(r)=\mu_o e^{-m(R^o-r)} \qquad (l,\ m=\text{const}). \tag{20}$$

The parameters *l* and *m* define inhomogeneity as the difference between the parameters λ^o and μ^o on the outer surface and at the center of the rod:

$$\lambda(0)=\lambda_o e^{-lR^o}, \qquad \mu(0)=\mu_o e^{-mR^o}. \tag{21}$$

Since the Young modulus is strongly dependent on the radius, the compressive stress in the fixed cross-section will also be strongly dependent of the radius:

$$\sigma_{zz}(r)=\frac{\mu_o(3\lambda_o+2\mu_o e^{-(m-l)r})}{\lambda_o+\mu_o e^{-(m-l)r}}e^{-mr}\varepsilon_{zz}(r). \tag{22}$$

The increase in the radius of the rod cross-section (swell) is insignificant and different from that in a homogeneous material:

$$\upsilon(r)=-\frac{\varepsilon_{rr}(r)}{\varepsilon_{zz}(r)}=\frac{\lambda_o}{2(\lambda_o+\mu_o e^{-(m-l)r})}. \tag{23}$$

The above-mentioned features of deformation of a straight circular rod with inhomogeneous mechanical properties under uniaxial compression may be used as a landmark for more adequate models of the deformation of a single-rooted tooth as a complex structure with inhomogeneous properties varying from the surface of the tooth to its middle.

4. Numerical Modeling of the Universal Compressive Deformation of a Rod as Applied to a Single-Rooted Tooth. Let the rod length L = 30 mm and diameter $2R^o$ = 6 mm. Assume that the Lame moduli of the upper layer of the tooth (enamel) and of the layer closest to its middle differ by a factor of 7 to 10 [1, 2]. We also assume that the distinction of the variable Lame moduli in the exponential description is 10% (at l = 0.9 m) and the values of these moduli for the enamel are μ^o and λ^o.

To determine approximately the Young modulus E for the enamel, we employ the experimental curves presented in Fig. 4 from [2]. Approximate recalculation of the curve *1* for stress–strain relationship for the enamel shows that the $\sigma \sim \varepsilon$ relationship remains constant until $\sigma \approx 78$ MPa which corresponds to $\varepsilon \approx 6 \cdot 10^{-4}$. Then $E \approx 13 \cdot 10^{10}$ Pa, which is similar to the modulus of tungsten. If the average statistical value of Poisson's ratio for all layers form the tooth $\upsilon = 0.3$, the Lame moduli $\mu^o = 5 \cdot 10^{10}$, $\lambda^o = 7.5 \cdot 10^{10}$.

With formula (20), we determine the values of the parameters l and m provided that $R^o = 0.003$ m and the moduli at a distance from the surface of the cylinder to its center decrease by a factor of 7.4:

$$\lambda(r) = \lambda_o e^{-l(0.003-r)}, \quad \mu(r) = \mu_o e^{-m(0.003-r)}.$$

If the equalities $e^{-0.003l} = 0.1$ and $e^{-0.003m} = 0.1$ are satisfied, then

$$\lambda(r) = \lambda_o e^{-741(0.003-r)}, \quad \mu(r) = \mu_o e^{-667(0.003-r)}, \tag{24}$$

$$E(r) = \frac{\mu_o (3\lambda_o + 2\mu_o e^{74(0.003-r)})}{\lambda_o + \mu_o e^{74(0.003-r)}} e^{-667(0.003-r)}, \tag{25}$$

$$\upsilon(r) = \frac{\lambda_o}{2(\lambda_o + \mu_o e^{74(0.003-r)})}. \tag{26}$$

Formulas (25) and (26) make it possible to determine the decrease in two parameters important in engineering: Young's modulus decreases by a factor of 7.55 (from $E \approx 13 \cdot 10^{10}$ to $1.72 \cdot 10^{10}$), while Poisson's ratio decreases by a factor of 0.09 (from $\nu \approx 0.3$ to 0.273). Thus, the expansion of the cylinder in the transverse direction under compression is rather small across the cylinder thickness, while the variation in Young's modulus in the radial direction is substantial.

5. More General Models of Inhomogeneous Elasticity Theory in Tension–Compression Problems for a Cylinder within the Framework of the Theory of Functionally Graded Materials. Consider a circular cylinder under certain constraints. The cylinder is acted upon by forces applied at the ends, while the lateral surface is free of loads. It is assumed that the rod is transversely isotropic and its anisotropy axis coincides with that of the rod as well as with the $0z$-axis in the cylindrical coordinate system $0r\vartheta z$. Moreover, we assume that the material characteristics vary only in the cross-sectional plane (isotropy plane), are constant along the length, and depend on the radial coordinate only.

The general system of equations of the inhomogeneous linear elasticity theory becomes [3, 18]:

Cauchy's relations

$$\varepsilon_{rr}(r,z) = u_{r,r}(r,z), \quad \varepsilon_{\vartheta\vartheta}(r,z) = [u_r(r,z)/r], \quad \varepsilon_{zz}(r,z) = u_{z,z}(r,z),$$

$$\varepsilon_{rz}(r,z) = (1/2)(u_{r,z}(r,z) + u_{z,r}(r,z)), \quad \varepsilon_{r\vartheta}(r,z) = 0, \quad \varepsilon_{\vartheta z}(r,z) = 0,$$

$$e(r,z) = \varepsilon_{rr}(r,z) + \varepsilon_{\vartheta\vartheta}(r,z) + \varepsilon_{zz}(r,z) = u_{r,r}(r,z) + u_{z,z}(r,z) + (u_r(r,z)/r) = (1/r)[ru_r(r,z)]_{,r} + u_{z,z}(r,z). \tag{27}$$

If the elastic parameters are radius-dependent (which corresponds to the assumption of problem axisymmetry), the constitutive equations for a transversely isotropic material are valid.

Remark 2. Although the five independent elastic parameters $C_{IK}(r)$ of a transversely isotropic material are represented in the form of a matrix:

$$C_{IK} = \begin{vmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (1/2)(C_{11} - C_{12}) \end{vmatrix},$$

these equations are most commonly written as dependence of strains on stresses:

$$\varepsilon_{rr}(r,z)=A_{rr}(r)\sigma_{rr}(r,z)+A_{r\vartheta}(r)\sigma_{\vartheta\vartheta}(r,z)+A_{rz}(r)\sigma_{zz}(r,z),$$

$$\varepsilon_{\vartheta\vartheta}(r,z)=A_{\vartheta r}(r)\sigma_{\vartheta r}(r,z)+A_{\vartheta\vartheta}(r)\sigma_{\vartheta\vartheta}(r,z)+A_{\vartheta z}(r)\sigma_{zz}(r,z),$$

$$\varepsilon_{zz}(r,z)=A_{zr}(r)\sigma_{rr}(r,z)+A_{z\vartheta}(r)\sigma_{\vartheta\vartheta}(r,z)+A_{zz}(r)\sigma_{zz}(r,z), \tag{28}$$

$$\varepsilon_{r\vartheta}(r,z)=G_{r\vartheta}(r)\sigma_{r\vartheta}(r,z),$$

$$\varepsilon_{\vartheta z}(r,z)=G_{\vartheta z}(r)\sigma_{\vartheta z}(r,z),$$

$$\varepsilon_{rz}(r,z)=G_{rz}(r)\sigma_{rz}(r,z). \tag{29}$$

Of the nine mechanical parameters $A_{rr}(r),\ldots,G_{rz}(r)$, only five of them are independent and should be determined experimentally. Written in the terms of technical elastic parameters, Eqs. (28) and (29) become

$$\varepsilon_{rr}(r,z)=\frac{1}{E(r)}\sigma_{rr}(r,z)-\frac{\nu(r)}{E(r)}\sigma_{\vartheta\vartheta}(r,z)+\frac{\nu'(r)}{E'(r)}\sigma_{zz}(r,z),$$

$$\varepsilon_{22}=-\frac{\nu(r)}{E(r)}\sigma_{11}+\frac{1}{E(r)}\sigma_{22}-\frac{\nu'(r)}{E'(r)}\sigma_{33},$$

$$\varepsilon_{33}=-\frac{\nu'}{E'}\sigma_{11}-\frac{\nu'}{E'}\sigma_{22}+\frac{1}{E'}\sigma_{33}, \tag{30}$$

$$\varepsilon_{r\vartheta}(r,z)=\frac{1}{G'(r)}\sigma_{r\vartheta}(r,z),$$

$$\varepsilon_{zr}(r,z)=\frac{1}{G(r)}\sigma_{zr}(r,z),$$

$$\varepsilon_{\vartheta z}(r,z)=\frac{1}{G(r)}\sigma_{\vartheta z}(r,z), \tag{31}$$

where E and G are Young's and shear moduli for any direction in the isotropy plane; ν is Poisson's ratio which characterizes the decrease in the rod cross-section in the isotropy plane when subject to tension in the same plane; E' and G' are Young's and shear moduli in the direction perpendicular to the isotropy plane; ν' is Poisson's ratio characterizing the decrease in the rod cross-section in the isotropy plane when subject to tension in the direction perpendicular to the isotropy plane; ν'' is Poisson's ratio characterizing the decrease in the rod cross-section in the direction perpendicular to the isotropy plane when subject to tension in the direction perpendicular to the isotropy plane and the identities $\nu'/E''=\nu''/E$ and $E=2G(1+\nu)$ hold.

Thus, the transversely isotropic material is defined by the five independent technical elastic parameters E,G,E',G',ν'.

It should be noted that the technical elastic parameters are usually used in the mechanics of materials. In solving elasticity problems, the classical elastic parameters are more convenient. The classical and technical parameters are related as

$$c_{1111}=C_{11}=\frac{1-(\nu')^2(E/E')}{1-\nu^2+(1+2\nu)(\nu')^2(E/E')}E,$$

$$c_{1122}=C_{12}=\frac{\nu-(\nu')^2(E/E')}{1-\nu^2+(1+2\nu)(\nu')^2(E/E')}E,$$

$$c_{1133}=C_{13}=\frac{\nu'(1-\nu)}{1-\nu^2+(1+2\nu)(\nu')^2(E/E')}E,$$

$$c_{3333}=C_{33}=\frac{1-(\nu)^2}{1-\nu^2+(1+2\nu)(\nu')^2(E/E')}E',$$

$$c_{1111}-c_{1122}=C_{44}=\frac{1-\nu}{1-\nu^2+(1+2\nu)(\nu')^2(E/E')}E=(1/2)G,$$

$$c_{2323}=c_{3131}=(1/2)G'. \tag{32}$$

The constitutive equations of the inhomogeneous elasticity theory including all the independent elastic parameters $C_{11}(r), C_{12}(r), C_{13}(r), C_{33}(r)$, and $C_{44}(r)$ become:

$$\sigma_{rr}(r,z)=C_{11}(r)\varepsilon_{rr}(r,z)+C_{12}(r)\varepsilon_{\vartheta\vartheta}(r,z)+C_{13}(r)\varepsilon_{zz}(r,z),$$

$$\sigma_{\vartheta\vartheta}(r,z)=C_{12}(r)\varepsilon_{rr}(r,z)+C_{11}(r)\varepsilon_{\vartheta\vartheta}(r,z)+C_{13}(r)\varepsilon_{zz}(r,z),$$

$$\sigma_{zz}(r,z)=C_{13}(r)\varepsilon_{rr}(r,z)+C_{13}(r)\varepsilon_{\vartheta\vartheta}(r,z)+C_{33}(r)\varepsilon_{zz}(r,z),$$

$$\sigma_{rz}(r,z)=2C_{44}(r)\varepsilon_{rz}(r,z). \tag{33}$$

The system of equations of the axisymmetric inhomogeneous elasticity theory is closed by the system of equations of motion (3).

In what follows, we will consider two problem-solving methods commonly used in the elasticity theory: (i) the complete system of equations is written in terms of displacements (Lame-type system of equations) and (ii) the complete system of equations is represented by the Beltrami–Michell equation for stresses.

5.1. Problem-Solving Method Based on Analysis of Equations for Displacements (Lame-type Equations). To derive the Lame-type system of equations, we will substitute the Cauchy relations and constitutive equations (28)–(31) into the equilibrium equations (6). This transformation from the equilibrium equations (3) with unknown stresses to the Lame-type system of equations with unknown displacements is complicated due to the necessity to differentiate the radius-dependent elastic parameter functions $c_{11}(r), c_{12}(r), c_{13}(r), c_{33}(r), c_{44}(r)$. This system includes two equations:

$$c_{11}(r)(u_{r,rr}+(1/r)u_{r,r}-(1/r^2)u_r)+\underline{c_{11,r}(r)u_{r,r}}+\underline{c_{12,r}(r)(1/r)}+c_{44}(r)u_{r,zz}$$

$$+(c_{13}(r)+c_{44}(r))u_{z,rz}+\underline{c_{13,r}(r)u_{z,z}}=0, \tag{34}$$

$$(c_{13}(r)+c_{44}(r))(u_{r,rz}+(1/r)u_{r,z})+\underline{c_{44,r}(r)u_{r,z}}$$

$$+c_{44}(r)\Delta u_z+(c_{33}(r)-c_{44}(r))u_{z,zz}+\underline{c_{44,r}(r)u_{z,r}}=0. \tag{35}$$

The five terms underlined in (34) and (35) are those new terms that are absent in the Lame equations of the homogeneous elasticity theory and describe special features of the inhomogeneous theory. They contain the derivatives of elastic parameters that are assumed to depend on the radius.

Let us introduce the Love function (potential) to reduce Eqs. (34) and (35) to a new equation for the Love function. To this end, it is necessary to repeat the classical way from the system of equations (34) and (35) for the two unknown functions $u_r(r,z)$ and $u_z(r,z)$ to the analysis of Love functions. At the first step, two new functions are introduced:

$$u_r(r,z)=R_{,r}(r,z), \qquad u_z(r,z)=Z(r,z). \tag{36}$$

Substituting Eqs. (36) into system (34), (35), we get

$$\{c_{11}(r)\Delta R_{,r}+\underline{c'_{11}(r)}R_{,rr}+\underline{c'_{12}(r)(1/r)R_{,r}}-[c_{11}(r)-c_{44}(r)]R_{,rzz}\}$$

$$+(c_{13}(r)+c_{44}(r))Z_{,rz}+\underline{c'_{13}(r)Z_{,z}}=0, \tag{37}$$

$$(c_{13}(r)+c_{44}(r))(\Delta R-R_{,zz})+\underline{c'_{44}(r)R_{,rz}}$$

$$+c_{44}(r)\Delta Z+(c_{33}(r)-c_{44}(r))Z_{,zz}+\underline{c'_{44}(r)Z_{,r}}=0. \tag{38}$$

Remark 3. System (37) corresponds to that of the homogenous theory. If the functions $c_{11}(z), c_{12}(z), c_{13}(z), c_{44}(z)$ are transformed to constants, the second system follows from the first one (the underlined terms are equal to zero).

Let us introduce a Love-type function $\chi(r,z)$ in a standard way. Then Eqs. (37) are satisfied identically by introducing the Love function

$$R = -[c_{13}(r)+c_{44}(r)]\chi_{,rz} - \underline{c'_{13}(r)\chi_{,z}},$$

$$Z = c_{11}(r)\Delta\chi_{,r} + \underline{c'_{11}(r)}\chi_{,rr} + \underline{c'_{12}(r)(1/r)}\chi_{,r} - (c_{11}(r)-c_{44}(r))\chi_{,rz}, \tag{39}$$

while Eqs. (38) are transformed to the equations

$$-\left\{\left[(c_{44}(z)+c_{13}(z))(\Delta\chi_{,z} - \chi_{,zzz})\right] + \cancel{c'_{13}(z)(\Delta\chi - \chi_{,zz})}\right\}$$

$$\times\left[(c_{13}(z)+c_{44}(z))\chi_{,z} + \cancel{c'_{44}(z)\chi}\right]$$

$$+\{c_{44}(z)(\Delta\chi - \chi_{,zz})\Delta + c_{33}(z)\chi_{,zz} + \cancel{c'_{33}(z)\chi_{,z}}\}$$

$$\times\{c_{11}(z)(\Delta\chi - \chi_{,zz}) + c_{44}(z)\chi_{,zz} + \cancel{c'_{44}(z)\chi_{,z}}\} = 0 \tag{40}$$

for determining the Love functions.

After laborious transformations, Eq. (40) takes a form that has new terms compared with the similar equation for an isotropic material [3]:

$$\Delta\Delta\chi_{,r} + k_{5rz}(r)\chi_{,rzzzz} + k_{\Delta 3rz}\Delta\chi_{,rzz} + k_{4rz}(r)\chi_{,rrzz} + k_{\Delta\Delta}(r)\Delta\Delta\chi + k_{\Delta r}(r)\Delta\chi_{,rr}$$

$$+k_{3rz}(r)\chi_{,rzz} + k_{\Delta}(r)\Delta\chi + k_{2z}(r)\chi_{,zz} + k_{2r}(r)\chi_{,rr} + k_{1r}(r)\chi_{,r} = 0, \tag{41}$$

where

$$k_{5rz} = -\frac{[c_{33}(r)-c_{44}(r)][c_{11}(r)-c_{44}(r)]}{c_{44}(r)c_{11}(r)},$$

$$k_{\Delta 3rz} = -\frac{2c_{44}(r)[c_{11}(r)+c_{13}(r)] + c_{11}(r)c_{33}(r) - [c_{13}(r)]^2}{c_{44}(r)c_{11}(r)},$$

$$k_{4rz}(r) = -2\frac{\lambda'(r)+\mu'(r)}{\lambda(r)+2\mu(r)}, \qquad k_{\Delta\Delta}(r) = \frac{\lambda'(r)}{\lambda(r)+2\mu(r)},$$

$$k_{\Delta r}(r) = \frac{\lambda(r)\mu'(r) + 2\lambda(r)\lambda'(r) + 3\mu(r)\lambda'(r) + 8\mu(r)\mu'(r)}{\mu(r)[\lambda(r)+2\mu(r)]},$$

$$k_{3rz}(r) = -\frac{3\mu(\lambda''-\mu'') + 4\lambda'\mu' + \mu(\lambda'+\mu')\frac{1}{r} + (\mu+\lambda)(\lambda+2\mu)\frac{1}{r^2}}{\mu(r)[\lambda(r)+2\mu(r)]},$$

$$k_{\Delta}(r) = \frac{\mu(r)\lambda'''(r) + \mu'(r)\lambda''(r) + \frac{1}{r}\mu(r)\lambda''(r)}{\mu(r)[\lambda(r)+2\mu(r)]}, \qquad k_{2z}(r) = -\frac{\mu(r)\lambda'''(r) + \frac{1}{r}\mu(r)\lambda''(r) - 2\mu'(r)\lambda''(r)}{\mu(r)[\lambda(r)+2\mu(r)]},$$

$$k_{2r}(r) = 2\frac{\mu'\mu'' + (\lambda\mu' + 2\mu\lambda' + 4\mu\mu') + \mu\mu''' + \frac{1}{r}\mu\mu''}{\mu(r)[\lambda(r)+2\mu(r)]},$$

$$k_{1r}(r) = -\frac{\mu(3\lambda'' + 2\mu'')\frac{1}{r^2} + \mu'(\lambda' + 2\mu')\frac{1}{r^2} + 3\mu(\lambda' + 2\mu')\frac{1}{r^3}}{\mu(r)[\lambda(r) + 2\mu(r)]}. \tag{42}$$

Remark 4. Of the ten coefficients appearing in (42), only the two first ones (k_{5rz}, $k_{\Delta 3rz}$) are independent of the derivatives of the elastic parameters. Of the eight remaining coefficients, only the sixth one $k_{3rz}(r)$ is nonzero in going to the homogeneous elasticity theory. These three coefficients define the part of Eq. (42) obtained in the homogeneous elasticity theory. The other seven coefficients include derivatives linearly and are equal to zero in the homogeneous elasticity theory. In this case, the third, fourth, and fifth coefficients $k_{4rz}(r)$, $k_{\Delta\Delta}(r)$, and $k_{\Delta r}(r)$ appearing in (42) contain only the first derivative, the tenth coefficient $k_{1r}(r)$ contains the first and second derivatives, the seventh, eighth, and ninth coefficients $k_{\Delta}(r)$, $k_{2z}(r)$, and $k_{2r}(r)$ include the first, second, and third derivatives.

Equation (41) contains a Laplacian. Sometimes, in applying the procedure of separation of variables, it is convenient to expand the Laplacian as $\Delta u = u_{,rr} + (1/r)u_{,r} + u_{,zz} = \Delta_r u + u_{,zz}$. Then the operators containing the Laplacian become:

$$\Delta\Delta u = \Delta_r \Delta_r u + 2\Delta_r u_{,zz} + u_{,zzzz},$$

$$\Delta u_{,rr} = \Delta_r u_{,rr} + u_{,rrzz}, \qquad \Delta u_{,r} = \Delta_r u_{,r} + u_{,rzz}$$

and Eq. (41) takes the form

$$\begin{gathered}\Delta_r \Delta_r \chi_{,r} + 2\Delta_r \chi_{,rzz} + \chi_{,rzzzz} + (k_{4rz}(r) + k_{\Delta 2r}(r))\chi_{,rrzz} \\ + k_{\Delta\Delta}(r)(\Delta_r \Delta_r \chi + 2\Delta_r \chi_{,zz} + \chi_{,zzzz}) \\ + k_{\Delta 2r}(r)\Delta_r \chi_{,rr} + k_{\Delta r}\Delta_r \chi_{,r} + (k_{3rz}(r) + k_{\Delta r})\chi_{,rzz} \\ + k_{\Delta}(r)\Delta_r \chi + (k_{2z}(r) + k_{\Delta}(r))\chi_{,zz} + k_{2r}(r)\chi_{,rr} + k_{1r}(r)\chi_{,r} = 0.\end{gathered} \tag{43}$$

This equation can also be written in the form of three groups, where the first one contains operators with respect to *r* (first and second rows), the second group contains only operators with respect to *z* (third row), and the third one contains mixed operators (fourth row):

$$\begin{gathered}\Delta_r \Delta_r \chi_{,r} + k_{\Delta\Delta}(r)\Delta_r \Delta_r \chi + k_{\Delta 2r}(r)\Delta_r \chi_{,rr} + k_{\Delta r}\Delta_r \chi_{,r} \\ + k_{\Delta}(r)\Delta_r \chi + k_{2r}(r)\chi_{,rr} + k_{1r}(r)\chi_{,r} \\ + k_{\Delta\Delta}(r)(2\Delta_r \chi_{,zz} + \chi_{,zzzz}) + (k_{2z}(r) + k_{\Delta}(r))\chi_{,zz} \\ + 2\Delta_r \chi_{,rzz} + \chi_{,rzzzz} + (k_{4rz}(r) + k_{\Delta 2r}(r))\chi_{,rrzz} + (k_{3rz}(r) + k_{\Delta r}(r))\chi_{,rzz} = 0.\end{gathered} \tag{44}$$

Remark 5. The complication appearing in Eqs. (41), (43), and (44) compared with the equation in the homogeneous elasticity theory is typical for the inhomogeneous theory [3, 4, 18, 28].

Thus, any problem can be solved in several steps. First, we find the solution of Eq. (44) with a specific potential $\chi(r, z)$. Next, considering this potential and formulas (36), we determine the displacements $u_r(r, z)$ and $u_z(r, z)$. With (27), we determine the components of the strain tensor $\varepsilon_{rr}(r, z)$, $\varepsilon_{\vartheta\vartheta}(r, z)$, $\varepsilon_{zz}(r, z)$, and $\varepsilon_{rz}(r, z)$. At the last step, with (33) we find the stresses $\sigma_{rr}(r, z)$, $\sigma_{\vartheta\vartheta}(r, z)$, $\sigma_{zz}(r, z)$, and $\sigma_{rz}(r, z)$. In so doing, it is necessary to keep in mind that the constitutive relations (33) additionally include the radius-dependences of the elastic parameters.

5.2. Problem-Solving Method Based on Analysis of the Equations for Stresses. In solving the equilibrium problem using the above algorithm for the cylindrical rod formulated in Sec. 5, a potential (function) in terms of which stresses are expressed is introduced in the homogeneous elasticity theory. The same procedure can be employed in the case of inhomogeneous axisymmetric elasticity with respect to the stress tensor $\sigma = (\sigma_{rr}, \sigma_{\vartheta\vartheta}, \sigma_{zz}, \sigma_{rz}, \sigma_{r\vartheta} = 0, \sigma_{\vartheta z} = 0)$. A special feature of this way for the inhomogeneous elasticity theory is very useful due to the fact that the procedure of introduction of the potential is somewhat different.

In what follows, we will consider the equilibrium problem for a hollow cylindrical rod of finite length $0 < z < Z_L$ and $r_0 < r < r_1$. The rod is subject to compression by constant forces *P* applied to the ends. This problem has a special feature: the mechanical fields in cross-sections in a very long rod are identical, and the analysis is reduced to a plane axisymmetric problem in which all quantities depend on the radius alone.

The equilibrium equations are simplified and contain only Poisson's ratio that characterize shear in the isotropy plane:

$$\Delta_r \sigma_{rr} - \frac{2}{r^2}(\sigma_{rr} - \sigma_{\vartheta\vartheta}) + \frac{1}{1+\nu}\hat{\sigma}_{,rr} = 0,$$

$$\Delta_r \sigma_{\vartheta\vartheta} + \frac{2}{r^2}(\sigma_{rr} - \sigma_{\vartheta\vartheta}) + \frac{1}{1+\nu}\left[\frac{1}{r}\hat{\sigma}_{,r}\right] = 0,$$

$$\Delta_r \sigma_{zz} + \frac{1}{1+\nu}\hat{\sigma}_{,zz} = 0. \tag{45}$$

The problem can be solved with the classical method by introducing a potential [3,6, 7, 24, 25]:

$$\sigma_{rr} = \frac{1}{r}F_{,r}, \quad \sigma_{\vartheta\vartheta} = F_{,rr}, \quad \sigma_{zz} = \frac{1}{\nu}(C - a(\sigma_{rr} - \sigma_{\vartheta\vartheta})). \tag{46}$$

The potential $\varphi(r) = F'(r)$ should be determined from the Beltrami–Michell equation [3, 6, 7, 24, 25], which in this case is transformed to an ordinary differential inhomogeneous equation with variable radius-dependent coefficients and right-hand side

$$\varphi_{,rr} + \left(\frac{1}{r} + \frac{B_{12}}{B_{13}}\right)\varphi_{,r} + \left(\frac{B_{12}}{B_{22}r} - \frac{B_{11}}{B_{22}r^2}\right)\varphi = \frac{C}{B_{22}r}\left[\frac{A_{12}}{A_{33}} - \left(\frac{A_{23}}{A_{33}}r\right)_{,r}\right]. \tag{47}$$

The coefficients are defined by

$$\frac{A_{12}}{A_{33}} = \frac{E'(r)}{E(r)}, \quad \frac{A_{23}}{A_{33}} = \frac{E'(r)}{E(r)}\nu'(r),$$

$$\frac{B_{12}}{B_{22}} = -\frac{\nu + \nu'^2}{1 - \nu'^2}, \quad \frac{B_{12}}{B_{13}} = \frac{B_{11}}{B_{22}} = 1. \tag{48}$$

With (48), Eq. (47) becomes

$$\varphi_{,rr} + \left(\frac{1}{r} + 1\right)\varphi_{,r} + \left(\frac{\nu(r) + (\nu'(r))^2}{1 - (\nu'(r))^2}\frac{1}{r} - \frac{1}{r^2}\right)\varphi = \frac{C}{r}\left[\frac{E'(r)}{E(r)} - \left(\frac{E'(r)}{E(r)}\nu'(r)r\right)_{,r}\right]. \tag{49}$$

As can be seen, this equation includes only four elastic parameters, while in the case of constant Poisson's ratio, the solution of the homogeneous equation is independent of the variable elastic parameters. However, these parameters will be present in the general solution since the right-hand side of the equation depends on the ratio of tensile moduli $E'(r)/E(r)$ in the isotropy plane and in the direction of the symmetry axis.

The solution of Eq. (49) takes the form

$$\varphi(r) = K_1\varphi_1(r) + K_2\varphi_2(r) + K_0\varphi_0(r), \tag{50}$$

where $\varphi_1(r)$ and $\varphi_2(r)$ are the general solutions of the inhomogeneous equation (49); $\varphi_0(r)$ is a partial solution of the inhomogeneous equation (49). The three arbitrary constants K_0, K_1, and K_2 are determined with allowance for the boundary condition that assumes absence of the normal stresses on the lateral surface of the cylinder

$$\sigma_{rr}(r_o) = 0 \tag{51}$$

in the case of presence of integral condition at the cylinder ends, which is written as follows:

$$\int_{r_0}^{r_1} \sigma_{zz}(r) r dr = (1/2\pi)P. \tag{52}$$

Next, we will consider a simpler variant of an isotropic material. In this case, Eq. (49) becomes

$$\varphi_{,rr} + \frac{1+r}{r}\varphi_{,r} + \left(\frac{\nu(r)}{1-\nu(r)}\frac{1}{r} - \frac{1}{r^2}\right)\varphi = \frac{K_0}{r}\left[1-(\nu(r)r)_{,r}\right]. \tag{53}$$

If Poisson's ratio is constant, Eq. (53), since $K_0/r[1-(\nu r)_{,r}]=0$, is transformed from inhomogeneous into homogeneous:

$$r^2\varphi_{,rr} + (r^2+r)\varphi_{,r} + \left(\frac{\nu}{1-\nu}r - 1\right)\varphi = 0. \tag{54}$$

Equation (54) is the classical Bessel-type equations. If its solution is substituted into (46), only the stress $\sigma_{zz}(r)$ becomes nonzero. Moreover, the equality $K_1 = K_2 = 0$ follows from the boundary conditions.

In what follows, we will consider the variant of a transversely isotropic material and Eqs. (49) where the elastic parameters E, G, E', G', ν' exponentially depend on the radius as

$$E(r) = E_0 e^{-mr}, \quad G(r) = G_0 e^{-mr}, \quad E'(r) = E'_0 e^{-mr},$$

$$G'(r) = G'_0 e^{-mr}, \quad \nu'(r) = \nu'_0 e^{-mr}. \tag{55}$$

If Poisson's ratios are constant, then

$$\nu'(r) = \nu'_0 = \text{const}, \quad \nu(r) = (E_0 - 2G_0)/2G_0 = \text{const}. \tag{56}$$

Since

$$E'(r)/E(r) = E'_0/E_0 = \text{const}, \tag{57}$$

to determine the potential, it is necessary to solve the linear equation with variable coefficients

$$r^2\varphi_{,rr} + r(r+1)\varphi_{,r} + \left(\frac{\nu + (\nu')^2}{1-(\nu')^2}r - 1\right)\varphi = 0, \tag{58}$$

which is somewhat complicated compared with formula (54) for an isotropic material.

The solutions of Eqs. (54) are described in various handbooks devoted to differential equations (including [11]). For example, the equation

$$r^2\varphi_{,rr} + r(r+1)\varphi_{,r} + (3r-1)\varphi = 0 \tag{59}$$

has the solution

$$\varphi(r) = r(r-3)e^{-r}\left[C_1 + C_2\int \frac{e^r dr}{r^3(r-3)^2}\right]. \tag{60}$$

The stresses are determined by

$$\sigma_{rr}(r) = \frac{1}{r}\varphi(r), \quad \sigma_{\theta\theta}(r) = \varphi_{,r}(r),$$

$$\sigma_{zz} = \frac{1}{\nu}(C - a(\sigma_{rr}(r) - \sigma_{\theta\theta}(r))). \tag{61}$$

If the stresses are known, the strains are defined by formulas (30) and (31), while the displacements are determined by well-known elasticity formulas.

Conclusions. Analysis of the simple problem of the compression of a rod under tension–compression shows that the stress considerably decreases with decrease in the Lame moduli with distance from the surface to the rod center, while the change in the cross-section remains almost constant in all cross-sections. The model adopted does not describe change in the tooth intersection under compression, but demonstrates substantial decrease in the compressive stress in the tooth.

From the analysis of two more complicated equilibrium problems for a cylindrical transversely isotropic body with radius-dependent elastic parameters, it follows that it is necessary to use more complicated models and mathematical apparatus under similar statement. Within the framework of both models, we have obtained theoretical representations of potentials in terms of which all the mechanical fields including stresses, strains, and displacements are determined. It has been established that all the fields strongly depend on the radial coordinate.

Thus, applying the inhomogeneous elasticity theory to the analysis of the stress state of a tooth, expressing its real inhomogeneity in the radial direction, is promising for a number of specific problems.

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